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INVARIANT SUBSPACES FOR THE SHIFT ON THE VECTOR-VALUED L^2 SPACE OF AN ANNULUS

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ABSTRACT. In this paper we study the invariant subspaces of the shift operator acting on the vector-valued L^2 space of an annulus, following an approach which originates in the work of Sarason. We obtain a Wiener-type result characterizing the reducing subspaces, and we give a description of all the invariant and doubly-invariant subspaces generated by a single function. We prove that every doubly-invariant subspace contained in the Hardy space of the annulus with values in $\mathbb{C}^m$ is the orthogonal direct sum of at most m doubly-invariant subspaces, each generated by a single function. As a corollary we prove that a doubly-invariant subspace that is also the graph of an operator is singly generated.

KEYWORDS: *Invariant subspace, vector-valued Hardy space, shift operator, multiply-connected domains.*

MSC (2000): Primary 46E20, 47B38; Secondary 30D55, 47A15.

1. INTRODUCTION

The purpose of this paper is to study the shift operator (multiplication by the independent variable) on certain Hardy spaces, consisting of vector-valued analytic functions on the annulus $A = \{r_0 < |z| < 1\}$, where r_0 is a positive real number less than unity. In the scalar case, significant contributions to the theory have been made by several authors, including Sarason [9], Royden [8], Hitt [4], Yakubovich [10] and Aleman–Richter [1].

The vectorial case has not been much considered, and presents difficulties of its own. We shall consider questions to do with reducing subspaces and singly and doubly-invariant subspaces, which are defined below. One important special case is when the functions take values in $\mathbb{C}^2$, and there is then the question of characterizing graphs of closed (possibly unbounded) shift-invariant operators.

We now introduce some necessary definitions and notation, after which we shall summarise the main contributions of the paper.

The boundary ∂A of A consists of two circles $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $r_0\mathbb{T}$. We let $\mathbb{D}$ denote the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ and Ω_0 the set $\{z \in \mathbb{C} : r_0 < |z|\} \cup \{\infty\}$, so that $A = \mathbb{D} \cap \Omega_0$. For $1 \leqslant p < \infty$, let $L^p(\partial A)$ be the complex Banach space of Lebesgue measurable functions f on ∂A that are pth-power integrable with respect to Lebesgue measure, the norm of f being defined by

$$\|f\|_p = \Big(\frac{1}{2\pi} \int_0^{2\pi} |f(\mathrm{e}^{\mathrm{i}t})|^p \mathrm{d}t + \frac{1}{2\pi} \int_0^{2\pi} |f(r_0\mathrm{e}^{\mathrm{i}t})|^p \mathrm{d}t \Big)^{1/p}.$$

The complex Banach space $L^\infty(\partial A)$ is the set of bounded Lebesgue measurable functions f on ∂A. Obviously, for $1 \leqslant p \leqslant \infty$, we have:

$$L^p(\partial A) = L^p(\mathbb{T}) \oplus L^p(r_0\mathbb{T}),$$

where $L^p(\mathbb{T})$ and $L^p(r_0\mathbb{T})$ are endowed with normalized Lebesgue measure. For $1 \leqslant p < \infty$ the Hardy space $H^p(\partial A)$ denotes the closure in $L^p(\partial A)$ of $R(A)$, the set of rational functions with poles off $\overline{A} = \{z \in \mathbb{C} : r_0 \leqslant |z| \leqslant 1\}$ (it is convenient to employ an abuse of language by saying that a function in $L^p(\partial A)$ "belongs to $R(A)$" if it is a restriction of a function in $R(A)$). These functions are analytic in A, and so we shall also use the notation $H^p(A)$ when we wish to emphasise this. A useful characterization of functions in $H^p(\partial A)$ is the following (see Lemma 1 in [9]): a function $f \in L^p(\partial A)$ belongs to $H^p(\partial A)$ if and only if, for all $n \in \mathbb{Z}$:

$$\int_0^{2\pi} f(r_0\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t = r_0^n \int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t. \tag{1.1}$$

As for the Hardy spaces on $\mathbb{D}$, there exists an inner-outer factorization for functions in the Hardy spaces defined on A. Following Royden [8], the *inner* functions in $H^2(A)$ are the holomorphic functions u such that $|u|$ is constant on each circle, whereas the *outer* functions in $H^2(A)$ are the holomorphic functions ϕ such that

$$\log|\phi(z)| = \frac{1}{2\pi} \int_{\partial A} \log|\phi(\xi)| \frac{\partial g(\xi, z)}{\partial n} \mathrm{d}s(\xi)$$

for $z \in A$, where g is the Green's function (normalized so that the constant 2π is correct). The *units* are the functions that are both inner and outer, e.g. z^k. Contrary to the case of the unit disc, if $f \in H^2(\partial A)$, we cannot always find an outer function ϕ with $|\phi| = |f|$ on ∂A. The best we can do is to define v on A in the following way:

$$v(z) = \frac{1}{2\pi} \int_{\partial A} \log|f(\xi)| \frac{\partial g(\xi, z)}{\partial n} \mathrm{d}s(\xi).$$

Now v is real and harmonic so that we can find a constant c and a real harmonic function h such that $\psi(z) := v(z) - c\log|z| + \mathrm{i}h(z)$ is holomorphic (we need the $\log|z|$ term as the annulus is not simply connected). Now $\phi(z) := \exp(\psi(z))$ is an outer function whose non-tangential boundary values satisfy $|\phi(\xi)| = \frac{|f(\xi)|}{|\xi|^c}$, and then $\frac{f}{\phi}$ is inner since $|u|$ is constant on each circle.

As usual $\mathrm{supp}(f)$ denotes the support of the function f.

Denote by S the operator of multiplication by z on $L^p(\partial A)$. A closed subspace M in $L^2(\partial A)$ is said to be *invariant* for S if $SM \subset M$, *doubly invariant* for S if M is both invariant for S and S^{-1} and *reducing* for S if M is both invariant for S and S^*.

For $f \in L^2(\partial A)$,

(i) $I_S[f]$ will denote the smallest closed subspace M in $L^2(\partial A)$ containing f and invariant for S.

(ii) $D_S[f]$ will denote the smallest closed subspace M in $L^2(\partial A)$ containing f and doubly invariant for S.

(iii) $R_S[f]$ will denote the smallest closed subspace M in $L^2(\partial A)$ containing f and reducing for S.

In other words

$$\begin{aligned} I_S[f] &= \mathrm{Span}\{S^n f : n \geqslant 0\}, \\ D_S[f] &= \mathrm{Span}\{S^n f : n \in \mathbb{Z}\}, \\ R_S[f] &= \mathrm{Span}\{p(S, S^*) f : p \in \mathbb{C}[z_1, z_2]\}, \end{aligned}$$

where Span is the closed linear hull.

If N is a set of functions $I_S(N)$, (respectively $D_S(N)$ and $R_S(N)$) denotes the smallest closed subspace containing $I_S(f)$ (respectively $D_S(f)$ and $R_S(f)$) for all $f \in N$.

Note that for $f = f_1 \oplus f_0 \in L^2(\mathbb{T}) \oplus L^2(r_0\mathbb{T})$,

$$\begin{aligned} &Sf = g_1 \oplus g_0 \text{ where } g_1(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}t} f_1(\mathrm{e}^{\mathrm{i}t}) \text{ and } g_0(r_0\mathrm{e}^{\mathrm{i}t}) = r_0\mathrm{e}^{\mathrm{i}t} f_0(r_0\mathrm{e}^{\mathrm{i}t}), \\ &S^{-1} f = h_1 \oplus h_0 \text{ where } h_1(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{-\mathrm{i}t} f_1(\mathrm{e}^{\mathrm{i}t}) \text{ and } h_0(r_0\mathrm{e}^{\mathrm{i}t}) = \tfrac{1}{r_0}\mathrm{e}^{-\mathrm{i}t} f_0(r_0\mathrm{e}^{\mathrm{i}t}), \\ &S^* f = k_1 \oplus k_0 \text{ where } k_1(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{-\mathrm{i}t} f_1(\mathrm{e}^{\mathrm{i}t}) \text{ and } k_0(r_0\mathrm{e}^{\mathrm{i}t}) = r_0\mathrm{e}^{-\mathrm{i}t} f_0(r_0\mathrm{e}^{\mathrm{i}t}). \end{aligned}$$

It follows that the operators S, S^* and S^{-1} commute and then

$$R_S[f] = \mathrm{Span}\{S^n S^{*m} f : n, m \geqslant 0\}.$$

We employ an analogous notation for subspaces of $L^2(\partial A, \mathbb{C}^m)$; in general we use lower case letters for scalar functions and capital letters for vector-valued functions.

