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GENERATORS AND RELATIONS FOR CERTAIN SPLITTING TREE ALGEBRAS

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ABSTRACT. A splitting tree algebra which splits into copies of $\mathbb{C}$ at each singular point is shown to be the universal C^*-algebra generated by finitely many projections with respect to certain relations. Moreover, these generators and relations are described explicitly.

KEYWORDS: *Generators and relations for C^*-algebras, splitting tree algebras.*

MSC (2000): 46L35, 46L99.

1. INTRODUCTION

Generating sets and relations for C^*-algebras play an important role in the classification program of amenable C^*-algebras (to show the existence of certain $*$-homomorphisms from building block C^*-algebras to either other building blocks or more general (even simple) C^*-algebras, see, for example, [2] and [5]), and it has been studied extensively by several authors (see the monograph [4] of T. Loring, for example).

Splitting tree algebras (first considered by H. Su in [6]) are certain unital sub-C^*-algebras of homogeneous C^*-algebras with 1-dimensional spectra. More precisely, let T be a tree (as a topological space) with finitely many vertices $\{v_i\}_{i=1}^n$, k be a natural number, and $\{\{k_{i1},\ldots,k_{ij_i}\}\}_{i=1}^n$ be n partitions of k; then a splitting tree algebra A is defined to be

$$A := \{f \in M_k(C(T)) : f(v_i) \in M_{k_{i1}}(\mathbb{C}) \oplus \cdots \oplus M_{k_{ij_i}}(\mathbb{C}) \text{ for all } i\}.$$

We call the vertices $\{v_i\}$ the singular points of A. In the case T only consists of two vertices, we also call A a splitting interval algebra. (The most special case of this — beyond just an ordinary interval algebras — is well known as the universal unital C^*-algebra generated by two projections.)

Certain classes of inductive limits of splitting tree algebras were shown to be classified by their K-theory information by Su in [6], and X. Jiang and Su in [3].

Splitting tree algebras also play an important role in the present author's Ph.D. Dissertation [5], in which a certain class of tracially approximated splitting tree algebras is classified by the Elliott invariant.

In [1], the authors studied a large class of sub-homogeneous C^*-algebras, including splitting tree algebras, and showed that they are finitely presented with respect to stable relations without giving concrete descriptions of generators and relations. In this paper, based on the universal unital C^*-algebra generated by two projections, we give explicit sets of generators and relations for the splitting tree algebras which have direct sums of $\mathbb{C}$ at their singular points, i.e. $k_{ij} = 1$ for all i and j, and moreover, the generating sets are shown to consist of minimal projections. These sets of generators and relations has been used in [5] to prove the existence theorem for the homomorphisms from splitting interval algebras to certain abstract algebras. (Another approach to the existence theorem, which is more direct, was found when the author was finishing writing the thesis.)

2. GENERATORS AND RELATIONS FOR CERTAIN SPLITTING TREE ALGEBRAS

Let us start with the elementary case when the tree T only consists two vertices and the functions split into two copies of $\mathbb{C}$ at these two vertices. It is well known that this C^*-algebra is the universal unital C^*-algebra generated by two projections. For the convenience of readers, a proof is given below.

LEMMA 2.1. *The C^*-algebra $S_2 = \{f \in M_2(C[0,1]) : f(0) \in \mathbb{C} \oplus \mathbb{C}, f(1) \in \mathbb{C} \oplus \mathbb{C}\}$ is the universal unital C^*-algebra generated by two projections.*

Proof. Set

$$p' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad q' = \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

Then p' and q' are projections in S_2. It is easy to see that, together with the identity, p' and q' generate S_2.

Let us consider the universal unital C^*-algebra $\mathcal{A}$ generated by two projections p and q. Let us prove that $\mathcal{A}$ is isomorphic to S_2 by calculating the irreducible representations of $\mathcal{A}$, and showing that any irreducible representation of $\mathcal{A}$ factors through the concrete C^*-algebra S_2.

Let π be an irreducible representation of $\mathcal{A}$. To save notation, still denote by p and q the images of the two generators respectively. One can verify that $(p-q)^2$ belongs to the center of the image of $\mathcal{A}$ under π, and therefore we get

$$(p-q)^2 = (1-t)I$$

for some $t \in [0,1]$.

In the case $t \neq 0,1$, consider the operator $v' = (pq - pqp)$. A calculation shows that $v'v'^* = t(1-t)p$ and $v'^*v' = t(1-t)(I-p)$. Therefore,

$v := v'/\sqrt{t(1-t)}$ is a partial isometry with

$$vv^* = p, \quad v^*v = I - p.$$

So, the C^*-algebra generated by $\{p, v\}$ is isomorphic to the 2×2 matrix algebra. Moreover, we also have

$$\begin{aligned} tp+(1-t)(I-p)+\sqrt{t(1-t)}v+\sqrt{t(1-t)}v^* &= pq-pqp+qp-pqp+tp+(1-t)(I-p) \\ &= pq+qp-2pqp+(p-q)^2-2(p-q)^2p = q. \end{aligned}$$

Therefore, we conclude that π is a 2-dimensional representation, which maps p to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and q to $\begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}$. It agrees with the point evaluation of S_2 at the point t.

