J. OPERATOR THEORY
62:1(2009), 33–64

© Copyright by THETA, 2009

INDUCED IDEALS AND PURELY INFINITE SIMPLE TOEPLITZ ALGEBRAS

QINGXIANG XU

Communicated by Kenneth R. Davidson

ABSTRACT. Let (G, G_+) be a quasi-lattice ordered group, $\mathbf{\Omega}$ be the collection of hereditary and directed subsets of G_+, and $\mathbf{\Omega}_\infty$ be the collection of the maximal elements of $\mathbf{\Omega}$. For any $H \in \mathbf{\Omega}$, let $S(H)$ be the closed θ-invariant subset of $\mathbf{\Omega}$ generated by H, and denote by $\mathcal{T}^{G_H}$ the associated Toeplitz algebra, where $G_H = G_+ \cdot H^{-1}$. In this paper, the concrete structure of $S(H)$ is clarified. As a result, it is proved that the induced ideals of the Toeplitz algebra $\mathcal{T}^{G_+}$ studied by Laca, Nica et al. can be expressed as the intersections of such kernels as $\mathrm{Ker}\gamma^{G_H,G_+}$ for some $H \in \mathbf{\Omega}$, where γ^{G_H,G_+} is the natural morphism from the Toeplitz algebra $\mathcal{T}^{G_+}$ onto $\mathcal{T}^{G_H}$. A condition is given under which the Toeplitz algebras $\mathcal{T}^{G_H}$ $(H \in \mathbf{\Omega}_\infty)$ become purely infinite simple. When applied to the free groups with finite or countably infinite generators, this gives a unified proof that the simplicity of the Cuntz algebras $\mathcal{O}_n$ $(n \geqslant 2)$, $\mathcal{O}_\infty$ implies the purely infinite simplicity of their tensor products.

KEYWORDS: *Quasi-lattice ordered group, hereditary and directed set, θ-invariant set, α-invariant ideal, induced ideal, Toeplitz algebra, Cuntz algebra, purely infinite simplicity.*

MSC (2000): 47B35.

INTRODUCTION

More than a decade ago, Nica [10] initiated the study of Toeplitz operators or Wiener–Hopf operators on quasi-lattice ordered groups. Since then much progress has been made, and the theory of the Toeplitz algebras associated with quasi-lattice ordered groups has been applied to quite a few fields of modern mathematics. For instance, these Toeplitz algebras serve as typical examples of crossed products of C^*-algebras by semigroups of endomorphisms [7], and of topologically graded C^*-algebras in the context of Hilbert C^*-modules [4].

The purpose of this paper is to give a detailed description of certain aspects of Toeplitz algebras on quasi-lattice ordered groups. In Section 1, we will recall some facts about quasi-lattice ordered groups. Basic examples of quasi-lattice ordered groups are $(\mathbb{Z}^n, \mathbb{Z}^n_+)$ $(n \geqslant 2)$ and the free groups. Apart from these known examples, it might be a rather tough work to find out other kinds of concrete quasi-lattice ordered groups. By choosing certain 2 by 2 upper triangular invertible matrices, we have managed to construct an example of non-commutative quasi-lattice ordered group (see Example 1.4). A special class of quasi-lattice ordered groups are ordered groups. Let (G, G_+) be an ordered group, denote by $\mathcal{T}^{G_+}$ the corresponding Toeplitz algebra. It was proved in [8] that $\mathcal{T}^{G_+}$ contains a smallest ideal if and only if there exists a smallest semigroup of G strictly containing G_+. An ordered group with such a property is also constructed in this section (see Example 1.7).

Let (G, G_+) be a quasi-latticed ordered group, $\mathbf{\Omega}$ be the collection of hereditary and directed subsets of G_+, and $\mathbf{\Omega}_\infty$ be the collection of the maximal elements of $\mathbf{\Omega}$. Given such a pair (G, G_+), two C^*-algebras, namely $\mathcal{T}^{G+}$ and $\mathcal{D}^{G_+}$ can be induced, and it was shown in [6] and [10] that there is a close relationship between the closed θ-invariant subsets of $\mathbf{\Omega}$, α-invariant ideals of $\mathcal{D}^{G_+}$ and the induced ideals of $\mathcal{T}^{G_+}$. In Sections 2 and 3, we focus on the study of the θ-invariant subsets of $\mathbf{\Omega}$. Among other things, we have clarified the detailed structure of the closed θ-invariant subset $S(H)$ generated by an element $H \in \mathbf{\Omega}$ (see Theorem 2.2). As a result, in Section 4 we prove that the induced ideals of the Toeplitz algebra $\mathcal{T}^{G_+}$ studied by Laca, Nica et al. all can be expressed as the intersections of such kernels as $\mathrm{Ker}\gamma^{G_H, G_+}$ for some $H \in \mathbf{\Omega}$, where $G_H = G_+ \cdot H^{-1}$ and γ^{G_H, G_+} is the natural morphism from the Toeplitz algebra $\mathcal{T}^{G_+}$ onto $\mathcal{T}^{G_H}$ (see Corollary 4.6). In Sections 5, 6, we study the maximal ideals and the largest ideals of the Toeplitz algebra $\mathcal{T}^{G_+}$ respectively. Specifically, a condition is given under which the Toeplitz algebras $\mathcal{T}^{G_H}$ $(H \in \mathbf{\Omega}_\infty)$ become purely infinite simple. When applied to the free groups with finite or countably infinite generators, this gives a unified proof that the simplicity of the Cuntz algebras $\mathcal{O}_n (n \geqslant 2)$, $\mathcal{O}_\infty$ implies the purely infinite simplicity of their tensor products.

