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A SEMI-FREDHOLM THEORY FOR WIENER–HOPF–HANKEL OPERATORS WITH PIECEWISE ALMOST PERIODIC FOURIER SYMBOLS

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ABSTRACT. We present a semi-Fredholm theory for Wiener–Hopf plus/minus Hankel operators acting between L^2 Lebesgue spaces, and having piecewise almost periodic Fourier symbols. This means conditions to ensure the Fredholm property and one-sided invertibility of these operators. This is based on some mean values of the representatives at infinity of the Fourier symbols as well as on the discontinuities of certain auxiliary functions. A formula for the sum of the Fredholm indices of these Wiener–Hopf plus and minus Hankel operators is also obtained, and interpreted upon different cases of symmetries of the discontinuities of the Fourier symbols. Several examples are presented, and the (both-sided) invertibility of the operators in study is also discussed.

KEYWORDS: *Fredholm property, invertibility, Wiener–Hopf–Hankel operators, piecewise almost periodic function.*

MSC (2000): 47A53, 47B35, 42A75.

INTRODUCTION

Motivated by the known semi-Fredholm theory for Wiener–Hopf operators with piecewise almost periodic Fourier symbols (see Theorem 3.1 below, and [6], [7], [34]), and by the attention that has been devoted to Wiener–Hopf plus/minus Hankel operators (cf. [2], [4], [9], [11], [12], [14], [17], [18], [19], [20], [21], [23], [25], [26], [27], [28], [32]), in the present paper we obtain a semi-Fredholm theory for both Wiener–Hopf plus Hankel and Wiener–Hopf minus Hankel operators (with such kind of Fourier symbols).

We will call *Wiener–Hopf–Hankel operators* both Wiener–Hopf plus Hankel and Wiener–Hopf minus Hankel operators (cf. also [23], [24], [33]).

Most of the interest in Wiener–Hopf–Hankel operators is due to the mathematical physics applications where these operators arise (cf. [12], [13], [24], [33]). Although we may say that the theory of Wiener–Hopf–Hankel operators is well developed for some classes of Fourier symbols (like the case of continuous or piecewise continuous functions), this is not the case for elements in the piecewise almost periodic class.

In Section 3, Theorem 3.2, conditions are provided to ensure the Fredholm property, and the lateral invertibility of the operators in study. This may be viewed as a generalization of the *Sarason's type theorem* for Wiener–Hopf plus Hankel operators. To achieve that result, we make use of several types of operator relations, a Fourier symbol decomposition, and appropriate mean motions and geometric mean values.

In this way, after the first section where several necessary notions are introduced, Section 2 presents a relation between Wiener–Hopf–Hankel operators and Wiener–Hopf operators, while in Section 3 we develop the above mentioned semi-Fredholm criterion for Wiener–Hopf–Hankel operators with piecewise almost periodic Fourier symbols. In the last section, a formula for the sum of the Fredholm indices of those Wiener–Hopf plus Hankel and Wiener–Hopf minus Hankel operators is obtained (based on some geometrical concepts). Moreover, the initial formula is also interpreted and simplified in three distinct cases (depending on the discontinuities of the Fourier symbol of the operators). Examples are provided in the last section to help in the interpretation of such index formula. At the end, a condition that ensures the invertibility of the operators in study is also exhibited.

1. PRELIMINARIES

In this section the formal definitions of our main objects are introduced.

We will consider operators defined in the framework of the $L^2(\mathbb{R})$ Banach space (of complex-valued Lebesgue measurable functions φ on $\mathbb{R}$ for which $|\varphi|^2$ is integrable). The *Wiener–Hopf plus/minus Hankel operators* in study have the form

$$W_\phi \pm H_\phi : L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+)$$

where W_ϕ and H_ϕ are *Wiener–Hopf* and *Hankel operators* defined by

$$W_\phi := r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F} : L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+), \tag{1.1}$$

$$H_\phi := r_+\mathcal{F}^{-1}\phi \cdot \mathcal{F}J : L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+), \tag{1.2}$$

respectively. Here $L^2_+(\mathbb{R})$ denotes the subspace of $L^2(\mathbb{R})$ formed by all the functions supported in the closure of $\mathbb{R}_+ := (0, +\infty)$, r_+ represents the *restriction operator* from $L^2_+(\mathbb{R})$ into $L^2(\mathbb{R}_+)$, $\mathcal{F}$ denotes the *Fourier transformation*, J is the *reflection operator* given by the rule $J\varphi(x) = \widetilde{\varphi}(x) := \varphi(-x)$, $x \in \mathbb{R}$, and ϕ belongs to the algebra of the piecewise almost periodic functions on $\mathbb{R}$. For defining this algebra we first need to introduce several notions.

Let AP be the algebra of *almost periodic functions* (see [3], [5], [16]), defined as the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions e_λ $(\lambda \in \mathbb{R})$

where $\mathrm{e}_\lambda(x) := \mathrm{e}^{\mathrm{i}\lambda x}$, $x \in \mathbb{R}$. Consider the C^*-algebra of all *(bounded) piecewise continuous functions* on $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ (usually denoted by PC or $PC(\dot{\mathbb{R}})$) as being the algebra of all functions $\psi \in L^\infty(\mathbb{R})$ for which the one-sided limits

$$\psi(x_0 - 0) := \lim_{x \to x_0 - 0} \psi(x), \quad \psi(x_0 + 0) := \lim_{x \to x_0 + 0} \psi(x)$$

exist for each $x_0 \in \dot{\mathbb{R}}$. By convention, we take

$$\psi(\infty - 0) := \psi(+\infty) = \lim_{x \to +\infty} \psi(x), \quad \psi(\infty + 0) := \psi(-\infty) = \lim_{x \to -\infty} \psi(x).$$

Since we identify functions which differ only on null measure sets, for $\psi \in PC$ we can assume that $\psi(x - 0) = \psi(x)$, for all $x \in \dot{\mathbb{R}}$. This means that it is enough to consider piecewise continuous functions which are always continuous from the left. Furthermore, we will also use the sub-class $PC_0 := \{\psi \in PC : \psi(\pm\infty) = 0\}$.

The C^*-algebra of *piecewise almost periodic functions* on $\mathbb{R}$ (denoted by PAP) is by definition the collection of all functions of the form

$$\phi = (1 - u)\phi_{\mathrm{l}} + u\phi_{\mathrm{r}} + \phi_0, \tag{1.3}$$

where $\phi_{\mathrm{l}}, \phi_{\mathrm{r}} \in AP$, $\phi_0 \in PC_0$ and $u \in C(\overline{\mathbb{R}})$ satisfying $u(-\infty) = 0$ and $u(+\infty) = 1$. $C(\overline{\mathbb{R}})$ denotes the set of all (bounded) continuous (complex-valued) functions on $\mathbb{R}$ with a possible jump at ∞. From the definition of piecewise almost periodic functions, it can be shown that PAP is the algebra generated by AP and PC, i.e.,

$$PAP = \mathrm{alg}(AP, PC).$$

Since the functions ϕ_{l} and ϕ_{r} given by (1.3) belong to AP, they are called the *almost periodic representatives of* ϕ at $-\infty$ and $+\infty$, respectively. Having in mind that some of the further results will be obtained due to certain characteristics of the almost periodic representatives of the Fourier symbol of the operators, we need to introduce some additional well-known notions about AP functions. Namely, for $\phi \in AP$, the *Bohr mean value of* ϕ is defined as

$$M(\phi) := \lim_{\alpha \to \infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} \phi(x) \mathrm{d}x,$$

where $\{I_\alpha\}_{\alpha \in A} = \{(x_\alpha, y_\alpha)\}_{\alpha \in A}$ is a family of intervals $I_\alpha \subset \mathbb{R}$ such that $|I_\alpha| = y_\alpha - x_\alpha \to \infty$, as $\alpha \to \infty$ (for an unbounded set $A \subset \mathbb{R}_+$). The Bohr mean value of a function in AP exists always, is finite, and is independent of the particular choice of the family $\{I_\alpha\}_{\alpha \in A}$.

From now on, we will use the notation $\mathcal{G}B$ for the group of all invertible elements of a Banach algebra B. By Bohr's theorem, for each $\phi \in \mathcal{G}AP$ there exist a real number $\kappa(\phi)$, and a function $\psi \in AP$ such that

$$\phi = \mathrm{e}_{\kappa(\phi)} \mathrm{e}^{\psi}.$$

The real number $\kappa(\phi)$ is uniquely determined, and usually called the *mean motion of* ϕ. The number

$$\mathbf{d}(\phi) := \mathrm{e}^{M(\psi)}$$

is called the *geometric mean value of* ϕ. If $\kappa(\phi) = 0$, we may represent $\mathbf{d}(\phi)$ as

$$\mathbf{d}(\phi) = \mathrm{e}^{M(\log \phi)}$$

where $\log \phi$ is any function in AP for which $\phi = \mathrm{e}^{\log \phi}$.

The C^*-algebra of the *semi-almost periodic functions* on $\mathbb{R}$ (denoted by SAP) is by definition the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains AP and $C(\overline{\mathbb{R}})$. For the case of invertible elements in PAP, we can find in Proposition 3.15 of [6] the following alternative representation to (1.3) (cf. also Section 9.26 of [7]): If $\phi \in \mathcal{G}PAP$, then there exist $\varphi \in \mathcal{G}SAP$ and $\psi \in \mathcal{G}PC$ satisfying $\psi(-\infty) = \psi(+\infty) = 1$ such that

$$\phi = \varphi\psi. \tag{1.4}$$

On the other hand, taking into account the characterization (1.3) of $\phi \in PAP$, we define by

$$\kappa_{\mathrm{l}}(\phi) := \kappa(\phi_{\mathrm{l}}), \quad \kappa_{\mathrm{r}}(\phi) := \kappa(\phi_{\mathrm{r}}), \quad \mathbf{d}_{\mathrm{l}}(\phi) := \mathbf{d}(\phi_{\mathrm{l}}), \quad \mathbf{d}_{\mathrm{r}}(\phi) := \mathbf{d}(\phi_{\mathrm{r}}) \tag{1.5}$$

the *left and right mean motions, and the left and right geometric mean values* of ϕ, respectively.

2. RELATIONS BETWEEN WIENER–HOPF–HANKEL AND WIENER–HOPF OPERATORS, AND SOME OF THEIR CONSEQUENCES

In this section, following the spirit of the *Gohberg–Krupnik–Litvinchuk identity* (cf. [19], [21], [22], [28]), we will describe some relations between Wiener–Hopf–Hankel operators and Wiener–Hopf operators (based also on certain paired operators). In the next section these relations will be very important in obtaining the semi-Fredholm criterion for the Wiener–Hopf–Hankel operators in study.

We will first recall different types of operator relations.

Let us consider two bounded linear operators $T : X_1 \to X_2$ and $S : Y_1 \to Y_2$, acting between Banach spaces. The operators T and S are said to be *equivalent*, and we will denote this by $T \sim S$, if there are two boundedly invertible linear operators, $E : Y_2 \to X_2$ and $F : X_1 \to Y_1$, such that

$$T = ESF. \tag{2.1}$$

It directly follows from (2.1) that if two operators are equivalent, then they belong to the same *regularity class* [8], [10], [31]. Namely, one of these operators is invertible, one-sided invertible, Fredholm, n-normal, d-normal or (only) normally solvable, if and only if the other operator enjoys the same property. Recall that a bounded linear operator $A : X \to Y$, acting between Banach spaces, is said to be *normally solvable* if $\mathrm{Im}A$ is closed. In this case, the *cokernel of* A is defined as $\mathrm{Coker}A := Y/\mathrm{Im}A$. For a normally solvable operator A, the *deficiency numbers* of A are defined by

$$n(A) := \dim \mathrm{Ker}A, \quad d(A) := \dim \mathrm{Coker}A.$$

If at least one of the deficiency numbers is finite, the (normally solvable) operator A is said to be a *semi-Fredholm* operator. A semi-Fredholm operator is said to be a *Fredholm* operator if $n(A)$ and $d(A)$ are finite, *n-normal* if $n(A)$ is finite, and *d-normal* if $d(A)$ is finite. In the case where only one of the deficiency numbers is finite, the (normally solvable) operator A is said to be a *properly semi-Fredholm* operator. In this case, the operator A is said to be *properly n-normal* if $n(A)$ is finite and $d(A)$ is infinite, and *properly d-normal* if $d(A)$ is finite and $n(A)$ is infinite. If A is a Fredholm operator, the *Fredholm index* of A is defined by

$$\mathrm{Ind} A := n(A) - d(A).$$

Concerning additional operator relations, in [10] was introduced the so-called *Δ-relation after extension* for bounded linear operators acting between Banach spaces, e.g. $T : X_1 \to X_2$ and $S : Y_1 \to Y_2$. Namely, we say that T is *Δ-related after extension with* S if there is a bounded linear operator acting between Banach spaces $T_\Delta : X_{1\Delta} \to X_{2\Delta}$, and invertible bounded linear operators E and F, such that

$$\begin{bmatrix} T & 0 \\ 0 & T_\Delta \end{bmatrix} = E \begin{bmatrix} S & 0 \\ 0 & I_Z \end{bmatrix} F, \tag{2.2}$$

where Z is an additional Banach space and I_Z represents the *identity operator in* Z. In the particular case where $T_\Delta : X_{1\Delta} \to X_{2\Delta} = X_{1\Delta}$ is the identity operator, we say that T and S are *equivalent after extension operators*, and we will denote this by $T \overset{*}{\sim} S$ (cf. [1]).

