J. OPERATOR THEORY
61:2(2009), 239–251

ON THE MULTIPLICITY OF SINGULAR VALUES OF HANKEL OPERATORS WHOSE SYMBOL IS A CAUCHY TRANSFORM ON A SEGMENT

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Communicated by Nikolai K. Nikolski

ABSTRACT. We derive a result on the boundedness of the multiplicity of the singular values for Hankel operator, whose symbol is of the form

$$F(z) := \int \frac{\mathrm{d}\lambda(t)}{z-t} + R(z),$$

where λ is a complex measure with infinitely many points in its support which is contained in the interval $(-1,1)$, and whose argument has bounded variation there, while R is a rational function with all its poles inside of the unit disk. For that we use results on the zero distribution of polynomials satisfying the orthogonality relations of the form

$$\int t^j q_n(t) Q(t) \frac{w_n(t)}{\widetilde{q}_n^2(t)} \mathrm{d}\lambda(t) = 0, \quad j = 0, \dots, n-s-1,$$

where Q is the denominator of R, $s = \deg(Q)$, $\widetilde{q}_n(z) = z^n \overline{q_n(1/\overline{z})}$ is the reciprocal polynomial of q, and $\{w_n\}$ is the outer factor of an n-th singular vector of $\mathcal{H}_F$.

KEYWORDS: *Hankel operators, singular values, Cauchy transform, meromorphic approximation.*

MSC (2000): 47B35, 30E25, 41A30.

1. INTRODUCTION

In the most general setting a Hankel operator is an operator acting on ℓ_2 given in the canonical basis by a matrix of the form $\{\alpha_{j+k}\}_{j,k\geqslant 0}$ with $\alpha_j \in \mathbb{C}$. Such a definition admits numerous realizations which, in turn, imply a wide range of applications of Hankel operators. In particular, they appeared to be an extremely important class of operators in approximation theory. The elaboration of the properties of Hankel operators from the approximation view point initiated with the celebrated AAK-Theory that showed the link between meromorphic approximation of L^∞ functions and singular numbers of the corresponding Hankel operators ([1], see also Chapter 4 of [16]). Later, this theory was generalized to L^p functions on the unit circle, $2 \leqslant p \leqslant \infty$ (see [5], [20], and [8]), and to more general domains of approximation (see [19]). Moreover, these methods turned out to be instrumental for investigating the degree of rational approximation of analytic functions (see [17], [12], and [21]) and helped to describe classes of analytic functions in the disk (Besov spaces) in terms of the rate of rational approximation (see [14], [15], [13], and [22]). In another connection, Hankel operators also play a significant role in operator theory. In particular, G. Pisier [18] (see also Theorem 15.3.1 in [16]) showed that there are polynomially bounded operators on a Hilbert space that are not similar to a contraction by using Hankel operators techniques. Further, it was shown that geometric problems in the theory of stationary Gaussian processes can be reduced to the question of describing those bounded linear operators on a Hilbert space that are unitarily equivalent to Hankel operators. A program of N.K. Nikolski to characterize such bounded linear operators in spectral terms (see [10]) was successfully completed in the self-adjoint case (see [9]). In the course of the proof it was shown that the absolute value of the difference of the multiplicities of symmetric eigenvalues of a Hankel operator (self-adjoint or not) is bounded by one. Nevertheless the question of the boundedness of the multiplicities themselves remained open. The modest objective of the present paper is to prove that the multiplicities of the singular values of Hankel operators whose symbol is the Cauchy transform of a complex measure with argument of bounded variation is bounded in terms of that variation. If moreover the measure is sufficiently nonvanishing, the singular values are asymptotically simple.

This paper is organized as follows. In the next section we introduce some notation and state the main results. The third section is devoted to known results that are crucial for the proofs that are given in the last section.

2. STATEMENTS OF THE RESULTS

Denote by $\mathrm{Hol}(D)$ the set of analytic functions on a domain $D \subset \mathbb{C}$. Among these functions we shall distinguish some special classes, namely, the Hardy spaces. Let $L^p(\mathbb{T}_r)$ stand for the space of p-summable functions on $\mathbb{T}_r := \{|z| = r : \ z \in \mathbb{C}\}$, $r > 0$, with the usual norm

$$\|h\|_{p,r}^p := \frac{1}{2\pi} \int_{\mathbb{T}} |h(r\xi)|^p |\mathrm{d}\xi| < \infty, \quad \text{if } \ p \in [1, \infty),$$
$$\|h\|_{\infty,r} := \operatorname{ess.sup}_{\xi \in \mathbb{T}} |h(r\xi)| < \infty, \quad \text{if } \ p = \infty.$$

Hereafter, we shall omit the subindex r for the case of the unit circle, $\mathbb{T}$. The Hardy space of exponent p, $p \in [1, \infty]$, of the open unit disk, $\mathbb{D}$, and the complement of the closed unit disk, $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$, are defined as

$$
\begin{aligned}
(2.1) \qquad H^p &:= \Big\{ h \in \mathrm{Hol}(\mathbb{D}) : \sup_{r<1} \|h\|_{p,r} < \infty \Big\}, \\
\overline{H}^p &:= \Big\{ h \in \mathrm{Hol}(\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}) : \sup_{r>1} \|h\|_{p,r} < \infty \Big\},
\end{aligned}
$$

respectively. Sometimes it is necessary to consider only those functions from $\overline{H}^p$ that vanish at infinity. We shall denote this subspace of $\overline{H}^p$ by $\overline{H}^p_0$.

