J. OPERATOR THEORY
61:2(2009), 347–367

KREĬN SPACES INDUCED BY SYMMETRIC OPERATORS

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Communicated by Nikolai K. Nikolski

ABSTRACT. We introduce the notion of Kreĭn space induced by a densely defined symmetric operator in a Hilbert space, as an abstract notion of indefinite energy spaces. Characterizations of existence and uniqueness, as well as certain canonical representations, are obtained. We exemplify these by the free and certain perturbed Dirac operators.

KEYWORDS: *Kreĭn space, induced Kreĭn space, Dirac operator.*

MSC (2000): 47B25, 47B50.

1. INTRODUCTION

According to the classical approach of K. Friedrichs [12], the problem of estimation of the spectrum of a nonnegative (or, more generally, semi-bounded) linear operator A associated to a partial differential equation leads naturally to Hilbert spaces that are obtained by a quotient-completion process performed on the quadratic form $\xi \mapsto \langle A\xi, \xi \rangle$. The Hilbert space obtained in this way is called the *energy space* due to a certain quantum mechanical interpretation of the spectral points (in particular, eigenvalues) of A as possible values of the energy of the system. This construction can be made abstract by the notion of *induced Hilbert spaces* as in [3], where we have exemplified it on different linear operators associated to partial differential equations. The induced Hilbert spaces are in general Sobolev type spaces and the main result in [3], see Theorem 2.2, shows that, under certain intertwining assumptions, estimation of the spectra of linear operators on the original Hilbert space yields an estimation of the spectra of the operators lifted to the energy space (for the case of bounded operators, cf. [16], [19], [18], and [8]).

In this paper, we are interested in performing similar constructions and obtaining similar results in case the operator A is indefinite, without assumptions of semi-boundedness. The corresponding induced space can no longer be a Hilbert space due to the fact that the quadratic form $\xi \mapsto \langle A\xi, \xi \rangle$ is indefinite. The natural (and most tractable) generalization of Hilbert space, and appropriate to the present situation, is that of Kreĭn space, and we expect the geometrical-topological difficulties of operator theory in Kreĭn space to show up. Indeed, the first difficulty comes from the fact that an indefinite inner product, with both positive and negative indices infinite, may not be associated to any Kreĭn space, e.g. see [2] and the literature cited there. Second, the uniqueness modulo unitary equivalence, that holds naturally for the positive case, does not exist, in the genuine cases of indefiniteness. In this respect, it is natural to ask for some "canonical" representations of induced Kreĭn spaces, when they exist, and in these cases, to look for necessary and sufficient conditions of uniqueness, modulo unitary equivalence. These are the main goals of this paper.

The idea of induced Kreĭn space is simple and comes from the following observation: let $\mathcal{H}$ be a Hilbert space and A a densely defined symmetric operator in $\mathcal{H}$. We consider $\operatorname{Dom}(A)$, the domain of A, and its factorization by the kernel of A, $\operatorname{Ker}(A)$. The (indefinite) inner product $\langle Ax, y\rangle$ factors to a non-degenerate inner product space $\operatorname{Dom}(A)/\operatorname{Ker}(A)$ and let us assume, for the moment, that this can be isometrically embedded into a Kreĭn space $\mathcal{K}$ with inner product $[\cdot,\cdot]$. Modulo the identification of $\operatorname{Dom}(A)/\operatorname{Ker}(A)$ with its image, this means that $\langle Ax, y\rangle = [\widehat{x}, \widehat{y}]$ for all $x, y \in \operatorname{Dom}(A)$, where $\widehat{x} = x + \operatorname{Ker}(A)$ denotes the corresponding equivalence class in $\operatorname{Dom}(A)/\operatorname{Ker}(A)$. We let Π be the operator obtained from the composition of the canonical projection $\operatorname{Dom}(A) \to \operatorname{Dom}(A)/\operatorname{Ker}(A)$ with the embedding of $\operatorname{Dom}(A)/\operatorname{Ker}(A)$ in $\mathcal{K}$, and call $(\mathcal{K}, \Pi)$ a Kreĭn space induced by the symmetric operator A. This construction, which is a natural generalization of the quotient-completion to a Hilbert space when A is nonnegative, can be put into an axiomatic framework as in Section 3. What we do is actually to look, formally, for factorizations $A = \Pi^* J \Pi$, where Π is a linear operator from $\mathcal{H}$ into $\mathcal{K}$ and J is a *symmetry* (or, in other terminology, a *unitary involution*) on some Hilbert space $\mathcal{K}$, and under certain minimality conditions. The difficulty comes from giving a sense to this factorization, taking into account that we deal with unbounded operators. In the bounded case (that is, when A is bounded) and, additionally, we require that Π is also bounded (see Remark 3.1), this construction was first performed in [4] and used successfully in dilation theory in [4], [5], and [6].

As a first motivation for our investigations, we started with the free Dirac operator which is a satisfactory model for a $\frac{1}{2}$-spin free electron in relativistic quantum theory. When considered on its natural domain, the free Dirac operator is selfadjoint and has a spectral gap in the neighbourhood of 0. It turns out that the Kreĭn space induced by the free Dirac operator exists and is unique, modulo unitary equivalence. This is a generalization of the Friedrichs energy space and has the interpretation of the existence of states with positive energy, corresponding to electrons, and of other states with negative energy, corresponding to positrons. These considerations have some overlapping with the supersymmetry of the free Dirac operator, e.g. see [22] for definitions and basic properties. The notion of induced Kreĭn space gets more consistency when applied to perturbed Dirac operators.

Let us briefly describe the contents of this article. In Section 2 we briefly recall the similar construction in the (positive) definite case, and also the lifting theorem. There is a slight modification of the induced Hilbert space with respect to that given in [3] due to an anomaly that was pointed to us by K.-H. Förster, which we acknowledge now. However, this does not change any of the results in [3], except the possibility of taking the operator Π closed, if A is closed, that is used in Proposition 2.2 in the cited paper. Then we recall a few things on Kreĭn spaces and their linear operators that we need in this paper.

In Section 3 we define Kreĭn spaces induced by symmetric densely defined operators and then give a variety of characterizations of existence. A particularly interesting condition of existence is when the operator A has selfadjoint extensions and we show by an example that there exist operators admitting induced Kreĭn spaces but having no selfadjoint extensions.

In Section 4 we describe two canonical representations of Kreĭn spaces induced by selfadjoint operators and prove the lifting theorem for this case (the bounded indefinite case was obtained in [9]). In the next section we give equivalent characterizations of uniqueness of the induced Kreĭn space, modulo unitary equivalence, both in spectral and geometric terms. We conclude the paper by exemplifying these on the free and certain perturbed Dirac operators.

2. SOME PRELIMINARY CONSIDERATIONS

2.1. HILBERT SPACES INDUCED BY NONNEGATIVE OPERATORS. We consider a Hilbert space $\mathcal{H}$ and A a densely defined nonnegative operator in $\mathcal{H}$ (in this paper, the nonnegativity of an operator A means $\langle Ax, x\rangle_{\mathcal{H}} \geqslant 0$ for all $x \in \mathrm{Dom}(A)$). A pair $(\mathcal{K}, \Pi)$ is called a *Hilbert space induced* by A if:

(i) $\mathcal{K}$ is a Hilbert space;
(ii) Π is a linear operator with domain $\mathrm{Dom}(\Pi) \supseteq \mathrm{Dom}(A)$ and range in $\mathcal{K}$;
(iii) $\Pi\,\mathrm{Dom}(A)$ is dense in $\mathcal{K}$;
(iv) $\langle \Pi x, \Pi y\rangle_{\mathcal{K}} = \langle Ax, y\rangle_{\mathcal{H}}$ for all $x \in \mathrm{Dom}(A)$ and all $y \in \mathrm{Dom}(\Pi)$.

Such an object always exists by an obvious quotient-completion procedure. In addition, they are essentially unique in the following sense: two Hilbert spaces $(\mathcal{K}_i, \Pi_i)$, $i = 1, 2$, induced by the same operator A, are called *unitary equivalent* if there exists a unitary operator $U \in \mathcal{B}(\mathcal{K}_1, \mathcal{K}_2)$ such that $U\Pi_1 = \Pi_2$.

REMARK 2.1. In the case of a nonnegative selfadjoint operator, the quotient-completion construction can be made more explicit. Thus, if A is a nonnegative selfadjoint operator in the Hilbert space $\mathcal{H}$, then $A^{1/2}$ exists as a nonnegative selfadjoint operator in $\mathcal{H}$, $\mathrm{Dom}(A^{1/2}) \supseteq \mathrm{Dom}(A)$ and $\mathrm{Dom}(A)$ is a core of $A^{1/2}$.