LEMMA 2.1. *Let $\phi \in \mathcal{G}L^\infty(\mathbb{R})$. The Wiener–Hopf plus Hankel operator $W_\phi + H_\phi$ is Δ-related after extension with the Wiener–Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$.*

Proof. We start by extending $W_\phi + H_\phi$ on the left by the zero extension operator $\ell_0 : L^2(\mathbb{R}_+) \to L^2_+(\mathbb{R})$, and therefore obtaining

$$W_\phi + H_\phi \sim \ell_0(W_\phi + H_\phi) : L^2_+(\mathbb{R}) \to L^2_+(\mathbb{R}). \tag{2.3}$$

Choosing the notation $P_+ = \ell_0 r_+$ and $P_- = I_{L^2(\mathbb{R})} - P_+$, we will now extend

$$\ell_0(W_\phi + H_\phi) = P_+ \mathcal{F}^{-1}(\phi \cdot + \phi \cdot J)\mathcal{F}_{|P_+ L^2(\mathbb{R})}$$

to the full $L^2(\mathbb{R})$ space by using the identity in $L^2_-(\mathbb{R})$. Here and in what follows, $L^2_-(\mathbb{R})$ will denote the subspace of $L^2(\mathbb{R})$ formed by all the functions supported in the closure of $\mathbb{R}_- := (-\infty, 0)$. Next, we will extend the obtained operator to $[L^2(\mathbb{R})]^2$ with the help of the auxiliary paired operator

$$\mathcal{T}_\phi = \mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F}P_+ + P_- : L^2(\mathbb{R}) \to L^2(\mathbb{R}). \tag{2.4}$$

Altogether, we have

$$\left[\begin{array}{cc|c} \ell_0(W_\phi + H_\phi) & 0 & 0 \\ 0 & I_{P_- L^2(\mathbb{R})} & 0 \\ \hline 0 & 0 & \mathcal{T}_\phi \end{array}\right] = E_1 \mathcal{W}_1 F_1 \tag{2.5}$$

with

$$
\begin{aligned}
E_1 &= \frac{1}{2}\begin{bmatrix} I_{L^2(\mathbb{R})} & J \\ I_{L^2(\mathbb{R})} & -J \end{bmatrix}, \\
F_1 &= \begin{bmatrix} I_{L^2(\mathbb{R})} & I_{L^2(\mathbb{R})} \\ J & -J \end{bmatrix} \begin{bmatrix} I_{L^2(\mathbb{R})} - P_-\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F}P_+ & 0 \\ 0 & I_{L^2(\mathbb{R})} \end{bmatrix}, \\
\mathcal{W}_1 &= \begin{bmatrix} \mathcal{F}^{-1}\phi \cdot \mathcal{F} & 0 \\ \mathcal{F}^{-1}\widetilde{\phi} \cdot \mathcal{F} & 1 \end{bmatrix} P_+ + \begin{bmatrix} 1 & \mathcal{F}^{-1}\phi \cdot \mathcal{F} \\ 0 & \mathcal{F}^{-1}\widetilde{\phi} \cdot \mathcal{F} \end{bmatrix} P_- = \begin{bmatrix} 1 & \mathcal{F}^{-1}\phi \cdot \mathcal{F} \\ 0 & \mathcal{F}^{-1}\widetilde{\phi} \cdot \mathcal{F} \end{bmatrix} (\mathcal{F}^{-1}\Psi \cdot \mathcal{F}P_+ + P_-) \\
&= \begin{bmatrix} 1 & \mathcal{F}^{-1}\phi \cdot \mathcal{F} \\ 0 & \mathcal{F}^{-1}\widetilde{\phi} \cdot \mathcal{F} \end{bmatrix} (P_+\mathcal{F}^{-1}\Psi \cdot \mathcal{F}P_+ + P_-)(I_{[L^2(\mathbb{R})]^2} + P_-\mathcal{F}^{-1}\Psi \cdot \mathcal{F}P_+),
\end{aligned}
$$

where in this definition of operator $\mathcal{W}_1$ we are using $P_\pm$ defined in $[L^2(\mathbb{R})]^2$ and

$$
\Psi := \begin{bmatrix} 0 & -\phi\widetilde{\phi^{-1}} \\ 1 & \widetilde{\phi^{-1}} \end{bmatrix}.
$$

We point out that the paired operator

$$
I_{[L^2(\mathbb{R})]^2} + P_-\mathcal{F}^{-1}\Psi \cdot \mathcal{F}P_+ : [L^2(\mathbb{R})]^2 \to [L^2(\mathbb{R})]^2
$$

used above is an invertible operator with inverse given by

$$
I_{[L^2(\mathbb{R})]^2} - P_-\mathcal{F}^{-1}\Psi \cdot \mathcal{F}P_+ : [L^2(\mathbb{R})]^2 \to [L^2(\mathbb{R})]^2.
$$

Therefore, we have just explicitly demonstrated that $W_\phi + H_\phi$ is Δ-related after extension with

$$
\mathcal{W}_\Psi := r_+\mathcal{F}^{-1}\Psi \cdot \mathcal{F} : [L^2_+(\mathbb{R})]^2 \to [L^2(\mathbb{R}_+)]^2.
$$

Furthermore, we have

$$
\begin{bmatrix} W_{\phi\widetilde{\phi^{-1}}}\ell_0 & 0 \\ 0 & I_{L^2(\mathbb{R}_+)} \end{bmatrix} = \mathcal{W}_\Psi \ell_0 r_+ \mathcal{F}^{-1} \begin{bmatrix} \widetilde{\phi^{-1}} & 1 \\ -1 & 0 \end{bmatrix} \mathcal{F}\ell_0 : [L^2(\mathbb{R}_+)]^2 \to [L^2(\mathbb{R}_+)]^2
$$

which shows an explicit equivalence after extension relation between $W_{\phi\widetilde{\phi^{-1}}}$ and $\mathcal{W}_\Psi$. This together with the Δ-relation after extension between $W_\phi + H_\phi$ and $\mathcal{W}_\Psi$ concludes the proof. ∎

From the definition of Δ-relation after extension, we already know that being $W_\phi + H_\phi$ Δ-related after extension with $W_{\phi\widetilde{\phi^{-1}}}$, there will be a transfer of regularity properties from the Wiener–Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ to the Wiener–Hopf plus Hankel operator $W_\phi + H_\phi$, and to the operator $\mathcal{T}_\phi$. Additionally, from an equivalence after extension relation between $\mathcal{T}_\phi$ and $W_\phi - H_\phi$, it is also possible to transfer regularity properties from the Wiener–Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ to the Wiener–Hopf minus Hankel operator $W_\phi - H_\phi$ as it is stated in the next result.

COROLLARY 2.2. *Let $\phi \in \mathcal{G}L^\infty(\mathbb{R})$. If the Wiener–Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ is invertible, left-invertible, right-invertible, Fredholm or normally solvable, then the Wiener–Hopf plus Hankel operator $W_\phi + H_\phi$ and the Wiener–Hopf minus Hankel operator $W_\phi - H_\phi$ have the same property as $W_{\phi\widetilde{\phi^{-1}}}$.*

Proof. As a direct consequence of the Δ-relation after extension between the Wiener–Hopf plus Hankel operator $W_\phi + H_\phi$ and the Wiener–Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ (presented in Lemma 2.1), we have that $W_\phi + H_\phi$ and $\mathcal{T}_\phi$ are invertible, left-invertible, right-invertible, Fredholm or normally solvable if the Wiener–Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ has the same property. To complete the proof, it is enough to show that $\mathcal{T}_\phi$ and $W_\phi - H_\phi$ are equivalent after extension operators (as mentioned above), and so they will have the same regularity properties. To do that, we start by observing that $\mathcal{T}_\phi = \mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F}P_+ + P_-$ and $P_+\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F}P_+ + P_-$ are equivalent operators:

$$\mathcal{T}_\phi = (P_+\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F}P_+ + P_-)(I_{L^2(\mathbb{R})} + P_-\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F}P_+).$$

In fact, this is the case because the operator

$$I_{L^2(\mathbb{R})} + P_-\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F}P_+ : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

used above is an invertible operator with inverse given by

$$I_{L^2(\mathbb{R})} - P_-\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F}P_+ : L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

Attending now to the direct sum $L^2(\mathbb{R}) = L^2_+(\mathbb{R}) \oplus L^2_-(\mathbb{R})$, we may write

$$P_+\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F}P_+ + P_- : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

in the matrix form

$$\begin{bmatrix} P_+\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F}P_+ & 0 \\ 0 & P_- \end{bmatrix} : L^2_+(\mathbb{R}) \oplus L^2_-(\mathbb{R}) \to L^2_+(\mathbb{R}) \oplus L^2_-(\mathbb{R}).$$

Using $P_- = I_{P_-L^2(\mathbb{R})} : L^2_-(\mathbb{R}) \to L^2_-(\mathbb{R})$, it follows that

$$P_+\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F}P_+ : L^2_+(\mathbb{R}) \to L^2_+(\mathbb{R})$$

is equivalent after extension to

$$P_+\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F}P_+ + P_- : L^2(\mathbb{R}) \to L^2(\mathbb{R}).$$

Recalling that $P_+ = \ell_0 r_+$, and taking into consideration the space where the operator $P_+\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F}P_+$ is acting, we have

$$P_+\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F}P_+ = \ell_0 r_+\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F} : L^2_+(\mathbb{R}) \to L^2_+(\mathbb{R}).$$

Finally, due to the fact that ℓ_0 is invertible, it follows that

$$\ell_0 r_+\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F} : L^2_+(\mathbb{R}) \to L^2_+(\mathbb{R})$$

is equivalent to

$$r_+\mathcal{F}^{-1}(\phi \cdot - \phi \cdot J)\mathcal{F} = W_\phi - H_\phi : L^2_+(\mathbb{R}) \to L^2(\mathbb{R}_+).$$

Altogether, it holds the announced equivalence after extension relation:

$$\mathcal{T}_\phi \overset{*}{\sim} W_\phi - H_\phi. \quad \blacksquare \tag{2.6}$$

3. A FREDHOLM AND ONE-SIDED INVERTIBILITY CRITERION

From now on, consider $[c_1, c_2]$ to be the line segment in the complex plane between (and including) the endpoints $c_1, c_2 \in \mathbb{C}$.

For Wiener–Hopf operators with semi-almost periodic symbols, Sarason developed a semi-Fredholm theory — known as the *Sarason's Theorem* [29]. In the recent times, several generalizations of this result have been made. For instance, in [6] we find the following generalization of Sarason's Theorem for Wiener–Hopf operators with piecewise almost periodic symbols:

THEOREM 3.1 (cf. Theorem 3.16 in [6], and Section 9.27 in [7]). *Let $\phi \in PAP$ and suppose ϕ is not identically zero.*

(i) *If $\phi \in \mathcal{G}PAP$, $\kappa_l(\phi) = \kappa_r(\phi) = 0$ and we have the following, then W_ϕ is a Fredholm operator:*

$$0 \notin [\mathbf{d}_l(\phi), \mathbf{d}_r(\phi)] \cup \bigcup_{x \in \mathbb{R}} [\phi(x-0), \phi(x+0)].$$

(ii) *If $\phi \in \mathcal{G}PAP$, $\kappa_l(\phi) \cdot \kappa_r(\phi) \geqslant 0$, $\kappa_l(\phi) + \kappa_r(\phi) > 0$ and we have the following, then W_ϕ is properly n-normal and left-invertible:*

$$0 \notin \bigcup_{x \in \mathbb{R}} [\phi(x-0), \phi(x+0)].$$

(iii) *If $\phi \in \mathcal{G}PAP$, $\kappa_l(\phi) \cdot \kappa_r(\phi) \geqslant 0$, $\kappa_l(\phi) + \kappa_r(\phi) < 0$ and we have the following, then W_ϕ is properly d-normal and right-invertible:*

$$0 \notin \bigcup_{x \in \mathbb{R}} [\phi(x-0), \phi(x+0)].$$

(iv) *In all the other cases, the operator W_ϕ is not normally solvable.*

Motivated by this last result, we obtain here a semi-Fredholm theory for Wiener–Hopf–Hankel operators with piecewise almost periodic symbols ϕ (based on the values of $\kappa_l(\phi)$ and $\kappa_r(\phi)$). As we will see below, the addition and the subtraction of the Hankel operator to the Wiener–Hopf operator introduce several changes in the regularity properties of the Wiener–Hopf–Hankel operators.

THEOREM 3.2. *Let $\phi \in \mathcal{G}PAP$.*

(i) *If $\kappa_l(\phi) + \kappa_r(\phi) = 0$ and*

$$0 \notin \left[\frac{\mathbf{d}_l(\phi)}{\mathbf{d}_r(\phi)}, \frac{\mathbf{d}_r(\phi)}{\mathbf{d}_l(\phi)}\right] \cup \bigcup_{x \in \mathbb{R}} [(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)],$$

then $W_\phi + H_\phi$ and $W_\phi - H_\phi$ are Fredholm operators.