By the Fatou theorem any function from H^p or $\overline{H}^p$, $p \in [1, \infty]$, has nontangential boundary values almost everywhere on $\mathbb{T}$, which define the *trace function*. The trace of any such function belongs to $L^p(\mathbb{T})$ for the corresponding index p, uniquely defines the function, and has the $L^p(\mathbb{T})$ norm equal to the supremum in (2.1). Therefore we may treat Hardy spaces as special classes of integrable functions on $\mathbb{T}$. In particular, we have that H^2, $\overline{H}^2 \subset L^2(\mathbb{T})$ and $L^2(\mathbb{T}) = H^2 \oplus \overline{H}^2_0$.

Throughout the paper the capital letters H and K shall be reserved for the notation of Hilbert spaces. Further, $\mathcal{L}(H; K)$ will stand for the space of linear operators from H to K.

Now we are ready to give a formal definition of Hankel operators acting on the Hardy class H^2. Let $f \in L^\infty(\mathbb{T})$. The *Hankel operator with symbol f*, denoted by $\mathcal{H}_f \in \mathcal{L}(H^2; \overline{H}^2_0)$, is defined by the rule

$$\mathcal{H}_f(h) := \mathcal{P}_-(fh),$$

where $\mathcal{P}_-$ is the antianalytic projection, i.e. the projection of $L^2(\mathbb{T})$ onto $\overline{H}^2_0$.

For $n \in \mathbb{Z}_+$, the *n-th singular number* of the operator $\mathcal{H}_f$ is defined as

$$s_n(\mathcal{H}_f) := \inf\{\|\mathcal{H}_f - \mathcal{O}\| : \mathcal{O} : H^2 \to \overline{H}^2_0 \text{ a linear operator of rank } \leqslant n\},$$

where $\|\cdot\|$ stands for the operator norm between two Hilbert spaces. Clearly $\{s_n(\mathcal{H}_f)\}_{n\in\mathbb{N}}$ is nonincreasing sequence. By $s_\infty(\mathcal{H}_f)$ we shall denote the distance from $\mathcal{H}_f$ to compact operators, i.e.

$$s_\infty(\mathcal{H}_f) = \lim_{n\to\infty} s_n(\mathcal{H}_f).$$

By the well-known theory of E. Schmidt (Theorem 7.1.1, Vol. I of [11]), s is a singular number of a compact operator $\mathcal{O} \in \mathcal{L}(H; K)$ if and only if s^2 is an eigenvalue of the operator $\mathcal{O}^*\mathcal{O}$, where $\mathcal{O}^*$ is the *adjoint* operator to $\mathcal{O}$.

Although $\{s_n(\mathcal{H}_f)\}$ is nonincreasing, it is not necessarily strictly decreasing. Let $\mu_n(\mathcal{H}_f)$ stand for the multiplicity of $s_n(\mathcal{H}_f)$, i.e. $\mu_n(\mathcal{H}_f)$ is an integer such that there exist constants $k, m \in \mathbb{Z}_+$ for which $\mu_n(\mathcal{H}_f) = m - k - 1$ and

$$s_k(\mathcal{H}_f) > s_{k+1}(\mathcal{H}_f) = \cdots = s_n(\mathcal{H}_f) = \cdots = s_{m-1}(\mathcal{H}_f) > s_m(\mathcal{H}_f).$$

The main objective of this paper is to investigate the behavior of the sequence $\{\mu_n(\mathcal{H}_f)\}$ for Hankel operators whose symbol assumes some special form.

The starting point for the investigation of the question above is the celebrated theorem of V.M. Adamyan, D.Z. Arov, and M.G. Krein, also known as the *AAK Theorem*, ([1], see also Chapter 4 of [16], and [5]) which establishes a connection between Hankel operators and problems of approximation by meromorphic functions. The set of *meromorphic functions in* $L^\infty(\mathbb{T})$ *with at most* n *poles in* $\mathbb{D}$ is defined as

$$H_n^\infty := H^\infty B_n^{-1},$$

where B_n is the *set of Blaschke products of degree at most* n, i.e. the set of rational functions of the form

$$b(z) = \mathrm{e}^{\mathrm{i}c} \prod_{j=1}^{m} \frac{z - z_j}{1 - \bar{z}_j z}, \quad m \leqslant n,\ z_j \in \mathbb{D},\ c \in \mathbb{R}.$$

This way of writing $b_n^{-1} g = h \in H_n^\infty$ is just a trick to express that h is the ratio of an analytic function which is bounded in $\mathbb{D}$ and a polynomial of degree at most n, while $\|g\|_\infty = \|h\|_\infty$ since $|b| = 1$ everywhere on $\mathbb{T}$.