In particular we have

$$\langle Ax, y\rangle_{\mathcal{H}} = \langle A^{1/2}x, A^{1/2}y\rangle_{\mathcal{H}}, \quad x \in \mathrm{Dom}(A),\ y \in \mathrm{Dom}(A^{1/2}),$$

which shows that we can consider the seminorm $\|A^{1/2}\cdot\|$ on $\mathrm{Dom}(A)$ and make the quotient completion with respect to this seminorm in order to get a Hilbert space $\mathcal{K}_A$. We denote by Π_A the corresponding canonical operator. Then it is easy to see that $(\mathcal{K}_A, \Pi_A)$ is a Hilbert space induced by A.

The main result of [3] is the following lifting theorem:

THEOREM 2.2. *Let A and B be nonnegative selfadjoint operators in the Hilbert spaces $\mathcal{H}_1$ and respectively $\mathcal{H}_2$, and let $(\mathcal{K}_A, \Pi_A)$ and $(\mathcal{K}_B, \Pi_B)$ be the Hilbert spaces induced by A and respectively B. For any operators $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $S \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that*

$$\langle Bx, Ty\rangle_{\mathcal{H}_2} = \langle Sx, Ay\rangle_{\mathcal{H}_1}, \quad x \in \mathrm{Dom}(B),\ y \in \mathrm{Dom}(A), \tag{2.1}$$

there exist uniquely determined operators $\widetilde{T} \in \mathcal{B}(\mathcal{K}_A, \mathcal{K}_B)$ and $\widetilde{S} \in \mathcal{B}(\mathcal{K}_B, \mathcal{K}_A)$ such that $\widetilde{T}\Pi_A x = \Pi_B T x$ for all $x \in \mathrm{Dom}(A)$, $\widetilde{S}\Pi_B y = \Pi_A S y$ for all $y \in \mathrm{Dom}(B)$, and

$$\langle \widetilde{S}h, k\rangle_{\mathcal{K}_A} = \langle h, \widetilde{T}k\rangle_{\mathcal{K}_B}, \quad h \in \mathcal{K}_B,\ k \in \mathcal{K}_A. \tag{2.2}$$

Among other results, in this paper we obtain a generalization of this theorem, see Theorem 4.2.

2.2. KREĬN SPACES AND THEIR LINEAR OPERATORS. We recall that a *Kreĭn space* $\mathcal{K}$ is a complex linear space on which it is defined an indefinite scalar product $[\cdot,\cdot]$ such that $\mathcal{K}$ is decomposed in a direct sum

$$\mathcal{K} = \mathcal{K}_+ [\dot{+}] \mathcal{K}_- \tag{2.3}$$

in such a way that $\mathcal{K}_\pm$ are Hilbert spaces with scalar products $\pm[\cdot,\cdot]$, respectively and the direct sum in (2.3) is orthogonal with respect to the indefinite scalar product $[\cdot,\cdot]$, i.e. $\mathcal{K}_+ \cap \mathcal{K}_- = \{0\}$ and $[x_+, x_-] = 0$ for all $x_\pm \in \mathcal{K}_\pm$. The decomposition (2.3) gives rise to a positive definite scalar product $\langle\cdot,\cdot\rangle$ by setting $\langle x, y\rangle := \langle x_+, y_+\rangle - \langle x_-, y_-\rangle$, where $x = x_+ + x_-$, $y = y_+ + y_-$, and $x_\pm, y_\pm \in \mathcal{K}_\pm$. The scalar product $\langle\cdot,\cdot\rangle$ defines on $\mathcal{K}$ a structure of Hilbert space. Subspaces $\mathcal{K}_\pm$ are orthogonal with respect to the scalar product $\langle\cdot,\cdot\rangle$, too. We denote by $P_\pm$ the corresponding orthogonal projections onto $\mathcal{K}_\pm$, and let $J = P_+ - P_-$. The operator J is a *symmetry*, i.e. a selfadjoint and unitary operator, $J^*J = JJ^* = J^2 = I$.

Given a Kreĭn space $(\mathcal{K}, [\cdot,\cdot])$ the cardinal numbers

$$\kappa^+(\mathcal{K}) = \dim(\mathcal{K}^+), \quad \kappa^-(\mathcal{K}) = \dim(\mathcal{K}^-), \tag{2.4}$$

do not depend on the fundamental decomposition and they are called, respectively, the *geometric ranks of positivity/negativity* of $\mathcal{K}$.

The operator J is called a *fundamental symmetry* of the Kreĭn space $\mathcal{K}$. Note that $[x, y] = \langle Jx, y\rangle$, $(x, y \in \mathcal{K})$. If T is a densely defined operator from a Kreĭn space $\mathcal{K}_1$ to another $\mathcal{K}_2$, it can be defined *the adjoint* of T as an operator $T^\sharp$ defined on the set of all $y \in \mathcal{K}_2$ for which there exists $h_y \in \mathcal{K}_1$ such that $[Tx, y] = [x, h_y]$, and $T^\sharp y = h_y$. We remark that $T^\sharp = J_1 T^* J_2$, where T^* denotes the adjoint operator of T with respect to the Hilbert spaces $(\mathcal{K}_1, \langle\cdot,\cdot\rangle_{J_1})$ and $(\mathcal{K}_1, \langle\cdot,\cdot\rangle_{J_1})$. We will use $\sharp$ to denote the adjoint when at least one of the spaces $\mathcal{K}_1$ or $\mathcal{K}_2$ is indefinite. In the case of an operator T defined on the Kreĭn space $\mathcal{K}$, T is called *symmetric* if $T \subset T^\sharp$, i.e. if the relation $[Tx, y] = [x, Ty]$ holds for each $x, y \in \mathrm{Dom}(T)$ and T is called *selfadjoint* if $T = T^\sharp$.

In this paper we will use a bit of the geometry of Kreĭn spaces. Thus, a (closed) subspace $\mathcal{L}$ of a Kreĭn space $\mathcal{K}$ is called *regular* if $\mathcal{K} = \mathcal{L} + \mathcal{L}^\perp$, where $\mathcal{L}^\perp = \{x \in \mathcal{K} : [x, y] = 0 \text{ for all } y \in \mathcal{L}\}$. Regular spaces of Kreĭn spaces are important since they are exactly the analog of Kreĭn subspaces, that is, if we want $\mathcal{L}$ be a Kreĭn space with the restricted indefinite inner product and the same strong topology, then it should be regular.

In addition, let us recall that, given a subspace $\mathcal{L}$ of a Kreĭn space, we call $\mathcal{L}$ *non-negative* (*positive*) if the inequality $[x, x] \geqslant 0$ holds for $x \in \mathcal{L}$ (respectively, $[x, x] > 0$ for all $x \in \mathcal{L} \setminus \{0\}$). Similarly we define *non-positive* and *negative* subspaces. A subspace $\mathcal{L}$ is called *degenerate* if $\mathcal{L} \cap \mathcal{L}^\perp \neq \{0\}$. Regular subspaces are non-degenerate. As a consequence of the Schwarz inequality, if a subspace $\mathcal{L}$ is either positive or negative it is nondegenerate. A remarkable class of subspaces are those regular spaces that are either positive or negative, for which the terms *uniformly positive*, respectively, *uniformly negative* are used. These notions can be defined for linear manifolds also, that is, without assuming closedness.

A linear operator V defined from a subspace of a Kreĭn space $\mathcal{K}_1$ and valued into another Kreĭn space $\mathcal{K}_2$ is called *isometric* if $[Vx, Vy] = [x, y]$ for all x, y in the domain of V. Note that isometric operators between genuine Kreĭn spaces are unbounded and different criteria of boundedness are available, see [2]. One can even define *unbounded unitary* operators in Kreĭn spaces (e.g. see [13]). However, in this paper a *unitary* operator between Kreĭn spaces means that it is a bounded isometric operator that has a bounded inverse.

3. KREĬN SPACES INDUCED BY SYMMETRIC OPERATORS

If A is a symmetric densely defined linear operator in the Hilbert space $\mathcal{H}$ we can define a new inner product $[\cdot,\cdot]_A$ on $\mathrm{Dom}(A)$, the domain of A, by

$$[x, y]_A = \langle Ax, y\rangle_{\mathcal{H}}, \quad x, y \in \mathrm{Dom}(A). \tag{3.1}$$

In this section we investigate the existence and the properties of some Kreĭn spaces associated to this kind of inner product space.

A pair $(\mathcal{K}, \Pi)$ is called a *Kreĭn space induced by A* if:

(i) $\mathcal{K}$ is a Kreĭn space;
(ii) Π is a linear operator from $\mathcal{H}$ into $\mathcal{K}$ such that $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(\Pi)$;
(iii) $\Pi\,\mathrm{Dom}(A)$ is dense in $\mathcal{K}$;

(iv) $[\Pi x, \Pi y] = \langle Ax, y\rangle$ for all $x \in \mathrm{Dom}(A)$ and $y \in \mathrm{Dom}(\Pi)$.