The characteristic function associated with a measurable set E will be denoted by χ_E.

In Section 2 we obtain a Wiener-type result (Theorem 2.3) characterizing the reducing subspaces $M \subset L^2(\partial A, \mathbb{C}^m)$. In Section 3 we give a description of all the invariant and doubly-invariant subspaces $M \subset L^2(\partial A, \mathbb{C}^m)$ generated by a single function. Some tables at the end of this section summarise our results. Finally, in Section 4 we establish the main result of this paper, Theorem 4.7. We prove that every doubly-invariant subspace $M \subset H^2(\partial A, \mathbb{C}^m)$ is the orthogonal direct sum of at most m doubly-invariant subspaces, each generated by a single function. As a corollary we prove that a doubly-invariant subspace in $H^2(\partial A, \mathbb{C}^m)$ that is also the graph of a (not necessarily bounded) operator is singly generated (Theorem 4.8). The use of analyticity is essential in the proof of our main result and hence the description of doubly-invariant subspaces of $L^2(\partial A, \mathbb{C}^m)$ remains open. We give a partial result in this direction (Theorem 4.9) for operator graphs.

2. REDUCING SUBSPACES

In the scalar case, Sarason characterized reducing subspaces for S on $L^2(\partial A)$ by making use of the Wiener theorem that every doubly-invariant subspace of $L^2(\mathbb{T})$ has the form $\chi_E L^2(\mathbb{T})$ for some measurable set $E \subset \mathbb{T}$ (see [3], [6], [7]).

THEOREM 2.1 ([9], p. 52). *A closed subspace M of $L^2(\partial A)$ is reducing for S if and only if $M = \chi_E L^2(\partial A)$ for some measurable set $E \subset \partial A$.*

We now move on to a discussion of the vector-valued case. Instead of a function taking values in the set $\{0, 1\}$ almost everywhere, we now need to deal with functions whose values are orthogonal projections. Accordingly, we say that $P : r\mathbb{T} \to \mathcal{L}(\mathbb{C}^m)$ is a *measurable projection-valued function* if it satisfies the following:

(i) $P(re^{iw})$ is the orthogonal projection onto some closed subspace $\mathcal{I}(re^{iw})$ of $\mathbb{C}^m$ for almost all $re^{iw} \in r\mathbb{T}$.

(ii) The mappings $w \to \langle P(re^{iw})x, y\rangle$ are measurable for every $x, y \in \mathbb{C}^m$.

Since $P(re^{iw})$ can be regarded as an $m \times m$ matrix-valued function, the second property just says that $P \in L^\infty(r\mathbb{T}, \mathcal{L}(\mathbb{C}^m))$. The vectorial version of the Wiener theorem is the following (see for example Theorem 3.1.6 of [7] and [3]).

LEMMA 2.2. *Let $r > 0$, let M be a closed subspace of $L^2(r\mathbb{T}, \mathbb{C}^m)$ and let S be defined by $Sf(re^{it}) = re^{it}f(re^{it})$ on $L^2(r\mathbb{T}, \mathbb{C}^m)$. Then M is doubly invariant or reducing on $L^2(r\mathbb{T}, \mathbb{C}^m)$ if and only if $M = PL^2(r\mathbb{T}, \mathbb{C}^m)$ where P is a measurable projection-valued function on $r\mathbb{T}$.*

Proof. The space $L^2(r\mathbb{T})$ is unitarily equivalent to $L^2(\mathbb{T})$ by a simple change of variables, from which the operator S on $L^2(r\mathbb{T})$ is seen to be unitarily equivalent to the operator rS on $L^2(\mathbb{T})$. This has the same reducing subspaces as the bilateral shift on $L^2(\mathbb{T})$, and the result follows from Wiener's theorem. ∎

We obtain the following result for $L^2(\partial A, \mathbb{C}^m)$.

THEOREM 2.3. *A closed subspace M of $L^2(\partial A, \mathbb{C}^m)$ is reducing for S if and only if $M = PL^2(\partial A, \mathbb{C}^m)$, where P is a measurable projection-valued function on ∂A.*

Proof. It is clear that $PL^2(\partial A, \mathbb{C}^m)$ is a reducing subspace for S. Now, note that, for $F_1 \oplus F_0 \in L^2(\mathbb{T}, \mathbb{C}^m) \oplus L^2(r_0\mathbb{T}, \mathbb{C}^m)$, we have:

$$\frac{r_0^2 \mathrm{Id} - SS^*}{r_0^2 - 1}(F_1 \oplus F_0) = F_1 \oplus 0 \quad \text{and} \quad \frac{SS^* - \mathrm{Id}}{r_0^2 - 1}(F_1 \oplus F_0) = 0 \oplus F_0.$$

In other words, $P_{L^2(\mathbb{T},\mathbb{C}^m)} \oplus 0$ and $0 \oplus P_{L^2(r_0\mathbb{T},\mathbb{C}^m)}$ (where $P_{L^2(r\mathbb{T},\mathbb{C}^m)}$ is the orthogonal projection from $L^2(\partial A, \mathbb{C}^m)$ onto $L^2(r\mathbb{T}, \mathbb{C}^m)$, for $r \in \{r_0, 1\}$) are linear combinations of Id and SS^*. In particular $P_{L^2(r\mathbb{T},\mathbb{C}^m)} M$ is also a reducing subspace for S on $L^2(r\mathbb{T}, \mathbb{C}^m)$, for $r \in \{r_0, 1\}$. It follows that if M is a reducing subspace for S then

$$P_{L^2(\mathbb{T},\mathbb{C}^m)} M \oplus P_{L^2(r_0\mathbb{T},\mathbb{C}^m)} M \subset M.$$

Since the converse inclusion is true for any subspace M, it follows that if M is a reducing subspace for S then

$$P_{L^2(\mathbb{T},\mathbb{C}^m)} M \oplus P_{L^2(r_0\mathbb{T},\mathbb{C}^m)} M = M.$$

By Lemma 2.2, for $r = r_0$ and $r = 1$, $P_{L^2(r\mathbb{T},\mathbb{C}^m)} M = P_r L^2(r\mathbb{T}, \mathbb{C}^m)$ for some measurable projection-valued functions P_r defined on $r\mathbb{T}$. Therefore $M = PL^2(\partial A)$ where $P(re^{iw}) = P_r(re^{iw})$ for $r = r_0$ and $r = 1$. ∎

COROLLARY 2.4. *Let $F \in L^2(\partial A, \mathbb{C}^m)$. Then*

$$R_S(F) = \{G \in L^2(\partial A, \mathbb{C}^m) : G(\xi) \in \mathbb{C}F(\xi) \text{ for a.e. } \xi \in \partial A\}.$$

Proof. This follows from Theorem 2.3, on observing that the range of $P(\xi)$ must equal the subspace spanned by $F(\xi)$, for almost all ξ. ∎

3. INVARIANT AND DOUBLY-INVARIANT SUBSPACES GENERATED BY ONE FUNCTION

The log-integrability of the functions that we consider is at the centre of our classification.

DEFINITION 3.1. Let $r > 0$ and $F \in L^2(r\mathbb{T}, \mathbb{C}^m)$. We say that F is log-integrable on $r\mathbb{T}$ if $\int_0^{2\pi} \log \|F(re^{it})\|_{\mathbb{C}^m} \mathrm{d}t$ exists.

The next propositions show how we are allowed to modify the generators of singly generated invariant and doubly-invariant subspaces, provided that they are log-integrable.

PROPOSITION 3.2. *Let $F_1 \oplus F_0 \in L^2(\mathbb{T}, \mathbb{C}^m) \oplus L^2(r_0\mathbb{T}, \mathbb{C}^m)$ such that F_1 is log-integrable on $\mathbb{T}$. Then we have:*

$$I_S(F_1 \oplus F_0) = I_S\Big(\frac{F_1}{u_1} \oplus \frac{F_0}{u_1}\Big) \quad \textit{and} \quad D_S(F_1 \oplus F_0) = D_S\Big(\frac{F_1}{u_1} \oplus \frac{F_0}{u_1}\Big),$$

where u_1 is an outer function in $H^2(\mathbb{D})$ such that $|u_1(e^{it})| = \|F_1(e^{it})\|_{\mathbb{C}^m}$ almost everywhere on $\mathbb{T}$.