The AAK Theorem states that for any $f \in L^\infty(\mathbb{T})$ and $n \in \mathbb{Z}_+$ we have

$$\inf_{g \in H_n^\infty} \|f - g\|_\infty = s_n(\mathcal{H}_f)$$

and this infimum is attained for some function $g_n \in H_n^\infty$. Further, assume that $s_n(\mathcal{H}_f) > s_\infty(\mathcal{H}_f)$ (in particular, this holds whenever f belongs to the *Douglas algebra* $H^\infty + C(\mathbb{T})$, where $C(E)$ stands for the space of continuous functions on a compact set E), then g_n is unique,

$$|f - g_n| = s_n(\mathcal{H}_f) \quad \text{a.e. on } \mathbb{T}, \tag{2.2}$$

$$f - g_n = \frac{\mathcal{H}_f(v_n)}{v_n}, \tag{2.3}$$

where v_n is an arbitrary eigenvector of $\mathcal{H}_f^* \mathcal{H}_f$ associated to $s_n(\mathcal{H}_f)$. Any such function v_n, normalized to have unit norm in H^2, is called a *singular vector associated to* g_n. We point out that each best approximant g_n may have several associated singular vectors, but there always exists one with inner-outer factorization

$$v_n(z) = b_n(z) w_n(z), \quad z \in \mathbb{D}, \tag{2.4}$$

where b_n is a Blaschke product of exact degree n and w_n is an outer function. Here, one should recall the well-known fact [7] that any nonzero function in H^p can be uniquely factored as $f = jw$, where

$$w(z) = \exp\left\{ \frac{1}{2\pi} \int \frac{\xi + z}{\xi - z} \log|f(\xi)||\mathrm{d}\xi| \right\}$$

belongs to H^p and is called the *outer factor* of f, while j has modulus 1 a.e. on $\mathbb{T}$ and is called the *inner factor* of f. The latter may be further decompose as $j = bS$, where b is a Blaschke product, while

$$S(z) = \exp\left\{ -\int \frac{\xi + z}{\xi - z} \mathrm{d}\nu(\xi) \right\}$$

is the *singular inner factor* associated with ν, a positive measure on $\mathbb{T}$ which is singular with respect to the Lebesgue measure. For simplicity, we often say that a function is outer (respectively inner) if it is equal to its outer (respectively inner) factor. Equation (2.4) means, in particular, that v_n has no singular inner factor and that its inner factor is a Blaschke product of degree n.

Summarizing the preceding discussion we can see that the AAK Theorem not only describes the error of best approximation of an $L^\infty(\mathbb{T})$ function by meromorphic ones, but also provides a way to construct best approximants.

As mentioned before, we are going to consider only symbols of Hankel operators of some special type, namely Cauchy transform of complex Borel measures. Let ν be such a measure with $\operatorname{supp}(\nu) \subset (-1,1)$ that consists of an infinite number of points. We shall assume that ν has an *argument of bounded variation*, i.e. the Radon-Nikodym derivative with respect to $|\nu|$ is of bounded variation, where $|\nu|$ stands for the total variation of ν. In other words, ν is of the form

$$\mathrm{d}\nu(x) = \mathrm{e}^{\mathrm{i}\arg(\nu;x)}\mathrm{d}|\nu|(x),$$

for some real-valued argument function $\arg(\nu;\cdot)$ such that

$$V(\arg(\nu;\cdot);\operatorname{supp}(\nu)) := \sup\left\{\sum_{j=1}^{N} |\arg(\nu;x_j) - \arg(\nu;x_{j-1})|\right\} < \infty,$$

where the supremum is taken over all finite sets of points $x_0 < x_1 < \cdots < x_N$ from $\operatorname{supp}(\nu)$ as N ranges over $\mathbb{N}$. Note that we may extend $\arg(\nu;\cdot)$ to the whole convex hull of $\operatorname{supp}(\nu)$, say $[c,d]$, without increasing the variation. This is easy to see if we extend $\arg(\nu;\cdot)$ linearly in each component of $[c,d]\setminus\operatorname{supp}(\nu)$. In other words, we may arrange the extension of $\arg(\nu;\cdot)$ so that

$$V(\arg(\nu;\cdot);\operatorname{supp}(\nu)) = V(\arg(\nu;\cdot);[c,d]).$$

Let $u \in C(E)$. We put

$$\arg(u;z) = -\mathrm{i}\log\Big(\frac{u(z)}{|u(z)|}\Big), \quad z \in E.$$

Clearly, $\arg(u;\cdot)$ is a multi-valued function which is defined everywhere on E except at the zeros of u. Thus, we shall specify which branch is used on each particular occasion.