The operator Π is called *the canonical operator*.

REMARK 3.1. Let A be a symmetric densely defined linear operator in the Hilbert space $\mathcal{H}$.

(1) $(\mathcal{K}, \Pi)$ is a Kreĭn space induced by A if and only if it satisfies the axioms (i)–(iii) and, in addition,

(iv′) $\Pi^\sharp \Pi \supseteq A$,

in the sense that $\mathrm{Dom}(A) \subseteq \Pi^\sharp \Pi$ and $Ax = \Pi^\sharp \Pi x$ for all $x \in \mathrm{Dom}(A)$.

(2) Without loss of generality we can assume that Π is closed. This follows from the remark at item (1): axiom (iv) can be interpreted as $\Pi^\sharp \Pi \supseteq A$. Then, by axiom (iii) it follows that $\Pi^\sharp$ is densely defined, hence Π is closable. Finally, we note that by replacing Π with its closure, all the axioms are fulfilled.

(3) Let us consider a symmetric densely defined operator A that admits an induced Kreĭn space $(\mathcal{K}, \Pi)$ such that Π is bounded. Then A is bounded. If A is bounded then, in general, it does not follow that Π is bounded. This anomaly is explained by the existence of unbounded isometric operators in a Kreĭn space. However, if A is not only bounded but also everywhere defined, then the operator Π is bounded as well.

For the moment it is not clear why Kreĭn spaces induced by symmetric operators should exist. This is the first major difference when compared to the nonnegative definite case, see [3].

At this level of generality, we distinguish a general characterization of existence of induced Kreĭn spaces, in connection to Theorem 7.1 in [5]. It is remarkable that this can be done in terms of decompositions as a difference of two nonnegative operators, as well.

THEOREM 3.2. *Let A be a densely defined and symmetric operator in a Hilbert space $\mathcal{H}$. The following assertions are equivalent:*

(i) *There exists a nonnegative quadratic form q on* $\mathrm{Dom}(A)$ *such that*

$$-q(x) \leqslant \langle Ax, x\rangle \leqslant q(x), \quad x \in \mathrm{Dom}(A).$$

(i′) *There exists a nonnegative operator B in $\mathcal{H}$ such that* $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(B)$ *and* $-\langle Bx, x\rangle_{\mathcal{H}} \leqslant \langle Ax, x\rangle_{\mathcal{H}} \leqslant \langle Bx, x\rangle_{\mathcal{H}}$ *for all $x \in \mathrm{Dom}(A)$.*

(ii) *There exists a nonnegative quadratic form q on* $\mathrm{Dom}(A)$ *such that*

$$|\langle Ax, y\rangle|^2 \leqslant q(x)q(y), \quad x, y \in \mathrm{Dom}(A).$$

(ii′) *There exists a nonnegative operator B in $\mathcal{H}$ such that* $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(B)$ *and* $|\langle Ax, y\rangle| \leqslant |\langle Bx, x\rangle|^{1/2} |\langle By, y\rangle|^{1/2}$ *for all $x, y \in \mathrm{Dom}(A)$.*

(iii) *$A \subseteq A_+ - A_-$ for two nonnegative operators $A_\pm$ in $\mathcal{H}$, that is,* $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(A_+) \cap \mathrm{Dom}(A_-)$ *and $Ax = A_+x - A_-x$ for all $x \in \mathrm{Dom}(A)$.*

(iv) *There exists a Kreĭn space induced by A.*

Proof. The equivalences of (i) with (ii) and, respectively, of (i′) and (ii′), is a standard argument of quadratic forms, see e.g. [21]. If B is as in (ii′) then let $A_+ = \frac{1}{2}(B+A)$ and $A_- = \frac{1}{2}(B-A)$, both with domain $\mathrm{Dom}(A)$. Then clearly (iii) holds. Conversely, once we have (iii) we let $B = A_+ + A_-$ which is a nonnegative operator that satisfies condition (ii′).

Let us now assume that (i) holds. We consider $\mathcal{H}$ the Hilbert space obtained by quotient-completion with respect to q: we factor $\mathrm{Dom}(A)$ by the isotropic subspace of $\mathcal{J}(q)$ of q and then take the abstract completion to a Hilbert space. Due to the inequality in (i) we have $\mathcal{J} \subseteq \mathrm{Ker}(A)$, hence the operator A can be factored by $\mathcal{J}(q)$ and the same inequality implies that this operator can be extended by continuity to a bounded (actually contractive) and selfadjoint operator onto the whole space $\mathcal{H}$. We define the Kreĭn space $\mathcal{K}$ as $\mathcal{H}$ where the indefinite inner product is given by the symmetry $S_A = \mathrm{sgn}(A)$. The operator Π is defined to have the domain $\mathrm{Dom}(A)$ and acts as the composition of the factorization by $\mathcal{J}(q)$ and the embedding of the factor space into $\mathcal{K}(=\mathcal{H})$. Then $(\mathcal{K}, \Pi)$ is a Kreĭn space induced by A.

On the other hand, let us assume that there exists a Kreĭn space $(\mathcal{K}, \Pi)$ induced by A. Let $\mathcal{K} = \mathcal{K}^+[+]\mathcal{K}^-$ be a fundamental decomposition and the corresponding fundamental symmetry $J = J^+ - J^-$. Define $\Pi_\pm = J^\pm \Pi \colon \mathrm{Dom}(\Pi) \to \mathcal{K}^+$. Then $A_+ = \Pi_+^* \Pi_+$ and $A_- = \Pi_-^* \Pi_-$ are nonnegative operators in $\mathcal{H}$ such that $A_+ - A_- = \Pi_+^*\Pi_+ - \Pi_-^*\Pi_- = \Pi^* J \Pi = \Pi^\sharp \Pi \supseteq A$, by Lemma 3.1. Hence, (iv) implies (iii). ∎

As a consequence of Theorem 3.2 and the spectral theory of selfadjoint operators in Hilbert space it follows

COROLLARY 3.3. *For any densely defined symmetric operator A that admits a selfadjoint extension in $\mathcal{H}$, there exists a Kreĭn space induced by A.*

Proof. If A is selfadjoint, then by the spectral theory of selfadjoint operators, there exists the Jordan decomposition $A = A_+ - A_-$, where $A_\pm$ are nonnegative selfadjoint operators (e.g. see [15], [1], [23]) defined by borelian functional calculus. Then use Theorem 3.2.

If A is not selfadjoint but it admits a selfadjoint extension in $\mathcal{H}$, we use the Jordan decomposition of the extension to produce two nonnegative operators $A_\pm$ such that $A \subseteq A_+ - A_-$, and proceed as before. ∎

In connection with the previous corollary, it is natural to ask whether the assertions (i)–(iv) in Theorem 3.2 are actually equivalent with the assertion in the corollary, namely, that A has a selfadjoint extension. The following example shows that this is not the case.

EXAMPLE 3.4. Let A_- and A_+ be the differential operators on $L_2(\mathbb{R}_+)$ defined by the differential expressions

$$A_- = -\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^2, \quad A_+ = -\Big(\frac{\mathrm{d}}{\mathrm{d}x}\Big)^2 + 2\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + 1,$$

where $\mathrm{Dom}(A_-) = \mathrm{Dom}(A_+)$ is the Sobolev space $W_2^2(\mathbb{R}_+)$, with the Dirichlet boundary conditions at 0. Then both A_- and A_+ are nonnegative selfadjoint operators, but the operator

$$A := A_+ - A_- = 2\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + 1$$

is a symmetric operator in $L_2(\mathbb{R}_+)$ with defect indices $(1,0)$, and hence does not have selfadjoint extensions.

Another distinction with respect to the definite case, that is, when the symmetric operator A is nonnegative as in [3], is the problem of uniqueness, modulo unitary equivalence. Two Kreĭn spaces $(\mathcal{K}_i, \Pi_i)$, $i = 1,2$, induced by the same symmetric operator A, are called *unitarily equivalent* if there exists a bounded unitary operator $U\colon \mathcal{K}_1 \to \mathcal{K}_2$ such that

$$U\Pi_1 x = \Pi_2 x, \quad x \in \mathrm{Dom}(A). \tag{3.2}$$

Before considering the uniqueness problem, we first record a special case, very useful in applications, when both existence and uniqueness hold. Recall that $\kappa_-(A)$ and $\kappa_+(A)$ denote the number of the negative and, respectively, the positive squares of the quadratic form associated to the inner product $\langle\cdot,\cdot\rangle_A$ defined as in (3.1), more precisely, $\kappa_\pm(A)$ is the number of positive/negative squares of the quadratic form $\mathrm{Dom}(A) \ni x \mapsto \langle Ax, x\rangle$, if this is finite, and the symbol $+\infty$ in the opposite case. In a different formulation, $\kappa_\pm(A)$ is the (algebraic) dimension of the spectral subspace of A corresponding to the positive/negative semi-axis, when these spectral subspaces exist.