1. SOME EXAMPLES OF QUASI-LATTICE ORDERED GROUPS AND QUASI-ORDERED GROUPS

The classical Toeplitz algebra $\mathcal{T}$ on the Hardy space $H^2(T)$ can be viewed as the Toeplitz algebra defined on the ordered group $(\mathbb{Z}, \mathbb{Z}_+)$, where $\mathbb{Z}$ is the integer group and $\mathbb{Z}_+$ is the semigroup consisting of non-negative integers. The pair $(\mathbb{Z}, \mathbb{Z}_+)$ has been generalized in a number of different directions, and the most relevant to this paper are quasi-lattice ordered groups and quasi-ordered groups. The relationship between these "ordered groups" can be roughly described with the following diagram:

$$\begin{array}{c} \text{Abelian ordered groups} \to \text{ordered groups} \to \text{quasi-ordered groups} \\ \downarrow \\ \text{quasi-lattice ordered groups.} \end{array}$$

The concept of quasi-lattice ordered group was first introduced by Nica [10] in the study of Wiener–Hopf operators or Toeplitz operators on discrete groups. Let G be a discrete group, G_+ a subset of G such that

$$e \in G_+, G_+ \cdot G_+ \subseteq G_+ \quad \text{and} \quad G_+ \cap G_+^{-1} = \{e\},$$

where e is the unit of G and $G_+^{-1} = \{g^{-1} : g \in G_+\}$. For any $x, y \in G_+$, we define a partial order on G by

$$x \leqslant y \Longleftrightarrow x^{-1}y \in G_+.$$

Note that the order defined above is *left invariant* in the sense that

$$x \leqslant y \Longrightarrow tx \leqslant ty \quad \text{for any } x, y, t \in G.$$

DEFINITION 1.1. The pair (G, G_+) is said to be a *quasi-lattice ordered group* if every finite subset of G with an upper bound in G_+ has a least upper bound in G_+.

Equivalently, (G, G_+) is a quasi-lattice ordered group if and only if every element of G having an upper bound in G_+ has a least such, and every two elements in G_+ having a common upper bound have a least common upper bound ([10], Section 2.1).

If (G, G_+) is a quasi-lattice ordered group and $x_1, x_2, \dots, x_n$ ($n \geqslant 2$) in G have a common upper bound in G_+, then their least common upper bound is denoted by $x_1 \vee x_2 \vee \cdots \vee x_n$. Note for any $x \in G$, x has an upper bound in G_+ if and only if $x \in G_+ \cdot G_+^{-1}$, and when $x \in G_+ \cdot G_+^{-1}$ (so does for x^{-1}), its least upper bound in G_+ will be denoted by $\sigma(x)$. Following the notation as in [10] and [6], we also let $\tau(x)$ be the least upper bound of x^{-1} in G_+; in other words, $\tau(x) = \sigma(x^{-1})$ for $x \in G_+ \cdot G_+^{-1}$. It is easy to check that for any $x \in G_+ \cdot G_+^{-1}$, we have

$$\tau(x) = x^{-1}\sigma(x) \quad \text{and} \quad x = \sigma(x) \cdot \tau(x)^{-1}. \tag{1.1}$$

EXAMPLE 1.2. For any $n \in \mathbb{N}$, $(\mathbb{Z}^n, \mathbb{Z}^n_+)$ is a quasi-lattice ordered group.

EXAMPLE 1.3. Let F_n be the free group with n generators $a_1, a_2, \dots, a_n$, and denote by F_n^+ the semigroup generated by $a_1, a_2, \dots, a_n$, then (F_n, F_n^+) is a non-abelian quasi-lattice ordered group ([10], Section 2.3).

Another non-abelian quasi-lattice ordered group, which serves as a model throughout this paper, will be constructed by choosing certain matrices of order 2 over the real line $\mathbb{R}$, and the details are as follows:

EXAMPLE 1.4. Let $G=\left\{\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22}\end{pmatrix}: a_{11}>0, a_{22}>0, a_{12}\in\mathbb{R}\right\}$ be a class of invertible 2 by 2 upper triangular matrices over the real line $\mathbb{R}$, and put

$$G_+=\left\{\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22}\end{pmatrix}: a_{11}\geqslant 1, a_{22}\geqslant 1, a_{12}\geqslant 0\right\}.$$

Then (G,G_+) is a quasi-lattice ordered group.

Proof. It is obvious that G is a group, $e=\mathrm{diag}(1,1)\in G_+$ and G_+ is a semigroup. For any $x=\begin{pmatrix} x_{11} & x_{12} \\ 0 & x_{22}\end{pmatrix}\in G$, we have $x^{-1}=\begin{pmatrix} \frac{1}{x_{11}} & -\frac{x_{12}}{x_{11}x_{22}} \\ 0 & \frac{1}{x_{22}}\end{pmatrix}$. It follows that $G_+\cap G_+^{-1}=\{e\}$, and for any $x=(x_{ij}), y=(y_{ij})\in G$,

$$x\leqslant y \Longleftrightarrow x^{-1}y\in G_+ \Longleftrightarrow x_{11}\leqslant y_{11}, x_{22}\leqslant y_{22} \quad\text{and}\quad \frac{x_{12}}{x_{22}}\leqslant\frac{y_{12}}{y_{22}}. \tag{1.2}$$

Therefore, every two elements $x=(x_{ij}), y=(y_{ij})$ in G have a least common upper bound $z=(z_{ij})$ in G_+ with

$$z_{11}=\max(1,x_{11},y_{11}),\quad z_{22}=\max(1,x_{22},y_{22}),\quad z_{12}=\max\Big(0, z_{22}\frac{x_{12}}{x_{22}}, z_{22}\frac{y_{12}}{y_{22}}\Big), \tag{1.3}$$

so (G,G_+) is a quasi-lattice ordered group.