(ii) *If* $\kappa_{\rm l}(\phi)+\kappa_{\rm r}(\phi)>0$ *and*

$$0\notin\bigcup_{x\in\mathbb{R}}[(\phi\widetilde{\phi^{-1}})(x-0),(\phi\widetilde{\phi^{-1}})(x+0)],$$

then $W_\phi+H_\phi$ *and* $W_\phi-H_\phi$ *are left-invertible. Moreover, at least one of these operators is properly n-normal.*

(iii) *If* $\kappa_{\rm l}(\phi)+\kappa_{\rm r}(\phi)<0$ *and*

$$0\notin\bigcup_{x\in\mathbb{R}}[(\phi\widetilde{\phi^{-1}})(x-0),(\phi\widetilde{\phi^{-1}})(x+0)],$$

then $W_\phi+H_\phi$ *and* $W_\phi-H_\phi$ *are right-invertible. In addition, at least one of these operators is properly d-normal.*

(iv) *In all the remaining cases, at least one of the operators* $W_\phi+H_\phi$ *and* $W_\phi-H_\phi$ *is not normally solvable.*

Proof. From the definition of PAP we have

$$\phi=(1-u)\phi_{\rm l}+u\phi_{\rm r}+\phi_0,$$

where $\phi_{\rm l},\phi_{\rm r}\in AP$, $\phi_0\in PC_0$ and $u\in C(\overline{\mathbb{R}})$ satisfying $u(-\infty)=0$ and $u(+\infty)=1$. Additionally, since $\phi\in\mathcal{G}PAP$, it follows that $\phi_{\rm l},\phi_{\rm r}\in\mathcal{G}AP$. Consequently, taking into consideration Bohr's Theorem and the definition of the geometric mean value, we may represent $\phi_{\rm l}$ and $\phi_{\rm r}$ as

$$\phi_{\rm l}=e_{\kappa(\phi_{\rm l})}\mathbf{d}(\phi_{\rm l})e^{\omega_{\rm l}},\quad \phi_{\rm r}=e_{\kappa(\phi_{\rm r})}\mathbf{d}(\phi_{\rm r})e^{\omega_{\rm r}}$$

with $\omega_{\rm l},\omega_{\rm r}\in AP$ and $M(\omega_{\rm l})=M(\omega_{\rm r})=0$. Thus,

$$\phi=(1-u)\mathbf{d}(\phi_{\rm l})e_{\kappa(\phi_{\rm l})}e^{\omega_{\rm l}}+u\mathbf{d}(\phi_{\rm r})e_{\kappa(\phi_{\rm r})}e^{\omega_{\rm r}}+\phi_0.$$

In view of the transfer of regularity properties from the Wiener–Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ to the Wiener–Hopf plus Hankel operator $W_\phi+H_\phi$ and to the Wiener–Hopf minus Hankel operator $W_\phi-H_\phi$ (stated in Corollary 2.2), we will now study the Wiener–Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$. Since

$$\widetilde{\phi}=(1-\widetilde{u})\mathbf{d}(\phi_{\rm l})e_{-\kappa(\phi_{\rm l})}e^{\widetilde{\omega_{\rm l}}}+\widetilde{u}\mathbf{d}(\phi_{\rm r})e_{-\kappa(\phi_{\rm r})}e^{\widetilde{\omega_{\rm r}}}+\widetilde{\phi_0},$$

we have

$$\phi\widetilde{\phi^{-1}}=\frac{(1-u)\mathbf{d}(\phi_{\rm l})e_{\kappa(\phi_{\rm l})}e^{\omega_{\rm l}}+u\mathbf{d}(\phi_{\rm r})e_{\kappa(\phi_{\rm r})}e^{\omega_{\rm r}}+\phi_0}{(1-\widetilde{u})\mathbf{d}(\phi_{\rm l})e_{-\kappa(\phi_{\rm l})}e^{\widetilde{\omega_{\rm l}}}+\widetilde{u}\mathbf{d}(\phi_{\rm r})e_{-\kappa(\phi_{\rm r})}e^{\widetilde{\omega_{\rm r}}}+\widetilde{\phi_0}}. \tag{3.1}$$

By the definition of piecewise almost periodic function, we know that $\phi\widetilde{\phi^{-1}}$ is of the form

$$\phi\widetilde{\phi^{-1}}=(1-v)(\phi\widetilde{\phi^{-1}})_{\rm l}+v(\phi\widetilde{\phi^{-1}})_{\rm r}+(\phi\widetilde{\phi^{-1}})_0,$$

where $v\in C(\overline{\mathbb{R}})$ is such that $v(-\infty)=0$ and $v(+\infty)=1$, $(\phi\widetilde{\phi^{-1}})_0\in PC_0$ and $(\phi\widetilde{\phi^{-1}})_{\rm l},(\phi\widetilde{\phi^{-1}})_{\rm r}$ are the almost periodic representatives of $\phi\widetilde{\phi^{-1}}$ at $\pm\infty$, and given by

$$(\phi\widetilde{\phi^{-1}})_{\mathrm{l}} = \frac{\mathbf{d}(\phi_{\mathrm{l}})}{\mathbf{d}(\phi_{\mathrm{r}})}\mathrm{e}_{\kappa(\phi_{\mathrm{l}})+\kappa(\phi_{\mathrm{r}})}\mathrm{e}^{\omega_{\mathrm{l}}-\widetilde{\omega_{\mathrm{r}}}}, \quad (\phi\widetilde{\phi^{-1}})_{\mathrm{r}} = \frac{\mathbf{d}(\phi_{\mathrm{r}})}{\mathbf{d}(\phi_{\mathrm{l}})}\mathrm{e}_{\kappa(\phi_{\mathrm{l}})+\kappa(\phi_{\mathrm{r}})}\mathrm{e}^{\omega_{\mathrm{r}}-\widetilde{\omega_{\mathrm{l}}}}$$

(cf. (3.1)). Because $\omega_{\mathrm{l}}, \omega_{\mathrm{r}} \in AP$ are such that $M(\omega_{\mathrm{l}}) = M(\omega_{\mathrm{r}}) = 0$ (which additionally implies $M(\widetilde{\omega_{\mathrm{l}}}) = M(\widetilde{\omega_{\mathrm{r}}}) = 0$), it results that

$$\kappa((\phi\widetilde{\phi^{-1}})_{\mathrm{l}}) = \kappa((\phi\widetilde{\phi^{-1}})_{\mathrm{r}}) = \kappa(\phi_{\mathrm{l}}) + \kappa(\phi_{\mathrm{r}}), \tag{3.2}$$

$$\mathbf{d}((\phi\widetilde{\phi^{-1}})_{\mathrm{l}}) = \frac{\mathbf{d}(\phi_{\mathrm{l}})}{\mathbf{d}(\phi_{\mathrm{r}})}, \quad \mathbf{d}((\phi\widetilde{\phi^{-1}})_{\mathrm{r}}) = \frac{\mathbf{d}(\phi_{\mathrm{r}})}{\mathbf{d}(\phi_{\mathrm{l}})}. \tag{3.3}$$

According to the definitions of left and right mean motions, and left and right geometric mean values of the piecewise almost periodic functions, one obtains

$$\kappa_{\mathrm{l}}(\phi\widetilde{\phi^{-1}}) = \kappa_{\mathrm{r}}(\phi\widetilde{\phi^{-1}}) = \kappa_{\mathrm{l}}(\phi) + \kappa_{\mathrm{r}}(\phi), \tag{3.4}$$

$$\mathbf{d}_{\mathrm{l}}(\phi\widetilde{\phi^{-1}}) = \frac{\mathbf{d}_{\mathrm{l}}(\phi)}{\mathbf{d}_{\mathrm{r}}(\phi)}, \quad \mathbf{d}_{\mathrm{r}}(\phi\widetilde{\phi^{-1}}) = \frac{\mathbf{d}_{\mathrm{r}}(\phi)}{\mathbf{d}_{\mathrm{l}}(\phi)}. \tag{3.5}$$

Applying now Theorem 3.1 to the Wiener–Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$, it follows from (3.4) and (3.5) that: (a) if $\kappa_{\mathrm{l}}(\phi) + \kappa_{\mathrm{r}}(\phi) = 0$ and

$$0 \notin \left[\frac{\mathbf{d}_{\mathrm{l}}(\phi)}{\mathbf{d}_{\mathrm{r}}(\phi)}, \frac{\mathbf{d}_{\mathrm{r}}(\phi)}{\mathbf{d}_{\mathrm{l}}(\phi)}\right] \cup \bigcup_{x\in\mathbb{R}} [(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)],$$

then $W_{\phi\widetilde{\phi^{-1}}}$ is a Fredholm operator; (b) if $\kappa_{\mathrm{l}}(\phi) + \kappa_{\mathrm{r}}(\phi) > 0$ and

$$0 \notin \bigcup_{x\in\mathbb{R}} [(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)],$$

then $W_{\phi\widetilde{\phi^{-1}}}$ is properly n-normal and left-invertible; (c) if $\kappa_{\mathrm{l}}(\phi) + \kappa_{\mathrm{r}}(\phi) < 0$ and $0 \notin \bigcup_{x\in\mathbb{R}} [(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)]$, then $W_{\phi\widetilde{\phi^{-1}}}$ is properly d-normal and right-invertible; and finally, (d) in all the other cases, the operator $W_{\phi\widetilde{\phi^{-1}}}$ is not normally solvable. Applying now Corollary 2.2, we obtain that: $W_\phi + H_\phi$ and $W_\phi - H_\phi$ are Fredholm operators, under the conditions of case (a); $W_\phi + H_\phi$ and $W_\phi - H_\phi$ are left-invertible, for the conditions of case (b); $W_\phi + H_\phi$ and $W_\phi - H_\phi$ are right-invertible, for the conditions of case (c). To arrive at the final assertion, we can interpret the Δ-relation after extension between the Wiener–Hopf plus Hankel operator $W_\phi + H_\phi$ and the Wiener–Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ (presented in Lemma 2.1) as an equivalence after extension between $\mathrm{diag}[W_\phi + H_\phi, \mathcal{T}_\phi]$ and $W_{\phi\widetilde{\phi^{-1}}}$. In this way, we get in cases (b) and (c) that $\mathrm{diag}[W_\phi + H_\phi, \mathcal{T}_\phi]$ is properly n-normal or properly d-normal, respectively. This means that at least one the operators $W_\phi + H_\phi$ and $\mathcal{T}_\phi$ is properly n-normal or properly d-normal, in the case (b) or (c), respectively. Considering now (2.6), the last proposition tells us that at least one the operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ is properly n-normal or properly

d-normal, if in the conditions of case (b) or case (c), respectively. In case (d), we have that $\operatorname{diag}[W_\phi + H_\phi, \mathcal{T}_\phi]$ is not normally solvable, which implies that at least one the operators $W_\phi + H_\phi$ and $\mathcal{T}_\phi$ is not normally solvable. From the equivalence after extension (2.6), it therefore follows that at least one the operators $W_\phi + H_\phi$ and $W_\phi - H_\phi$ is not normally solvable (in this case (d)). ∎

REMARK 3.3. We would like to point out the following details concerning the last result:

(i) Since $\mathbf{d}_\mathrm{l}(\phi\widetilde{\phi^{-1}})$ and $\mathbf{d}_\mathrm{r}(\phi\widetilde{\phi^{-1}})$ are inverses of each other (cf. (3.5)), the condition

$$0 \notin \Big[\frac{\mathbf{d}_\mathrm{l}(\phi)}{\mathbf{d}_\mathrm{r}(\phi)}, \frac{\mathbf{d}_\mathrm{r}(\phi)}{\mathbf{d}_\mathrm{l}(\phi)}\Big]$$

is satisfied if and only if $\mathbf{d}_\mathrm{l}(\phi)/\mathbf{d}_\mathrm{r}(\phi)$ is such that $\Re e(\mathbf{d}_\mathrm{l}(\phi)/\mathbf{d}_\mathrm{r}(\phi)) \neq 0$.

(ii) From the above theorem and from Theorem 3.1, it follows that if $\phi \in \mathcal{G}PAP$ is such that $\kappa_\mathrm{l}(\phi) \cdot \kappa_\mathrm{r}(\phi) < 0$, $\kappa_\mathrm{l}(\phi) + \kappa_\mathrm{r}(\phi) \neq 0$,

$$0 \notin \bigcup_{x\in\mathbb{R}} [\phi(x-0), \phi(x+0)] \quad \text{and} \quad 0 \notin \bigcup_{x\in\mathbb{R}} [(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)],$$

then the Wiener–Hopf plus Hankel operator $W_\phi + H_\phi$ and the Wiener–Hopf minus Hankel operator $W_\phi - H_\phi$ are normally solvable although the Wiener–Hopf operator W_ϕ is not normally solvable. In particular, this exemplifies the changes in the regularity properties of the Wiener–Hopf plus/minus Hankel operators by adding or subtracting the Hankel operator to the Wiener–Hopf operator.