PROPOSITION 3.5. *Let A be a densely defined and symmetric operator such that either $\kappa_-(A) < \infty$ or $\kappa_+(A) < \infty$. Then there exists and it is unique, up to a unitary equivalence, a Kreĭn space induced by A.*

Proof. Assume that $\kappa_-(A) < \infty$. The inner product space $(\mathrm{Dom}(A), [\cdot,\cdot]_A)$ is decomposable, that is, there exists a decomposition

$$\mathrm{Dom}(A) = \mathcal{D}_- \dot{+} \operatorname{Ker} A \dot{+} \mathcal{D}_+, \tag{3.3}$$

where the inner product spaces $(\mathcal{D}_\pm, \pm[\cdot,\cdot])$ are positive definite and mutually orthogonal, e.g. see Theorem I.11.7 in [2]. We consider the nondegenerate inner product space $(\mathcal{D}_- \dot{+} \mathcal{D}_+, [\cdot,\cdot]_A)$ and, since $\dim \mathcal{D}_- = \kappa_-(A) < \infty$, there exists the completion of this space to a Pontryagin space $(\mathcal{K}, [\cdot,\cdot]_A)$ such that $\kappa^-(\mathcal{K}) = \kappa_-(A) < \infty$. Consider the linear mapping $\Pi\colon\ \mathrm{Dom}(A) \to \mathcal{K}$ defined by

$$\mathrm{Dom}(A) \ni x_- + x_0 + x_+ \mapsto x_- + x_+ \in \mathcal{K}, \quad x_\pm \in \mathcal{D}_\pm,\ x_0 \in \operatorname{Ker} A.$$

Then Π has dense range. Also Π has a densely defined adjoint, more precisely, this is an extension of the linear mapping $\mathcal{K} \supseteq \mathcal{D}_- + \mathcal{D}_+ \ni x \mapsto x \in \mathcal{H}$, and hence Π is closable. We denote by the same symbol Π its closure and then $(\mathcal{K}, \Pi)$ is a Kreĭn space induced by A.

Let $(\mathcal{K}_1, \Pi_1)$ be another Kreĭn space induced by A. Since Π_1 satisfies the axiom (iv), it follows that $\kappa^-(\mathcal{K}_1) \geqslant \kappa_-(A)$. Since Π_1 has dense range, we easily obtain the converse inequality and hence $\kappa^-(\mathcal{K}_1) = \kappa_-(A)$. Define a linear operator $U \colon \mathcal{D}_- + \mathcal{D}_+ \to \mathcal{R}(\Pi_1)$ by

$$Ux = \Pi_1 x, \quad x \in \mathcal{D}_- + \mathcal{D}_+. \tag{3.4}$$

Since both Π and Π_1 satisfy the axiom (iv), it follows that U is isometric and then (see Theorem VI.3.5 in [2]) it follows that U can uniquely be extended to a bounded unitary operator $U \colon \mathcal{K} \to \mathcal{K}_1$. The analog of (3.2) follows from (3.4) and the definition of Π. ∎

4. TWO CANONICAL REPRESENTATIONS OF KREĬN SPACES INDUCED BY SELFADJOINT OPERATORS

The existence of Kreĭn spaces induced by symmetric operators is guaranteed in case the operator A is selfadjoint, cf. Corollary 3.3. Since, even for selfadjoint operators (as will be seen in Theorem 5.3) we do not have in general uniqueness of the induced Kreĭn spaces, it is useful to point out some "canonical" constructions.

4.1. THE INDUCED KREĬN SPACE $(\mathcal{K}_A, \Pi_A)$. The first example starts with a selfadjoint operator A and describes a construction of a Kreĭn space induced by A, more or less the equivalent of the quotient completion method.

Let A be a selfadjoint operator in the Hilbert space $\mathcal{H}$. We consider the polar decomposition of A

$$A = S_A |A|, \tag{4.1}$$

where, by borelian functional calculus, there are defined $|A| = (A^*A)^{1/2} = (A^2)^{1/2}$, the *modulus* (or the *absolute value*) of the operator A, and $S_A = \operatorname{sgn}(A)$, that is a selfadjoint partial isometry on $\mathcal{H}$. It is known (e.g. see [23], [1]) that $\operatorname{Dom}(A) = \operatorname{Dom}(|A|)$ and that $|A|$ is a nonnegative selfadjoint operator. We now consider the quotient completion of $\operatorname{Dom}(A)$ with respect to the nonnegative selfadjoint operator $|A|$ as in Remark 2.1, and define $\mathcal{K}_A = \mathcal{K}_{|A|}$. To be more precise, we do the following: on $\operatorname{Dom}(A)$ we consider the semi-norm $\| |A|^{1/2} \cdot \|$, factor $\operatorname{Dom}(A)$ by the kernel of A (which coincides with the isotropic part of this semi-norm) and then complete the factor space $\operatorname{Dom}(A) / \operatorname{Ker}(A)$ to the Hilbert space that we denote by $\mathcal{K}_A$. Recall that $\operatorname{Dom}(A) \subseteq \operatorname{Dom}(|A|^{1/2})$ and that $\operatorname{Dom}(A)$ is a core for $|A|^{1/2}$. Further, $\operatorname{Ker}(S_A) = \operatorname{Ker}(A)$ and S_A leaves invariant $\operatorname{Dom}(A)$. Since S_A is a selfadjoint partial isometry, its spectrum coincides with its point spectrum and is contained in $\{-1,0,+1\}$. Hence $\mathrm{Dom}(A) = \mathcal{D}_+ \oplus \mathrm{Ker}(A) \oplus \mathcal{D}_-$ where

$$\mathcal{D}_\pm = \mathrm{Dom}(A) \cap \mathrm{Ker}(S_A \mp I). \tag{4.2}$$

This implies that we can identify naturally $\mathrm{Dom}(A)/\mathrm{Ker}(A)$ with $\mathcal{D}_+ \oplus \mathcal{D}_-$. Now observe that we can complete $\mathcal{D}_\pm$ with respect to the norm $\||A|^{1/2}\cdot\|$ and let these completions be denoted by $\mathcal{K}_A^\pm$. Then, $\mathcal{K}_A$ can be naturally identified with $\mathcal{K}_A^+ \oplus \mathcal{K}_A^-$ and, considering this as a fundamental decomposition,

$$\mathcal{K}_A = \mathcal{K}_A^+[+]\mathcal{K}_A^- \tag{4.3}$$

it yields an indefinite inner product $[\cdot,\cdot]$ with respect to which $\mathcal{K}_A$ becomes a Kreĭn space.

Equivalently, this construction of the Kreĭn space $(\mathcal{K}_A, [\cdot,\cdot])$ can be done as follows: we first recall that S_A commutes with all selfadjoint operators A, $|A|$, and $|A|^{1/2}$. For example, since S_A commutes with $|A|^{1/2}$ it follows that $\mathrm{Dom}(|A|^{1/2})$ is invariant under S_A and for all $x \in \mathrm{Dom}(|A|^{1/2})$ we have $S_A|A|^{1/2}x = |A|^{1/2}S_A x$. This implies that S_A is isometric with respect to this seminorm and hence, S_A factors by $\mathrm{Ker}(A)$ and extends uniquely by continuity to an isometric operator on the Hilbert space $\mathcal{K}_{|A|}$, that we denote also by S_A. We now observe that S_A is actually a symmetry (that is, both unitary and selfadjoint) on the Hilbert space $\mathcal{K}_{|A|}$. Indeed, for this we take into account that S_A commutes with $|A|$, that is,

$$S_A|A|x = |A|S_A x, \quad x \in \mathrm{Dom}(|A|) = \mathrm{Dom}(A),$$

and get

$$\langle |A|S_A x, y\rangle = \langle S_A|A|x, y\rangle = \langle |A|x, S_A y\rangle, \quad x,y \in \mathrm{Dom}(|A|) = \mathrm{Dom}(A),$$

which shows the selfadjointness of S_A in the Hilbert space $\mathcal{K}_{|A|}$. Since S_A is also isometric with respect to the seminorm $\||A|^{1/2}\cdot\|$, it follows that it is a symmetry in $\mathcal{K}_{|A|}$. Then we use this symmetry to introduce on $\mathcal{K}_{|A|} = \mathcal{K}_A$ an indefinite inner product that turns $\mathcal{K}_A$ into a Kreĭn space. It is easy to see that the fundamental decomposition in (4.3) is exactly that corresponding to the fundamental symmetry S_A.

Finally, let Π_A be the operator which is obtained by composing the canonical surjection $\mathrm{Dom}(A) \to \mathrm{Dom}(A)/\mathrm{Ker}(A)$ with the embedding of the space $\mathrm{Dom}(A)/\mathrm{Ker}(A)$ into its Hilbert space completion $\mathcal{K}_{|A|} = \mathcal{K}_A$.