Let us consider $E=[1,\infty)\times[1,\infty)\times[0,\infty)$, a subset of Euclidean space $\mathbb{R}^3$, endowed with the usual order, and define $\wedge: G_+\to E$ by

$$\wedge\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22}\end{pmatrix}=\Big(a_{11}, a_{22}, \frac{a_{12}}{a_{22}}\Big). \tag{1.4}$$

Then by (1.2) we know that $\wedge$ is an order-preserving isomorphism of sets in the sense that for any $x,y\in G_+$,

$$\wedge(x)\leqslant\wedge(y)\Longleftrightarrow x\leqslant y \quad\text{and}\quad (\wedge(x))\vee(\wedge(y))=\wedge(x\vee y). \qquad \blacksquare$$

We are somehow surprised to find that when we deal with the same kind of matrices of higher order, the method employed in the above example however fails to work as illustrated down below:

EXAMPLE 1.5. Let $n\in\mathbb{N}, n\geqslant 3$, then $(G_n,(G_n)_+)$ is not a quasi-lattice ordered group, where

$$G_n=\left\{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn}\end{pmatrix}: \begin{array}{ll} a_{ii}>0 & (1\leqslant i\leqslant n), \\ a_{ij}=0 & (i>j), \end{array}\right\},$$

$$(G_n)_+=\left\{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn}\end{pmatrix}: \begin{array}{ll} a_{ii}\geqslant 1 & (1\leqslant i\leqslant n), \\ a_{ij}=0 & (i>j), \\ a_{ij}\geqslant 0 & (i<j), \end{array}\right\}.$$

Proof. Since G_n can be embedded into G_m through $x \to \operatorname{diag}(x, I_{m-n})$ for $n < m$, where I_{m-n} is the identity matrix of order $m-n$ over the real line $\mathbb{R}$, we may as well assume that $n = 3$. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} \in G_3,$$

then

$$A^{-1} = \begin{pmatrix} \frac{1}{a_{11}} & -\frac{a_{12}}{a_{11}a_{22}} & -\frac{a_{13}a_{22}-a_{12}a_{23}}{a_{11}a_{22}a_{33}} \\ 0 & \frac{1}{a_{22}} & -\frac{a_{23}}{a_{22}a_{33}} \\ 0 & 0 & \frac{1}{a_{33}} \end{pmatrix}.$$

So for any $A = (a_{ij}), B = (b_{ij}) \in (G_3)_+$,

$$A \leqslant B \Longleftrightarrow \begin{cases} a_{ii} \leqslant b_{ii}, 1 \leqslant i \leqslant 3; \\ \frac{a_{12}}{a_{22}} \leqslant \frac{b_{12}}{b_{22}}, \frac{a_{23}}{a_{33}} \leqslant \frac{b_{23}}{b_{33}}; \\ \frac{b_{13}}{b_{33}} - \frac{a_{13}}{a_{33}} \geqslant \frac{a_{12}}{a_{22}}\left(\frac{b_{23}}{b_{33}} - \frac{a_{23}}{a_{33}}\right). \end{cases}$$

As in Example 1.4, we can define an isomorphism of sets $\wedge$ from $(G_3)_+$ onto $[1,\infty)^3 \times [0,\infty)^3$ such that

$$\wedge(a_{ij}) = \left(a_{11}, a_{22}, a_{33}, \frac{a_{12}}{a_{22}}, \frac{a_{23}}{a_{33}}, \frac{a_{13}}{a_{33}}\right) \quad \text{for } (a_{ij}) \in (G_3)_+.$$

Thereby a partial order on $[1,\infty)^3 \times [0,\infty)^3$ can be induced as

$$(x_1, x_2, x_3, x_4, x_5, x_6) \leqslant (y_1, y_2, y_3, y_4, y_5, y_6) \Longleftrightarrow \begin{cases} x_i \leqslant y_i, \quad 1 \leqslant i \leqslant 5; \\ y_6 - x_6 \geqslant x_4(y_5 - x_5). \end{cases}$$

Now let $x = (1,1,1,0,1,10), y = (1,1,1,1,0,0) \in [1,\infty)^3 \times [0,\infty)^3$, we prove that x, y have no least common upper bound in $[1,\infty)^3 \times [0,\infty)^3$. Suppose on the contrary that the least common upper bound of x and y exists in $[1,\infty)^3 \times [0,\infty)^3$, let $x \vee y = (u_1, u_2, u_3, u_4, u_5, u_6)$, then

$$u_i \geqslant 1 \quad (1 \leqslant i \leqslant 5), \quad u_6 - 10 \geqslant 0 \cdot (u_5 - 1), \quad u_6 - 0 \geqslant 1 \cdot (u_5 - 0),$$

or

$$u_i \geqslant 1 \quad (1 \leqslant i \leqslant 5), \quad u_6 \geqslant 10, u_6 \geqslant u_5. \tag{1.5}$$

Since $z_1 = (1,1,1,1,1,10)$ is a common upper bound of x and y, we have $z_1 \geqslant x \vee y$, so

$$u_i \leqslant 1 \quad (1 \leqslant i \leqslant 5), \quad 10 - u_6 \geqslant u_4 \cdot (1 - u_5). \tag{1.6}$$

By (1.5) and (1.6), we know that $x \vee y = (1,1,1,1,1,10)$.