(iii) Theorem 3.2 may be called a Sarason's type theorem for Wiener–Hopf–Hankel operators since it describes the Fredholm nature of $W_\phi + H_\phi$ and $W_\phi - H_\phi$ based on the values of $\kappa_\mathrm{l}(\phi)$ and $\kappa_\mathrm{r}(\phi)$ when $\phi \in \mathcal{G}PAP$ and

$$0 \notin \bigcup_{x\in\mathbb{R}} [(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)].$$

4. INDEX FORMULA, EXAMPLES AND INVERTIBILITY

4.1. INDEX FORMULA. In Example 3.25 of [18], T. Ehrhardt gave an example of a Toeplitz plus Hankel operator and a Toeplitz minus Hankel operator, with the same Fourier symbol but having different Fredholm indices. Considering the complex unit circle $\mathbb{T}$, and the usual isometric isomorphism B_0 from $L^\infty(\mathbb{R})$ onto $L^\infty(\mathbb{T})$, given by

$$(B_0\phi)(t) := \phi\Big(\mathrm{i}\frac{1+t}{1-t}\Big), \quad t \in \mathbb{T}\backslash\{1\},$$

it is possible to directly construct the corresponding examples in the framework of Wiener–Hopf–Hankel operators. This happens because it is possible to relate, through an equivalence relation, Toeplitz minus Hankel operators with Wiener–Hopf plus Hankel operators (cf., e.g., [25]), and similarly one can also relate

Toeplitz plus Hankel operators with Wiener–Hopf minus Hankel operators. In this way, considering

$$\phi_\beta(x) := \Big(\frac{x-\mathrm{i}}{x+\mathrm{i}}\Big)^{\beta} \mathrm{e}^{-\mathrm{i}\beta\pi}$$

defined for all $x \in \mathbb{R}$ and with $\beta \in \mathbb{C}$ such that $1/4 < \Re e\beta < 3/4$, the Wiener–Hopf minus Hankel operator $W_{\phi_\beta} - H_{\phi_\beta}$ is a Fredholm operator with $\mathrm{Ind}(W_{\phi_\beta} - H_{\phi_\beta}) = -1$, and the Wiener–Hopf plus Hankel operator $W_{\phi_\beta} + H_{\phi_\beta}$ is invertible (and therefore $\mathrm{Ind}(W_{\phi_\beta} + H_{\phi_\beta}) = 0$). In particular, this example shows that Fredholm Wiener–Hopf plus Hankel and Fredholm Wiener–Hopf minus Hankel operators may have different Fredholm indices.

In the present subsection we will achieve a formula for the sum of the indices of such kind of Fredholm Wiener–Hopf plus/minus Hankel operators. Recall that from Theorem 3.2, we have that if $\phi \in \mathcal{G}PAP$ is such that $\kappa_{\mathrm{l}}(\phi) + \kappa_{\mathrm{r}}(\phi) = 0$ and

$$0 \notin \Big[\frac{\mathbf{d}_{\mathrm{l}}(\phi)}{\mathbf{d}_{\mathrm{r}}(\phi)}, \frac{\mathbf{d}_{\mathrm{r}}(\phi)}{\mathbf{d}_{\mathrm{l}}(\phi)}\Big] \cup \bigcup_{x\in\mathbb{R}} [(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)],$$

then $W_\phi + H_\phi$ and $W_\phi - H_\phi$ are Fredholm operators. In this case, we will obtain a formula which relates the Fredholm index of the Wiener–Hopf plus Hankel operator $W_\phi + H_\phi$ with the Fredholm index of the Wiener–Hopf minus Hankel operator $W_\phi - H_\phi$ based on the winding number of a piecewise almost periodic function (constructed from the initial Fourier symbol of the operators). In addition, we will be able to simplify the formula of the winding number of this piecewise almost periodic function accordingly with the following three situations: (1) ϕ has no symmetric discontinuities; (2) ϕ has symmetric discontinuities and ϕ is continuous at 0; and finally, (3) ϕ is discontinuous at 0. We say that $\phi \in PAP$ has *symmetric discontinuities* if ϕ has discontinuities at x_0 and $-x_0$ for some $x_0 \in \mathbb{R}$. Thus, a piecewise almost periodic function ϕ has symmetric discontinuities if and only if ϕ and $\widetilde{\phi}$ have common discontinuities. Moreover, $\phi \in \mathcal{G}PAP$ has symmetric discontinuities if and only if ϕ and $\widetilde{\phi^{-1}}$ have common discontinuities.

For having all the necessary instruments to the Fredholm index formula, we will start with the definitions of Cauchy index and winding number of a continuous and piecewise continuous function, and the definition of winding number of a semi-almost periodic function (in the context of corresponding Wiener–Hopf–Hankel operators having such functions as their Fourier symbols). Additionally, we will introduce a generalization of the known definition of winding number of a piecewise almost periodic function from the context of Wiener–Hopf operators to the framework of Wiener–Hopf–Hankel operators.

For $\varrho \in \mathcal{G}C(\overline{\mathbb{R}})$, the *Cauchy index* of ϱ is defined by

$$\mathrm{ind}\varrho := \frac{1}{2\pi}((\arg\varrho)(+\infty) - (\arg\varrho)(-\infty)), \tag{4.1}$$

where $\arg \varrho$ is any continuous argument of ϱ. The *winding number* of ϱ ($\mathrm{wind}\varrho$) is given by the number of times that the curve traced out by the point $\varrho(x)$ surrounds the origin counter-clockwise, as x moves from $-\infty$ to $+\infty$. If $\varrho \in \mathcal{G}C(\mathring{\mathbb{R}})$, then $\mathrm{wind}\varrho = \mathrm{ind}\varrho$. Here $C(\mathring{\mathbb{R}})$ denotes the space of all (bounded) continuous (complex-valued) functions on $\mathbb{R}$ for which both limits at $\pm\infty$ exist and coincide.

For all $\psi \in PC$, one defines the function $\psi^{\#} : \mathring{\mathbb{R}} \times [0,1] \to \mathbb{C}$ by

$$\psi^{\#}(x,\mu) := (1-\mu)\psi(x-0) + \mu\psi(x+0).$$

The range of $\psi^{\#}$ is a continuous closed curve with a natural orientation induced by the orientation of $\mathbb{R}$ from $-\infty$ to $+\infty$. Basically, $\psi^{\#}$ is obtained from ψ by joining, in each jump of ψ, the points $\psi(x_0-0)$ and $\psi(x_0+0)$ with a line segment. In this way, the line segment $[\psi(x_0-0), \psi(x_0+0)]$ is oriented from $\psi(x_0-0)$ to $\psi(x_0+0)$. If $\psi^{\#}(x,\mu) \neq 0$ for all $(x,\mu) \in \mathring{\mathbb{R}} \times [0,1]$, the *winding number* of ψ is defined as the number of times that the curve $\psi^{\#}(\mathring{\mathbb{R}}, [0,1])$ surrounds the origin counter-clockwise. Consider now $\psi \in PC$ having finitely many jumps, denote by $\Lambda_\psi \subset \mathring{\mathbb{R}}$ the set of points at which ψ is discontinuous, and let Θ be the set of all connected components l of $\mathring{\mathbb{R}} \backslash \Lambda_\psi$. For each $l \in \Theta$, we define $\mathrm{ind}_l \psi$ as $(2\pi)^{-1}$ times the increment of the argument of ψ on l. If $\psi^{\#}(x,\mu) \neq 0$ for all $(x,\mu) \in \mathring{\mathbb{R}} \times [0,1]$, we define the *Cauchy index* of ψ by

$$\begin{aligned}\mathrm{ind}\psi &:= \sum_{l\in\Theta} \mathrm{ind}_l \psi + \sum_{x \in \Lambda_\psi \backslash \infty} \Big(-\frac{1}{2} + \Big\{ \frac{1}{2} + \frac{1}{2\pi} \arg \frac{\psi(x+0)}{\psi(x-0)} \Big\} \Big) \\ &= \sum_{l\in\Theta} \mathrm{ind}_l \psi + \sum_{x \in \Lambda_\psi \backslash \infty} \Big(\frac{1}{2} - \Big\{ \frac{1}{2} - \frac{1}{2\pi} \arg \frac{\psi(x+0)}{\psi(x-0)} \Big\} \Big),\end{aligned}$$

where $\{x\}$ denotes the *fractional part* $\tau \in [0,1)$ of the real number $x = n + \tau$, with $n \in \mathbb{Z}$. Considering $\arg(\psi(x+0)/\psi(x-0)) \in (-\pi, \pi)$, we have

$$\mathrm{ind}\psi = \sum_{l\in\Theta} \mathrm{ind}_l \psi + \frac{1}{2\pi} \sum_{x \in \Lambda_\psi \backslash \infty} \arg \frac{\psi(x+0)}{\psi(x-0)}.$$

The Cauchy index of ψ is therefore $(2\pi)^{-1}$ times the increment of the argument of z when z moves along the curve $\psi^{\#}(\mathring{\mathbb{R}}, [0,1])$ from $\psi(-\infty)$ to $\psi(+\infty)$. If $\psi \in PC$ has countably many jumps and $\psi^{\#}(x,\mu) \neq 0$ for all $(x,\mu) \in \mathring{\mathbb{R}} \times [0,1]$, we can also define the Cauchy index of ψ. For this purpose, we can uniformly approximate ψ by $\psi_n \in PC$ with finitely many jumps and such that $\psi_n^{\#}(x,\mu) \neq 0$ for all $(x,\mu) \in \mathring{\mathbb{R}} \times [0,1]$, with $\psi_n(\pm\infty) = \psi(\pm\infty)$. Then the Cauchy index of ψ is defined by

$$\mathrm{ind}\psi := \lim_{n\to+\infty} \mathrm{ind}\psi_n.$$

Considering the definitions of winding number and Cauchy index of a piecewise continuous function, we have

$$\begin{aligned} \operatorname{wind}\psi &= \operatorname{ind}\psi - \frac{1}{2} + \left\{\frac{1}{2} + \frac{1}{2\pi}\arg\frac{\psi(-\infty)}{\psi(+\infty)}\right\} \\ &= \operatorname{ind}\psi + \frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{2\pi}\arg\frac{\psi(-\infty)}{\psi(+\infty)}\right\} = \operatorname{ind}\psi + \frac{1}{2\pi}\arg\frac{\psi(-\infty)}{\psi(+\infty)}, \end{aligned} \tag{4.2}$$

where the last equality is valid if and only if $\arg\left(\psi(-\infty)/\psi(+\infty)\right) \in (-\pi,\pi)$.

In [25], one needed to introduce an appropriated definition of winding number of semi-almost periodic functions in the context of Wiener–Hopf plus Hankel operators (which also applies to the Wiener–Hopf minus Hankel case). There, if $\varphi \in \mathcal{G}SAP$ is such that $\kappa(\varphi_\mathrm{l}) = \kappa(\varphi_\mathrm{r}) = 0$ and $0 \notin [\mathbf{d}(\varphi_\mathrm{l}), \mathbf{d}(\varphi_\mathrm{r})]$ or if $\varphi \in \mathcal{G}SAP$ is such that $\kappa(\varphi_\mathrm{l}) + \kappa(\varphi_\mathrm{r}) = 0$ and $\Re e(\mathbf{d}(\varphi_\mathrm{l})/\mathbf{d}(\varphi_\mathrm{r})) \neq 0$ (cf. Remark 3.3 (i)) then the *winding number* of φ was defined as

$$\operatorname{wind}\varphi := \operatorname{ind}\varphi - \frac{1}{2} + \left\{\frac{1}{2} + \frac{1}{2\pi}\arg\frac{\mathbf{d}(\varphi_\mathrm{l})}{\mathbf{d}(\varphi_\mathrm{r})}\right\} = \operatorname{ind}\varphi + \frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{2\pi}\arg\frac{\mathbf{d}(\varphi_\mathrm{l})}{\mathbf{d}(\varphi_\mathrm{r})}\right\}.$$

Similarly to the case of the Cauchy index of a piecewise continuous function, if we consider $\arg(\mathbf{d}(\varphi_\mathrm{l})/\mathbf{d}(\varphi_\mathrm{r})) \in (-\pi,\pi)$, we have

$$\operatorname{wind}\varphi = \operatorname{ind}\varphi + \frac{1}{2\pi}\arg\frac{\mathbf{d}(\varphi_\mathrm{l})}{\mathbf{d}(\varphi_\mathrm{r})}.$$

Such generalization of winding number of a semi-almost periodic function will help to the generalization of the existent definition of winding number of a piecewise almost periodic function, in the sense of the following definition.

DEFINITION 4.1. For $\phi \in \mathcal{G}PAP$, consider that ϕ is represented as in (1.4), $\phi = \varphi\psi$, and with $\psi^{\#}(x,\mu) \neq 0$ for all $(x,\mu) \in \mathring{\mathbb{R}} \times [0,1]$. If $\kappa_\mathrm{l}(\phi) = \kappa_\mathrm{r}(\phi) = 0$ and $0 \notin [\mathbf{d}_\mathrm{l}(\phi), \mathbf{d}_\mathrm{r}(\phi)]$ or if $\kappa_\mathrm{l}(\phi) + \kappa_\mathrm{r}(\phi) = 0$ and $0 \notin \left[\frac{\mathbf{d}_\mathrm{l}(\phi)}{\mathbf{d}_\mathrm{r}(\phi)}, \frac{\mathbf{d}_\mathrm{r}(\phi)}{\mathbf{d}_\mathrm{l}(\phi)}\right]$, then the winding number of ϕ is defined by

$$\operatorname{wind}\phi := \operatorname{wind}\varphi + \operatorname{wind}\psi.$$

After having the winding number notion in the framework of Definition 4.1, we are now able to present a Fredholm index formula for the sum of Fredholm Wiener–Hopf plus Hankel and Wiener–Hopf minus Hankel operators with piecewise almost periodic Fourier symbols.