PROPOSITION 4.1. *If A is a selfadjoint operator on the Hilbert space $\mathcal{H}$ then, with the notation as before, $(\mathcal{K}_A, \Pi_A)$ is a Kreĭn space induced by A.*

Proof. We verify the axioms (i)–(iv) in the definition of the Kreĭn space induced by A. It was proved above that $\mathcal{K}_A$ is a Kreĭn space. By definition Π_A is a linear operator with domain $\mathrm{Dom}(\Pi_A) = \mathrm{Dom}(A)$ and range $\mathrm{Dom}(A)/\mathrm{Ker}(A)$ dense in $\mathcal{K}_A$. Thus only the axiom (iv) remains to be verified. For any $x, y \in$

$\mathrm{Dom}(A) = \mathrm{Dom}(\Pi)$ we have the following, which concludes the proof:

$$\begin{aligned}[\Pi x, \Pi y]_{\mathcal{K}_A} &= [x + \mathrm{Ker}(A), y + \mathrm{Ker}(A)]_{\mathcal{K}_A} = \langle S_A(x + \mathrm{Ker}(A)), y + \mathrm{Ker}(A)\rangle_{\mathcal{K}_{|A|}} \\ &= \langle |A|^{1/2} S_A x, |A|^{1/2} y\rangle_{\mathcal{H}} = \langle |A|^{1/2} S_A |A|^{1/2} x, y\rangle_{\mathcal{H}} = \langle Ax, y\rangle. \quad \blacksquare\end{aligned}$$

The previous result allows us to introduce the following definition: *the geometric positive/negative ranks* of the selfadjoint operator A are, by definition,

$$\kappa^+(A) = \kappa^+(\mathcal{K}_A), \quad \kappa^-(A) = \kappa^-(\mathcal{K}_A), \tag{4.4}$$

where $(\mathcal{K}_A, \Pi_A)$ is the Kreĭn space induced by A as in Example 2.1, and the geometric ranks of positivity/negativity are defined as in (2.4). It is not difficult to see that $\kappa^\pm(A)$ coincides with the (Hilbert space) dimension of the spectral subspace of A corresponding to the positive/negative semi-axis.

4.2. THE LIFTING PROPERTY OF THE SPACE $(\mathcal{K}_A, \Pi_A)$. In order to exploit the full power of induced Kreĭn spaces we need to know which linear operators can be lifted to induced Kreĭn spaces. Based on Theorem 2.2 we can answer positively this question for the Kreĭn spaces in the unitary orbit of $(\mathcal{K}_A, \Pi_A)$, that is, for any other Kreĭn space $(\mathcal{K}, \Pi)$ that is unitarily equivalent with $(\mathcal{K}_A, \Pi_A)$.

THEOREM 4.2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let A and B be selfadjoint operators in $\mathcal{H}_1$ and respectively $\mathcal{H}_2$. We consider the induced Kreĭn spaces $(\mathcal{K}_A, \Pi_A)$ and $(\mathcal{K}_B, \Pi_B)$. Then for any operators $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, and $S \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ such that*

$$\langle Bx, Ty\rangle_{\mathcal{H}_2} = \langle Sx, Ay\rangle_{\mathcal{H}_1}, \quad x \in \mathrm{Dom}(B),\ y \in \mathrm{Dom}(A), \tag{4.5}$$

there exist uniquely determined operators $\widetilde{T} \in \mathcal{L}(\mathcal{K}_A, \mathcal{K}_B)$ and $\widetilde{S} \in \mathcal{L}(\mathcal{K}_B, \mathcal{K}_A)$ such that $\widetilde{T}\Pi_A x = \Pi_B T x$ for all $x \in \mathrm{Dom}(A)$ and $\widetilde{S}\Pi_B y = \Pi_A S y$, for all $y \in \mathrm{Dom}(B)$ and

$$\langle \widetilde{S}h, k\rangle_{\mathcal{K}} = \langle h, \widetilde{T}k\rangle_{\mathcal{K}}, \quad h \in \mathcal{K}_B,\ k \in \mathcal{K}_A.$$

Proof. Let $A = S_A|A|$ and $B = S_B|B|$ be the polar decompositions of A and respectively B, then we note that (4.5) can be written

$$\langle |B|x, S_B T y\rangle_{\mathcal{H}_2} = \langle S_A S x, Ay\rangle_{\mathcal{H}_1}, \quad x \in \mathrm{Dom}(B),\ y \in \mathrm{Dom}(A), \tag{4.6}$$

and hence we can apply Theorem 2.2 to the operators $S_B T$ and $S_A S$ to obtain the lifted operators X and Y. Then note that S_B and S_A can be lifted to fundamental symmetries on $\mathcal{K}_B$ and respectively S_A, and hence they are invertible on $\mathcal{K}_B$ and, respectively, $\mathcal{K}_A$, and finally let $\widetilde{T} = S_B^{-1} X$ and $\widetilde{S} = S_A^{-1} Y$. $\blacksquare$

In Theorem 2.3 of [6], it is proven that in any infinite dimensional Hilbert space there exist bounded selfadjoint operators that admit induced Kreĭn spaces that do not have the lifting property. Of course, this implies that in the unbounded case the situation is not better.

4.3. THE INDUCED KREĬN SPACE $(\mathcal{H}_A, \pi_A)$. The construction of the Kreĭn space $(\mathcal{K}_A, \Pi_A)$ induced by A, when A is a selfadjoint operator in the Hilbert space $\mathcal{H}$, has the disadvantage that it is obtained by a completion procedure and hence some of the vectors in $\mathcal{K}_A$ can be outside of $\mathcal{H}$. In the following we present a different construction in which the induced Kreĭn space is actually a subspace of $\mathcal{H}$, the strong topology of this induced Kreĭn space is inherited from the strong topology of $\mathcal{H}$, but the cost is a more involved canonical mapping Π.

Let A be a selfadjoint operator in the Hilbert space $\mathcal{H}$. We consider the polar decomposition (4.1) of the operator A. The operator S_A is a selfadjoint partial isometry and consider the subspace $\mathcal{H}_A = \overline{\operatorname{Ran}(A)}$ that is invariant under S_A. $\mathcal{H}_A$ is a Hilbert space, as a closed subspace of $\mathcal{H}$. The restriction of this operator to $\mathcal{H}_A$ is a symmetry and let us define the inner product $[\cdot,\cdot]$ by

$$[x,y]_{S_A} = \langle S_A x, y\rangle_{\mathcal{H}}, \quad x,y \in \mathcal{H}_A. \tag{4.7}$$

We consider the Kreĭn space $\langle \mathcal{H}_A, [\cdot,\cdot]_{S_A})$. Since $\operatorname{Ran}(|A|^{1/2}) \subseteq \mathcal{H}_A$ we can define the operator $\pi_A \colon \operatorname{Dom}(|A|^{1/2}) \to \mathcal{H}_A$ by

$$\pi_A x = |A|^{1/2} x, \quad x \in \operatorname{Dom}(|A|^{1/2}). \tag{4.8}$$

PROPOSITION 4.3. *Let A be a selfadjoint operator on the Hilbert space $\mathcal{H}$. With the notation as above, $(\mathcal{H}_A, \pi_A)$ is a Kreĭn space induced by A. Moreover, $(\mathcal{K}_A, \Pi_A)$ is unitarily equivalent with $(\mathcal{H}_A, \pi_A)$.*

Proof. To prove that $(\mathcal{H}_A, \pi_A)$ is a Kreĭn space induced by A, note that we already proved above that $\mathcal{H}_A$ is a Kreĭn space. Then note that π_A is closed and densely defined, as $|A|^{1/2}$ has the same properties. Since $\operatorname{Dom}(A) = \operatorname{Dom}(|A|)$ is a core of $|A|^{1/2}$ it follows that $\pi_A \operatorname{Dom}(A)$ is dense in $\mathcal{H}_A$. In addition,

$$[\pi_A x, \pi_A y]_{S_A} = \langle S_A |A|^{1/2} x, |A|^{1/2} y) = [Ax, y], \quad x \in \operatorname{Dom}(A),\ y \in \operatorname{Dom}(|A|^{1/2}).$$

We prove now that the induced Kreĭn spaces $(\mathcal{H}_A, \pi_A)$ and $(\mathcal{K}_A, \Pi_A)$ are unitarily equivalent. To this end, we consider the operator U with $\operatorname{Dom}(U) = \operatorname{Dom}(A) \subseteq \mathcal{K}_A$ and range in $\mathcal{H}_A$, defined by

$$Ux = |A|^{1/2} x, \quad x \in |A|^{1/2} \operatorname{Dom}(A). \tag{4.9}$$

It follows that for all $x, y \in \operatorname{Dom}(A)$ we have

$$[Ux, Uy]_{S_A} = \langle S_A |A|^{1/2} x, |A|^{1/2} y\rangle_{\mathcal{H}} = \langle Ax, y\rangle_{\mathcal{H}} = [x,y]_A,$$

which proves that U is isometric with respect to the indefinite inner products on $\mathcal{H}_A$ and respectively $\mathcal{K}_A$. Taking into account how the strong topologies on these Kreĭn spaces are defined, more precisely, on $\mathcal{K}_A$ it is given by the (semi)norm $\| |A|^{1/2} \cdot \|$ while on $\mathcal{H}_A$ it is that inherited from $\mathcal{H}$, it follows that U is actually isometric with respect to these Hilbert space norms, and thus continuous. Since, by the definition of the space $\mathcal{H}_A$, U has dense range, it follows that it is a bounded unitary operator between the Kreĭn spaces $\mathcal{K}_A$ and $\mathcal{H}_A$. Using the definition of U it follows that $U\Pi_A x = \pi_A x$, for all $x \in \operatorname{Dom}(A)$. ∎

5. UNIQUENESS

We are now in a position to approach the uniqueness, modulo unitary equivalence, of the Kreĭn spaces induced by symmetric densely defined operators. First we record a sufficient condition.