On the other hand, let $z_2 = (1,1,1,1,9,10)$, then z_2 is also a common upper bound of x and y, therefore $z_2 \geqslant x \vee y$. It follows that $10 - 10 \geqslant 1 \cdot (9-1)$, which is a contradiction. ∎

DEFINITION 1.6. Let G be a discrete group, E a subset of G, we say that (G,E) is a *quasi-ordered group* if

$$e\in E, E\cdot E\subseteq E, \quad \text{and} \quad G=E\cup E^{-1}.$$

If in addition, $E\cap E^{-1}=\{e\}$, then (G,E) is referred to as an *ordered group*.

By definition, every ordered group is quasi-lattice ordered, and an example of non-abelian ordered group is as follows:

EXAMPLE 1.7. Let G be as in Example 1.4. Set $G_+=G_1\cup G_2\cup G_3$, where

$$G_1=\left\{\begin{pmatrix} a_{11} & a_{12}\\ 0 & a_{22}\end{pmatrix} : a_{11}>1\right\},\ G_2=\left\{\begin{pmatrix} 1 & a_{12}\\ 0 & a_{22}\end{pmatrix} : a_{12}>0\right\},\ G_3=\left\{\begin{pmatrix} 1 & 0\\ 0 & a_{22}\end{pmatrix} : a_{22}\geqslant 1\right\},$$

then (G,G_+) is an ordered group with a property that

$$x^{-1}y\in G_+ \quad \text{for any } x\in G_3, y\in G_1\cup G_2.$$

And for any $x\in G_3, y\in G_3\setminus\{e\}$, it is obvious that there exists $n\in\mathbb{N}$ such that $x^{-1}(y^n)\in G_3$. It follows that for any $x\in G_3$ and any $y\in G_+\setminus\{e\}$, there exists $n\in\mathbb{N}$ such that $x^{-1}(y^n)\in G_+$. So if we let G_y be the semigroup of G generated by G_+ and y^{-1}, then

$$x^{-1}=(x^{-1}y^n)\cdot(y^{-1})^n\subseteq G_+\cdot(y^{-1})^n\subseteq G_y,$$

hence $\quad G_3^{-1}\subseteq \bigcap\limits_{y\in G_+\setminus\{e\}} G_y$.

On the other hand, it is easy to check that $G_1\cup G_2\cup G_3\cup G_3^{-1}$ is a semigroup of G which contains every y^{-1} for $y\in G_3\setminus\{e\}$, which implies that

$$G_y\subseteq G_1\cup G_2\cup G_3\cup G_3^{-1} \quad \text{for } y\in G_3\setminus\{e\}.$$

It follows that for any $x\in G_+$, if $x^{-1}\in \bigcap\limits_{y\in G_+\setminus\{e\}} G_y$, then $x\in G_3$. Therefore,

$$\left\{x\in G_+ : x^{-1}\in \bigcap_{y\in G_+\setminus\{e\}} G_y\right\}=G_3. \tag{1.7}$$

Note that $\bigcap\limits_{y\in G_+\setminus\{e\}} G_y$ is the smallest semigroup strictly containing G_+, and the discussion above shows that this semigroup is equal to $G_+\cup G_3^{-1}$.

2. GENERAL CLOSED θ-INVARIANT SUBSETS OF $\mathbf{\Omega}$

Throughout out this section, (G,G_+) denotes a quasi-lattice ordered group. Given any $x,y\in G_+$, as in Section 1, the notation $x\vee y$ is used for the least common upper bound of x and y in G_+, with the convention that $x\vee y=\infty$ when there is no common upper bound in G_+. A subset H of G_+ is said to be *hereditary* if for any $x,y\in G_+, x\leqslant y\in H$ implies $x\in H$; and H is said to be *directed* if any two elements of H have a common upper bound in H. Let $\mathbf{\Omega}$ be the collection of hereditary and directed subsets of G_+. Then $\mathbf{\Omega}$ is a compact Hausdorff space when endowed with the topology inherited from $\{0,1\}^{G_+}$ in the following way:

Let $\{0,1\}$ be the subset of the real line $\mathbb{R}$ which contains only two elements 0 and 1. Denote by $\{0,1\}^{G_+}$ the product space, that is,

$$\{0,1\}^{G_+} = \{\varphi : G_+ \to \{0,1\} : \forall t \in G_+, \varphi(t) = 0 \text{ or } \varphi(t) = 1\}.$$