In the case $t = 1$, we have that $(p-q)^2 = 0$. Therefore, $p = q$, and p and q are in the center of the image of the algebra. Hence $p = q = 0$ or 1. It agrees with the point evaluation of S_2 at 0 and projection onto the lower-right corner or the upper-left corner, respectively.

A similar argument shows that the case $t = 0$ gives us the point evaluation of S_2 at 1 and projection onto the lower-right corner or the upper-left corner. Therefore, the irreducible representations of the quotient S_2 of $\mathcal{A}$ exhaust the irreducible representations of $\mathcal{A}$. Hence the canonical surjective map from $\mathcal{A}$ to S_2 is an isomorphism. This shows that S_2 is the universal unital C^*-algebra generated by two projections. ∎

REMARK 2.2. From the proof of the previous lemma, one has that if two projections p and q satisfy the relation

$$(p-q)^2 = (1-t)I,$$

then the unital C^*-algebra generated by p and q is isomorphic to $M_2(\mathbb{C})$ via the map

$$p \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad q \mapsto \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix}.$$

Using the elementary splitting interval algebra above, we have the generators and relations for a the splitting interval algebras which split into several copies of $\mathbb{C}$ at the singular points.

LEMMA 2.3. *The C^*-algebra*

$$S_n = \{f \in M_n(C[0,1]) : f(0) \in \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{n \text{ copies}}, f(1) \in \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{n \text{ copies}}\}$$

is the universal C^*-algebra generated by the projections $\{p_i\}_{i=1}^n$ and $\{p_{1i}\}_{i=2}^n$ with the following relations

$$\begin{aligned} &p_1 + \cdots + p_n = I, \\ &p_{1i}(p_1 + p_i) = p_{1i} \quad \textit{for any } 2 \leqslant i \leqslant n, \\ &p_1 p_{1i} p_1 = p_1 p_{1j} p_1 \quad \textit{for any } 2 \leqslant i, j \leqslant n. \end{aligned}$$

Proof. Set

$$p_i = \mathrm{diag}\{\underbrace{0, \cdots 0, 1}_{i}, 0, \cdots, 0\}$$

and

$$p_{1i} = \begin{pmatrix} t & 0 & \cdots & 0 & \sqrt{t(1-t)} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \sqrt{t(1-t)} & 0 & \cdots & 0 & 1-t & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then one can verify that these projections satisfy the relations of the lemma, and also generate S_n.

On the other hand, let $\mathcal{A}$ denote the universal unital C^*-algebra generated by these elements subject to the relations of the lemma. Let us verify that the sum

$$c := (p_1 - p_{12})^2 + \sum_{i=3}^{n} p_i p_{1i} p_i$$

belongs to the center of $\mathcal{A}$. Indeed, it is easy to verify that c commutes with $\{p_i\}$ and p_{12}. For any projection p_{1k} with $k = 3, \ldots, n$, in order to show that it commutes with c, it is enough to verify that p_{1k} commutes with $(p_1 - p_{12})^2 + p_k p_{1k} p_k$. Using the identity $p_1 p_{12} p_1 = p_1 p_{1k} p_1$, one can conclude

$$p_1(p_1 - p_{12})^2 p_1 = p_1(p_1 - p_{1k})^2 p_1.$$

Together with the observation that $p_k (= (p_1 + p_k) - p_1)$ commutes with $(p_1 - p_{1k})^2$, we have

$$\begin{aligned} (p_1 - p_{12})^2 + p_k p_{1k} p_k &= p_1(p_1 - p_{12})^2 p_1 + p_2(p_1 - p_{12})^2 p_2 + p_k p_{1k} p_k \\ &= p_1(p_1 - p_{1k})^2 p_1 + p_2(p_1 - p_{12})^2 p_2 + p_k(p_1 - p_{1k})^2 p_k \\ &= (p_1 - p_{1k})^2 + p_2 p_{12} p_2. \end{aligned}$$

It is clear that p_{1k} commutes with $(p_1 - p_{1k})^2 + p_2 p_{12} p_2$. In particular, p_{1k} commutes with $(p_1 - p_{12})^2 + p_k p_{1k} p_k$ and hence commutes with c.

Since c is a sum of mutually orthogonal elements with norm at most one, the norm of c is at most one. Therefore, for any irreducible representation π of $\mathcal{A}$, we have

$$\pi(c) = \pi\Big((p_1 - p_{12})^2 + \sum_{i=3}^{n} p_i p_{1i} p_i\Big) = (1-t)I$$

for some $t \in [0,1]$. For simplicity, let us use the same notation for the images of the generators under π.