PROPOSITION 5.1. *Let A be a densely defined symmetric operator in a Hilbert space $\mathcal{H}$ such that it admits a Kreĭn space induced by A, let this be $(\mathcal{K},\Pi)$ subject to the property that the linear manifold $\Pi\,\mathrm{Dom}(A)$ contains a maximal uniformly definite subspace of $\mathcal{K}$. Then the Kreĭn space induced by A is unique, modulo unitary equivalence.*

Proof. Let $(\mathcal{K}_i,\Pi_i)$, $i=1,2$, be Kreĭn spaces induced by A. The equation $U\Pi_1 x=\Pi_2 x$, $x\in\mathrm{Dom}(A)$, uniquely determines an isometric operator densely defined in $\mathcal{K}_1$ and with dense range in $\mathcal{K}_2$. If $\Pi_1\,\mathrm{Dom}(A)$ contains a maximal uniformly definite subspace then by Theorem VI.3.5 in [2] it follows that U has a unique extension to a bounded unitary operator and hence the two Kreĭn spaces induced by A are unitarily equivalent. ∎

REMARK 5.2. The question whether the sufficient condition in the previous proposition is also necessary is related to the study of dense operator ranges in Kreĭn spaces, as in [10]. For a dense operator range $\mathcal{K}$ in a Kreĭn space $\mathcal{D}$, according to [10], the following alternative holds: either $\mathcal{D}$ contains a maximal uniformly definite subspace or it is contained in a subspace of form $\mathcal{L}+\mathcal{L}^{\perp}$, where $\mathcal{L}$ is a maximal positive subspace that is not uniformly definite. According to [13], subspaces of the latter form are exactly the domains of unbounded unitary operators. Therefore, if additionally we require that the symmetric densely defined operator A admits an induced Kreĭn space $(\mathcal{K},\Pi)$ such that $\mathrm{Dom}(A)=\mathrm{Dom}(\Pi)$ and Π is closed, then the uniqueness of the Kreĭn space induced by A, modulo unitary equivalence, implies that $\Pi\,\mathrm{Dom}(A)$ contains a maximal uniformly definite subspace, equivalently, $\Pi\,\mathrm{Dom}(A)$ is not contained in any domain of unbounded unitary operators.

In the special case of a selfadjoint operator, we can obtain a characterization of uniqueness in spectral terms. The lateral spectral gap condition plays a role in similar uniqueness problems, as pointed out in [14], [7], [10], and [4]. In the following, $\rho(A)$ denotes the resolvent set of the operator A.

THEOREM 5.3. *Let A be a selfadjoint operator in the Hilbert space $\mathcal{H}$. The following statements are equivalent:*

(i) *The Kreĭn space induced by A is unique, modulo unitary equivalence.*

(ii) *A has a lateral spectral gap, that is, there exists an $\varepsilon>0$ such that either $(0,\varepsilon)\subset\rho(A)$ or $(-\varepsilon,0)\subset\rho(A)$.*

Proof. (i)⇒(ii) We actually show that the same idea as in Theorem 3.2 in [4] works in this unbounded case as well. Let us assume that the statement (ii) does not hold. Then there exists a decreasing sequence of values $\{\mu_n\}_{n\geqslant 1}\subseteq\sigma(A)$,

$0 < \mu_n < 1$ such that $\mu_n \to 0$ $(n \to \infty)$, and there exists a decreasing sequence of values $\{\nu_n\}_{n\geqslant 1} \subseteq \sigma(-A)$, $0 < \nu_n < 1$, such that $\nu_n \to 0$ $(n \to \infty)$. Then, letting $\mu_0 = \nu_0 = 1$ there exist sequences of orthonormal vectors $\{e_n\}_{n\geqslant 1}$ and $\{f_n\}_{n\geqslant 1}$ such that

$$e_n \in E((\mu_n, \mu_{n-1}])\mathcal{H}, \quad f_n \in E([-\nu_{n-1}, -\nu_n))\mathcal{H}, \quad n \geqslant 1. \tag{5.1}$$

where E denotes the spectral measure of A. As a consequence, we also have

$$\langle Ae_i, f_j\rangle = 0, \quad i,j \geqslant 1. \tag{5.2}$$

Define the sequence $\{\lambda_n\}_{n\geqslant 1}$ by

$$\lambda_n = \max\left\{\sqrt{1-\mu_n^2}, \sqrt{1-\nu_n^2}\right\}. \tag{5.3}$$

Then, $0 < \lambda_n \leqslant 1$ for all n and

$$\lim_{n\to\infty} \lambda_n = 1. \tag{5.4}$$

We now consider $(\mathcal{K}_A, \Pi_A)$, the Kreĭn space induced by A and defined as in Example 4.1, as well as the sequence $\{\mathcal{S}_n\}_{n\geqslant 1}$, of subspaces of the Kreĭn space $\mathcal{K}_A$, defined by

$$\mathcal{S}_n = \mathbb{C}e_n \dot{+} \mathbb{C}f_n, \quad n \geqslant 1,$$

and then define the operators $U_n \in \mathcal{L}(\mathcal{S}_n)$

$$U_n = \frac{1}{\sqrt{1-\lambda_n^2}} \begin{bmatrix} 1 & -\lambda_n \\ \lambda_n & -1 \end{bmatrix}, \quad n \geqslant 1. \tag{5.5}$$

Further, we define the linear manifold $\mathcal{D}_0$ in $\mathcal{K}_A$ by $\mathcal{D}_0 = \bigcup_{k\geqslant 1} \mathcal{S}_k$.

Recalling the notation in (4.2), the linear manifold

$$\mathcal{D} = \mathcal{D}_+ \dot{+} \mathcal{D}_- = \operatorname{Ran}(\Pi_A)$$

is dense in $\mathcal{K}_A$, where $A = A_+ - A_-$ is the Jordan decomposition of A and $\mathcal{D}_\pm = \operatorname{Dom}(A) \cap \overline{\operatorname{Ran}(A_\pm)}$. By construction, $\mathcal{D}_0 \subseteq \mathcal{D} = \mathcal{D}_0 \dot{+} (\mathcal{D} \cap \mathcal{D}_0^\perp)$. Letting

$$\mathcal{D}_{+0} = \operatorname{Span}\{e_n : n \geqslant 1\}, \quad \mathcal{D}_{-0} = \operatorname{Span}\{f_n : n \geqslant 1\}, \tag{5.6}$$

from (5.1) it follows that $\mathcal{D}_0 = \mathcal{D}_{+0} + \mathcal{D}_{-0}$, where $\mathcal{D}_{\pm 0}$ are mutually orthogonal uniformly positive/negative linear manifolds. Then define a linear operator U in $\mathcal{K}_A$, with domain $\mathcal{D}_0$ and the same range, by $U|\mathcal{S}_n = U_n$, $n \geqslant 1$, and then extend it to $\mathcal{D}$ by letting $U|(\mathcal{D} \cap \mathcal{D}_0^\perp) = I|(\mathcal{D} \cap \mathcal{D}_0^\perp)$. The operator U is isometric, it has dense range as well as dense domain. On the other hand, U is unbounded because it maps uniformly definite subspaces $\mathcal{D}_{\pm 0}$ into subspaces that are not uniformly definite. Indeed, considering the sequence $x_n = U_n e_n$, we observe that

$$\langle |A| x_n, x_n\rangle = \frac{1+\lambda_n}{\sqrt{1-\lambda_n^2}} \langle Ae_n, e_n\rangle \geqslant \frac{\mu_n(1+\lambda_n)}{\sqrt{1-\lambda_n^2}} \geqslant 1,$$

where the inequality follows by (5.4). On the other hand, by (5.4),

$$[Ax_n, x_n] = \frac{1-\lambda_n}{\sqrt{1-\lambda_n^2}}\langle Ae_n, e_n\rangle = \frac{\sqrt{1-\lambda_n}}{\sqrt{1+\lambda_n}} \to 0, \quad \text{as } n \to \infty,$$

hence $U\mathcal{D}_{+0}$ is not uniformly positive.