Since $\{0,1\}$ is a compact Hausdorff space, by Tychonoff Theorem $\{0,1\}^{G_+}$ is also a compact Hausdorff space when endowed with the product topology: a net $\{\varphi_\alpha\}$ in $\{0,1\}^{G_+}$ converges to φ if and only if $\varphi_\alpha(t) \to \varphi(t)$ for any $t \in G_+$. For any $\varphi \in \{0,1\}^{G_+}$, let $A = \{t \in G_+ : \varphi(t) = 1\}$. Then clearly, $\varphi(t) = \chi_A(t)$ for any $t \in G_+$, where χ_A is the characteristic function of A, so the product space $\{0,1\}^{G_+}$ can be regarded as the collection of the characteristic functions of the subsets of G_+. Therefore an injective morphism ρ of sets from $\mathbf{\Omega}$ into $\{0,1\}^{G_+}$ can be defined by $\rho(H) = \chi_H$ for $H \in \mathbf{\Omega}$. It is easy to show that the image of ρ is closed in $\{0,1\}^{G_+}$, which implies that $\mathbf{\Omega}$ is a compact Hausdorff space when endowed with the topology: a net $\{H_\alpha\}$ in $\mathbf{\Omega}$ converges to $H \in \mathbf{\Omega} \Longleftrightarrow \chi_{H_\alpha}(t) \to \chi_H(t)$ for any $t \in G_+$.

For any $t \in G_+$ and $A \in \mathbf{\Omega}$, we consider the smallest hereditary subset of G_+ containing tA, i.e.,

$$[e, tA] = \{x \in G_+ : \exists a \in A, \text{ such that } x \leqslant ta\}.$$

Note that $[e, tA]$ is also directed, for if $x, y \in [e, tA]$, then there exist $a_1, a_2 \in A$, such that $x \leqslant ta_1$ and $y \leqslant ta_2$. Since A is directed, we know that $a_1 \vee a_2 \in A$, hence $x \vee y \leqslant t(a_1 \vee a_2) \in tA$. Note also $t \in [e, tA]$ since e is contained in A. By Proposition 2.2 of [6] we know that for each $t \in G_+$, $\mathbf{\Omega}_t = \{B \in \mathbf{\Omega} : t \in B\}$ is a clopen subset of $\mathbf{\Omega}$, and the map $\theta_t : \mathbf{\Omega} \to \mathbf{\Omega}_t$ defined by $\theta_t(A) = [e, tA] (A \in \mathbf{\Omega})$ is a homeomorphism from $\mathbf{\Omega}$ onto $\mathbf{\Omega}_t$. Furthermore, we have $\theta_s \circ \theta_t = \theta_{st}$ for any $s, t \in G_+$.

For any $x, t \in G_+$ and $A \in \mathbf{\Omega}$, it is easy to verify that

$$x \in \theta_t(A) \Longleftrightarrow t \vee x \in G_+ \quad \text{and} \quad t^{-1}(t \vee x) \in A, \tag{2.1}$$

and when $A \in \mathbf{\Omega}_t$, $\theta_t^{-1}(A) = \{t^{-1}(t \vee x) : x \in A\}$ has the property that

$$a \in \theta_t^{-1}(A) \Longleftrightarrow ta \in A \quad \text{for any } a \in G_+. \tag{2.2}$$

DEFINITION 2.1. Let $\mathbb{K}$ be a *non-empty* subset of $\mathbf{\Omega}$, $\mathbb{K}$ is said to be θ-*invariant* if for any $t \in G_+$,

$$\theta_t(\mathbb{K}) = \{\theta_t(A) : A \in \mathbb{K}\} \subseteq \mathbb{K} \quad \text{and} \quad \theta_t(\mathbf{\Omega} \setminus \mathbb{K}) = \{\theta_t(B) : B \notin \mathbb{K}\} \subseteq \mathbf{\Omega} \setminus \mathbb{K}.$$

By definition, $\mathbb{K}$ is a θ-invariant subset of $\mathbf{\Omega}$ if and only if for any $t \in G_+$ and $A \in \mathbf{\Omega}$, we have

$$A \in \mathbb{K} \Longleftrightarrow \theta_t(A) \in \mathbb{K}.$$

Clearly, if $\mathbb{K}$ is a θ-invariant subset of $\mathbf{\Omega}$, then the closure of $\mathbb{K}$, $\mathrm{cl}(\mathbb{K})$ is also θ-invariant. Note that the intersection of any family of closed θ-invariant subsets of $\mathbf{\Omega}$ is also θ-invariant. Therefore, for any $\mathbb{K} \subseteq \mathbf{\Omega}$, there exists a smallest closed θ-invariant subset containing it, which is denoted by $S(\mathbb{K})$ and is called the closed θ-invariant subset generated by $\mathbb{K}$. In the peculiar case when $\mathbb{K}$ is a single point set $\{H\}$, we denote $S(\{H\})$ simply by $S(H)$, and its detailed structure is clarified as follows:

THEOREM 2.2. *Let (G, G_+) be a quasi-lattice ordered group, H be a directed and hereditary subset of G_+, and $S(H)$ be the closed θ-invariant subset of $\mathbf{\Omega}$ generated by $\{H\}$. Then $S(H)$ is the closure of $\mathbb{D}$, where*

$$\mathbb{D} = \{\theta_s(\theta_t^{-1}(H)) : s \in G_+, t \in H\}. \tag{2.3}$$

Proof. Let $\mathbb{K}$ be any closed θ-invariant subset of $\mathbf{\Omega}$ which contains H, then clearly $\mathbb{D} \subseteq \mathbb{K}$, therefore $\mathrm{cl}(\mathbb{D}) \subseteq \mathbb{K}$, it follows that $\mathrm{cl}(\mathbb{D}) \subseteq S(H)$. So the conclusion will hold if $H \in \mathbb{D}$ and $\mathrm{cl}(\mathbb{D})$ is θ-invariant.