In the case $t = 1$, we have that $p_i p_{1i} p_i = 0$ for $i \geqslant 3$, and $p_1 = p_{12}$. It follows that the image of $\mathcal{A}$ is a commutative C^*-algebra. Therefore, π is a 1-dimensional representation, and only one of the p_i's, say p_m, is non-zero. This projection must be the unit of the image. If $m = 1$, then we have

$$p_1 p_{1i} = p_{1i}, \;\; p_1 p_{1i} p_1 = p_1 p_{1j} p_1,$$

and hence $p_{1i} = p_{1j}$. In particular, $p_{1i} = p_{12} = p_1 = 1$. If $m \neq 1$, then we have that $p_{1m} = p_m p_{1m} = 0$ and $p_{1i} = p_{1i}(p_1 + p_i) = 0$ if $i \neq m$. Therefore, π agrees with the point evaluation of S_n at $t = 1$ and projection onto the mth coordinate.

In the case $t = 0$, we have

$$(p_1 - p_{12})^2 + \sum_{i=3}^{n} p_i p_{1i} p_i = 1.$$

Therefore, one concludes that

$$p_1 = \Big((p_1 - p_{12})^2 + \sum_{n=3}^{n} p_i p_{1i} p_i\Big) p_1 = (p_1 - p_{12})^2 p_1 = p_1 + p_1 p_{1i} p_1, \quad 1 \leqslant i \leqslant n,$$

which implies $p_1 p_{1i} p_1 = 0$. Hence the projections p_1 and p_{1i} are orthogonal. Moreover, the projections p_{1i} and p_{1j} are also orthogonal if $i \neq j$. The image of $\mathcal{A}$ is then a commutative C^*-algebra, and only one of $\{p_1, \ldots, p_m\}$ is non-zero (and hence to be the identity). Denote this non-zero projection by p_m. Using an argument similar to the case of $t = 1$, we can show that π agrees with the point evaluation of S_n at $t = 0$ and projection onto the mth coordinate.

If $t \neq 0, 1$, consider the element $v_i' := p_1 p_{1i} - p_1 p_{1i} p_1$. A calculation shows that $v_i' v_i'^* = t(1-t)p_1$ and $v_i'^* v_i' = t(1-t)p_i$. Then by setting $e_{1i} = v_i' / \sqrt{t(1-t)}$, one gets a system of matrix units $\{e_{ij} : i, j = 1, \ldots, m\}$ such that $p_1 = e_{1i} e_{1i}^* = e_{11}$, $p_i = e_{1i}^* e_{1i} = e_{ii}$ and

$$p_{1i} = t e_{11} + (1-t) e_{ii} + \sqrt{t(1-t)} e_{1i} + \sqrt{t(1-t)} e_{1i}^*.$$

Therefore, the representation π is n-dimensional, and we have

$$p_i \mapsto \mathrm{diag}\{\underbrace{0, \ldots 0, 1}_{i}, 0, \ldots, 0\}$$

and

$$p_{1i} \mapsto \begin{pmatrix} t & 0 & \cdots & 0 & \sqrt{t(1-t)} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \sqrt{t(1-t)} & 0 & \cdots & 0 & 1-t & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

in this system of matrix units. It agrees with the point evaluation of S_n at the point t. Hence, the irreducible representations of the quotient S_n of $\mathcal{A}$ exhaust the irreducible representations of $\mathcal{A}$, and therefore the concrete C^*-algebra S_n is isomorphic to $\mathcal{A}$. In other words, S_n is the universal C^*-algebra of the generators and relations of the lemma. ∎

In the following, let us consider certain splitting tree algebras with more than two singular points. Let us first consider the special case when the tree is the interval $[0, n]$ with integers i, $0 \leqslant i \leqslant n$, as vertices. For a natural number n, denote by A_n the C^*-algebra

$$A_n := \{f \in M_2(C[0,n]) : f(i) \in \mathbb{C} \oplus \mathbb{C} \text{ for all } i \in \mathbb{Z}, 0 \leqslant i \leqslant n\}.$$

Then we have the following lemma.

LEMMA 2.4. *The C^*-algebra A_n defined above is the universal C^*-algebra generated by the projections $\{p, q, p_i : i = 0, \dots, n-1\}$ with the relations*

$$p + q = 1,$$
$$(p - p_i)(q - p_j) = 0 \quad \textit{for any } j > i.$$

Proof. Let p denote the projection in A_n which is the constant function

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and let q denote its complement. Denote by p_i the projection in A_n which takes the constant $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ between 0 and i, takes value

$$\begin{pmatrix} t-i & \sqrt{(t-i)(i+1-t)} \\ \sqrt{(t-i)(i+1-t)} & i+1-t \end{pmatrix}$$

at the point t between i and $i+1$ and takes $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ between $i+1$ and n. Then $p, q, p_0, \dots, p_{n-1}$ generate A_n and satisfy the relations of the lemma.