Proof. Since u_1 is an outer scalar function in $H^2(\mathbb{D})$, applying Beurling's theorem, there exists a sequence of polynomials $(p_n)_n$ such that

$$\lim_{n\to\infty} \|u_1 p_n - \mathbb{1}\|_{L^2(\mathbb{T})} = 0,$$

where $\mathbb{1}$ is the function identically equal to 1 on ∂A. Since $\frac{F_1}{u_1} \in L^\infty(\mathbb{T}, \mathbb{C}^m)$, it follows that

$$\Big\|(u_1 p_n - \mathbb{1})\frac{F_1}{u_1}\Big\|_{L^2(\mathbb{T},\mathbb{C}^m)} = \Big\|p_n F_1 - \frac{F_1}{u_1}\Big\|_{L^2(\mathbb{T},\mathbb{C}^m)}$$

tends to 0 as n tends to infinity. Moreover, $\lim_{n\to\infty} \|u_1 p_n - \mathbb{1}\|_{L^2(\mathbb{T})} = 0$ implies that $\lim_{n\to\infty} \|u_1 p_n - \mathbb{1}\|_{L^\infty(r_0\mathbb{T})} = 0$. Now, since u_1 is outer, $\frac{1}{u_1} \in L^\infty(r_0\mathbb{T})$ and so $\lim_{n\to\infty} \Big\|\frac{u_1 p_n - \mathbf{1}}{u_1}\Big\|_{L^\infty(r_0\mathbb{T})} = 0$. It follows that

$$\Big\|\Big(\frac{u_1 p_n - \mathbb{1}}{u_1}\Big)F_0\Big\|_{L^2(r_0\mathbb{T})} = \Big\|p_n F_0 - \frac{F_0}{u_1}\Big\|_{L^2(r_0\mathbb{T})}$$

tends to 0 as n tends to infinity. Therefore we have

$$I_S\Big(\frac{F_1}{u_1} \oplus \frac{F_0}{u_1}\Big) \subset I_S(F_1 \oplus F_0) \quad \text{and} \quad D_S\Big(\frac{F_1}{u_1} \oplus \frac{F_0}{u_1}\Big) \subset D_S(F_1 \oplus F_0).$$

In order to prove the converse inclusions, note that since $u_1 \in H^2(\mathbb{D})$, there exists a sequence of polynomials $(q_n)_n$ such that $\lim_{n\to\infty} \|u_1 - q_n\|_{L^2(\mathbb{T})} = 0$. Since $\frac{F_1}{u_1} \in L^\infty(\mathbb{T}, \mathbb{C}^m)$, we get

$$\Big\|(u_1 - q_n)\frac{F_1}{u_1}\Big\|_{L^2(\mathbb{T},\mathbb{C}^m)} = \Big\|F_1 - q_n\frac{F_1}{u_1}\Big\|_{L^2(\mathbb{T},\mathbb{C}^m)}$$

tends to 0 as n tends to infinity. Moreover, $\lim_{n\to\infty} \|u_1 - q_n\|_{L^2(\mathbb{T})} = 0$ implies that $\lim_{n\to\infty} \|u_1 - q_n\|_{L^2(r_0\mathbb{T})} = 0$, and then $\lim_{n\to\infty} \Big\|\frac{u_1 - q_n}{u_1}\Big\|_{L^2(r_0\mathbb{T})} = 0$ since u_1 is bounded below on $r_0\mathbb{T}$. It follows that

$$\Big\|\Big(\frac{u_1 - q_n}{u_1}\Big)F_0\Big\|_{L^2(r_0\mathbb{T})} = \Big\|F_0 - \frac{q_n F_0}{u_1}\Big\|_{L^2(r_0\mathbb{T})}$$

tends to 0 as n tends to infinity. This proves the converse inclusions and ends the proof of the proposition. ∎

The natural dual version of the above proposition is the following.

PROPOSITION 3.3. *Let $F_1 \oplus F_0 \in L^2(\mathbb{T}, \mathbb{C}^m) \oplus L^2(r_0\mathbb{T}, \mathbb{C}^m)$ be such that F_0 is log-integrable on $r_0\mathbb{T}$. Then we have:*

$$I_{S^{-1}}(F_1 \oplus F_0) = I_{S^{-1}}\Big(\frac{F_1}{u_0} \oplus \frac{F_0}{u_0}\Big) \quad \textit{and} \quad D_S(F_1 \oplus F_0) = D_S\Big(\frac{F_1}{u_0} \oplus \frac{F_0}{u_0}\Big),$$

where u_0 *is an outer function in* $H^2(\widehat{\mathbb{C}}\backslash\overline{r_0\mathbb{D}})$ *with* $|u_0(r_0\mathrm{e}^{\mathrm{i}t})| = \|F_0(r_0\mathrm{e}^{\mathrm{i}t})\|_{\mathbb{C}^m}$ *almost everywhere on* $r_0\mathbb{T}$*.*

Proof. Set $G_1(\mathrm{e}^{\mathrm{i}t}) = F_0(r_0\mathrm{e}^{-\mathrm{i}t})$ and $G_0(r_0\mathrm{e}^{\mathrm{i}t}) = F_1(\mathrm{e}^{-\mathrm{i}t})$. Considering the unitary map $\Psi : L^2(\partial A, \mathbb{C}^m) \to L^2(\partial A, \mathbb{C}^m)$ defined by $\Psi(F_1 \oplus F_0) = G_1 \oplus G_0$, along the same lines as the proof of the previous proposition, we get the desired equalities. ∎

Combining those two first results we get the following theorem.

THEOREM 3.4. *Let* $F_1 \oplus F_0 \in L^2(\mathbb{T}, \mathbb{C}^m) \oplus L^2(r_0\mathbb{T}, \mathbb{C}^m)$ *such that* F_1 *is log-integrable on* $\mathbb{T}$ *and* F_0 *is log-integrable on* $r_0\mathbb{T}$*. Then there is a function* $W_1 \oplus W_0 \in L^\infty(\mathbb{T}, \mathbb{C}^m) \oplus L^\infty(r_0\mathbb{T}, \mathbb{C}^m)$ *such that* $\|W_1(\mathrm{e}^{\mathrm{i}t})\|_{\mathbb{C}^m} = 1$ *almost everywhere on* $\mathbb{T}$*,* $\frac{1}{\|W_0\|_{\mathbb{C}^m}} \in L^\infty(r_0\mathbb{T})$ *and satisfying*

$$D_S(F_1 \oplus F_0) = D_S(W_1 \oplus W_0) = H^2(\partial A)(W_1 \oplus W_0).$$

Proof. Using Proposition 3.3, the doubly-invariant subspace for S generated by $F_1 \oplus F_0$ is equal to the one generated by $\frac{F_1}{u_0} \oplus \frac{F_0}{u_0}$, where u_0 is a scalar outer function in $H^2(\widehat{\mathbb{C}}\backslash\overline{r_0\mathbb{D}})$ such that $|u_0(r_0\mathrm{e}^{\mathrm{i}t})| = \|F_0(r_0\mathrm{e}^{\mathrm{i}t})\|_{\mathbb{C}^m}$ almost everywhere on $r_0\mathbb{T}$. Note that, since u_0 is outer, $\frac{F_1}{u_0}$ is also log-integrable on $\mathbb{T}$ whenever F_1 is. Using Proposition 3.2, the doubly-invariant subspace for S generated by $\frac{F_1}{u_0} \oplus \frac{F_0}{u_0}$ is equal to the one generated by $W_1 \oplus W_0$ where $W_1 = \frac{F_1}{u_0 u_1}$ and $W_0 = \frac{F_0}{u_0 u_1}$, with u_1 a scalar outer function on $\mathbb{T}$ satisfying $|u_1(\mathrm{e}^{\mathrm{i}t})| = \frac{\|F_1(\mathrm{e}^{\mathrm{i}t})\|_{\mathbb{C}^m}}{|u_0(\mathrm{e}^{\mathrm{i}t})|}$ almost everywhere on $\mathbb{T}$. Since u_1 and $\frac{1}{u_1}$ belong to $L^\infty(r_0\mathbb{T})$, W_0 satisfies the desired hypothesis.