Now, we are ready to state the main theorems.

THEOREM 2.1. *Let $F \in C(\mathbb{T})$ be of the form*

$$F(z) := \int \frac{\mathrm{d}\lambda(x)}{z-x} + R(z), \tag{2.5}$$

where the measure λ has infinitely many points in its support contained in the interval $(-1,1)$ and an argument of bounded variation, while $R = P/Q$ is a rational function with no poles on $\mathbb{T}$. Then the sequence of multiplicities of singular values of the Hankel *operator $\mathcal{H}_F$, $\{\mu_n(\mathcal{H}_F)\}_{n\in\mathbb{Z}_+}$, is uniformly bounded. More precisely, the following upper bound holds for any $n \in \mathbb{Z}_+$:*

$$\mu_n(\mathcal{H}_F) \leqslant \frac{2}{\pi}[V(\arg(\lambda;\cdot);[a,b]) + V(\arg(Q;\cdot);[a,b]) + \pi\deg(Q) + V_{\mathcal{W}}] + N_{\mathcal{W}} + 1, \tag{2.6}$$

where $[a,b]$ is the convex hull of $\operatorname{supp}(\lambda)$*,*

$$N_{\mathcal{W}} := \max_{n\in\mathbb{Z}_+} \#\{\xi \in \mathbb{T} : \ w_n(\xi) = 0\}, \tag{2.7}$$

$$V_{\mathcal{W}} := \sup_{n\in\mathbb{Z}_+} V(\arg(w_n;\cdot);[a,b]), \tag{2.8}$$

and w_n is the outer factor of a singular vector v_n with exactly n zeros in $\mathbb{D}$ associated to g_n, the best meromorphic approximant to F of order n given by the AAK*-Theory.*

The finiteness of the constants $N_{\mathcal{W}}$ and $V_{\mathcal{W}}$ will be shown during the proof of the theorem.

It is worth noting that in the case where F is just a *Markov function*, i.e. the Cauchy transform of a positive measure supported on the real line, all the singular values of the corresponding Hankel operator are simple (see [4]). This phenomenon is due to the positivity of the measure and cannot be expected to hold in the complex case. Nevertheless, in the case where F is the Cauchy transform of a complex measure supported on an interval that has a Dini-continuous nonvanishing Radon-Nykodim derivative with respect to the logarithmic equilibrium distribution on this interval, it is possible to deduce more detailed information on the sequence of outer factors $\{w_n\}$, which, in turn, can be used to show that singular values of the corresponding Hankel operator are asymptotically simple.

THEOREM 2.2. *Let $F\in C(\mathbb{T})$ be of the form* (2.5)*, where the measure λ is such that*

$$d\lambda(x) = \frac{\ell(x)dx}{(x-a)^\alpha(b-x)^\beta}, \quad \alpha,\beta \in (0,1/2], \ x \in [a,b] \subset (-1,1), \tag{2.9}$$

with ℓ being a complex-valued Dini-continuous nonvanishing function on $[a,b]$ having an argument of bounded variation, while R is a rational function with no poles on $\mathbb{T}$ or $[a,b]$. Then

$$\lim_{n\to\infty} \mu_n(\mathcal{H}_F) = 1. \tag{2.10}$$

The proofs of the theorems rely on two known results that are significant on their own. For the ease of the reader we present them separately in the next section.

3. KERNELS OF TOEPLITZ OPERATORS AND MEASURES ORTHOGONAL TO POLYNOMIALS

This section is devoted to known results on the spectrum of Toeplitz operators and on the size of the variation of an argument of a measure orthogonal to polynomials up to some fixed degree.

Before we state these results we need to introduce several additional concepts. Let, as before, $f \in L^\infty(\mathbb{T})$. Recall that the *Toeplitz operator with symbol f*, $\mathcal{T}_f \in \mathcal{L}(H^2)$, is defined as

$$\mathcal{T}_f(h) := \mathcal{P}_+(fh).$$

It is easy to see that

$$\mathcal{T}_f + \mathcal{H}_f = \mathcal{M}_f,$$

where $\mathcal{M}_f : H^2 \to L^2(\mathbb{T})$ is the operator of multiplication by f. Recall also that an operator $\mathcal{O} \in \mathcal{L}(H)$ is called *Fredholm* if it is invertible modulo compact operators. The *index* of a Fredholm operator $\mathcal{O}$ is defined by

$$\operatorname{ind}(\mathcal{O}) := \dim\ker(\mathcal{O}) - \dim\ker(\mathcal{O}^*).$$