On the other hand, denote by $\mathcal{A}_n$ the universal C^*-algebra generated by the generators and relations of the lemma. For each $i = 0, \dots, n-1$, let us show that

$(p-p_i)^2$ commutes with $\{p,q,p_j : j=0,\dots,n-1\}$. For any projection p_j with $j>i$, since $(p-p_i)(q-p_j)=0$, we have

$$(p-p_i)q=(p-p_i)p_j.$$

Therefore

$$(p-p_i)^2p_j=(p-p_i)^2q=q(p-p_i)^2=p_j(p-p_i)^2,$$

and hence the projection p_j commutes with $(p-p_i)^2$. For any projection p_j with $j<i$, a similar argument shows p_j commutes with $(q-p_i)^2$. Since $(p-p_i)^2=1-(q-p_i)^2$, we have that p_j commutes with $(p-p_i)^2$. Therefore, the element $(p-p_i)^2$ is a central element of $\mathcal{A}$.

Consider the central elements $\{(p-p_i)^2(1-(p-p_i)^2) : i=0,\dots,n-1\}$. Let us first show that they are orthogonal to each other. In fact, for any pair of elements $(p-p_i)^2(1-(p-p_i)^2)$ and $(p-p_j)^2(1-(p-p_j)^2)$ with $i<j$, since $1-(p-p_j)^2=(q-p_j)^2$ and $(p-p_i)(q-p_j)=0$, we have

$$\begin{aligned}(p-p_i)^2(1-(p-p_i)^2)\cdot(p-p_j)^2(1-(p-p_j)^2)&\\ =(1-(p-p_i)^2)(p-p_i)^2(q-p_j)^2(p-p_j)^2&=0.\end{aligned}$$

Let π be a irreducible representation of $\mathcal{A}_n$. For simplicity of notation, let us use the same notation for the images of the generators under π. Since π is irreducible, any central element must be a scalar multiple of 1. Moreover, since $\{(p-p_i)^2(1-(p-p_i)^2) : i=0,\dots,n-1\}$ are mutually orthogonal central elements, at most one of them, say $(p-p_k)^2(1-(p-p_k)^2)$, is non-zero. Note that if $(p-p_i)^2(1-(p-p_i)^2)=0$, then $(p-p_i)^2$ is a projection. Moreover, since $(p-p_i)^2$ is a scalar multiple of 1, it must be the trivial projection, i.e., $(p-p_i)^2$ must be 0 or 1.

If $(p-p_k)^2=(1-t)1$ with $t\in(0,1)$, since $(p-p_i)(q-p_j)=0$ for any pair i,j with $j>i$ and $(p-p_k)^2=1-(q-p_k)^2$, we have that $(p-p_i)^2=0$ for all $i<k$, and $(q-p_i)^2=0$ for all $i>k$. Therefore, we have that $p_i=p$ if $i<k$, and $p_i=1-p=q$ if $i>k$. The image of $\mathcal{A}_n$ is then generated by the projections p and p_k with the relation $(p-p_k)^2=(1-t)1$. By Remark 2.2, the image of $\mathcal{A}_n$ is isomorphic to $M_2(\mathbb{C})$ with the map

$$p\mapsto\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad p_k\mapsto\begin{pmatrix}t&\sqrt{t(1-t)}\\ \sqrt{t(1-t)}&1-t\end{pmatrix}.$$

Hence π agrees with the point evaluation of the concrete algebra A_n at the point $i+t$.

If $(p-p_i)^2(1-(p-p_i)^2)=0$ for any $i=0,\dots,n-1$, then $(p-p_i)^2=0$ or 1. Set

$$k=\min\{i : (p-p_i)^2=1\},$$

and set $k=n$ if $(p-p_i)^2=0$ for any $i=0,\dots,n-1$. Then we assert that $(q-p_i)^2=0$ for any $i>k$. Indeed, since $(p-p_k)^2=1$ by the choice of k, for any

$i > k$, we have

$$(q - p_i)^2 = (q - p_i)^2 (p - p_k)^2 = 0.$$

Also by the choice of k, we have that $(p - p_i)^2 = 0$ for any $i < k$. Hence, one has that $p_i = p$ for any $i < k$, and $p_i = q$ for all $i \geqslant k$. Therefore, the image of $\mathcal{A}_n$ is generated by p and q. Since π is irreducible and p and q are orthogonal, one of the projections p and q must be 0 under π, and π is one-dimensional. Therefore the irreducible representation π of $\mathcal{A}_n$ coincides with the point evaluations of A_n at k and projection onto the coordinate corresponding to the non-zero projection of $\{p, q\}$.

This shows that the irreducible representations of A_n exhaust the irreducible representations of $\mathcal{A}_n$ which implies that $\mathcal{A}_n$ is isomorphic to the concrete algebra A_n. ∎

Based on the argument above, we have the generators and relations for the splitting tree algebra $A_{n,m}$ defined as follows:

$$A_{n,m} := \{f \in M_m(C[0,n]) : f(i) \in \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{\text{m copies}} \text{ for all } i \in \mathbb{Z}, 0 \leqslant i \leqslant n\}.$$