It remains to prove that $D_S(W_1 \oplus W_0) = H^2(\partial A)(W_1 \oplus W_0)$. Obviously a reformulation of (1.1) is $H^2(\partial A) = D_S(\mathbb{1})$. Therefore

$$H^2(\partial A)(W_1 \oplus W_0) \subset D_S(W_1 \oplus W_0).$$

Now consider $T : L^2(\partial A) \to L^2(\partial A, \mathbb{C}^m)$ defined by $Tf = f(W_1 \oplus W_0)$. Since $\|W_1\|_{\mathbb{C}^m}$ and $\|W_0\|_{\mathbb{C}^m}$ are essentially bounded above and below on $\mathbb{T}$ and $r_0\mathbb{T}$, the linear mapping T is both bounded and bounded below. Therefore $TH^2(\partial A) = H^2(\partial A)(W_1 \oplus W_0)$ is a closed subspace of $L^2(\partial A, \mathbb{C}^m)$. It follows that the previous inclusion is, by the definition of D_S, an equality. ∎

In the case where we only have information on F_1, we have the following result on the smallest closed invariant subspace for S generated by $F_1 \oplus F_0$.

PROPOSITION 3.5. *Let* $F_1 \oplus F_0 \in L^2(\mathbb{T}, \mathbb{C}^m) \oplus L^2(r_0\mathbb{T}, \mathbb{C}^m)$*.*

(i) *If* F_1 *is log-integrable on* $\mathbb{T}$*, then*

$$I_S(F_1 \oplus F_0) = H^2(\mathbb{D})\Big(\frac{F_1}{u_1} \oplus \frac{F_0}{u_1}\Big),$$

where u_1 *is a scalar outer function on* $\mathbb{T}$ *satisfying* $|u_1(\mathrm{e}^{\mathrm{i}t})| = \|F_1(\mathrm{e}^{\mathrm{i}t})\|_{\mathbb{C}^m}$ *almost everywhere on* $\mathbb{T}$*.*

(ii) *If* F_1 *is not log-integrable on* $\mathbb{T}$, *then*

$$I_S(F_1 \oplus F_0) = R_S(F_1) \oplus I_S(F_0).$$

Proof. (i) We have $I_S(F_1 \oplus F_0) = I_S\left(\frac{F_1}{u_1} \oplus \frac{F_0}{u_1}\right)$ by Proposition 3.2. Since $\frac{F_1}{u_1} \in L^\infty(\mathbb{T}, \mathbb{C}^m)$ and $f_{|r_0\mathbb{T}} \in L^\infty(r_0\mathbb{T})$ whenever $f \in H^2(\mathbb{D})$, $I_S\left(\frac{F_1}{u_1} \oplus \frac{F_0}{u_1}\right)$ contains the invariant subspace for S defined by $H^2(\mathbb{D})\left(\frac{F_1}{u_1} \oplus \frac{F_0}{u_1}\right)$. Moreover, the subspace $H^2(\mathbb{D})\left(\frac{F_1}{u_1} \oplus \frac{F_0}{u_1}\right)$ is closed in $L^2(\partial A, \mathbb{C}^m)$ as the image of the closed subspace $H^2(\mathbb{D})$ by the bounded-below operator T defined by $T : H^2(\mathbb{D}) \to L^2(\partial A, \mathbb{C}^m)$, $Tf = f\left(\frac{F_1}{u_1} \oplus \frac{F_0}{u_1}\right)$. It follows that

$$I_S(F_1 \oplus F_0) = I_S\left(\frac{F_1}{u_1} \oplus \frac{F_0}{u_1}\right) = H^2(\mathbb{D})\left(\frac{F_1}{u_1} \oplus \frac{F_0}{u_1}\right).$$

(ii) Let $H_1 \oplus H_0$ in $L^2(\mathbb{T}, \mathbb{C}^m) \oplus L^2(r_0\mathbb{T}, \mathbb{C}^m)$ be orthogonal to $I_S(F_1 \oplus F_0)$. In other words,

$$\langle H_1, \mathrm{e}^{int} F_1 \rangle_{\mathbb{T}} + \langle H_0, r_0^n \mathrm{e}^{int} F_0 \rangle_{r_0\mathbb{T}} = 0, \ n \geqslant 0. \tag{3.1}$$

Therefore, $\langle H_1, \mathrm{e}^{int} F_1 \rangle_{\mathbb{T}} = O(r_0^n)$, $n \geqslant 0$. If one denotes by f_1 the scalar function on $\mathbb{T}$ defined by $\langle F_1, H_1 \rangle_{\mathbb{T}}$, then f_1 extends to a function in $H^1(\mathbb{T} \cup r\mathbb{T})$ where $r_0 < r < 1$. Indeed, denote by f_r the function in $L^2(r\mathbb{T})$ (and thus in $L^1(r\mathbb{T})$) defined by $f_r(r\mathrm{e}^{\mathrm{i}t}) = \sum_{n \in \mathbb{Z}} r^n \widehat{f_1}(n) \mathrm{e}^{int}$. Then $f_1 \oplus f_r \in H^1(\mathbb{T} \cup r\mathbb{T})$ using (1.1). This implies that $f_1 = \langle F_1, H_1 \rangle_{\mathbb{T}}$ is log-integrable, and therefore, since $\log |f_1(\mathrm{e}^{\mathrm{i}t})| \leqslant \log \|F_1(\mathrm{e}^{\mathrm{i}t})\|_{\mathbb{C}^m} + \log \|H_1(\mathrm{e}^{\mathrm{i}t})\|_{\mathbb{C}^m}$, this forces F_1 to be log-integrable, a contradiction. So f_1 is identically equal to 0 and then

$$\langle H_1, z^n F_1 \rangle_{\mathbb{T}} = 0, \ n \in \mathbb{Z}.$$

By (3.1), we have also $\langle H_0, z^n F_0 \rangle_{\mathbb{T}} = 0$ for all $n \geqslant 0$. Thus $H_1 \oplus H_0$ is orthogonal to $R_S(F_1) \oplus I_S(F_0)$, and then $R_S(F_1) \oplus I_S(F_0) \subset I_S(F_1 \oplus F_0)$. Since the converse inclusion is always true, we get the desired equality. ∎

The natural dual version of the above proposition is the following. We omit the proof, which can be deduced along the same lines as Proposition 3.5 via the changes detailed in the proof of Proposition 3.3.

PROPOSITION 3.6. *Let* $F_1 \oplus F_0 \in L^2(\mathbb{T}, \mathbb{C}^m) \oplus L^2(r_0\mathbb{T}, \mathbb{C}^m)$.

(i) *If* F_0 *is log-integrable on* $r_0\mathbb{T}$, *then*

$$I_{S^{-1}}(F_1 \oplus F_0) = H^2(\widehat{\mathbb{C}} \setminus r_0\overline{\mathbb{D}})\left(\frac{F_1}{u_0} \oplus \frac{F_0}{u_0}\right),$$

where u_0 *is a scalar outer function on* $r_0\mathbb{T}$ *satisfying* $|u_0(r_0\mathrm{e}^{\mathrm{i}t})| = \|F_0(r_0\mathrm{e}^{\mathrm{i}t})\|_{\mathbb{C}^m}$ *almost everywhere on* $r_0\mathbb{T}$.

(ii) *If* F_0 *is not log-integrable on* $r_0\mathbb{T}$, *then* $I_{S^{-1}}(F_1 \oplus F_0) = I_{S^{-1}}(F_1) \oplus R_S(F_0)$.

We are now ready to describe the invariant subspaces for S generated by a single function.

THEOREM 3.7. *Let $F_1 \in L^2(\mathbb{T}, \mathbb{C}^m)$ and $F_0 \in L^2(r_0\mathbb{T}, \mathbb{C}^m)$. Then we have:*

(i) *If F_1 is not log-integrable on $\mathbb{T}$ and if F_0 is log-integrable on $r_0\mathbb{T}$, then*

$$I_S(F_1 \oplus F_0) = P_1 L^2(\mathbb{T}, \mathbb{C}^m) \oplus H^2(r_0\mathbb{D}) \frac{F_0}{u_0}$$

where u_0 is an outer function in $H^2(r_0\mathbb{D})$ with $|u_0(r_0 \mathrm{e}^{\mathrm{i}t})| = \|F_0(r_0\mathrm{e}^{\mathrm{i}t})\|_{\mathbb{C}^m}$ almost everywhere on $r_0\mathbb{T}$ and where P_1 is a measurable projection-valued function on $\mathbb{T}$.