THEOREM 4.2. *If* $\phi \in \mathcal{G}PAP$, $\kappa_\mathrm{l}(\phi) + \kappa_\mathrm{r}(\phi) = 0$ *and*

$$0 \notin \left[\frac{\mathbf{d}_\mathrm{l}(\phi)}{\mathbf{d}_\mathrm{r}(\phi)}, \frac{\mathbf{d}_\mathrm{r}(\phi)}{\mathbf{d}_\mathrm{l}(\phi)}\right] \cup \bigcup_{x\in\mathbb{R}} [(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)],$$

then

$$\operatorname{Ind}(W_\phi + H_\phi) + \operatorname{Ind}(W_\phi - H_\phi) = -\operatorname{wind}(\phi\widetilde{\phi^{-1}}).$$

Moreover:

(i) *If ϕ does not have symmetric discontinuities, then*

$$\operatorname{Ind}(W_\phi + H_\phi) + \operatorname{Ind}(W_\phi - H_\phi) = -2\operatorname{wind}\phi.$$

(ii) *If ϕ has symmetric discontinuities and ϕ is continuous at* 0, *then*

$$\operatorname{Ind}(W_\phi + H_\phi) + \operatorname{Ind}(W_\phi - H_\phi) = -2\big(\operatorname{wind}\varphi + \operatorname{wind}\rho_\psi\big),$$

considering $\phi = \varphi\psi$ (cf. the representation (1.4) *of ϕ), and*

$$\rho_\psi(x) := \begin{cases} \psi(0)(\psi\widetilde{\psi^{-1}})(x) & \text{if } x \leqslant 0, \\ \psi(0) & \text{if } x > 0. \end{cases} \tag{4.3}$$

(iii) *If ϕ is discontinuous at* 0, *then*

$$\operatorname{Ind}(W_\phi+H_\phi)+\operatorname{Ind}(W_\phi-H_\phi)=-2(\operatorname{wind}\varphi+\operatorname{ind}((\psi\widetilde{\psi^{-1}})_-))-\frac{1}{2\pi}\arg\Big(\Big(\frac{\psi(0+0)}{\psi(0-0)}\Big)^2\Big),$$

considering $\phi = \varphi\psi$ (in the sense of the representation (1.4)*),*

$$(\psi\widetilde{\psi^{-1}})_-(x) := \begin{cases} (\psi\widetilde{\psi^{-1}})(x) & \text{if } x \leqslant 0, \\ (\psi\widetilde{\psi^{-1}})(0-0) & \text{if } x > 0, \end{cases} \quad \text{and} \quad \arg\Big(\Big(\frac{\psi(0+0)}{\psi(0-0)}\Big)^2\Big) \in (-\pi, \pi).$$

Proof. Under the hypothesis, Theorems 3.1 and 3.2 ensure that $W_{\phi\widetilde{\phi^{-1}}}$, $W_\phi + H_\phi$ and $W_\phi - H_\phi$ are all Fredholm operators. Recalling now that $W_\phi + H_\phi$ is Δ-related after extension with $W_{\phi\widetilde{\phi^{-1}}}$ (cf. Lemma 2.1), it holds that

$$\operatorname{Ind}W_{\phi\widetilde{\phi^{-1}}} = \operatorname{Ind}(W_\phi + H_\phi) + \operatorname{Ind}\mathcal{T}_\phi.$$

From the relation $\mathcal{T}_\phi \overset{*}{\sim} W_\phi - H_\phi$ (see (2.6)), we have

$$\operatorname{Ind}\mathcal{T}_\phi = \operatorname{Ind}(W_\phi - H_\phi).$$

Combining the two last identities, it results that

$$\operatorname{Ind}(W_\phi + H_\phi) + \operatorname{Ind}(W_\phi - H_\phi) = \operatorname{Ind}W_{\phi\widetilde{\phi^{-1}}}. \tag{4.4}$$

Applying the index formula presented in Theorem 3.16 of [6], it follows

$$\operatorname{Ind}W_{\phi\widetilde{\phi^{-1}}} = -\operatorname{wind}(\phi\widetilde{\phi^{-1}}). \tag{4.5}$$

Thus, combining (4.4) and (4.5), we have

$$\operatorname{Ind}(W_\phi + H_\phi) + \operatorname{Ind}(W_\phi - H_\phi) = -\operatorname{wind}(\phi\widetilde{\phi^{-1}}). \tag{4.6}$$

Having in mind the representation (1.4) of ϕ (i.e., $\phi = \varphi\psi$ with φ and ψ in the indicated classes), it follows $\phi\widetilde{\phi^{-1}} = \varphi\widetilde{\varphi^{-1}}\,\psi\widetilde{\psi^{-1}}$ where $\varphi\widetilde{\varphi^{-1}} \in \mathcal{G}SAP$, and $\psi\widetilde{\psi^{-1}} \in \mathcal{G}PC$ is such that $\psi\widetilde{\psi^{-1}}(-\infty) = \psi\widetilde{\psi^{-1}}(+\infty) = 1$. According to Definition 4.1, one gets

$$\operatorname{wind}(\phi\widetilde{\phi^{-1}}) = \operatorname{wind}(\varphi\widetilde{\varphi^{-1}}) + \operatorname{wind}(\psi\widetilde{\psi^{-1}}). \tag{4.7}$$

In this case, $\mathrm{wind}(\varphi\widetilde{\varphi^{-1}})$ is well defined because:

$$\kappa((\varphi\widetilde{\varphi^{-1}})_{\mathrm{l}}) = \kappa((\varphi\widetilde{\varphi^{-1}})_{\mathrm{r}}) = \kappa_{\mathrm{l}}(\phi) + \kappa_{\mathrm{r}}(\phi) = 0,$$
$$0 \notin [\mathbf{d}((\varphi\widetilde{\varphi^{-1}})_{\mathrm{l}}), \mathbf{d}((\varphi\widetilde{\varphi^{-1}})_{\mathrm{r}})] = \Big[\frac{\mathbf{d}_{\mathrm{l}}(\phi)}{\mathbf{d}_{\mathrm{r}}(\phi)}, \frac{\mathbf{d}_{\mathrm{r}}(\phi)}{\mathbf{d}_{\mathrm{l}}(\phi)}\Big].$$

In addition, we know how to relate $\mathrm{wind}\varphi$ with $\mathrm{wind}(\varphi\widetilde{\varphi^{-1}})$ in the form

$$\mathrm{wind}(\varphi\widetilde{\varphi^{-1}}) = 2\mathrm{wind}\varphi. \tag{4.8}$$

Let us now look for an identity involving $\mathrm{wind}(\psi\widetilde{\psi^{-1}})$ and $\mathrm{wind}\psi$. For this purpose we will analyze the functions $\psi^{\#}, (\widetilde{\psi^{-1}})^{\#}, (\psi\widetilde{\psi^{-1}})^{\#} : \mathring{\mathbb{R}} \times [0,1] \to \mathbb{C}$. For all $(x,\mu) \in \mathring{\mathbb{R}} \times [0,1]$, we have

$$(\psi\widetilde{\psi^{-1}})^{\#}(x,\mu) - \psi^{\#}(x,\mu)(\widetilde{\psi^{-1}})^{\#}(x,\mu)$$
$$= (1-\mu)\mu(\psi(x+0) - \psi(x-0))(\widetilde{\psi^{-1}}(x+0) - \widetilde{\psi^{-1}}(x-0)).$$

We will now consider three different situations: (1) ϕ has no symmetric discontinuities; (2) ϕ has symmetric discontinuities and ϕ is continuous at 0; and finally, (3) ϕ is discontinuous at 0. Since the discontinuities of ϕ arise from the factor ψ (in the factorization $\phi = \varphi\psi$), we are in fact facing the following cases: (1) ψ has no symmetric discontinuities; (2) ψ has symmetric discontinuities and ψ is continuous at 0; and, (3) ψ is discontinuous at 0.

Case 1. Since in this case ψ and $\widetilde{\psi^{-1}}$ have no common discontinuities, we conclude that

$$(\psi\widetilde{\psi^{-1}})^{\#}(x,\mu) = \psi^{\#}(x,\mu)(\widetilde{\psi^{-1}})^{\#}(x,\mu) \quad \text{for all } (x,\mu) \in \mathring{\mathbb{R}} \times [0,1].$$

Thus, observing that $\psi^{\#}$ and $(\widetilde{\psi^{-1}})^{\#}$ are closed continuous curves away from zero, it follows

$$\mathrm{wind}(\psi\widetilde{\psi^{-1}})^{\#} = \mathrm{wind}\psi^{\#} + \mathrm{wind}(\widetilde{\psi^{-1}})^{\#}. \tag{4.9}$$

Furthermore, $\mathrm{wind}\psi^{\#} = \mathrm{ind}\psi^{\#}$ and $\mathrm{wind}(\widetilde{\psi^{-1}})^{\#} = \mathrm{ind}(\widetilde{\psi^{-1}})^{\#}$. Computing now the Cauchy index of $(\widetilde{\psi^{-1}})^{\#}$, one gets

$$\begin{aligned}\mathrm{ind}(\widetilde{\psi^{-1}})^{\#} &= \frac{1}{2\pi}[(\arg(\widetilde{\psi^{-1}})^{\#})(+\infty) - (\arg(\widetilde{\psi^{-1}})^{\#})(-\infty)]\\ &= \frac{1}{2\pi}[(\arg\widetilde{\psi^{-1}})(+\infty) - (\arg\widetilde{\psi^{-1}})(-\infty)] = \frac{1}{2\pi}[-(\arg\psi)(-\infty) + (\arg\psi)(+\infty)]\\ &= \frac{1}{2\pi}[(\arg\psi^{\#})(+\infty) - (\arg\psi^{\#})(-\infty)] = \mathrm{ind}\psi^{\#},\end{aligned}$$

i.e.,

$$\mathrm{wind}(\widetilde{\psi^{-1}})^{\#} = \mathrm{wind}\psi^{\#}. \tag{4.10}$$

Combining (4.9) and (4.10), it results that

$$\mathrm{wind}(\psi\widetilde{\psi^{-1}})^{\#} = 2\mathrm{wind}\psi^{\#}.$$

Applying the definition of winding number for piecewise continuous functions, we obtain

$$\operatorname{wind}(\psi\widetilde{\psi^{-1}}) = 2\operatorname{wind}\psi. \tag{4.11}$$

From (4.7), (4.8), (4.11), and Definition 4.1, we have

$$\operatorname{wind}(\phi\widetilde{\phi^{-1}}) = 2\operatorname{wind}\varphi + 2\operatorname{wind}\psi = 2\operatorname{wind}\phi.$$

According to (4.6), it follows that

$$\operatorname{Ind}(W_\phi + H_\phi) + \operatorname{Ind}(W_\phi - H_\phi) = -2\operatorname{wind}\phi.$$

Case 2. In this case, we may write

$$\psi = \psi_-\psi_+ \tag{4.12}$$

in such a way that $\psi_-, \psi_+ \in PC$ do not have common discontinuities. For that, consider

$$\psi_-(x) = \begin{cases} \psi(x) & \text{if } x \leqslant 0, \\ \psi(0) & \text{if } x > 0, \end{cases} \quad \text{and} \quad \psi_+(x) = \begin{cases} 1 & \text{if } x \leqslant 0, \\ \dfrac{\psi(x)}{\psi(0)} & \text{if } x > 0, \end{cases}$$

where ψ_- has only discontinuities in $\mathbb{R}_-$ and ψ_+ has only discontinuities in $\mathbb{R}_+$. From (4.12), it follows

$$\psi\widetilde{\psi^{-1}} = \psi_-\psi_+\widetilde{\psi_+^{-1}}\widetilde{\psi_-^{-1}} = (\psi_-\widetilde{\psi_+^{-1}})(\psi_+\widetilde{\psi_-^{-1}}).$$

Computing $\psi_-\widetilde{\psi_+^{-1}}$ and $\psi_+\widetilde{\psi_-^{-1}}$, one gets

$$(\psi_-\widetilde{\psi_+^{-1}})(x) = \begin{cases} \psi(0)\dfrac{\psi(x)}{\psi(-x)} & \text{if } x \leqslant 0, \\ \psi(0) & \text{if } x > 0; \end{cases} \qquad (\psi_+\widetilde{\psi_-^{-1}})(x) = \begin{cases} \dfrac{1}{\psi(0)} & \text{if } x \leqslant 0, \\ \dfrac{\psi(x)}{\psi(0)\psi(-x)} & \text{if } x > 0. \end{cases}$$

Since $\rho_\psi = \psi_-\widetilde{\psi_+^{-1}}$ is a function with discontinuities in $\mathbb{R}_-$ and $\psi_+\widetilde{\psi_-^{-1}} = \widetilde{\rho_\psi^{-1}}$ is a function with discontinuities in $\mathbb{R}_+$, we see that ρ_ψ and $\widetilde{\rho_\psi^{-1}}$ do not have common discontinuities. Additionally, from the hypothesis

$$0 \notin \bigcup_{x\in\mathbb{R}} [(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)],$$

it follows that

$$0 \notin \bigcup_{x\in\mathbb{R}} [(\psi\widetilde{\psi^{-1}})(x-0), (\psi\widetilde{\psi^{-1}})(x+0)],$$

i.e., $(\psi\widetilde{\psi^{-1}})^\#(x,\mu) \neq 0$ for all $(x,\mu) \in \mathring{\mathbb{R}} \times [0,1]$. Therefore, we have $\rho_\psi^\#(x,\mu) \neq 0$ for all $(x,\mu) \in \mathring{\mathbb{R}} \times [0,1]$ and $\widetilde{\rho_\psi^{-1}}^{\#}(x,\mu) \neq 0$ for all $(x,\mu) \in \mathring{\mathbb{R}} \times [0,1]$, which means that wind$\rho_\psi$ and wind$\widetilde{\rho_\psi^{-1}}$ are well defined. Following now the same reasoning as in Case 1, we obtain

$$\operatorname{wind}(\psi\widetilde{\psi^{-1}}) = 2\operatorname{wind}\rho_\psi. \tag{4.13}$$