The *essential spectrum*, $\sigma_{\mathrm{e}}(\mathcal{O})$, of a bounded operator $\mathcal{O}$ is, by definition,

$$\sigma_{\mathrm{e}}(\mathcal{O}) := \{z \in \mathbb{C} : \ \mathcal{O} - z\mathcal{I} \ \text{ is not Fredholm}\}.$$

The next notion that we need is the notion of the *winding number with respect to the origin* of a continuous function on $\mathbb{T}$. Let $u \in C(\mathbb{T})$ and assume that u does not vanish on $\mathbb{T}$. Further, let $\arg(u;\cdot)$ be any continuous branch of the argument of u. Then the winding number of u with respect to the origin is defined by

$$\operatorname{wind}(u) := \frac{1}{2\pi}[\arg(u;2\pi) - \arg(u;0)].$$

Clearly $\operatorname{wind}(u)$ does not depend on the choice of the branch of the argument of u.

In general, let u be an *invertible* function in $H^\infty + C(\mathbb{T})$, i.e. $1/u \in H^\infty + C(\mathbb{T})$. Denote also by u the harmonic extension of u into $\mathbb{D}$. Then it is known (see Theorem 3.3.5 in [16]) that there exists $r_0 \in (0,1)$ such that $|u|$ is bounded away from zero on the annulus $\{z : r_0 < |z| < 1\}$ and the functions $u_r(\xi) := u(r\xi)$, $\xi \in \mathbb{T}$, have the same winding number for any $r \in (r_0, 1)$. Thus, for any invertible function u in $H^\infty + C(\mathbb{T})$ we define the winding number as

$$\operatorname{wind}(u) := \operatorname{wind}(u_r), \quad r \in (r_0, 1).$$

Now we can describe the essential spectrum of a Toeplitz operator (see Theorem 3.3.8 in [16]).

THEOREM P. *Let $u \in H^\infty + C(\mathbb{T})$. Then for any $z_0 \notin \sigma_{\mathrm{e}}(\mathcal{T}_u)$*

$$\operatorname{ind}(\mathcal{T}_u - z_0\mathcal{I}) = -\operatorname{wind}(u - z_0). \tag{3.1}$$

Moreover, if u is a continuous function then $\sigma_{\mathrm{e}}(\mathcal{T}_u) = u(\mathbb{T})$.

We continue this section with the result on a size of the variation of a measure. It was stated in Lemma 3.1(a) in [6] and shown in the course of the proof of Lemma 3.2 in [2].

LEMMA B. *Let ν be a measure with an argument of bounded variation, $[c,d]$ be the convex hull of* $\operatorname{supp}(\nu)$*, and ψ be a real-valued function of bounded variation on $[c,d]$. Suppose further that for some $l \in \mathbb{N}$ holds*

$$\int x^j e^{i\psi(x)} d\nu(x) = 0, \quad j = 0,\dots,l-1.$$

Then

$$V(\psi + \arg(\nu;\cdot);[c,d]) \geqslant l\pi. \tag{3.2}$$

Recall that any branch of the argument of ν can be continued to the whole interval $[c,d]$ without increasing its variation and therefore $V(\psi + \arg(\nu;\cdot);[c,d])$ is well-defined.

4. PROOFS

For the upcoming proofs we need to define one more concept, namely, the *angle in which an interval is seen at a point*. For any $\xi \neq 0 \in \mathbb{C}$, we let $\operatorname{Arg}(\xi) \in (-\pi,\pi]$ be the principal branch of the argument and for $\xi = 0$ we set $\operatorname{Arg}(0) = \pi$. Under such a definition, $\operatorname{Arg}(\cdot)$ becomes a left continuous function on $\mathbb{R}$. Now, for any interval $[a,b] \subset \mathbb{R}$ we define the angle in which this interval is seen at $\xi \in \mathbb{C}$ by

$$\operatorname{Angle}(\xi,[a,b]) := |\operatorname{Arg}(a-\xi) - \operatorname{Arg}(b-\xi)|.$$