Using all these, define the operator Π from $\mathcal{H}$ into $\mathcal{K}_A$ by $\Pi = U\Pi_A$. We claim that $(\mathcal{K}_A, \Pi)$ is a Kreĭn space induced by A.

Indeed, $\Pi \operatorname{Dom}(A) = U\Pi_A \operatorname{Dom}(A) \supseteq \mathcal{D}$ which is dense in $\mathcal{K}_A$. Further,

$$[\Pi x, \Pi y] = [U\Pi_A x, U\Pi_A y] = [\Pi_A x, \Pi_A y] = [Ax, y], \quad x \in \operatorname{Dom}(A),\ y \in \operatorname{Dom}(\Pi).$$

This concludes the proof of the claim. Since U is unbounded it follows that $(\mathcal{K}_A, \Pi_A)$ is not unitarily equivalent with $(\mathcal{K}_A, \Pi)$.

(ii)⇒(i). Let $A = A_+ - A_-$ be the Jordan decomposition of the operator A. Denoting $\mathcal{H}_\pm = \overline{\operatorname{Ran}(A_\pm)}$, the following decomposition holds

$$\mathcal{H} = \mathcal{H}_+ \oplus \operatorname{Ker} A \oplus \mathcal{H}_-. \tag{5.7}$$

The operators $A_\pm$ yield selfadjoint operators in the Hilbert spaces $\mathcal{H}_\pm$, respectively, with domains $\mathcal{D}_\pm = \mathcal{H}_\pm \cap \operatorname{Dom}(A)$. As in Example 4.1 it follows that the strong topology of $\mathcal{K}$ is determined by the norms $\mathcal{D}_\pm \ni x \mapsto \|(A_\pm)^{1/2}x\|$.

To make a choice, let us assume that there exists $\varepsilon > 0$ such that $(-\varepsilon, 0) \subseteq \rho(A)$, equivalently A_- has closed range. Since A_- is closed this implies that the normed space $(\mathcal{D}_-, \|(A_-)^{1/2}\cdot\|)$ is complete and hence, by the definition of the Kreĭn space $\mathcal{K}_A$, $\mathcal{D}_-$ is a maximal uniformly negative subspace of $\mathcal{K}_A$. In case it is assumed that $(0, \varepsilon) \subseteq \rho(A)$, in a similar way we prove that $\mathcal{D}_+$ is a maximal uniformly positive subspace of $\mathcal{K}_A$. Then we use Proposition 5.1. ∎

6. EXAMPLES

In this section we consider some concrete realizations of Kreĭn spaces induced by linear operators associated to partial differential equations. Before doing this we point out another abstract but useful construction.

6.1. REPRESENTATIONS IN TERMS OF THE CANONICAL MAPPING Π. As pointed out in Remark 3.1, given a densely defined symmetric operator A in a Hilbert space $\mathcal{H}$, the possibility of getting a Kreĭn space induced by A is more or less related to getting a factorization of A of type $\Pi^* J\Pi$, where J is a symmetry on a Hilbert space $\mathcal{K}$ and Π satisfies the axiom (i)–(iii). In this section we adopt a different point of view, when compared to the previous Section 4. Our interest is justified because both of the representations $(\mathcal{K}_A, \Pi_A)$ and $(\mathcal{H}_A, \pi_A)$ heavily depend on the modulus $|A|$ and its square root $|A|^{1/2}$, that are difficult to calculate.

PROPOSITION 6.1. *Let $T \in \mathcal{C}(\mathcal{H}, \mathcal{H}_1)$, that is, T is a closed linear operator with domain* $\mathrm{Dom}(T)$ *dense in the Hilbert space $\mathcal{H}$ and range* $\mathrm{Ran}(T)$ *in the Hilbert space $\mathcal{H}_1$ such that for some $c > 0$ we have*

$$\|Tu\|_{\mathcal{H}_1} \geqslant c\|u\|_{\mathcal{H}}, \quad u \in \mathrm{Dom}(T). \tag{6.1}$$

Let also J be a symmetry on $\mathrm{Ran}(T)$.

Then, the operator $A = T^*JT$ is selfadjoint that has a spectral gap in the neighbourhood of 0, *and* $(\mathrm{Ran}(T), T)$ *is a Kreĭn space induced by A.*

Proof. From (6.1) it follows that $\mathrm{Ran}(T)$ is closed and T is boundedly invertible, that is there exists the bounded linear operator $S = T^{-1} \colon \mathrm{Ran}(T) \to \mathcal{H}$. Therefore, the operator $B = S^*JS$ is a bounded selfadjoint operator on the Hilbert space $\mathrm{Ran}(T)$. In addition, B is injective and hence its inverse $A = B^{-1} = T^*JT$ is a selfadjoint operator in $\mathcal{H}$ and has a spectral gap in the neighbourhood of 0. The Kreĭn space structure of $\mathrm{Ran}(T)$ is given by the strong topology (inherited from that of $\mathcal{H}_1$) and the symmetry J. It is clear now that $(\mathrm{Ran}(T), T)$ is a Kreĭn space induced by A. ∎

6.2. THE FREE DIRAC OPERATOR. We first consider the standard free Dirac operator which describes the free electron in relativistic quantum mechanics (e.g. see [11], [17], [22]). To simplify the notation, we assume the mass $m = 1$ and the light speed $c = 1$. The free Dirac operator is defined in the space $\mathcal{H} = L_2(\mathbb{R}^3; \mathbb{C}^4)$ identified with $\mathbb{C}^4 \otimes L_2(\mathbb{R}^3)$ as the following

$$H_0 = \sum_{j=1}^{3} \alpha_j \otimes D_j + \alpha_0 \otimes I_{L_2(\mathbb{R}^3)},$$

where $D_j = \mathrm{i}\frac{\partial}{\partial x_j}$ $(j = 1, 2, 3)$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, α_j $(j = 1, 2, 3, 4)$ are the Dirac matrices, i.e. 4×4 Hermitian matrices which satisfy the anticommutation relations

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}, \quad j, k = 0, 1, 2, 3.$$

In the standard representation, see e.g. [22], the Dirac matrices α_j $(j = 0, 1, 2, 3)$ are chosen as follows

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \text{ for } j = 1, 2, 3; \quad \alpha_0 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices ($\sigma_0 = I_2$ designates the 2×2 identity matrix). We consider the operator H_0 defined on its maximal domain, i.e. on the Sobolev space $\mathrm{Dom}(H_0) = W_2^1(\mathbb{R}^3; \mathbb{C}^4)$. It is known that on this domain H_0 is a self-adjoint operator. Note that by applying the Fourier transformation to the elements of the space $L_2(\mathbb{R}^3;\mathbb{C}^4)$ the operator H_0 is transformed (in the momentum space) into a multiplication operator by the following matrix-valued function

$$h_0(\xi) = \begin{bmatrix} \sigma_0 & \sigma(\xi) \\ \sigma(\xi) & -\sigma_0 \end{bmatrix},$$

where

$$\sigma(\xi) = \xi_1\sigma_1 + \xi_2\sigma_2 + \xi_3\sigma_3, \quad \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3.$$

The Fourier transformation is defined by the formula

$$\widehat{u}(\xi) = (Fu)(\xi) = \frac{1}{(2\pi)^{3/2}} \int u(x) \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle} \mathrm{d}x, \quad u \in L_2(\mathbb{R}^3)$$

in which $\langle x, \xi \rangle$ designates the scalar product of the elements $x, \xi \in \mathbb{R}^3$ (here and in what follows $\int := \int\limits_{\mathbb{R}^3}$). The matrix $h_0(\xi)$ is the symbol of the operator H_0 considered as a matrix differential operator with constant coefficients. This matrix has the following eigenvalues, where $r(\xi) = (1 + |\xi|^2)^{1/2}$:

$$\lambda_1(\xi) = \lambda_2(\xi) = r(\xi), \quad \lambda_3(\xi) = \lambda_4(\xi) = -r(\xi).$$