Finally, from (4.7), (4.8), (4.13), and Definition 4.1, we have

$$\operatorname{wind}(\phi\widetilde{\phi^{-1}}) = 2(\operatorname{wind}\varphi + \operatorname{wind}\rho_\psi),$$

which, by (4.6), yields that

$$\operatorname{Ind}(W_\phi + H_\phi) + \operatorname{Ind}(W_\phi - H_\phi) = -2(\operatorname{wind}\varphi + \operatorname{wind}\rho_\psi).$$

Case 3. Since ψ is discontinuous at 0, we may identify ψ with

$$\psi(x) = \begin{cases} \psi_1(x) & \text{if } x \leqslant 0, \\ \psi_2(x) & \text{if } x > 0, \end{cases}$$

where $\psi_1, \psi_2 \in \mathcal{G}PC$ are such that $\psi_1(0) \neq \psi_2(0+0)$ and $\psi_1(-\infty) = \psi_2(+\infty) = 1$ (recall that from (1.4) we have $\psi(-\infty) = \psi(+\infty) = 1$). Thus, it follows that $\psi\widetilde{\psi^{-1}}$ may be written in the form

$$(\psi\widetilde{\psi^{-1}})(x) = \begin{cases} (\psi_1\widetilde{\psi_2^{-1}})(x) & \text{if } x \leqslant 0, \\ (\widetilde{\psi_1^{-1}}\psi_2)(x) & \text{if } x > 0. \end{cases} \tag{4.14}$$

Due to the equality $(\psi\widetilde{\psi^{-1}})(-\infty) = (\psi\widetilde{\psi^{-1}})(+\infty)$, and according to (4.2), it follows that

$$\operatorname{wind}(\psi\widetilde{\psi^{-1}}) = \operatorname{ind}(\psi\widetilde{\psi^{-1}}). \tag{4.15}$$

In order to compute the Cauchy index of $\psi\widetilde{\psi^{-1}}$, let us assume (without lost of generality) that $\psi\widetilde{\psi^{-1}}$ has finitely many jumps. It is clear that the discontinuities of $\psi\widetilde{\psi^{-1}}$ are symmetric. Note however that ψ admits a discontinuity at 0 but $\psi\widetilde{\psi^{-1}}$ is not always discontinuous at 0. Hence

$$\Lambda_{\psi\widetilde{\psi^{-1}}} = \{-x_n, \ldots, -x_1, x_0, x_1, \ldots, x_n\} \quad \text{or} \quad \Lambda_{\psi\widetilde{\psi^{-1}}} = \{-x_n, \ldots, -x_1, x_1, \ldots, x_n\},$$

with $n \in \mathbb{N}_0 := \{0,1,2,3,\ldots\}$, $x_j \in \mathbb{R}_+$ (for all $j = 1, \ldots, n$), and $x_0 = 0$. Let us consider first the case $\Lambda_{\psi\widetilde{\psi^{-1}}} = \{-x_n, \ldots, -x_1, x_0, x_1, \ldots, x_n\}$. Taking into account the definition of Cauchy index of a piecewise continuous function, we get

$$\begin{aligned}\operatorname{ind}(\psi\widetilde{\psi^{-1}}) = {} & \operatorname{ind}_{]-\infty,-x_n[}(\psi\widetilde{\psi^{-1}}) + \sum_{j=1}^{n} \operatorname{ind}_{]-x_j,-x_{j-1}[}(\psi\widetilde{\psi^{-1}}) \\ & + \sum_{j=1}^{n} \operatorname{ind}_{]x_{j-1},x_j[}(\psi\widetilde{\psi^{-1}}) + \operatorname{ind}_{]x_n,+\infty[}(\psi\widetilde{\psi^{-1}}) + \sum_{x \in \Lambda_{\psi\widetilde{\psi^{-1}}}} \frac{1}{2\pi} \arg \frac{(\psi\widetilde{\psi^{-1}})(x+0)}{(\psi\widetilde{\psi^{-1}})(x-0)},\end{aligned}$$

i.e.,

$$\begin{aligned}\operatorname{ind}(\widetilde{\psi\psi^{-1}}) =&\operatorname{ind}_{]-\infty,-x_n[}(\widetilde{\psi\psi^{-1}}) + \sum_{j=1}^{n} \operatorname{ind}_{]-x_j,-x_{j-1}[}(\widetilde{\psi\psi^{-1}}) \\ &+ \sum_{x\in\Lambda_{\widetilde{\psi\psi^{-1}}}\cap\mathbb{R}_-} \frac{1}{2\pi}\arg\frac{(\widetilde{\psi\psi^{-1}})(x+0)}{(\widetilde{\psi\psi^{-1}})(x-0)} + \frac{1}{2\pi}\arg\frac{(\widetilde{\psi\psi^{-1}})(0+0)}{(\widetilde{\psi\psi^{-1}})(0-0)} \\ &+ \sum_{x\in\Lambda_{\widetilde{\psi\psi^{-1}}}\cap\mathbb{R}_+} \frac{1}{2\pi}\arg\frac{(\widetilde{\psi\psi^{-1}})(x+0)}{(\widetilde{\psi\psi^{-1}})(x-0)} \\ &+ \sum_{j=1}^{n} \operatorname{ind}_{]x_{j-1},x_j[}(\widetilde{\psi\psi^{-1}}) + \operatorname{ind}_{]x_n,+\infty[}(\widetilde{\psi\psi^{-1}}),\end{aligned}$$

where $\arg\left((\widetilde{\psi\psi^{-1}})(x+0)/(\widetilde{\psi\psi^{-1}})(x-0)\right) \in (-\pi,\pi)$, for all $x \in \Lambda_{\widetilde{\psi\psi^{-1}}}$. Using now (4.14), it follows that

$$\begin{aligned}\operatorname{ind}(\widetilde{\psi\psi^{-1}}) =& \operatorname{ind}_{]-\infty,-x_n[}(\widetilde{\psi_1\psi_2^{-1}}) + \sum_{j=1}^{n} \operatorname{ind}_{]-x_j,-x_{j-1}[}(\widetilde{\psi_1\psi_2^{-1}}) \\ &+ \sum_{x\in\Lambda_{\widetilde{\psi\psi^{-1}}}\cap\mathbb{R}_-} \frac{1}{2\pi}\arg\frac{(\widetilde{\psi_1\psi_2^{-1}})(x+0)}{(\widetilde{\psi_1\psi_2^{-1}})(x-0)} + \frac{1}{2\pi}\arg\frac{(\widetilde{\psi\psi^{-1}})(0+0)}{(\widetilde{\psi\psi^{-1}})(0-0)} \\ &+ \sum_{x\in\Lambda_{\widetilde{\psi\psi^{-1}}}\cap\mathbb{R}_+} \frac{1}{2\pi}\arg\frac{(\widetilde{\psi_1^{-1}\psi_2})(x+0)}{(\widetilde{\psi_1^{-1}\psi_2})(x-0)} \\ &+ \sum_{j=1}^{n} \operatorname{ind}_{]x_{j-1},x_j[}(\widetilde{\psi_1^{-1}\psi_2}) + \operatorname{ind}_{]x_n,+\infty[}(\widetilde{\psi_1^{-1}\psi_2}).\end{aligned} \tag{4.16}$$

Noticing that $(\widetilde{\psi_1^{-1}\psi_2})(x) = (\widetilde{\widetilde{\psi_1\psi_2^{-1}})^{-1}}(x)$, for all $x > 0$, we have

$$(\arg(\widetilde{\psi_1^{-1}\psi_2}))(x) = -(\arg(\widetilde{\psi_1\psi_2^{-1}}))(-x)$$

for all $x > 0$. Therefore, we obtain the following identities:

$$\begin{aligned}\operatorname{ind}_{]x_n,+\infty[}(\widetilde{\psi_1^{-1}\psi_2}) &= \frac{1}{2\pi}((\arg(\widetilde{\psi_1^{-1}\psi_2}))(+\infty) - (\arg(\widetilde{\psi_1^{-1}\psi_2}))(x_n)) \\ &= \frac{1}{2\pi}((\arg(\widetilde{\psi_1\psi_2^{-1}}))(-x_n) - (\arg(\widetilde{\psi_1\psi_2^{-1}}))(-\infty)) \\ &= \operatorname{ind}_{]-\infty,-x_n[}(\widetilde{\psi_1\psi_2^{-1}}),\end{aligned} \tag{4.17}$$

$$\begin{aligned}(4.18)\quad \mathrm{ind}_{]x_{j-1},x_j[}(\widetilde{\psi_1^{-1}\psi_2}) &= \frac{1}{2\pi}((\arg(\widetilde{\psi_1^{-1}\psi_2}))(x_j) - (\arg(\widetilde{\psi_1^{-1}\psi_2}))(x_{j-1}))\\ &= \frac{1}{2\pi}((\arg(\psi_1\widetilde{\psi_2^{-1}}))(-x_{j-1}) - (\arg(\psi_1\widetilde{\psi_2^{-1}}))(-x_j))\\ &= \mathrm{ind}_{]-x_j,-x_{j-1}[}(\psi_1\widetilde{\psi_2^{-1}}),\end{aligned}$$

for all $j = 1, \ldots, n$. Moreover, it holds

$$(4.19)\qquad \frac{(\widetilde{\psi_1^{-1}\psi_2})(x+0)}{(\widetilde{\psi_1^{-1}\psi_2})(x-0)} = \frac{(\psi_1^{-1}\widetilde{\psi_2})(-x-0)}{(\psi_1^{-1}\widetilde{\psi_2})(-x+0)} = \frac{(\psi_1\widetilde{\psi_2^{-1}})(-x+0)}{(\psi_1\widetilde{\psi_2^{-1}})(-x-0)},$$

for all $x > 0$. Inserting (4.17), (4.18) and (4.19) in (4.16), we obtain

$$\begin{aligned}(4.20)\quad \mathrm{ind}(\psi\widetilde{\psi^{-1}}) = 2\Big(&\mathrm{ind}_{]-\infty,-x_n[}(\psi_1\widetilde{\psi_2^{-1}}) + \sum_{j=1}^{n}\mathrm{ind}_{]-x_j,-x_{j-1}[}(\psi_1\widetilde{\psi_2^{-1}})\\ &+ \sum_{x\in\Lambda_{\psi\widetilde{\psi^{-1}}}\cap\mathbb{R}_-}\frac{1}{2\pi}\arg\frac{(\psi_1\widetilde{\psi_2^{-1}})(x+0)}{(\psi_1\widetilde{\psi_2^{-1}})(x-0)}\Big) + \frac{1}{2\pi}\arg\frac{(\psi\widetilde{\psi^{-1}})(0+0)}{(\psi\widetilde{\psi^{-1}})(0-0)}.\end{aligned}$$

Considering the function $(\psi\widetilde{\psi^{-1}})_-$ given by

$$(\psi\widetilde{\psi^{-1}})_-(x) = \begin{cases}(\psi\widetilde{\psi^{-1}})(x) & \text{if } x \leqslant 0,\\ (\psi\widetilde{\psi^{-1}})(0-0) & \text{if } x > 0,\end{cases} = \begin{cases}(\psi_1\widetilde{\psi_2^{-1}})(x) & \text{if } x \leqslant 0,\\ (\psi_1\widetilde{\psi_2^{-1}})(0-0) & \text{if } x > 0,\end{cases}$$

we can represent the Cauchy index of $\psi\widetilde{\psi^{-1}}$ in the form

$$\mathrm{ind}(\psi\widetilde{\psi^{-1}}) = 2\mathrm{ind}((\psi\widetilde{\psi^{-1}})_-) + \frac{1}{2\pi}\arg\frac{(\psi\widetilde{\psi^{-1}})(0+0)}{(\psi\widetilde{\psi^{-1}})(0-0)}.$$

Due to the simplification

$$\frac{(\psi\widetilde{\psi^{-1}})(0+0)}{(\psi\widetilde{\psi^{-1}})(0-0)} = \Big(\frac{\psi(0+0)}{\psi(0-0)}\Big)^2,$$

we get

$$(4.21)\qquad \mathrm{ind}(\psi\widetilde{\psi^{-1}}) = 2\mathrm{ind}((\psi\widetilde{\psi^{-1}})_-) + \frac{1}{2\pi}\arg\Big(\Big(\frac{\psi(0+0)}{\psi(0-0)}\Big)^2\Big).$$

In the case where $\Lambda_{\psi\widetilde{\psi^{-1}}} = \{-x_n, \ldots, -x_1, x_1, \ldots, x_n\}$, i.e., $\psi\widetilde{\psi^{-1}}$ is not discontinuous at 0, the last formula also holds, and so we have in this case

$$\mathrm{ind}(\psi\widetilde{\psi^{-1}}) = 2\mathrm{ind}((\psi\widetilde{\psi^{-1}})_-).$$

From (4.15) and (4.21), it follows

$$(4.22)\qquad \mathrm{wind}(\psi\widetilde{\psi^{-1}}) = 2\mathrm{ind}((\psi\widetilde{\psi^{-1}})_-) + \frac{1}{2\pi}\arg\Big(\Big(\frac{\psi(0+0)}{\psi(0-0)}\Big)^2\Big).$$

Taking into account (4.7), (4.8), (4.22) and Definition 4.1, we obtain the formula for the winding number of $\phi\widetilde{\phi^{-1}}$:

$$\operatorname{wind}(\phi\widetilde{\phi^{-1}}) = 2(\operatorname{wind}\varphi + \operatorname{ind}((\psi\widetilde{\psi^{-1}})_{-})) + \frac{1}{2\pi}\arg\left(\left(\frac{\psi(0+0)}{\psi(0-0)}\right)^2\right).$$

Finally, from (4.6) it holds

$$\operatorname{Ind}(W_\phi+H_\phi)+\operatorname{Ind}(W_\phi-H_\phi) = -2(\operatorname{wind}\varphi+\operatorname{ind}((\psi\widetilde{\psi^{-1}})_{-}))-\frac{1}{2\pi}\arg\left(\left(\frac{\psi(0+0)}{\psi(0-0)}\right)^2\right)$$

and this concludes the proof. ∎

REMARK 4.3. Although in the case where ψ does not have symmetric discontinuities we have $\operatorname{wind}(\psi\widetilde{\psi^{-1}}) = 2\operatorname{wind}\psi$ (and consequently $\operatorname{wind}(\phi\widetilde{\phi^{-1}}) = 2\operatorname{wind}\phi$), this is, in general, not true for ψ with symmetric discontinuities (as we will see in Example 4.6 of the next subsection). In this sense, we need to find another way to compute $\operatorname{wind}(\psi\widetilde{\psi^{-1}})$ based on a winding number or a Cauchy index of a function related with ψ. The obtained formula in the case where ψ is continuous at 0, $\operatorname{wind}(\psi\widetilde{\psi^{-1}}) = 2\operatorname{wind}\rho_\psi$, can also be applied when ψ does not have symmetric discontinuities. In the case where ψ is discontinuous at 0, we obtain a general formula that also applies to the previous first and second cases.