Step 1. If we set $s = t = e$, then $H = \theta_e(\theta_e^{-1}(H)) \in \mathbb{D}$.

Step 2. If $A \in \mathrm{cl}(\mathbb{D})$, then there exists a net $\{B_\alpha\}$ in $\mathbb{D}$ such that $B_\alpha \to A$. Let $B_\alpha = \theta_{s_\alpha}(\theta_{t_\alpha}^{-1}(H))$ for some $s_\alpha \in G_+$ and $t_\alpha \in H$. So for each $t \in G_+$, we have

$$\theta_t(B_\alpha) = \theta_{ts_\alpha}(\theta_{t_\alpha}^{-1}(H)) \to \theta_t(A),$$

hence $\theta_t(A) \in \mathrm{cl}(\mathbb{D})$.

Step 3. If $A \in \mathbf{\Omega}, t \in G_+$ such that $\theta_t(A) \in \mathrm{cl}(\mathbb{D})$, then there exists a net $\{B_\alpha\}$ in $\mathbb{D}$ which converges to $\theta_t(A)$. Since $t \in \theta_t(A)$, there exists an α_0 such that $t \in B_\alpha$ for any $\alpha \geqslant \alpha_0$. Since $\{B_\alpha\} \subseteq \mathbb{D}$, we may put B_α as follows:

$$B_\alpha = \theta_{s_\alpha}(\theta_{t_\alpha}^{-1}(H)), \quad s_\alpha \in G_+,\ t_\alpha \in H,$$

and since the morphism $\theta_t^{-1} : \mathbf{\Omega}_t \to \mathbf{\Omega}$ is continuous, we have

$$\{\theta_t^{-1}(B_\alpha)\}_{\alpha \geqslant \alpha_0} = \{\theta_t^{-1}(\theta_{s_\alpha}(\theta_{t_\alpha}^{-1}(H)))\}_{\alpha \geqslant \alpha_0} \longrightarrow \theta_t^{-1}(\theta_t(A)) = A. \tag{2.4}$$

Note when $\alpha \geqslant \alpha_0$,

$$t \in \theta_{s_\alpha}(\theta_{t_\alpha}^{-1}(H)) \Rightarrow s_\alpha^{-1}(s_\alpha \vee t) \in \theta_{t_\alpha}^{-1}(H) \Rightarrow t_\alpha(s_\alpha^{-1}(s_\alpha \vee t)) \in H. \tag{2.5}$$

Since both s_α and t belong to $\theta_{s_\alpha}(\theta_{t_\alpha}^{-1}(H))$ for $\alpha \geqslant \alpha_0$, we know that $s_\alpha \vee t \in \theta_{s_\alpha}(\theta_{t_\alpha}^{-1}(H))$ for $\alpha \geqslant \alpha_0$. So when $\alpha \geqslant \alpha_0$, by (2.5) we have

$$\begin{aligned}\theta_t^{-1} \circ \theta_{s_\alpha} \circ \theta_{t_\alpha}^{-1}(H) &= (\theta_t^{-1} \circ \theta_{s_\alpha \vee t}) \circ (\theta_{s_\alpha \vee t}^{-1} \circ \theta_{s_\alpha} \circ \theta_{t_\alpha}^{-1})(H)\\ &= \theta_{t^{-1}(s_\alpha \vee t)} \circ \theta^{-1}_{s_\alpha^{-1}(s_\alpha \vee t)} \circ \theta_{t_\alpha}^{-1}(H) = \theta_{t^{-1}(s_\alpha \vee t)} \circ \theta^{-1}_{t_\alpha s_\alpha^{-1}(s_\alpha \vee t)}(H) \in \mathbb{D}.\end{aligned}$$

By (2.4) we know $A \in \mathrm{cl}(\mathbb{D})$, so $\mathrm{cl}(\mathbb{D})$ is really θ-invariant. ∎

REMARK 2.3. Let (G, G_+) be a quasi-lattice ordered group, $\mathbf{\Omega}$ be the collection of hereditary and directed subsets of G_+. For any $x \in G_+$, let $[e, x] = \{y \in$

$G_+ : y \leqslant x\}$, then the closure of $\{[e,x] : x \in G_+\}$ is equal to $\mathbf{\Omega}$ ([10], Section 6.2) and hence $S([e,x]) = \mathbf{\Omega}$ by Theorem 2.2.

PROPOSITION 2.4. *Let (G_1, G_1^+), (G_2, G_2^+) be two quasi-lattice ordered groups, denote by $\mathbf{\Omega}_1, \mathbf{\Omega}_2$ the collection of hereditary and directed subsets of G_1^+ and G_2^+ respectively. Put $G = G_1 \times G_2$ and $G_+ = G_1^+ \times G_2^+$, then (G, G_+) is a quasi-lattice ordered group with*

$$\mathbf{\Omega} = \mathbf{\Omega}_1 * \mathbf{\Omega}_2 \stackrel{\text{def}}{=} \{H_1 \times H_2 : H_1 \in \mathbf{\Omega}_1, H_2 \in \mathbf{\Omega}_2\},$$

where $\mathbf{\Omega}$ is the collection of hereditary and directed subsets of G_+. Furthermore, for any $H_1 \in \mathbf{\Omega}_1, H_2 \in \mathbf{\Omega}_2$, we have