THEOREM 2.5. *The C^*-algebra $A_{n,m}$ is the universal algebra generated by the projections $\{p_k\}_{k=1}^m$ and $\{p_{ij}\}$, where $0 \leqslant i \leqslant n-1$ and $2 \leqslant j \leqslant m$, with the relations:*

$$\begin{aligned} &p_1 + \cdots + p_m = 1, \\ &p_{ij}(p_1 + p_j) = p_{ij}, \quad \textit{for any } i, j, \\ &p_1 p_{ij_1} p_1 = p_1 p_{ij_2} p_1 \quad \textit{for any } i, j_1, j_2, \textit{and} \\ &(p_1 - p_{i_1 j_1})(p_{j_2} - p_{i_2 j_2}) = 0 \quad \textit{for any } i_1 < i_2. \end{aligned}$$

Proof. We only sketch the proof here. Denote by p_k the constant-valued function

$$\mathrm{diag}\{\underbrace{0, \ldots, 0, 1}_{k}, 0, \ldots, 0\}.$$

Let p_{ij} denote the function which takes the constant value

$$\mathrm{diag}\{\underbrace{0, \ldots, 0, 1}_{j}, 0, \ldots, 0\}$$

between 0 and i, takes the value

$$\begin{pmatrix} t-i & 0 & \cdots & 0 & \sqrt{(t-i)(i+1-t)} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \sqrt{(t-i)(i+1-t)} & 0 & \cdots & 0 & i+1-t & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

at the point t between i and $i+1$, and takes the constant value

$$\operatorname{diag}\{1,\ldots,0,0,0,\ldots,0\}$$

between $i+1$ and n. These projections generate the C^*-algebra $A_{n,m}$ and satisfy the relations of the theorem.

On the other hand, consider the universal algebra $\mathcal{A}_{n,m}$ generated by the generators and relations of the theorem. For each $0 \leqslant i \leqslant n-1$, set

$$b_i = (p_1 - p_{i2})^2 + \sum_{j=3}^{m} p_j p_{ij} p_j,$$

and let us verify that b_i is a central element. Indeed, it is clear that b_i commutes with $\{p_k\}$ and p_{i2}. In order to show the projection p_{jk} commutes with b_i, it is enough to verify that p_{jk} commutes with $(p_1 - p_{i2})^2 + p_k p_{ij} p_k$.

The same argument as Lemma 2.3 shows

$$(p_1 - p_{i2})^2 + p_k p_{ij} p_k = (p_1 - p_{ik})^2 + p_2 p_{i2} p_2.$$

Therefore, the projections $\{p_{ij} : j = 2, \ldots, m\}$ commutes with b_i. For any projection p_{jk} with $j > i$, since $(p_1 - p_{ik})(p_k - p_{jk}) = 0$, we have

$$(p_1 - p_{ik})p_k = (p_1 - p_{ik})p_{jk}.$$

Hence

$$p_{jk}(p_1 - p_{ik})^2 = p_k(p_1 - p_{ik})^2 = (p_1 - p_{ik})^2 p_k = (p_1 - p_{ik})^2 p_{jk}.$$

Therefore p_{jk} commutes with b_i. A similar argument shows that p_{jk} commutes with b_i for any $j < i$. Therefore, b_i is a central element. Moreover, if we set $c_i = b_i(1 - b_i)$, then $\{c_i\}$ are mutually orthogonal.

Let π be an irreducible representation of $\mathcal{A}_{n,m}$, and let us still use the same notation for the image under π. Since π is irreducible, each b_i and c_i must be a scale multiple of 1, and at most one of $\{c_i\}$, say c_k, is non-zero. Note that if $c_i = 0$, then $b_i = 0$ or $b_i = 1$. In each case, similar arguments as of Lemma 2.3 show that p_{ij} is in the C^*-algebra generated by $\{p_1, \ldots, p_m\}$.

If $b_k = (p_i - p_{k2})^2 + \sum\limits_{j=3}^{m} p_j p_{kj} p_j = (1-t)1$ for some $t \in (0,1)$, a calculation as that of Lemma 2.3 shows that π agrees with the point evaluation map of $A_{n,m}$ at $k+t$.

If $b_i = 0$ or $b_i = 1$ for any $i = 0, \ldots, n-1$, then set

$$k := \min\{i : b_i = 1\},$$

and set $k = n$ if $b_i = 0$ for any $i = 0, \ldots, n-1$. Similar arguments as those of Lemma 2.3 and Lemma 2.4 show that π agrees with the point evaluation map of $A_{n,m}$ at k and projection onto the coordinate corresponding to the non-zero projection of $\{p_1, \ldots, p_m\}$.

Therefore, the C^*-algebra $A_{n,m}$ is the universal C^*-algebra of the generators and relations of the theorem. ∎

Let us study the generating sets for certain splitting tree algebras over a general tree. Let T be a rooted tree, and denote by $\{v_i\}_{i=1}^n$ its vertices. Then there is a natural partial order on the vertices defined as the following.

Fix one vertex, say v_1, as the root of the tree. Then, for any two vertices v_i and v_j, we define $v_i > v_j$ if the the minimal path between v_j and v_1 contains the minimal path between v_i and v_1. We say v_j is a child of v_i, or v_i is an ancestor of v_j. It is clear that v_1 is the maximal element, and any finite subset of $\{v_i\}_{i=1}^n$ has a unique minimal upper bound.