(ii) *If F_0 is not log-integrable on $r_0\mathbb{T}$ and if F_1 is log-integrable on $\mathbb{T}$, then*

$$I_{S^{-1}}(F_1 \oplus F_0) = H^2(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}) \frac{F_1}{u_1} \oplus P_2 L^2(r_0\mathbb{T}, \mathbb{C}^m)$$

where u_1 is an outer function in $H^2(\mathbb{D})$ such that $|u_1(\mathrm{e}^{\mathrm{i}t})| = \|F_1(\mathrm{e}^{\mathrm{i}t})\|_{\mathbb{C}^m}$ almost everywhere on $r_0\mathbb{T}$ and where P_2 is a measurable projection-valued function on $r_0\mathbb{T}$.

(iii) *If neither F_0 nor F_1 are log-integrable, then*

$$I_S(F_1 \oplus F_0) = PL^2(\partial A, \mathbb{C}^m) = I_{S^{-1}}(F_1 \oplus F_0)$$

where P is a measurable projection-valued function on ∂A.

Proof. (i) The second assertion of Proposition 3.5 implies that $I_S(F_1 \oplus F_0) = R_S(F_1) \oplus I_S(F_0)$. By Lemma 2.2, $R_S(F_1) = P_1 L^2(\mathbb{T}, \mathbb{C}^m)$ where P_1 is a measurable projection-valued function on $\mathbb{T}$. Since F_0 is log-integrable, then $I_S(F_0) = I_S\Big(\frac{F_0}{u_0}\Big)$ where u_0 is an outer function in $H^2(r_0\mathbb{D})$ such that $|u_0(r_0\mathrm{e}^{\mathrm{i}t})| = \|F_0(r_0\mathrm{e}^{\mathrm{i}t})\|_{\mathbb{C}^m}$ almost everywhere on $r_0\mathbb{T}$. Since $I_S\Big(\frac{F_0}{u_0}\Big)$ contains $H^2(r_0\mathbb{D})\frac{F_0}{u_0}$ and since this last subspace is closed as the range of a bounded-below operator, it follows that $I_S\Big(\frac{F_0}{u_0}\Big) = H^2(r_0\mathbb{D})\frac{F_0}{u_0}$.

(ii) If F_0 is not log-integrable, the second assertion of Proposition 3.6 implies that $I_{S^{-1}}(F_1 \oplus F_0) = I_{S^{-1}}(F_1) \oplus R_S(F_0)$. Since F_1 is log-integrable, $I_{S^{-1}}(F_1) = H^2(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})\frac{F_1}{u_1}$, where u_1 is an outer function in $H^2(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})$ such that $|u_1(\mathrm{e}^{\mathrm{i}t})| = \|F_1(\mathrm{e}^{\mathrm{i}t})\|_{\mathbb{C}^m}$ almost everywhere on $\mathbb{T}$. Since $I_{S^{-1}}\Big(\frac{F_1}{u_1}\Big)$ contains $H^2(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})\frac{F_1}{u_1}$, and since this last subspace is closed as the range of a bounded below operator, it follows that $I_{S^{-1}}(F_1 \oplus F_0) = H^2(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})\frac{F_1}{u_1} \oplus P_2 L^2(r_0\mathbb{T}, \mathbb{C}^m)$.

(iii) Since F_1 is not log-integrable, $I_S(F_1 \oplus F_0) = R_S(F_1) \oplus I_S(F_0)$. It remains to prove that if F_0 is not log-integrable then $I_S(F_0) = D_S(F_0)$. To prove this, it is sufficient to check that whenever $H_0 \perp I_S(F_0)$, then $H_0 \perp D_S(F_0)$. Now, $H_0 \perp I_S(F_0)$ implies that the negative Fourier coefficients of the scalar $L^1(r_0\mathbb{T})$-function $f_0 := \langle F_0, H_0 \rangle$ are equal to 0. Therefore, f_0 extends to a function in $H^1(r_0\mathbb{D})$ and thus f_0 is log-integrable. This forces F_0 to be log-integrable, a contradiction. Thus we have f_0 identically equal to 0 and then $H_0 \perp D_S(F_0)$. By Lemma 2.2, $D_S(F_0) = R_S(F_0) = P_0 L^2(r_0\mathbb{T}, \mathbb{C}^m)$ where P_0 is a measurable projection-valued function on $r_0\mathbb{T}$. Now, taking $P = P_1 \oplus P_0$, we get the desired result. Using similar arguments we easily prove that $I_{S^{-1}}(F_1) = D_S(F_1)$ whenever F_1 is not log-integrable. The last equality follows. ∎

It remains to describe the doubly-invariant subspace for S generated by $F = F_1 \oplus F_0$ in the case where F_1 or F_0 is not log-integrable.

THEOREM 3.8. *Let $F_1 \in L^2(\mathbb{T}, \mathbb{C}^m)$ and $F_0 \in L^2(r_0\mathbb{T}, \mathbb{C}^m)$. Suppose that F_1 or F_0 is not log-integrable. Then*

$$D_S(F_1 \oplus F_0) = D_S(F_1) \oplus D_S(F_0) = PL^2(\partial A, \mathbb{C}^m)$$

where P is a measurable projection-valued function on ∂A.

Proof. Suppose that F_1 is not log-integrable. The second assertion of Proposition 3.5 asserts that $I_S(F_1 \oplus F_0) = D_S(F_1) \oplus I_S(F_0)$. In particular $0 \oplus I_S(F_0) \subset I_S(F_1 \oplus F_0)$. Therefore $D_S(F_1 \oplus F_0)$ contains $0 \oplus I_S(F_0)$ and then contains $0 \oplus D_S(F_0)$. Then we get $D_S(F_1 \oplus F_0) = D_S(F_1) \oplus D_S(F_0)$ since $D_S(F_1 \oplus F_0)$ is always contained in $D_S(F_1) \oplus D_S(F_0)$. If F_0 is not log-integrable, the second assertion of Proposition 3.6 asserts that $I_{S^{-1}}(F_1 \oplus F_0) = I_{S^{-1}}(F_1) \oplus D_S(F_0)$. As previously, since $D_{S^{-1}}(F_1 \oplus F_0) = D_S(F_1 \oplus F_0)$, we get $D_S(F_1 \oplus F_0) = D_S(F_1) \oplus D_S(F_0)$. The vector-valued Wiener theorem implies the existence of P a measurable projection-valued function on ∂A such that $D_S(F_1) \oplus D_S(F_0) = PL^2(\partial A, \mathbb{C}^m)$. ∎

We may summarise the structure theorems that we have derived, by means of the following tables.

Description of $I_S(F)$ and $I_{S^{-1}}(F)$, where $F = F_1 \oplus F_0$:

F_1	F_0 is log-integrable :	
log-int.	Yes	No
Yes	$I_S(F) = H^2(\mathbb{D})F/u_1$ $I_{S^{-1}}(F) = H^2(\widehat{\mathbb{C}} \setminus r_0\overline{\mathbb{D}})F/u_0$	$I_{S^{-1}}(F) = H^2(\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}})(\frac{F_1}{u_1}) \oplus P_2L^2(r_0\mathbb{T})$ $I_S(F) = H^2(\mathbb{D})F/u_1$
No	$I_S(F) = P_1L^2(\mathbb{T}, \mathbb{C}^m) \oplus H^2(r_0\mathbb{D})(\frac{F_0}{u_0})$ $I_{S^{-1}}(F) = H^2(\widehat{\mathbb{C}} \setminus r_0\overline{\mathbb{D}})F/u_0$	$I_S(F) = I_{S^{-1}}(F)$ $= PL^2(\partial A, \mathbb{C}^m)$

Description of $D_S(F_1 \oplus F_0)$:

F_1	F_0 is log-integrable :	
log-int.	Yes	No
Yes	$H^2(\partial A)(W_1 \oplus W_0)$	$PL^2(\partial A, \mathbb{C}^m)$
No	$PL^2(\partial A, \mathbb{C}^m)$	$PL^2(\partial A, \mathbb{C}^m)$

The following result is a simple consequence of the above classification theorem. We write $\sigma_{\mathrm{p}}(T)$ for the point spectrum of an operator T, i.e., the set of eigenvalues.