The unitary transformation $U(\xi)$ which brings $h_0(\xi)$ to the diagonal form is given explicitly by

$$U(\xi) = \begin{bmatrix} a(\xi)I_2 & -b(\xi)\sigma(\xi) \\ b(\xi)\sigma(\xi) & -a(\xi)I_2 \end{bmatrix},$$

where $a(\xi) = \left(\frac{1}{2}(1 + r(\xi))^{-1}\right)^{1/2}$ and $b(\xi) = a(\xi)(1 + \gamma(\xi))^{-1}$. Thus, we have

$$U(\xi)h_0(\xi)U(\xi)^* = \alpha_0 r(\xi). \tag{6.2}$$

Now, we let

$$T(\xi) = r(\xi)^{1/2} U(\xi),$$

and denote by $T = T(D)$ the pseudo-differential operator corresponding to its symbol $T(\xi)$. The operator T is defined in the space $\mathcal{H} = L_2(\mathbb{R}^3;\mathbb{C}^4)$ by

$$(Tu)(x) = \frac{1}{(2\pi)^{3/2}} \int T(\xi)\widehat{u}(\xi) \mathrm{e}^{-\mathrm{i}\langle x,\xi\rangle} \mathrm{d}\xi, \ x \in \mathbb{R}^n, \tag{6.3}$$

on the domain

$$\mathrm{Dom}(T) = \{u \in L_2(\mathbb{R}^3;\mathbb{C}^4) : T(\xi)\widehat{u}(\xi) \in L_2(\mathbb{R}^3;\mathbb{C}^4)\}.$$

Obviously, $u \in \mathrm{Dom}(T)$ if and only if $\widehat{u} \in L_{2,r}(\mathbb{R}^3;\mathbb{C}^4)$, where $L_{2,r}(\mathbb{R}^3;\mathbb{C}^4)$ stands for the space weighted by $r(\xi) = (1 + |\xi|^2)^{1/2}$, i.e. the space of all functions $f \in L_2(\mathbb{R}^3;\mathbb{C}^4)$ such that $rf \in L_2(\mathbb{R}^3;\mathbb{C}^4)$. Note that $F^* L_{2,r}(\mathbb{R}^3;\mathbb{C}^4) = W_2^1(\mathbb{R}^3;\mathbb{C}^4)$ (the Fourier transformation in the space $L_2(\mathbb{R}^3;\mathbb{C}^4)$ is again denoted by F).

It follows from (6.3) the factorization

$$H_0 = T^*(\alpha_0 \otimes I_{L_2(\mathbb{R}^3)})T.$$

Since

$$\|Tu\|^2 = \int |T(\xi)\widehat{u}(\xi)|^2 \mathrm{d}\xi = \int r(\xi)|\widehat{u}(\xi)|^2 \mathrm{d}\xi \geqslant \int |\widehat{u}(\xi)|^2 \mathrm{d}\xi = \|u\|$$

for all $u \in \mathrm{Dom}(T)$, the condition (6.1) from Proposition 6.1 is fulfilled. In particular, the range $\mathrm{Ran}(T)$ is closed in the space $L_2(\mathbb{R}^3;\mathbb{C}^4)$, and so we have the Hilbert space

$$\mathcal{G}_T = (\mathrm{Ran}(T), \|\cdot\|_{L_2(\mathbb{R}^3;\mathbb{C}^4)}).$$

On the space $\mathcal{H} = L_2(\mathbb{R}^3;\mathbb{C}^4)$ we consider the symmetry given by

$$Ju = \alpha_0 \otimes I_{L_2(\mathbb{R}^r)} u, \quad u \in \mathcal{L}_2(\mathbb{R}^r), \tag{6.4}$$

and hence the Hilbert space $\mathcal{G}_T$ equipped with the indefinite scalar product defined by J becomes a Kreĭn space that we denote by $\mathcal{K}$. We have the decomposition

$$\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-,$$

where the orthogonal projection operators from $\mathcal{K}$ onto $\mathcal{K}_\pm$ are given by

$$P_\pm = \frac{1}{2}(I \pm \alpha_0) \otimes I_{L_2(\mathbb{R}^r)}.$$

We conclude that the pair $(\mathcal{K}, \Pi)$, where $\Pi = T$ (recall that T is the pseudo-differential operator defined in the space $L_2(\mathbb{R}^3;\mathbb{C}^4)$ by (6.3)), is a Kreĭn space induced by the free Dirac operator H_0, by Proposition 6.1.

Further on, denote by E_0 the spectral measure associated with H_0 and put

$$\mathrm{sgn}(H_0) = \int \mathrm{sgn}(\lambda) \mathrm{d}E_0(\lambda).$$

Next we consider the symmetry $J_0 = \mathrm{sgn}(H_0)$ (in the space $\mathcal{H} = L_2(\mathbb{R}^3;\mathbb{C}^4)$). With respect to the symmetry J_0 the space $\mathcal{H}$ decomposes into an orthogonal direct sum

$$\mathcal{H} = \mathcal{H}_+^0 \oplus \mathcal{H}_-^0,$$

where $\mathcal{H}_\pm^0 = P_\pm^0 \mathcal{H}$ and $P_\pm^0 = \frac{1}{2}(I \pm J_0)$. In the theory of quantum mechanics $\mathcal{H}_+^0$ (respectively, $\mathcal{H}_-^0$) is known as the subspace of positive (respectively, negative) energies.

We note the relation between the symmetries J_0 and J defined as in (6.4)

$$J_0 = W^* J W,$$

where $W = UF$ and U denotes the operator (in the momentum space) of multiplication by the unitary matrix $U(\xi)$, and F is the Fourier operator.

We have the polar decomposition of the free Dirac operator $H_0 = J_0|H_0|$. Since

$$P_+^0 J_0 = \frac{1}{2}(I + J_0) J_0 = \frac{1}{2}(J_0 + I) = P_+^0,$$

and, similarly, $P_-^0 J_0 = -P_-^0$, it follows that

$$H_+^0 = P_+^0 H_0 P_+^0 = P_+^0 J_0 |H_0| P_+^0 = P_+^0 |H_0| P_+^0 \geqslant 0, \text{ and}$$
$$H_-^0 = P_-^0 H_0 P_-^0 = P_-^0 J_0 |H_0| P_-^0 = -P_-^0 |H_0| P_-^0 \leqslant 0.$$

Thus, H_0 acts as a positive operator on the positive energy subspace $\mathcal{H}_+$, and similarly, H_0 is negative on the corresponding negative energy subspace $\mathcal{H}_-$. Therefore, we see (by Theorem 5.3) that the Kreĭn space induced by the free Dirac operator H_0 is unique, modulo unitary equivalence. In this respect we note that $\sigma(H_-^0) = (-\infty, -1]$, $\sigma(H_+^0) = [1, +\infty)$, and $\sigma(H_0) = \sigma(H_-^0) \cup \sigma(H_+^0) = (-\infty, -1] \cup [1, +\infty)$ or, in other words, the interval $(-1, 1)$ is a spectral gap for the free Dirac operator.

6.3. THE PERTURBED DIRAC OPERATOR. We consider now the perturbed Dirac operator $H = H_0 + Q$, where Q is the operator of multiplication by a given 4×4 Hermitian matrix-valued function $Q(x), x \in \mathbb{R}^3$, relatively compact with respect to H_0. We assume that the entries of $Q(x)$ are bounded and measurable functions on $\mathbb{R}^3$. Due to the fact that the operator Q is a bounded operator the perturbed Dirac operator H is defined on the Sobolev space $W_2^1(\mathbb{R}^3; \mathbb{C}^4)$ as the unperturbed operator H_0. Moreover, the operator H is self-adjoint in the space $\mathcal{H} = L_2(\mathbb{R}^3; \mathbb{C}^4)$. It is known (e.g. see [15], [20], [23]) via the Weyl theory that, if assuming in addition that the entries of the matrix-valued function $Q(x)$ vanish at infinity, then the essential spectra of the perturbed Dirac operator $H = H_0 + Q$ and H_0 are the same, i.e. $\sigma_{\text{ess}}(H) = (-\infty, -1] \cup [1, +\infty)$, and the perturbation Q can add a non-trivial set of eigenvalues in the spectral gap $(-1, 1)$, but their possible points of accumulation can be only the endpoints ± 1. Thus, again arguing as in the case of the free Dirac operator, we can define the subspace of positive energies $\mathcal{H}_+ \subset \mathcal{H} (= L_2(\mathbb{R}^3; \mathbb{C}^4))$ and the subspace $\mathcal{H}_- = \mathcal{H} \ominus \mathcal{H}_+$ of negative energies for the perturbed Dirac operators. Obviously, $\mathcal{H}_\pm = P_\pm \mathcal{H}$, where $P_\pm = \frac{1}{2}(I \pm J)$ with $J = \operatorname{sgn}(H)$. By applying Theorem 5.3 we conclude that the Kreĭn space induced by the perturbed Dirac operator (of course, under our hypotheses) is unique, modulo unitary equivalence.

Acknowledgements. The first named author's research partially supported by TODEQ MTKD-CT-2005-030042. Research of the second named author is partially supported by the CNCSIS grant GR202/2006 code 813.

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Received October 4, 2006.