On the other hand, for any partial order on a finite set $\{v_i\}_{i=1}^n$ with a maximal element v_1, if it has the property that the minimal upper bound of any subset is unique, then there is a canonical way to construct a rooted tree T which has $\{v_i\}_{i=1}^n$ as its vertices and the partial order induced by T is the given order. It can be described as follows. To construct such a tree with vertices $\{v_i\}_{i=1}^n$, it is enough to determine edges connecting the vertices. For a vertex v_i, define the set E_i to be

$$E_i = \{v_j : v_j < v_i \text{ and there does not exist a vertex } v_k \text{ such that } v_j < v_k < v_i\}.$$

In other words, E_i is the set of the closest children of v_i. Then one can put an edge between v_i and each point of E_i. Starting with the maximal element v_1, one can get a graph which is determined by E_i and therefore by the given partial order. This graph is actually a tree, since the minimal upper bound of any subset is unique. Denote by $[v_i, v_j]$ the edge connecting v_i and v_j if there exists one.

LEMMA 2.6. *Let $\{v_1, v_2, \dots, v_n\}$ be a partially ordered set with the partial order induced by a rooted tree T, i.e., there exists a maximal element, say v_1, and the minimal upper bound of any subset is unique. Then the C^*-algebra*

$$A := \{f \in M_2(C(T)) : f(v_i) \in \mathbb{C} \oplus \mathbb{C}, i = 1, \dots, n\}$$

is the universal C^*-algebra generated by the projections $\{p, q, p_i : i = 2, \dots, n\}$ with respect to the following relations:

$$\begin{aligned} & p + q = 1, \\ & (p - p_i)(p - p_j) = 0 \quad \textit{if } v_j \textit{ and } v_i \textit{ can not be compared}, \\ & (p - p_i)(q - p_j) = 0 \quad \textit{if } v_i < v_j. \end{aligned}$$

Proof. The proof goes along the same line as that of Lemma 2.4 by calculating the irreducible representations. Let p be the projection in A which is the constant function

$$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

and let q be its complement.

For each vertex v_i, $i = 2, \dots, n$, denote by v_i' the closest ancestor of v_i. Denote by E_i the set consisting of v_i itself and all the children of v_i, and denote by D_i the set of the vertices which are not in E_i. (Note that the only edge connecting D_i and E_i is the edge $[v_i', v_i]$.) Let us define the projection p_i in A piecewisely. The projection p_i takes the constant value $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ on the edges connecting vertices in D_i, takes the value $\begin{pmatrix} 1-t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & t \end{pmatrix}$ at the point t on edges connecting v_i' and v_i (regarding the edge $[v_i', v_i]$ as the interval $[0,1]$), and takes the constant value $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ on the remaining edges. In the construction of p_i, the values of p_i agree with each other on the vertex v_i' (which corresponds to $t = 0$) and the vertex v_i (which corresponds to $t = 1$), therefore p_i is a continuous projection valued function on T. Moreover, the set of projections $\{p, q, p_2, \ldots, p_n\}$ generates A and satisfies the relations of the lemma.

On the other hand, let $\mathcal{A}$ be the universal C^*-algebra generated by

$$\{p, q, p_i : i = 2, \ldots, n\}$$

with respect to the relations of the lemma. We shall show that $(p - p_i)^2$ is a central element of $\mathcal{A}$ for all i.

Fix $(p - p_i)^2$. It is clear that $(p - p_i)^2$ commutes with the projections p, q, and p_i. For any $v_j > v_i$, we have $(p - p_i)(q - p_j) = 0$. Therefore,

$$(p - p_i)q = (p - p_i)p_j.$$

Hence we have

$$(p - p_i)^2 p_j = (p - p_i)^2 q = q(p - p_i)^2 = p_j(p - p_i)^2,$$

and therefore the projection p_j commutes with $(p - p_i)^2$. Interchanging p_i with p_j, the same argument shows that any projection p_j with $v_j < v_i$ commutes with $(q - p_i)^2$. Since $(p - p_i)^2 = 1 - (q - p_i)^2$, we have that $(p - p_i)^2$ commutes with p_j for any child v_j of v_i. If v_i and v_j are not comparable, then we have $(p - p_i)(p - p_j) = 0$, and hence

$$(p - p_i)p = (p - p_i)p_j.$$

A direct calculation also shows that $(p - p_i)^2$ commutes with p_j. Thus the element $(p - p_i)^2$ commutes with all projections $\{p, q, p_i : i = 2, \ldots, n\}$, and it is a central element of $\mathcal{A}$.