COROLLARY 3.9. *For each doubly-invariant subspace $M \subset L^2(\partial A, \mathbb{C}^m)$ one has*

$$\sigma_{\mathrm{p}}((S_{|M})^*) \subset A.$$

Proof. This is equivalent to the statement that $(S - \lambda \mathrm{Id})M$ is dense in M whenever $\lambda \notin A$, which follows since then $D_S((S - \lambda \mathrm{Id})F) = D_S(F)$ for all $F \in M$. ∎

4. DOUBLY-INVARIANT SUBSPACES

4.1. COMPLETELY NON-REDUCING SUBSPACES.

DEFINITION 4.1. A closed subspace M in $L^2(\partial A, \mathbb{C}^m)$ is called *completely non-reducing* if it contains no trivial reducing subspaces.

LEMMA 4.2. *Let M be a doubly-invariant subspace for S and let M_1 be a reducing subspace for S. Then $M_2 := M \cap M_1^{\perp}$ is doubly invariant for S.*

Proof. First we check that M_2 is invariant for S. Indeed, for $F_1 \in M_1$ and $F_2 \in M_2$, we have:

$$\langle SF_2, F_1 \rangle = \langle F_2, S^* F_1 \rangle = 0,$$

since M_1 is reducing. Therefore $SM_2 \subset M_2$. Now we check that M_2 is invariant for S^{-1}, that is that $M_2 \perp (S^{-1})^* M_1$. Since M_1 is reducing for S, using the vector-valued Wiener theorem, there exists a measurable projection-valued function P such that for almost all $\xi \in \partial A$, $P(\xi) : \mathbb{C}^m \to \mathcal{I}(\xi)$ where $\mathcal{I}(\xi) = \{F(\xi) : F \in M_1\}$. Since $(S^{-1})^* F(\mathrm{e}^{\mathrm{i}t}) = \mathrm{e}^{\mathrm{i}t} F(\mathrm{e}^{\mathrm{i}t}) \in P(\mathrm{e}^{\mathrm{i}t})\mathbb{C}^m$ and $(S^{-1})^* F(r_0 \mathrm{e}^{\mathrm{i}t}) = \frac{\mathrm{e}^{\mathrm{i}t}}{r_0} F(r_0 \mathrm{e}^{\mathrm{i}t}) \in P(r_0 \mathrm{e}^{\mathrm{i}t})\mathbb{C}^m$, $(S^{-1})^* F \in PL^2(\partial A, \mathbb{C}^m) = M_1$ for $F \in M_1$, and thus we get the desired result. ∎

Using Lemma 4.2, in the sequel we study the doubly-invariant subspaces that are completely non-reducing.

LEMMA 4.3. *In the scalar case the doubly-invariant subspaces that are completely non-reducing coincide with the doubly-invariant subspaces that are non-reducing.*

Proof. Suppose that M is doubly invariant but contains a nontrivial reducing subspace M_1. It follows, via Wiener's theorem, that $M_1 = \chi_E L^2(\partial A)$ where E and its complement are of positive measure. Now for any $f \in M$, write $f = \chi_E f + \chi_{\partial A \setminus E} f$, where $\chi_E f \in M_1 \subset M$. Then, $\chi_{\partial A \setminus E} f \in M$ and $D_S(\chi_{\partial A \setminus E} f) \subset M$ for all $f \in M$. Since $\chi_E f$ and $\chi_{\partial A \setminus E} f$ are not log-integrable, we get $D_S(\chi_{\partial A \setminus E} f) = R_S(\chi_{\partial A \setminus E} f)$ and $D_S(\chi_E f) = R_S(\chi_E f)$. Therefore the subspace M is reducing. ∎

It is easily seen by taking direct sums that the above result does not hold in $L^2(\partial A, \mathbb{C}^m)$ when $m > 1$.

4.2. ANALYTIC DOUBLY-INVARIANT SUBSPACES. In this section we restrict ourselves to closed subspaces of analytic functions in the Hardy spaces $H^2(\partial A, \mathbb{C}^m)$. In [8], Royden proved that the nontrivial closed subspaces of $H^2(\partial A)$ that are doubly invariant for S have the form $\phi H^2(\partial A)$, where $\phi \in H^\infty(\partial A)$ and is inner (constant in modulus on each component of ∂A). His proof is based on the inner-outer factorization of functions in Hardy spaces of multiply connected domains (cf. [5]). Note that it follows from Sarason's result that every non-reducing doubly-invariant subspace M of $L^2(\partial A)$ has the form $H^2(\partial A)(w_1 \oplus w_0)$, where w_1 is unimodular on $\mathbb{T}$ and w_0 is bounded and bounded below on $r_0\mathbb{T}$ (scalar version of Theorem 3.4). Obviously, if $M \subset H^2(\partial A)$, then $(w_1 \oplus w_0) \in H^\infty(\partial A)$ and ϕ is obtained by taking its inner factor.

First of all we prove that if M is a nontrivial closed subspace in $H^2(\partial A, \mathbb{C}^m)$ (with $m \geqslant 2$), then there exist at most m functions in M "generating" the smallest closed reducing subspace containing M. Recall that, in the scalar case, using Wiener's theorem, every function f in $M \setminus \{0\}$ satisfies $R_S(f) = M = L^2(\partial A)$.

THEOREM 4.4. *Let M be a nontrivial closed subspace in $H^2(\partial A, \mathbb{C}^m)$. Then, there exists a set of functions $G^1, \dots, G^k$ in M with $k \leqslant m$, such that*

$$R_S(M) = R_S(G^1) + \cdots + R_S(G^k).$$

Proof. Let $G^1 = \begin{pmatrix} g_1^1 \\ \vdots \\ g_m^1 \end{pmatrix}$ be a nonconstant function in M. For any $G^2 = \begin{pmatrix} g_1^2 \\ \vdots \\ g_m^2 \end{pmatrix}$ in M, consider the $H^1(\partial A)$-functions $h_j = g_j^1 g_1^2 - g_1^1 g_j^2$ for $2 \leqslant j \leqslant m$. Then either every h_j is identically equal to 0, and then $R_S(G^2) \subset R_S(G^1)$, or else there exists a function h_{j_0} with $2 \leqslant j_0 \leqslant m$ which is nonzero almost everywhere on ∂A, and then we consider the reducing subspace $R_S(G^1) + R_S(G^2) = P_2 L^2(\partial A, \mathbb{C}^m)$, where for almost all $\xi \in \partial A$, the rank of $P_2(\xi)$ is equal to 2. Either $R_S(M) = R_S(G^1) + R_S(G^2)$, or we take a third function $G^3 = \begin{pmatrix} g_1^3 \\ \vdots \\ g_m^3 \end{pmatrix} \in M$.

Then we consider the $H^{2/3}(\partial A)$-functions

$$h_j = \begin{vmatrix} g_1^1 & g_1^2 & g_1^3 \\ g_{j_0}^1 & g_{j_0}^2 & g_{j_0}^3 \\ g_j^1 & g_j^2 & g_j^3 \end{vmatrix}$$

for $2 \leqslant j \leqslant m$, $j \neq j_0$. Then either every h_j is identically equal to 0, and then $R_S(G^3) \subset R_S(G^1) + R_S(G_2)$, or else there exists a function h_j with $3 \leqslant j \leqslant m$ which is nonzero almost everywhere on ∂A, and then we consider the reducing subspace $R_S(G^1) + R_S(G^2) + R_S(G^3) = P_3 L^2(\partial A, \mathbb{C}^m)$, where for almost all $\xi \in \partial A$, the rank of $P_3(\xi)$ is equal to 3. We continue in this way, until either $R_S(M) = R_S(G^1) + \cdots + R_S(G^k)$ for some $k < m$, or there exist $m-1$ functions in M such that $R_S(G^l)$ does not belong to $R_S(G^1) + \cdots + R_S(G^{l-1})$ for all $2 \leqslant l \leqslant m-1$. Then $R_S(G^1) + \cdots + R_S(G^{m-1}) = P_{m-1} L^2(\partial A, \mathbb{C}^m)$, where for almost all $\xi \in \partial A$, the rank of $P_{m-1}(\xi)$ is equal to $m-1$. Take $G^m = \begin{pmatrix} g_1^m \\ \vdots \\ g_m^m \end{pmatrix} \in M$, and consider the $H^{2/m}(\partial A)$-functions

$$h = \begin{vmatrix} g_1^1 & \cdots & g_1^m \\ \vdots & \vdots & \vdots \\ g_m^1 & \cdots & g_m^m \end{vmatrix}.$$