In the next subsection we will provide some examples that illustrate different possibilities for the Fredholm property and index of the operators in study. Namely, Example 4.5 presents the case of Fredholm operators $W_\phi + H_\phi$, $W_\phi - H_\phi$ and W_ϕ such that the sum of the Fredholm indices of $W_\phi + H_\phi$ and $W_\phi - H_\phi$ is not equal to two times the Fredholm index of W_ϕ. In addition, combining Theorem 3.1 and Theorem 3.2, we observe that the case of Fredholm operators $W_\phi + H_\phi$, $W_\phi - H_\phi$ and W_ϕ only occurs when $\kappa_{\mathrm{l}}(\phi) = \kappa_{\mathrm{r}}(\phi) = 0$, and

$$0 \notin [\mathbf{d}_{\mathrm{l}}(\phi), \mathbf{d}_{\mathrm{r}}(\phi)] \cup \bigcup_{x\in\mathbb{R}} [\phi(x-0), \phi(x+0)],$$

$$0 \notin \left[\frac{\mathbf{d}_{\mathrm{l}}(\phi)}{\mathbf{d}_{\mathrm{r}}(\phi)}, \frac{\mathbf{d}_{\mathrm{r}}(\phi)}{\mathbf{d}_{\mathrm{l}}(\phi)}\right] \cup \bigcup_{x\in\mathbb{R}} [(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)].$$

4.2. EXAMPLES AND INVERTIBILITY.

EXAMPLE 4.4. Consider the function ϕ given by

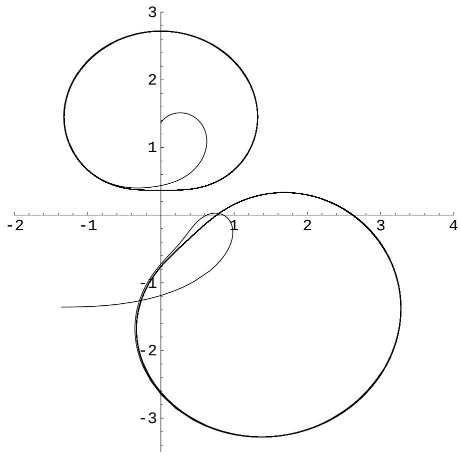
$$\phi(x) := (1-u(x))(1+\mathrm{i})\mathrm{e}^{2\mathrm{i}x} + u(x)5\mathrm{e}^{-2\mathrm{i}x} + \phi_0(x), \tag{4.23}$$

where

$$u(x) := \begin{cases} \frac{1}{2}\mathrm{e}^{x} & \text{if } x \leqslant 0, \\ 1-\frac{1}{2}\mathrm{e}^{-x} & \text{if } x > 0, \end{cases} \quad \text{and} \quad \phi_0(x) := \begin{cases} \mathrm{e}^{x} & \text{if } x \leqslant 0, \\ \operatorname{arc\,cot} x & \text{if } x > 0. \end{cases}$$

From the above results, we have that the Wiener–Hopf plus Hankel operator $W_\phi + H_\phi$ and the Wiener–Hopf minus Hankel operator $W_\phi - H_\phi$ are Fredholm operators while the Wiener–Hopf operator W_ϕ is not a Fredholm operator (since it is not normally solvable).

FIGURE 1. The range of $\phi(x)$ for x between -1000 and 1000.



EXAMPLE 4.5. Consider the function ϕ (see Figure 1) given by

$$\phi(x) := (1-u(x))\mathrm{i}\mathrm{e}^{\mathrm{e}^{\mathrm{i}x}} + u(x)(1-\mathrm{i})\mathrm{e}^{\mathrm{e}^{\mathrm{i}x}} + \phi_0(x), \tag{4.24}$$

where

$$u(x) := \frac{1}{2}(1+\tanh x) \quad \text{and} \quad \phi_0(x) := \begin{cases} \frac{1}{2}(\mathrm{i}-1)\mathrm{e}^{x+1} & \text{if } x \leqslant 0, \\ -\left(1+\frac{\mathrm{i}}{2}\right)\mathrm{e}^{-x+1} & \text{if } x > 0. \end{cases}$$

Since $\phi \in \mathcal{G}PAP$, we may look for a representation of ϕ as (1.4). For instance, consider φ and ψ given by

$$\varphi(x) = (1-u(x))\mathrm{i}\mathrm{e}^{\mathrm{e}^{\mathrm{i}x}} + u(x)(1-\mathrm{i})\mathrm{e}^{\mathrm{e}^{\mathrm{i}x}}$$

and

$$\psi(x) = 1 + (\phi_0\varphi^{-1})(x) = \begin{cases} 1 + \dfrac{(\mathrm{i}-1)\mathrm{e}^{x+1}}{2\varphi(x)} & \text{if } x \leqslant 0, \\ 1 - \dfrac{\left(1+\frac{\mathrm{i}}{2}\right)\mathrm{e}^{-x+1}}{\varphi(x)} & \text{if } x > 0, \end{cases}$$

respectively. In this way, we get

$$\phi = \varphi\psi. \tag{4.25}$$

FIGURE 2. The range of $\varphi(x)$ for x between -500 and 500.

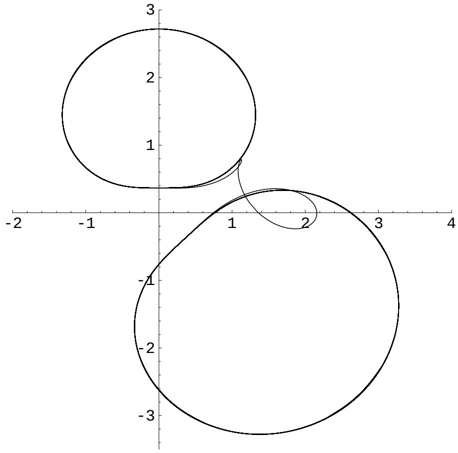
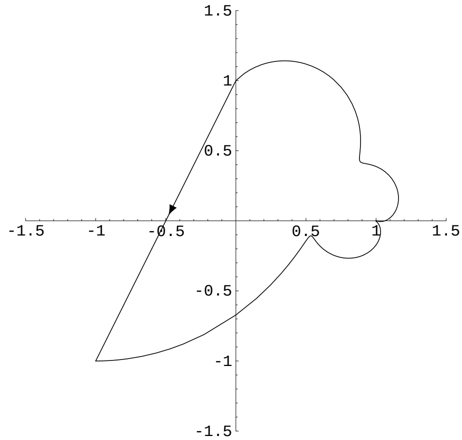


FIGURE 3. The oriented graph of $\psi^{\#}$.



We see that $\varphi \in \mathcal{G}SAP$ and $\psi \in \mathcal{G}PC$ is such that $\psi(-\infty) = \psi(+\infty) = 1$. Computing the winding number of φ, ψ, $\varphi\widetilde{\varphi^{-1}}$ and $\psi\widetilde{\psi^{-1}}$ (see Figures 2, 3, 4 and 5), we have

$$\operatorname{wind}\varphi = 0, \quad \operatorname{wind}(\varphi\widetilde{\varphi^{-1}}) = -1 \quad \text{and} \quad \operatorname{wind}\psi = \operatorname{wind}(\psi\widetilde{\psi^{-1}}) = 1.$$

From Definition 4.1, we obtain

$$\operatorname{wind}\phi = 1 \quad \text{and} \quad \operatorname{wind}(\phi\widetilde{\phi^{-1}}) = 0.$$

Additionally, by the index formula presented in (4.5) and (4.6), this means that

$$\operatorname{Ind} W_\phi = -1 \quad \text{and} \quad \operatorname{Ind}\big(W_\phi + H_\phi\big) + \operatorname{Ind}\big(W_\phi - H_\phi\big) = 0.$$

In order to exemplify the simplification for the formula of the winding number of $\psi\widetilde{\psi^{-1}}$ presented in the second case of Theorem 4.2, we will now present an example of a particular piecewise continuous function ψ (with symmetric discontinuities) for which we compute the winding number of $\psi\widetilde{\psi^{-1}}$ based on the winding number of a piecewise continuous function which depends on ψ, but has discontinuities only in $\mathbb{R}_-$.

EXAMPLE 4.6. Consider the piecewise continuous function ψ given by

$$\psi(x) := \begin{cases} 1 + \dfrac{1-\mathrm{i}}{x} & \text{if } x \leqslant -1, \\ -x + \mathrm{i}\dfrac{x+1}{2} & \text{if } -1 < x \leqslant 1, \\ 1 + \dfrac{10-\mathrm{i}}{x^3} & \text{if } x > 1. \end{cases}$$

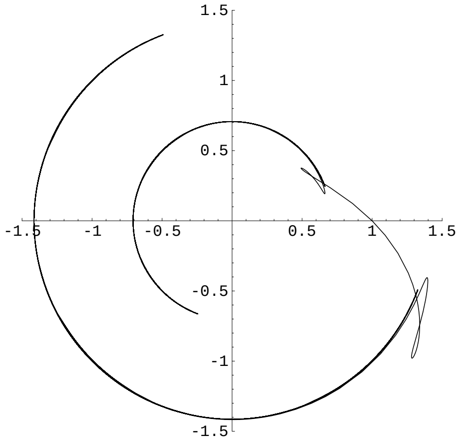


FIGURE 4. The range of $(\widetilde{\varphi\varphi^{-1}})(x)$ for x between -1000 and 1000.

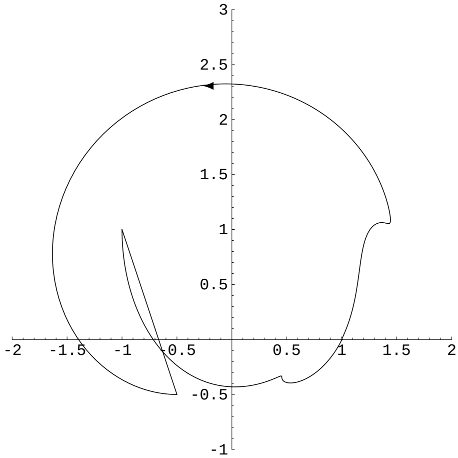


FIGURE 5. The oriented graph of $(\widetilde{\psi\psi^{-1}})^{\#}$.

It is clear that ψ is a continuous function at 0, and has symmetric discontinuities at -1 and 1. Additionally,

$$(\widetilde{\psi\psi^{-1}})(x) = \begin{cases} \dfrac{x^3 + x^2(1-\mathrm{i})}{x^3 - 10 + \mathrm{i}} & \text{if } x < -1, \\ \dfrac{\mathrm{i}}{\mathrm{i} - 1} & \text{if } x = -1, \\ \dfrac{x(\mathrm{i}-2) + \mathrm{i}}{x(2-\mathrm{i}) + \mathrm{i}} & \text{if } -1 < x < 1, \\ 1 + \mathrm{i} & \text{if } x = 1, \\ \dfrac{x^3 + 10 - \mathrm{i}}{x^3 + x^2(\mathrm{i}-1)} & \text{if } x > 1, \end{cases}$$

and therefore we may identify $\widetilde{\psi\psi^{-1}}$ with

$$(\widetilde{\psi\psi^{-1}})(x) = \begin{cases} \dfrac{x^3 + x^2(1-\mathrm{i})}{x^3 - 10 + \mathrm{i}} & \text{if } x \leqslant -1, \\ \dfrac{x(\mathrm{i}-2) + \mathrm{i}}{x(2-\mathrm{i}) + \mathrm{i}} & \text{if } -1 < x \leqslant 1, \\ \dfrac{x^3 + 10 - \mathrm{i}}{x^3 + x^2(\mathrm{i}-1)} & \text{if } x > 1. \end{cases}$$

From Figures 6, 7, 8, and 9, we may observe that

$$\operatorname{wind}(\widetilde{\psi\psi^{-1}})^{\#} \neq \operatorname{wind}(\psi^{\#}(\widetilde{\psi^{-1}})^{\#}),$$

FIGURE 6. The graph of $\psi^{\#}$.