Consider any pair of elements $(p - p_i)^2(1 - (p - p_i)^2)$ and $(p - p_j)^2(1 - (p - p_j)^2)$. Note that $(p - p_j)^2 = 1 - (q - p_j)^2$. If $v_j < v_i$ or $v_i < v_j$, we then have

$$\begin{aligned}(p - p_i)^2(1 - (p - p_i)^2) \cdot (p - p_j)^2(1 - (p - p_j)^2) \\ = (1 - (p - p_i)^2)(p - p_i)^2(q - p_j)^2(p - p_j)^2 = 0.\end{aligned}$$

If v_i and v_j can not be compared, we then have

$$\begin{aligned}(p - p_i)^2(1 - (p - p_i)^2) \cdot (p - p_j)^2(1 - (p - p_j)^2) \\ = (1 - (p - p_i)^2)(p - p_i)^2(p - p_j)^2(1 - (p - p_j)^2) = 0.\end{aligned}$$

Therefore, the positive elements $(p-p_i)^2(1-(p-p_i)^2), i = 1, \ldots, n$ are orthogonal to each other.

We shall show that the irreducible representations of $\mathcal{A}$ can be parameterized by the tree T (with splitting at the vertices.). Let π be an irreducible representation of $\mathcal{A}$, and let us use the same notation for the image of $\{p, q, p_i\}$. Since π is irreducible and $(p-p_i)^2(1-(p-p_i)^2), i = 1, \ldots, n$ are mutually orthogonal central elements, we have that at most one of them, say $(p-p_k)^2(1-(p-p_k)^2)$, is non-zero. Note that if $(p-p_i)^2(1-(p-p_i)^2) = 0$ for some i, then the central element $(p-p_i)^2$ is a projection, and hence must be 0 or 1 by the irreducibility of π.

If $(p-p_k)^2 = t1$ with $t \in (0,1)$, since $(p-p_i)(q-p_k) = 0$ for all $v_k > v_i$, and $(p-p_i)(p-p_k) = 0$ if v_i and v_k can not be compared, we have that $(p-p_i)^2 = 0$ for any v_i that is not comparable to v_k or is an ancestor of v_k, and $(q-p_i)^2 = 0$ for any child v_i of v_k. Therefore, we get $p_i = p$ for any v_i that is not comparable to v_k or is an ancestor of v_k, and $p_i = 1 - p = q$ for any child v_i of v_k. Hence the image of $\mathcal{A}$ is generated by the projections p and p_k. with the relation $(p-p_k)^2 = t1$. By Remark 2.2, the representation π agrees with the point evaluation of A at the point t between v_k' and v_k (recall that we regard the edge $[v_k', v_k]$ as $[0,1]$).

If $(p-p_i)^2 = 0, 1$ for all v_i, denote by

$$S := \{v_i : (p-p_i)^2 = 1\}.$$

We assert that if $v_k \in S$, then any ancestor of v_k belongs to S, and $(p-p_i)^2 = 0$ for any v_i which is not comparable to v_k. Indeed, if $v_k \in S$, then $(p-p_k)^2 = 1$. One has

$$(p-p_i)^2 = 1-(q-p_i)^2 = 1-(q-p_i)^2(p-p_k)^2 = 1$$

for all $v_i > v_k$, and

$$(p-p_i)^2 = (p-p_i)^2(p-p_k)^2 = 0$$

for all v_i which can not be compared with v_k. Therefore, if the set S is non-empty, the minimal element of the set S exists and is unique. Let us denote by v_k this minimal element if it exists, and set $v_k = v_1$ if S is empty. Then one has

$$(p-p_i)^2 = 0$$

for any v_i a child of v_k or v_i is not comparable to v_k, and

$$(q-p_i)^2 = 1-(p-p_i)^2 = 0$$

otherwise. Therefore, we have that $p_i = p$ if v_i is a child of v_k or v_i is not comparable to v_k, and $p_i = q$ otherwise. Hence the image of $\mathcal{A}$ under π is generated by the orthogonal projections p and q. Since π is irreducible, one of the projections p and q must be sent to zero. Then the irreducible representation π of $\mathcal{A}$ coincides with the point evaluation of A at v_k and projection onto the coordinate corresponding to the non-zero projection of $\{p, q\}$.

Therefore the irreducible representations of A exhaust the irreducible representation of $\mathcal{A}$, and hence $\mathcal{A}$ is isomorphic to A, as desired. ∎

Based on the lemma above, we have the following description of the generators and relations for a splitting tree algebra which splits into copies of $\mathbb{C}$ at the singular points.

THEOREM 2.7. *Let $\{v_1, v_2, \dots, v_n\}$ be a partially ordered set with the partial order induced by a rooted tree T, i.e., there exists a maximal element, say v_1, and the minimal upper bound of any subset is unique. Let m be a natural number. Then the C^*-algebra*

$$A := \{f \in M_m(C(T)) : f(v_i) \in \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{m \text{ copies}}, i = 1, \dots, n\}$$

is the universal C^*-algebra generated by the projections $\{p_k\}_{k=1}^m$ and $\{p_{ij}\}$, where $2 \leqslant i \leqslant n$ and $2 \leqslant j \leqslant m$, with respect to the following relations:

$$\begin{aligned}
&p_1 + \cdots + p_m = 1, \\
&p_{ij}(p_1 + p_j) = p_{ij}, \quad \text{for any } i, j, \\
&p_1 p_{ij_1} p_1 = p_1 p_{ij_2} p_1 \quad \text{for any } i, j_1, j_2, \\
&(p_1 - p_{i_1 j_1})(p_1 - p_{i_2 j_2}) = 0 \quad \text{if } v_{i_1} \text{ and } v_{i_2} \text{ can not be compared}, \\
&(p_1 - p_{i_1 j_1})(p_{j_2} - p_{i_2 j_2}) = 0 \quad \text{if } v_{i_1} < v_{i_2}.
\end{aligned}$$

Proof. The proof is the routine calculation of the irreducible representations. We only sketch it here.