Then either h is identically equal to 0, and then

$$R_S(G^m) \subset R_S(G^1) + \cdots + R_S(G_{m-1}),$$

or else the function h is nonzero almost everywhere on ∂A, and then we consider the reducing subspace $R_S(G^1) + \cdots + R_S(G^m) = P_m L^2(\partial A, \mathbb{C}^m)$, where for almost all $\xi \in \partial A$, the rank of $P_m(\xi)$ is equal to m. It follows that P_m is the identity map and thus

$$R_S(G^1) + \cdots + R_S(G^m) = L^2(\partial A, \mathbb{C}^m).$$

Note that the analyticity has been used to show that the rank of a measurable projection-valued function of ξ is almost everywhere independent of ξ. ∎

PROPOSITION 4.5. *Let $F \in H^2(\partial A, \mathbb{C}^m) \setminus \{0\}$. Then there exists a positive constant c and $W \in H^\infty(\partial A, \mathbb{C}^m)$ satisfying $\|W(\xi)\|_{\mathbb{C}^m} = 1$ a.e. on $\mathbb{T}$ and $\|W(\xi)\|_{\mathbb{C}^m} = c$ a.e. on $r_0\mathbb{T}$, such that we have*

$$D_S(F) = H^2(\partial A)W \quad \textit{and} \quad R_S(F) = L^2(\partial A)W.$$

Proof. Since $F \in H^2(\partial A, \mathbb{C}^m) \setminus \{0\}$, it follows that $\log \|F\| \in L^1(\partial A)$. Then we define the function v on A by

$$v(z) = \int_{\partial A} \log \|F(\xi)\|_{\mathbb{C}^m} \frac{\partial g(z, \xi)}{\partial n} \mathrm{d}s(\xi).$$

Then, although A is not simply connected, there exist a constant s and a real harmonic function h such that

$$\psi(z) = v(z) - s \log |z| + \mathrm{i}h(z)$$

is holomorphic. Now $\phi(z) := \exp(\psi(z))$ is an outer function whose non-tangential boundary values satisfy $|\phi(\xi)| = \frac{\|F(\xi)\|_{\mathbb{C}^m}}{|\xi|^s}$. Set $W = \frac{F}{\phi}$ and observe that $W \in$

$H^\infty(\partial A, \mathbb{C}^m)$ with $\|W(\xi)\|_{\mathbb{C}^m} = 1$ a.e. on $\mathbb{T}$ and $\|W(\xi)\|_{\mathbb{C}^m} = r_0^s$ a.e. on $r_0\mathbb{T}$. Since $\phi \in H^2(\partial A)$, ϕ is the L^2-norm limit of a sequence of trigonometric polynomials $(p_n)_n$. Since

$$\|F - p_nW\|_2^2 = \|(\phi - p_n)W\|_2^2 \leqslant \max(c^2, 1)\|\phi - p_n\|_2^2$$

with $c = r_0^s$, it follows that $\|F - p_nW\|_2$ tends to 0 as n tends to ∞. Therefore we have $D_S(F) \subset D_S(W)$. Moreover, since ϕ is outer, there exists a sequence of trigonometric polynomials $(q_n)_n$ such that $\lim_{n\to\infty} \|q_n\phi - 1\|_2 = 0$. It follows that

$$\|q_nF - W\|_2^2 = \|(q_n\phi - 1)W\|_2^2 \leqslant \max(c^2, 1)\|q_n\phi - 1\|_2^2,$$

and thus $\|q_nF - W\|_2$ tends to 0 as n tends to ∞. Therefore we have $D_S(W) \subset D_S(F)$, and then $D_S(W) = D_S(F)$.

We can also check that $R_S(F) = R_S(W)$. By Corollary 2.4 $R_S(F) = \{G \in L^2(\partial A, \mathbb{C}^m) : G(\xi) \in \mathbb{C}F(\xi)$ for a.e. $\xi \in \partial A\}$. Since $F = \phi W$, where $\phi(\xi) \neq 0$ a.e. on ∂A, $R_S(F) = \{G \in L^2(\partial A, \mathbb{C}^m) : G(\xi) \in \mathbb{C}W(\xi)$ for a.e. $\xi \in \partial A\} = R_S(W)$. Since W is bounded and bounded below, $L^2(\partial A)W$ is closed and thus equal to $R_S(W)$. ∎

REMARK 4.6. Following Wiener's theorem, there exists a projection-valued function P such that $R_S(F) = PL^2(\partial A, \mathbb{C}^m)$. A natural choice for P is $J_{1/c}W \otimes e_1$ where e_1 is the first vector of the canonical orthonormal basis of $\mathbb{C}^m$ and where

$$J_{1/c} = \Big(P_{L^2(\mathbb{T},\mathbb{C}^m)} + \frac{1}{c}P_{L^2(r_0\mathbb{T},\mathbb{C}^m)}\Big)\ \Big(= \Big(\frac{r_0^2\mathrm{Id} - SS^*}{r_0^2 - 1} + \frac{1}{c}\frac{SS^* - \mathrm{Id}}{r_0^2 - 1}\Big)\Big).$$

The proof of the next result is based on the proof used by Sarason [9] in the scalar case. Using Theorem 4.4, we can prove that given a nontrivial doubly-invariant subspace M in $H^2(\partial A, \mathbb{C}^m)$, there exists a finite set of functions in M "generating" M.

THEOREM 4.7. *Let M be a nontrivial doubly-invariant subspace (completely not reducing) in $H^2(\partial A, \mathbb{C}^m)$. Then there exists a finite set of at most m bounded functions in M, say $F^1, \dots, F^r$, such that*

$$M = D_S(F^1) \oplus^\perp \cdots \oplus^\perp D_S(F^r).$$

Moreover, if $R_S(M) = PL^2(\partial A, \mathbb{C}^m)$ where P is a projection-valued function, the rank of $P(\xi)$ is constant and equal to r, for all $\xi \in \partial A$.

Proof. First we claim that there exists $\lambda_0 \in A$ such that $M \ominus (S - \lambda_0\mathrm{Id})M \neq \{0\}$. Indeed, if not, for all $\lambda \in A$ and all $e \in \mathbb{C}^m$, we have $P_M(k_\lambda e) = 0$, where P_M is the orthogonal projection onto M and where k_λ is the reproducing kernel in $H^2(A)$ at λ. Since $\mathrm{Span}\{(k_\lambda e) : \lambda \in A, e \in \mathbb{C}^m\}$ is equal to $H^2(\partial A, \mathbb{C}^m)$, it follows that $M = \{0\}$, a contradiction.

Take $F^1 \in M \ominus (S - \lambda_0\mathrm{Id})M$. By Proposition 4.5, there exists a function $W_1 \in H^\infty(\partial A, \mathbb{C}^m)$ such that $\|W_1(\xi)\|_{\mathbb{C}^m}$ is constant almost everywhere on each circle of ∂A and such that $D_S(F^1) = H^2(\partial A)W_1$ and $R_S(F^1) = L^2(\partial A)W_1$. Now, consider $M_1 := M \cap R_S(F^1)$ which contains $N_1 := D_S(F^1)$, and take $N_2 := D_{S^*}((S^* - \overline{\lambda_0}\mathrm{Id})F^1)$. Since $S^{*m}S^n$ is a linear combination of S^{n-m} and $S^{*(m-n)}$ for $n \neq m$ in $\mathbb{Z}$, and $S^{*n}S^n$ is a linear combination of Id and S^*S, it follows that

$$R_S(F^1) \subset N_1 + N_2 + \mathbb{C}S^*SF^1.$$

Since $N_2 \subset R_S(F^1) \cap M^\perp$, it follows that $M_1 \subset N_1 + M \cap \mathbb{C}S^*SF^1$. We get that $\dim(M_1 \ominus D_S(F^1)) \leqslant 1$. In other words,

$$M \cap R_S(F^1) = D_S(F^1) \quad \text{or} \quad M \cap R_S(F^1) = D_S(F^1) + \mathbb{C}S^*SF^1.$$

Now, let us check that there exists a function G^1 in M such that $M_1 = D_S(G^1)$. If $\dim(M_1 \ominus D_S(F^1)) = 0$, take $G^1 = F^1$. It remains to consider the case when $M_1 = D_S(F^1) + \mathbb{C}S^*SF^1$, i.e., when

$$\dim(M_1 \ominus D_S(F^1)) = 1. \tag{4.1}$$

Take $G \in M_1 \ominus D_S(F^1)$, with $G \neq 0$. Then $P_{M_1}S^*G \perp D_S(F^1)$, and since $\dim(M_1 \ominus D_S(F^1)) = 1$, there exists a unique $\mu_0 \in \mathbb{C}$ such that $P_{M_1}S^*G = \overline{\mu}_0 G$; equivalently, $\mu_0 \in \sigma_{\mathrm{p}}((S_{|M_1})^*)$. By Corollary 3.9, we see that $\mu_0 \in A$.