It is easy to see that for any $\xi \notin \mathbb{D}$ and any $[a,b] \subset (-1,1)$ there holds

$$\operatorname{Angle}(\xi,[a,b]) \leqslant \frac{\pi}{2}.$$

Proof of Theorem 2.1. Fix an arbitrary $n \in \mathbb{Z}_+$. Without loss of generality we may assume that $s_{n-1}(\mathcal{H}_F) > s_n(\mathcal{H}_F)$. Denote by g_n the best meromorphic approximant to F on $\mathbb{T}$ out of H_n^∞ (recall that g_n is unique by the compactness of $\mathcal{H}_F$). Then, by the circularity property (2.2), the function

$$u_n := s_n(\mathcal{H}_F)^{-1}(F - g_n)$$

is unimodular almost everywhere on the unit circle. It is known (Theorem 4.1.7 in [16]) that in this case

$$\dim\ker(\mathcal{T}_{u_n}) = 2n + \mu_n(\mathcal{H}_F).$$

It is also known (Theorem 3.1.4 in [16]) that either $\ker(\mathcal{T}_u) = \{0\}$ or $\ker(\mathcal{T}_u^*) = \{0\}$ for any nonzero function from $L^\infty(\mathbb{T})$. Thus,

$$\operatorname{ind}(\mathcal{T}_{u_n}) = \dim\ker(\mathcal{T}_{u_n}) = 2n + \mu_n(\mathcal{H}_F). \tag{4.1}$$

Therefore, upon showing that u_n is a continuous and nonvanishing function on $\mathbb{T}$, we will obtain from (4.1) and (3.1) that

$$\mu_n(\mathcal{H}_F) = -2n - \operatorname{wind}(u_n). \tag{4.2}$$

Indeed, in this case zero does not belong to $\sigma_{\mathrm{e}}(\mathbb{T}_{u_n}) = u(\mathbb{T})$ and we may apply Theorem P. To show continuity of u_n recall from (2.3) that by the AAK Theorem there exists a singular vector $v_n \in H^2$ with the inner-outer factorization

$$v_n = b_n w_n, \tag{4.3}$$

where b_n is a Blaschke product of exact degree n and w_n is an outer function, such that

$$u_n = \frac{\mathcal{H}_F(v_n)}{v_n}. \tag{4.4}$$

Moreover, it is known (see e.g. Sections 8 and 9 in [5]) that $\mathcal{H}_F(v_n)$ has the following representation, where j_n is some inner function:

$$\mathcal{H}_F(v_n)(z) = \frac{s_n(\mathcal{H}_F)}{z}\,\overline{(j_n b_n w_n)\Big(\frac{1}{\bar{z}}\Big)}, \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}}. \tag{4.5}$$

In another connection, by the definition of Hankel operators and (2.5) we have that

$$\begin{aligned}
\mathcal{H}_F(v_n)(z) = \mathcal{P}_-(Fv_n)(z) &= \frac{1}{2\pi \mathrm{i}} \int\limits_{\mathbb{T}} \frac{F(\xi)v_n(\xi)}{z-\xi}\mathrm{d}\xi \\
&= \frac{1}{2\pi \mathrm{i}} \int\limits_{\mathbb{T}} \int \frac{v_n(\xi)}{(z-\xi)(\xi - x)}\mathrm{d}\lambda(x)\mathrm{d}\xi + \frac{1}{2\pi \mathrm{i}} \int\limits_{\mathbb{T}} \frac{R(\xi)v_n(\xi)}{z-\xi}\mathrm{d}\xi \\
&= \int \frac{v_n(x)}{z-x}\mathrm{d}\lambda(x) + \frac{1}{2\pi \mathrm{i}} \int\limits_{\mathbb{T}} \frac{P(\xi)v_n(\xi)}{z-\xi}\,\frac{\mathrm{d}\xi}{Q(\xi)}, \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}},
\end{aligned} \tag{4.6}$$

where $R = P/Q$. Note that the second integral in (4.6) is, in fact, a rational function with denominator Q by the Cauchy integral formula. Combining (4.5) and (4.6) we get that

$$(j_n b_n w_n)(z) = s_n(\mathcal{H}_F)^{-1}\Big(\int \frac{\overline{v_n(x)}}{1-xz}\overline{\mathrm{d}\lambda(x)} + \overline{\frac{1}{2\pi \mathrm{i}} \int\limits_{\mathbb{T}} \frac{P(\xi)v_n(\xi)}{1-\xi\bar{z}}\,\frac{\mathrm{d}\xi}{Q(\xi)}}\Big), \quad z \in \mathbb{D}. \tag{4.7}$$

Observe that the right-hand side of (4.7) is well-defined for $z \in \Omega$, where

$$\Omega := \mathbb{C} \setminus \Big(\Big\{z \in \mathbb{C} : \ Q\Big(\frac{1}{\bar{z}}\Big) = 0\Big\} \bigcup \Big\{z \in \mathbb{C} : \ \frac{1}{\bar{z}} \in \operatorname{supp}(\lambda)\Big\}\Big). \tag{4.8}$$