$$S(H_1 \times H_2) = \{A \times B : A \in S(H_1), B \in S(H_2)\}. \tag{2.6}$$

Proof. Let us first prove that $\mathbf{\Omega} = \mathbf{\Omega_1} * \mathbf{\Omega_2}$. Clearly, $H_1 \times H_2$ is a hereditary subset of G_+ when $H_1 \in \mathbf{\Omega}_1$ and $H_2 \in \mathbf{\Omega}_2$, and for any $(x_1, y_1), (x_2, y_2) \in H_1 \times H_2$, we have $(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2) \in H_1 \times H_2$, therefore $H_1 \times H_2$ is also directed. On the other hand, for any $H \in \mathbf{\Omega}$, let

$$H_1 = \{x \in G_1^+ : \exists y \in G_2^+ \text{ such that } (x,y) \in H\},$$
$$H_2 = \{y \in G_2^+ : \exists x \in G_1^+ \text{ such that } (x,y) \in H\}.$$

Then it is easy to show that $H_1 \in \mathbf{\Omega}_1$, $H_2 \in \mathbf{\Omega}_2$ with $H = H_1 \times H_2$.

Next, we prove that $S(H) = \overline{\{A \times B : A \in S(H_1), B \in S(H_2)\}}$. For any $x \in G_1^+, y \in G_2^+, H_1 \in \mathbf{\Omega_1}, H_2 \in \mathbf{\Omega_2}, s \in H_1$ and $t \in H_2$, we have

$$\theta_{(x,y)}(H_1 \times H_2) = \theta_x(H_1) \times \theta_y(H_2), \quad \theta^{-1}_{(s,t)}(H_1 \times H_2) = \theta_s^{-1}(H_1) \times \theta_t^{-1}(H_2).$$

By Theorem 2.2 we know that

$$\begin{aligned} S(H) &= \overline{\{\theta_{(x,y)} \circ \theta^{-1}_{(s,t)}(H_1 \times H_2) : x \in G_1^+, y \in G_2^+, s \in H_1, t \in H_2\}} \\ &= \overline{\{(\theta_x(\theta_s^{-1}(H_1))) \times (\theta_y(\theta_t^{-1}(H_2))) : x \in G_1^+, y \in G_2^+, s \in H_1, t \in H_2\}} \\ &\subseteq \overline{\{A \times B : A \in S(H_1),\ B \in S(H_2)\}}. \end{aligned}$$

On the other hand, for any $A \in S(H_1), B \in S(H_2)$, there exist $\{x_\alpha\} \subseteq G_1^+, \{s_\alpha\} \subseteq H_1, \{y_\beta\} \subseteq G_2^+, \{t_\beta\} \subseteq H_2$ such that

$$\theta_{x_\alpha} \circ \theta_{s_\alpha}^{-1}(H_1) \to A, \quad \theta_{y_\beta} \circ \theta_{t_\beta}^{-1}(H_2) \to B.$$

It follows that

$$\theta_{(x_\alpha, y_\beta)} \circ \theta^{-1}_{(s_\alpha, t_\beta)}(H_1 \times H_2) = (\theta_{x_\alpha} \circ \theta_{s_\alpha}^{-1}(H_1)) \times (\theta_{y_\beta} \circ \theta_{t_\beta}^{-1}(H_2)) \to A \times B,$$

therefore $A \times B \in \overline{S(H)} = S(H)$, so $\overline{\{A \times B : A \in S(H_1),\ B \in S(H_2)\}} \subseteq S(H)$.

Finally, we prove that $\{A \times B : A \in S(H_1),\ B \in S(H_2)\}$ is closed in $\mathbf{\Omega}$, and thus

$$S(H) = \{A \times B : A \in S(H_1),\ B \in S(H_2)\}.$$

In fact, suppose $E\times F\in\overline{\{A\times B: A\in S(H_1), B\in S(H_2)\}}$ for $E\in\mathbf{\Omega_1}$ and $F\in\mathbf{\Omega_2}$, then there exist $A_\alpha\in S(H_1), B_\alpha\in S(H_2)$ such that the net $\{A_\alpha\times B_\alpha\}$ converges to $E\times F$ in $\mathbf{\Omega}$.

Given any $x_0\in E, y_0\in F$, since $\chi_{A_\alpha\times B_\alpha}(x_0,y_0)\to\chi_{E\times F}(x_0,y_0)=1$, there exists an α_0 such that $(x_0,y_0)\in A_\alpha\times B_\alpha$ for all $\alpha\geqslant\alpha_0$. So when $\alpha\geqslant\alpha_0$, we know for any $x\in G_1^+$,

$$\chi_{A_\alpha}(x)=\chi_{A_\alpha\times B_\alpha}(x,y_0)\to\chi_{E\times F}(x,y_0)=\chi_E(x),$$

hence $E\in\overline{S(H_1)}=S(H_1)$. Similarly, $F\in S(H_2)$. Since E and F are arbitrary, this completes the proof. ∎

EXAMPLE 2.5. Let $G=\mathbb{R}, G_+=\mathbb{R}_+=[0,\infty)$, then

(i) $\mathbf{\Omega}=\{[0,x]: x\in\mathbb{R}_+\}\cup\{[0,x): x\in(0,\infty)\}\cup\{G_+\}$;

(ii) $\forall x\in\mathbb{R}_+, S([0,x])=\mathbf{\Omega}$;

(iii) $\forall x\in(0,\infty), S([0,x))=\mathbf{\Omega}$;

(iv) $S(G_+)=\{G_+\}$.