Now $D_S(F_1) = H^2(\partial A)W_1$ as in Proposition 4.5, and so

$$\dim(D_S(F^1) \ominus (S - \mu_0\mathrm{Id})D_S(F^1)) = 1 \tag{4.2}$$

(note that the operator $S - \mu_0\mathrm{Id}$ is bounded below, and so $(S - \mu_0\mathrm{Id})D_S(F_1)$ is closed). We also have

$$\dim((S - \mu_0\mathrm{Id})M_1 \ominus (S - \mu_0\mathrm{Id})D_S(F^1)) = 1, \tag{4.3}$$

given that $\dim(M_1 \ominus D_S(F^1)) = 1$.

We summarise these observations in the following diagram.

$$\begin{array}{ccccc} & & M_1 = M \cap R_S(F^1) & & \\ & \swarrow & & \searrow 1 & \\ (S - \mu_0\mathrm{Id})M_1 & & & & D_S(F^1) = H^2(\partial A)W_1 \\ & 1 \searrow & & \swarrow 1 & \\ & & (S - \mu_0\mathrm{Id})D_S(F^1) & & \end{array}$$

Now it follows from (4.1), (4.2) and (4.3) that

$$\dim(M_1 \ominus (S - \mu_0\mathrm{Id})M_1) = 1,$$

with $G \in M_1 \ominus D_S(F^1)$ and $G \in M_1 \ominus (S - \mu_0\mathrm{Id})M_1$.

Hence $(S - \mu_0\mathrm{Id})M_1 = D_S(F^1)$, and so $F^1(\mu_0) = 0$; also F^1 is analytic, $(S - \mu_0\mathrm{Id})^{-1}F^1 \in M_1$, and then $M_1 = D_S(G^1)$, with $G^1 = (S - \mu_0\mathrm{Id})^{-1}F^1$.

At this stage we have proved that there exists a function $G^1 \in M$ such that $M = D_S(G^1) \oplus^\perp M'$, where $M' = M \cap R_S(F^1)^\perp$ is still doubly invariant, by Lemma 4.2.

By induction we may arrive at an expression

$$M = D_S(G^1) \oplus^\perp D_S(G^2) \oplus^\perp \cdots \oplus^\perp D_S(G^r) \oplus^\perp M'',$$

for functions $G^1, \dots, G^r \in M$ and where M'' is also doubly invariant for S. We wish to show that this procedure terminates with $M'' = \{0\}$ for some $r \leqslant m$.

Using Proposition 4.5, there exist functions $W_1, \dots, W_r$ in $H^\infty(\partial A, \mathbb{C}^m)$ such that $\|W_k(\xi)\|_{\mathbb{C}^m}$ is 1 on $\mathbb{T}$ and equal to a positive constant c_k on $r_0\mathbb{T}$, such that

$$\begin{cases} M = H^2(\partial A)W_1 \oplus^\perp \cdots \oplus^\perp H^2(\partial A)W_r \oplus^\perp M'', \\ R_S(M) = L^2(\partial A)W_1 + \cdots + L^2(\partial A)W_r + R_S(M''). \end{cases}$$

By Remark 4.6, taking $J_{1/c} = P_{L^2(\mathbb{T},\mathbb{C}^m)} + \frac{1}{c}P_{L^2(r_0\mathbb{T},\mathbb{C}^m)}$, we have

$$R_S(M) = J_{1/c_1}(L^2(\partial A)W_1) + \cdots + J_{1/c_r}(L^2(\partial A)W_r) + R_S(M'').$$

Now we consider the operator-valued function Q defined almost everywhere on ∂A by

$$Q(\xi) = r^{-1/2}(J_{1/c_1}W_1(\xi), \dots, J_{1/c_r}W_r(\xi)).$$

By construction we easily check that $Q(\xi)$ is an orthogonal projection and then

$$R_S(M) = QL^2(\partial A, \mathbb{C}^m) + R_S(M''),$$

where the rank of $Q(\xi)$ is equal to r for almost all $\xi \in \partial A$. Using Wiener's theorem, there exists a measurable projection-valued function P such that $R_S(M) = PL^2(\partial A, \mathbb{C}^m)$. Note that, by Theorem 4.4, since the rank k of $P(\xi)$ is independent of ξ and is less or equal than m, necessarily we have $r \leqslant k \leqslant m$; thus the induction must terminate with $M'' = \{0\}$ at some stage with $r \leqslant m$. As a consequence we also get $R_S(M) = QL^2(\partial A, \mathbb{C}^m)$, which implies that $k = r$. ∎

4.3. OPERATOR GRAPHS. One application of the study of shift-invariant subspaces is to the study of closed shift-invariant operators. For the Hardy space of the disc, this idea is due to Georgiou and Smith [2], who gave applications to control theory. Now for the annulus we have the following particular case of Theorem 4.7.

THEOREM 4.8. *Let M be a nontrivial closed subspace in $H^2(\partial A, \mathbb{C}^2)$. If M is both doubly invariant and the graph of a (not necessarily bounded) operator, then there exists a bounded function $\Theta \in M$ such that*

$$M = D_S(\Theta) = H^2(\partial A)\Theta.$$

Proof. By Theorem 4.7, the only case to consider is the case when there exist two functions $\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}$ in M such that

$$M = D_S \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \oplus^\perp D_S \begin{pmatrix} f_2 \\ g_2 \end{pmatrix},$$

with $|f_1|^2 + |g_1|^2$ and $|f_2|^2 + |g_2|^2$ equal to 1 on $\mathbb{T}$ and equal to a positive constant on $r_0\mathbb{T}$.

Note that

$$f_1 \begin{pmatrix} f_2 \\ g_2 \end{pmatrix} - f_2 \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} = \begin{pmatrix} 0 \\ f_1 g_2 - f_2 g_1 \end{pmatrix} \in M.$$

Since M is the graph of an operator, necessarily

$$f_1 g_2 - g_1 f_2 = 0. \tag{4.4}$$

Moreover, since $D_S \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \perp D_S \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}$, we have also

$$f_1 \overline{f_2} + g_1 \overline{g_2} = 0. \tag{4.5}$$

Multiplying (4.5) by f_2 and using (4.4), we obtain:

$$f_1 |f_2|^2 + f_1 |g_2|^2 = 0.$$

It follows that $f_1 = 0$ and then $g_1 = 0$ since M is the graph of an operator. Therefore M is "singly" generated. ∎

An analogous result holds for $L^2(\partial A)$, under slightly stronger hypotheses, but using more elementary methods. Note that using the analyticity was essential for us to deduce Theorem 4.7.

THEOREM 4.9. *Let M be a nontrivial closed subspace of $L^2(\partial A, \mathbb{C}^2)$. If M is both doubly invariant and the graph of a (not necessarily bounded) operator T whose spectrum is not the whole plane, then there exists a bounded function $\Theta \in M$ such that*

$$M = D_S(\Theta) = L^2(\partial A)\Theta.$$

Proof. Take $\lambda \in \mathbb{C}$ not in the spectrum of T. Then consider the bounded operator $V = (T - \lambda \mathrm{Id})^{-1}$ which commutes with S. Let $V(1 \oplus 0) = h_1 \oplus h_2$, so $V(S^n(1 \oplus 0)) = S^n(h_1 \oplus h_2)$ for all $n \in \mathbb{Z}$. Since V is bounded it implies that $h_2 = 0$ and $h_1 \in L^\infty(\mathbb{T})$ because $V(f \oplus 0) = h_1 f \oplus 0$ for $f \in L^2(\mathbb{T})$ (see Chapter 3 of [7]). Similarly, there exists $h_2' \in L^\infty(r_0\mathbb{T})$ such that $V(0 \oplus g) = (0 \oplus h_2' g)$ for $g \in L^2(r_0\mathbb{T})$. Thus the graph of V is $\left\{ \begin{pmatrix} f \\ hf \end{pmatrix} : f \in L^2(\partial A) \right\}$, where $h = h_1 \oplus h_2' \in L^\infty(\partial A)$. Now $y = Tx$ if and only if $(T - \lambda \mathrm{Id})^{-1}(y - \lambda x) = x$, so $M = L^2(\partial A) \begin{pmatrix} h \\ 1 + \lambda h \end{pmatrix}$. ∎

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