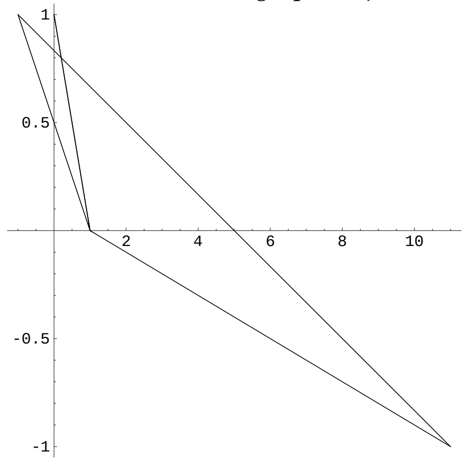


FIGURE 7. The graph of $(\widetilde{\psi^{-1}})^{\#}$.

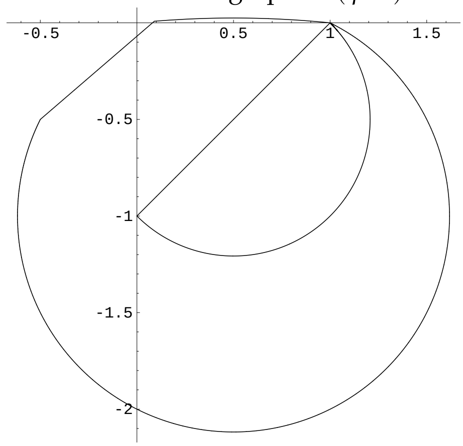


FIGURE 8. The oriented graph of $(\psi\widetilde{\psi^{-1}})^{\#}$.

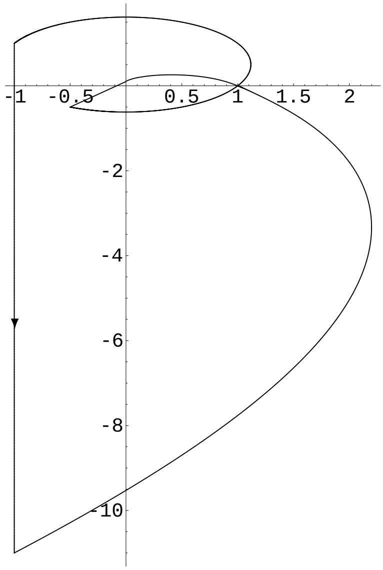
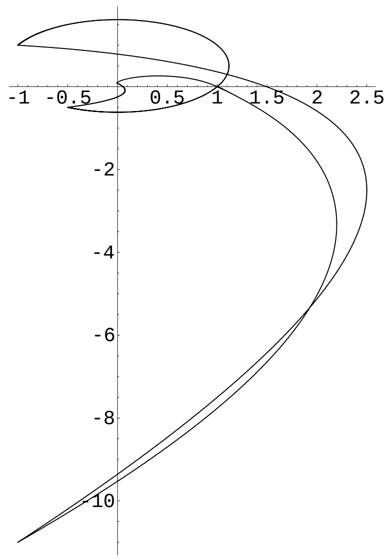


FIGURE 9. The graph of $\psi^{\#}(\widetilde{\psi^{-1}})^{\#}$.



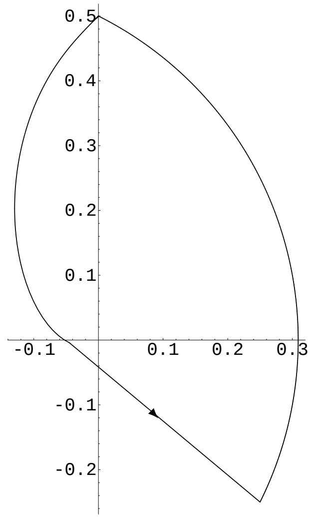
i.e.,

$$\text{wind}(\psi\widetilde{\psi^{-1}})^{\#} \neq \text{wind}\psi^{\#} + \text{wind}(\widetilde{\psi^{-1}})^{\#}.$$

Recalling that $\text{wind}(\widetilde{\psi^{-1}})^{\#} = \text{wind}\psi^{\#}$, it follows that

$$\text{wind}(\psi\widetilde{\psi^{-1}})^{\#} \neq 2\text{wind}\psi^{\#},$$

FIGURE 10. The oriented graph of $\rho_\psi^{\#}$.



which gives us

$$\operatorname{wind}(\psi\widetilde{\psi^{-1}}) \neq 2\operatorname{wind}\psi$$

(by using the definition of winding number of a piecewise continuous function). By the definition of ρ_ψ (see (4.3)), we have

$$\rho_\psi(x) = \begin{cases} \dfrac{\mathrm{i}}{2}\dfrac{x^3 + x^2(1-\mathrm{i})}{x^3 - 10 + \mathrm{i}} & \text{if } x \leqslant -1, \\ \dfrac{\mathrm{i}}{2}\dfrac{x(\mathrm{i}-2)+\mathrm{i}}{x(2-\mathrm{i})+\mathrm{i}} & \text{if } -1 < x \leqslant 0, \\ \dfrac{\mathrm{i}}{2} & \text{if } \quad x > 0. \end{cases}$$

According to Figure 10, we see that $\operatorname{wind}\rho_\psi = 1$. Thus, from (4.13) it follows that $\operatorname{wind}(\psi\widetilde{\psi^{-1}}) = 2$ (which is also confirmed by Figure 8).

We close the paper with a result that provides the invertibility of the Wiener–Hopf–Hankel operators $W_\phi \pm H_\phi$.

COROLLARY 4.7. *If $\phi \in \mathcal{G}PAP$, $\kappa_{\mathrm{l}}(\phi) + \kappa_{\mathrm{r}}(\phi) = 0$,*

$$0 \notin \Big[\frac{\mathbf{d}_{\mathrm{l}}(\phi)}{\mathbf{d}_{\mathrm{r}}(\phi)}, \frac{\mathbf{d}_{\mathrm{r}}(\phi)}{\mathbf{d}_{\mathrm{l}}(\phi)}\Big] \cup \bigcup_{x\in\mathbb{R}} [(\phi\widetilde{\phi^{-1}})(x-0), (\phi\widetilde{\phi^{-1}})(x+0)]$$

and $\operatorname{wind}(\phi\widetilde{\phi^{-1}}) = 0$, *then $W_\phi \pm H_\phi$ are invertible operators.*

Proof. Under the conditions of the statement, we have a scalar Wiener–Hopf operator $W_{\phi\widetilde{\phi^{-1}}}$ with zero Fredholm index. Thus, from the *Coburn–Simonenko Theorem* (see [15] and [30]), we derive that $W_{\phi\widetilde{\phi^{-1}}}$ is invertible. Therefore, Corollary 2.2 ensures that $W_\phi \pm H_\phi$ are invertible operators. ∎

From the last result we conclude now that the operators $W_\phi \pm H_\phi$ in Example 4.5 are invertible Wiener–Hopf–Hankel operators (although W_ϕ is not an invertible operator).

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REFERENCES

[1] H. BART, V.È. TSEKANOVSKIĬ, Matricial coupling and equivalence after extension, *Oper. Theory Adv. Appl.* **59**(1992), 143–160.

[2] E.L. BASOR, T. EHRHARDT, Factorization theory for a class of Toeplitz + Hankel operators, *J. Operator Theory* **51**(2004), 411–433.

[3] A.S. BESICOVITCH, *Almost Periodic Functions*, Cambridge Univ. Press, London 1932.

[4] G. BOGVERADZE, L.P. CASTRO, Wiener–Hopf plus Hankel operators on the real line with unitary and sectorial symbols, *Contemp. Math.* **414**(2006), 77–85.

[5] H. BOHR, *Almost Periodic Functions*, Chelsea Publ. Co., New York 1947.

[6] A. BÖTTCHER, Y.I. KARLOVICH, I.M. SPITKOVSKY, *Convolution Operators and Factorization of Almost Periodic Matrix Functions*, Birkhäuser, Basel 2002.

[7] A. BÖTTCHER, B. SILBERMANN, *Analysis of Toeplitz Operators*, Springer, Berlin 1990.

[8] L.P. CASTRO, Regularity of convolution type operators with PC symbols in Bessel potential spaces over two finite intervals, *Math. Nachr.* **261/262**(2003), 23–36.

[9] L.P. CASTRO, A.S. SILVA, Invertibility of matrix Wiener–Hopf plus Hankel operators with symbols producing a positive numerical range, *Z. Anal. Anwend.* **28**(2009), 119–127.

[10] L.P. CASTRO, F.-O. SPECK, Regularity properties and generalized inverses of delta-related operators, *Z. Anal. Anwend.* **17**(1998), 577–598.

[11] L.P. CASTRO, F.-O. SPECK, Inversion of matrix convolution type operators with symmetry, *Port. Math. (N.S.)* **62**(2005), 193–216.

[12] L.P. CASTRO, F.-O. SPECK, F.S. TEIXEIRA, Explicit solution of a Dirichlet–Neumann wedge diffraction problem with a strip, *J. Integral Equations Appl.* **15**(2003), 359–383.

[13] L.P. CASTRO, F.-O. SPECK, F.S. TEIXEIRA, On a class of wedge diffraction problems posted by Erhard Meister, *Oper. Theory Adv. Appl.* **147**(2004), 211–238.

[14] L.P. CASTRO, F.-O. SPECK, F.S. TEIXEIRA, A direct approach to convolution type operators with symmetry, *Math. Nachr.* **269–270**(2004), 73–85.

[15] L.A. COBURN, Weyl's theorem for nonnormal operators, *Michigan Math. J.* **13**(1966), 285–288.

[16] C. CORDUNEANU, *Almost Periodic Functions*, Chelsea Publ. Co., New York 1989.

[17] T. EHRHARDT, Invertibility theory for Toeplitz plus Hankel operators and singular integral operators with flip, *J. Funct. Anal.* **208**(2004), 64–106.

[18] T. EHRHARDT, *Factorization Theory for Toeplitz plus Hankel Operators and Singular Integral Operators with Flip*, Habilitation Thesis, Technischen Universtität Chemnitz, Chemnitz 2004.

[19] N. KARAPETIANTS, S. SAMKO, *Equations with Involutive Operators*, Birkhäuser, Boston 2001.

[20] V.G. KRAVCHENKO, A.B. LEBRE, G.S. LITVINCHUK, Spectrum problems for singular integral operators with Carleman shift, *Math. Nachr.* **226**(2001), 129–151.

[21] V.G. KRAVCHENKO, G.S. LITVINCHUK, *Introduction to the Theory of Singular Integral Operators with Shift*, Kluwer Academic Publ. Grp., Dordrecht 1994.

[22] N.Y. KRUPNIK, *Banach Algebras with Symbol and Singular Integral Operators*, Birkhäuser Verlag, Basel 1987.

[23] A.B. LEBRE, E. MEISTER, F.S. TEIXEIRA, Some results on the invertibility of Wiener–Hopf–Hankel Operators, *Z. Anal. Anwend.* **11**(1992), 57–76.

[24] E. MEISTER, F.-O. SPECK, F.S. TEIXEIRA, Wiener–Hopf–Hankel operators for some wedge diffraction problems with mixed boundary conditions, *J. Integral Equations Appl.* **4**(1992), 229–255.

[25] A.P. NOLASCO, *Regularity Properties of Wiener–Hopf–Hankel Operators*, Ph.D. Dissertation, Univ. of Aveiro, Aveiro 2007.

[26] A.P. NOLASCO, L.P. CASTRO, Factorization theory for Wiener–Hopf plus Hankel operators with almost periodic symbols, *Contemp. Math.* **414**(2006), 111–128.

[27] A.P. NOLASCO, L.P. CASTRO, A Duduchava–Saginashvili's type theory for Wiener–Hopf plus Hankel operators, *J. Math. Anal. Appl.* **331**(2007), 329–341.

[28] S. ROCH, B. SILBERMANN, Algebras of convolution operators and their image in the Calkin algebra, report MATH (90-05), Akademie der Wissenschaften der DDR, Karl-Weierstrass-Institut für Mathematik, Berlin 1990.

[29] D. SARASON, Toeplitz operators with semi-almost periodic symbols, *Duke Math. J.* **44**(1977), 357–364.

[30] I.B. SIMONENKO, Some general questions of the theory of the Riemann boundary value problem [Russian], *Izv. Akad. Nauk SSSR Ser. Mat.* **32**(1968), 1138–1146; English *Math. USSR Izv.* **2**(1968), 1091–1099.

[31] F.-O. SPECK, *General Wiener–Hopf Factorization Methods*, Pitman, London 1985.

[32] F.S. TEIXEIRA, On a class of Hankel operators: Fredholm properties and invertibility, *Integral Equations Operator Theory* **12**(1989), 592–613.

[33] F.S. TEIXEIRA, Diffraction by a rectangular wedge: Wiener–Hopf–Hankel formulation, *Integral Equations Operator Theory* **14**(1991), 436–454.

[34] J. XIA, Wiener–Hopf operators with piecewise continuous almost periodic symbol, *J. Operator Theory* **14**(1985), 147–171.

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