In other words, equation (4.7) provides an analytic continuation of the product $j_n b_n w_n$ outside of the unit disk. In particular, this means that j_n is a finite Blaschke product and the number of zeros of w_n on $\mathbb{T}$ is finite. Let $\{\zeta_{j,n}\}$ be the set of zeros of w_n on $\mathbb{T}$. Then w_n can be written as

$$w_n(z) = w_n^{\#}(z)P_n(z), \quad P_n(z) := \prod_j (z - \zeta_{j,n}), \tag{4.9}$$

where $w_n^\#$ is an analytic and zero-free function in some neighborhood of $\overline{\mathbb{D}}$. Then (4.4) with the help of (4.3), (4.5), and (4.9) yields

$$u_n(\xi)=\frac{s_n(\mathcal{H}_F)}{\xi}\frac{\overline{(j_nb_nw_n^\#)(\xi)}}{(b_nw_n^\#)(\xi)}\prod_j\left(\frac{-1}{\zeta_{j,n}\xi}\right)=\frac{s_n(\mathcal{H}_F)}{\xi(j_nb_n^2)(\xi)}\frac{\overline{w_n^\#(\xi)}}{w_n^\#(\xi)}\prod_j\left(\frac{-1}{\zeta_{j,n}\xi}\right),\quad \xi\in\mathbb{T}. \tag{4.10}$$

Equation (4.10) shows that u_n is a continuous nonvanishing function on $\mathbb{T}$ which, in turn, validates equation (4.2).

Denote by Q_n the numerator of the Blaschke product j_n. Then we obtain from (4.10) that

$$\operatorname{wind}(u_n) = -1-\deg(P_n)-\deg(Q_n)-2n, \tag{4.11}$$

since $w_n^\#$ is zero-free and analytic in some neighborhood of $\overline{\mathbb{D}}$. Combining (4.11) and (4.2) we get that

$$\mu_n(\mathcal{H}_F) = \deg(Q_n)+\deg(P_n)+1 \leqslant \deg(Q_n)+N_{\mathcal{W}}+1, \tag{4.12}$$

where $N_{\mathcal{W}}$ was defined in (2.7). Thus, to prove (2.6) it remains to show that

$$k_n:=\deg(Q_n)\leqslant\frac{2}{\pi}\left(V(\arg(\lambda;\cdot);[a,b])+V(\arg(Q;\cdot);[a,b])+\pi\deg(Q)+V_{\mathcal{W}}\right) \tag{4.13}$$

and that the constants $N_{\mathcal{W}}$ and $V_{\mathcal{W}}$ are finite. Recall that $V_{\mathcal{W}}$ was defined in (2.8).

It is known (Lemma 3.4 in [6], Proposition 6.3 in [3], and Theorem 10.1 in [5]) that the sequence $\{w_m\}_{m\in\mathbb{Z}_+}$ forms a normal family in Ω, where Ω was defined in (4.8). Moreover, the zero function is not a limit point of this family, since $\|w_m\|_2 = 1$ for each $m\in\mathbb{Z}_+$. This proves the finiteness of $N_{\mathcal{W}}$.

Now, recall that j_n can be represented as $j_n = Q_n/\widetilde{Q}_n$, where we set $\widetilde{p}(z) = z^k\overline{p(1/\bar{z})}$, $k=\deg(p)$, for any polynomial p. Similarly we can write $b_n = q_n/\widetilde{q}_n$, where q_n is a monic polynomial with all zeros in $\mathbb{D}$ and of exact degree n. Let $z_0\in\mathbb{D}$ be such that $(q_nQ_n)(z_0)=0$. Then we deduce from (4.7) that

$$\int\frac{v_n(x)}{1-x\bar{z}_0}\mathrm{d}\lambda(x)+\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}}\frac{P(\xi)v_n(\xi)}{1-\xi\bar{z}_0}\frac{\mathrm{d}\xi}{Q(\xi)}=0. \tag{4.14}$$

By taking linear combinations of equation (4.14) with different roots of q_n and Q_n we obtain that

$$\int\frac{p(x)v_n(x)}{\widetilde{q}_n(x)\widetilde{Q}_n(x)}\mathrm{d}\lambda(x)+\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}}\frac{p(\xi)P(\xi)v_n(\xi)}{\widetilde{q}_n(\xi)\widetilde{Q}_n(\xi)}\frac{\mathrm{d}\xi}{Q(\xi)}=0 \tag{4.15}$$

for any polynomial p of degree at most $n+k_n-1$. It can be readily verified that equation (4.15) and the Cauchy integral theorem imply the following orthogonality relations

$$\int\frac{x^jQ(x)v_n(x)}{\widetilde{q}_n(x)\widetilde{Q}_n(x)}\mathrm{d}\lambda(x)=0,\quad j=0,\dots,n+k_n-\deg(Q)-1. \tag{4.16}$$

By using the inner-outer factorization (4.3) we can rewrite (4.16) in the form

$$\int x^j q_n(x)\frac{Q(x)w_n(x)}{\widetilde{q}_n^2(x)\widetilde{Q}_n(x)}\mathrm{d}\lambda(x)=0,\quad j=0,\dots,n+k_n-\deg(Q)-1. \tag{4.17}$$