Set p_k to be the projection of A which is the constant function

$$p_k = \operatorname{diag}\{\underbrace{0, \dots, 0, 1}_{k}, 0, \dots, 0\}.$$

For each vertex v_i, $i = 2, \dots, n$, denote by v_i' be the closest ancestor of v_i. Denote by E_i the set consisting of v_i itself and all the children of v_i, and denote by D_i the set of the vertices which are not in E_i. (Note that the only edge connecting D_i and E_i is the edge $[v_i', v_i]$.) Let us define the projection p_{ij} in A piecewisely. The projection p_{ij} takes the constant value

$$\operatorname{diag}(1, 0, \dots, 0)$$

on the edges connecting vertices in D_i, takes the value

$$\begin{pmatrix}
1-t & 0 & \cdots & 0 & \sqrt{t(1-t)} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\sqrt{t(1-t)} & 0 & \cdots & 0 & t & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}$$

at the point t between v_i' and v_i (regarding the edge $[v_i', v_i]$ as the interval $[0,1]$), and takes the constant value

$$\mathrm{diag}\{\underbrace{0,\dots,0,1}_{j},0,\dots,0\}$$

on the remaining edges. Since the values of p_{ij} agree with each other on the vertex v_i' (which corresponds to $t=0$) and the vertex v_i (which corresponds to $t=1$), p_{ij} is a well defined continuous projection valued function on T. A direct calculation together with the Stone–Weierstrass theorem show that the projections $\{p_k, p_{ij}\}_{i,j,k}$ satisfy the relations of the theorem and generate A.

On the other hand, denote by $\mathcal{A}$ the universal C^*-algebra of the generators and relations of the theorem. Let us show that the element

$$b_i := (p_1 - p_{i2})^2 + \sum_{j=3}^{m} p_j p_{ij} p_j$$

is a central element of $\mathcal{A}$ for each $2 \leqslant i \leqslant n$. It is clear that b_i commutes with p_{i2} and each p_k, $k = 1,\dots,m$. For any $2 \leqslant k \leqslant m$ and $2 \leqslant j \leqslant n$, in order to show that the projection p_{jk} commutes with b_i, it is enough to verify that p_{jk} commutes with $(p_1 - p_{i2})^2 + p_k p_{ik} p_k$.

With a similar argument as that of Lemma 2.3, we have

$$(p_1 - p_{i2})^2 + p_k p_{ik} p_k = (p_1 - p_{ik})^2 + p_2 p_{i2} p_2.$$

If $j = i$, it is clear that p_{jk} commutes with $(p_1 - p_{ik})^2 + p_2 p_{i2} p_2$, and hence commutes with b_i. If v_j is not comparable to v_i, since $(p_1 - p_{jk})(p_1 - p_{ik}) = 0$, we have

$$p_1(p_1 - p_{ik}) = p_{jk}(p_1 - p_{ik}).$$

Therefore, p_{jk} commutes with $(p_1 - p_{ik})^2$ and hence commutes with b_i. A similar argument also shows that p_{jk} commutes with b_i if v_j is a child of v_i or v_i is a child of v_j. Therefore, the elements $\{b_i : i = 2,\dots,n\}$ are central in $\mathcal{A}$.

Moreover, it can be verified that the elements $c_i := b_i(1 - b_i)$, $i = 2,\dots,n$, are mutually orthogonal. Thus, for any irreducible representation π of $\mathcal{A}$, the images of $\{c_i : i = 2,\dots,n\}$ are scalar multiples, and we have at most one of them is non-zero.

If c_k is non-zero, then $b_k = t1$ for some $t \in (0,1)$. Arguments similar to those of Lemma 2.3 and Lemma 2.6 show that π coincides with the point evaluation of A at the point t in $[v_k', v_k]$ (recall we regard the edge $[v_k', v_k]$ as the unit interval $[0,1]$).

If c_k is zero for any $k = 2,\dots,n$, denote by

$$S = \{v_i : b_i = 1\}.$$

An argument similar to that of Lemma 2.6 shows that S has a unique minimal element if S is non-empty. Set v_k to be the minimal element of S if it exists, and set

$v_k = v_1$ if S is empty. Then it can be verified that the image of A under π is generated by $\{p_1, \ldots, p_m\}$, and π agrees with the evaluation of A at v_k and projection onto the coordinate corresponding to the non-zero projection of $\{p_1, \ldots, p_m\}$.

Therefore, the irreducible representations of A exhaust all the irreducible representations of $\mathcal{A}$, and hence A is isomorphic to $\mathcal{A}$, as desired. ∎

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