Then the bound (3.2) of Lemma B together with orthogonality relations (4.17) yields

$$(n+k_n-\deg(Q))\pi\leqslant V\Big(\arg\Big(\frac{q_n(x)Q(x)w_n(x)}{\widetilde{q}_n^2(x)\widetilde{Q}_n(x)};\cdot\Big)+\arg(\lambda;\cdot);[a,b]\Big), \tag{4.18}$$

where $[a,b]$ is the convex hull of the measure λ. It follows from the normality of the family $\{w_m\}_{m\in\mathbb{Z}_+}$ in Ω that the sequence $\{V(w_m;[a,b])\}_{m\in\mathbb{Z}_+}$ is uniformly bounded, i.e. $V_{\mathcal{W}}$ is finite. Therefore by (4.18) and the sublinearity of $V(\cdot;[a,b])$ we obtain

$$\begin{aligned}(n+k_n)\pi &\leqslant V(\arg(\lambda;\cdot);[a,b])+V(\arg(Q;\cdot);[a,b])+\pi\deg(Q)+V_{\mathcal{W}}\\ &\quad+V\Big(\arg\Big(\frac{q_n}{\widetilde{q}_n^2};\cdot\Big);[a,b]\Big)+V(\arg(\widetilde{Q}_n;\cdot);[a,b]).\end{aligned} \tag{4.19}$$

Write $q_n(z)=\prod_{j=1}^{n}(z-\xi_{j,n})$. It was shown in Lemma 5.2 of [2] that

$$V\Big(\arg\Big(\frac{q_n}{\widetilde{q}_n^2};\cdot\Big),[a,b]\Big)\leqslant\sum_{j=1}^{n}\mathrm{Angle}(\xi_{j,n},[a,b]). \tag{4.20}$$

By writing the polynomial Q_n in the form $Q_n(z)=\prod_{j=1}^{k_n}(z-\eta_{j,n})$ we obtain from the monotonicity of $\mathrm{Angle}(\cdot,[a,b])$ that

$$V(\arg(\widetilde{Q}_n;\cdot);[a,b])\leqslant\sum_{j=1}^{k_n}V\Big(\arg\Big(\cdot;\frac{-1}{\overline{\eta}_{j,n}}\Big),[a,b]\Big)=\sum_{j=1}^{k_n}\mathrm{Angle}\Big(\frac{1}{\overline{\eta}_{j,n}},[a,b]\Big)\leqslant\frac{k_n\pi}{2}, \tag{4.21}$$

since $\eta_{j,n}\in\mathbb{D}$ for all $j=1,\dots,k_n$. Combining (4.19), (4.20), and (4.21) we get that

$$\begin{aligned}&\sum_{j=1}^{n}(\pi-\mathrm{Angle}(\xi_{j,n},[a,b]))+\frac{k_n\pi}{2}\\ &\quad\leqslant V(\arg(\lambda;\cdot);[a,b])+V(\arg(Q;\cdot);[a,b])+\pi\deg(Q)+V_{\mathcal{W}}.\end{aligned} \tag{4.22}$$

The last inequality proves (4.13) and therefore the assertion of the theorem. ∎

Proof of Theorem 2.2. Let F be given by equation (2.5) and Λ be a subsequence of natural numbers defined by the rule

$$\Lambda:=\{n\in\mathbb{N}:\ s_{n-1}(\mathcal{H}_F)>s_n(\mathcal{H}_F)\},$$

where $s_n(\mathcal{H}_F)$ stands, as before, for the n-th singular value of the Hankel operator with symbol F. It is obvious that we may apply the preceding theorem for a measure of the form (2.9). Namely, it can be deduced from equations (4.9) and (4.12) that

$$\mu_n(\mathcal{H}_F) \leqslant \deg(Q_n) + \#\{\xi \in \mathbb{T} : w_n(\xi) = 0\} + 1, \quad n \in \Lambda, \tag{4.23}$$

where $j_n = Q_n/\widetilde{Q}_n$ and w_n were defined in (4.3)–(4.5), with v_n being a singular vector with exactly n poles associated to the best meromorphic approximant to F of order n.

It is shown in Theorem 1 of [23] that in the case where the measure λ is of the form (2.9), the sequence $\{j_n w_n\}$ is not only a normal family in Ω, where Ω was defined in (4.8), but also is locally uniformly convergent to the function

$$w(z) = \frac{c}{\sqrt{(1-az)(1-bz)}},$$

where c is some positive constant. This, in particular, means that

$$\lim_{n\to\infty} \#\{\xi \in \mathbb{T} : w_n(\xi) = 0\} = 0$$

and for n large enough $j_n \equiv 1$. Combining equation (4.23) with these two observations we get (2.10). This proves the theorem. ∎

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Received July 19, 2006.