Proof. (i) For any $H\in\mathbf{\Omega}$, let $\beta=\sup\{x: x\in H\}$. Then

$$H=\begin{cases}[0,\beta] & \text{if } \beta<\infty \text{ and } \beta\in H,\\ [0,\beta) & \text{if } \beta<\infty \text{ and } \beta\notin H,\\ G_+ & \text{if } \beta=\infty.\end{cases}$$

Therefore, $\mathbf{\Omega}=\{[0,x]: x\in\mathbb{R}_+\}\cup\{[0,x): x\in(0,\infty)\}\cup\{G_+\}$.

(ii) By Remark 2.3, we know that $S([0,x])=\mathbf{\Omega}$ for any $x\in\mathbb{R}_+$.

(iii) For $s\in(0,\infty)$ and $t\in\mathbb{R}_+$, it is easy to verify that

$$\theta_t([0,s))=[0,s+t),\quad \theta_t^{-1}([0,s))=[0,s-t)\quad (t<s).$$

Therefore $[0,y)=\theta_x^{-1}\circ\theta_y([0,x))$ for $x>0$ and $y>0$, hence

$$S([0,x))=\overline{\{[0,y): y>0\}}.$$

Now for any $a>0$, since the characteristic functions $\{\chi_{[0,a+\frac{1}{n})}\}_{n=1}^\infty$ converge pointwisely to $\chi_{[0,a]}$ on $[0,\infty)$, we know that $[0,a]\in\overline{\{[0,y): y>0\}}=S([0,x))$, it follows that $S([0,x))\supseteq\overline{\{[0,a]: a>0\}}=\mathbf{\Omega}$.

(iv) Since every single-point set is closed in any Hausdorff space, $\{G_+\}$ is a closed subset of $\mathbf{\Omega}$, which is also θ-invariant, and thus $S(G_+)=\{G_+\}$. ∎

EXAMPLE 2.6. Let $G=\mathbb{Z}^2$ and $G_+=\mathbb{Z}_+^2$. For any $m,n\in\mathbb{Z}_+$, put

$$H_{m,n}=\{(s,t)\in\mathbb{Z}_+^2: s\leqslant m, t\leqslant n\},\qquad H_{m,\infty}=\{(s,t)\in\mathbb{Z}_+^2: s\leqslant m\},$$
$$H_{\infty,n}=\{(s,t)\in\mathbb{Z}_+^2: t\leqslant n\},\qquad H_{\infty,\infty}=\mathbb{Z}_+^2.$$

Then the following conclusion holds:

(i) $\mathbf{\Omega}=\{H_{m,n}, H_{m,\infty}, H_{\infty,n}, H_{\infty,\infty}: m,n\in\mathbb{Z}_+\}$;

(ii) $\forall m,n\in\mathbb{N}, S(H_{m,n})=\mathbf{\Omega}$;

(iii) $\forall n\in\mathbb{N}, S(H_{n,\infty})=\{\mathbb{Z}_+^2\}\cup\{H_{m,\infty}: m\in\mathbb{Z}_+\}$;

(iv) $\forall n \in \mathbb{N}$, $S(H_{\infty,n}) = \{\mathbb{Z}_+^2\} \cup \{H_{\infty,m} : m \in \mathbb{Z}_+\}$;
(v) $S(\{\mathbb{Z}_+^2\}) = \{\mathbb{Z}_+^2\}$.

Proof. Let $G_1 = G_2 = \mathbb{Z}$, $G_1^+ = G_2^+ = \mathbb{Z}_+$ and $\mathbf{\Omega}_i$ be the collection of hereditary and directed subsets of $(G_i)_+$ for $i = 1, 2$. Obviously, $\mathbf{\Omega_1} = \mathbf{\Omega_2} = \{[0, n] : n \in \mathbb{Z}_+\} \cup \{\mathbb{Z}_+\}$, and $S(\{\mathbb{Z}_+\}) = \{\mathbb{Z}_+\}$, $S([0, n]) = \mathbf{\Omega_1} = \mathbf{\Omega_2}$ for any $n \in \mathbb{Z}_+$. The conclusion then follows from Proposition 2.4. ∎

EXAMPLE 2.7. Let (G, G_+) be as in Example 1.4. As shown in Section 1, there is an order-preserving isomorphism of sets $\wedge : G_+ \to E = [1, \infty) \times [1, \infty) \times [0, \infty)$, which satisfies

$$\wedge \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} = \Big(a_{11}, a_{22}, \frac{a_{12}}{a_{22}}\Big).$$

Let $\mathbf{\Omega}'$ be $\wedge(\mathbf{\Omega}) = \{\wedge(H) : H \in \mathbf{\Omega}\}$, and for the sake of convenience, the elements of $\mathbf{\Omega}'$ are also called hereditary and directed subsets of $E = [1, \infty) \times [1, \infty) \times [0, \infty)$. For any $H \in \mathbf{\Omega}'$, let

$$\begin{aligned}
\alpha &= \sup\{x \in [1, \infty) : \exists y \in [1, \infty), z \in [0, \infty), \text{ such that } (x, y, z) \in H\}, \\
\beta &= \sup\{y \in [1, \infty) : \exists x \in [1, \infty), z \in [0, \infty), \text{ such that } (x, y, z) \in H\}, \\
\gamma &= \sup\{z \in [0, \infty) : \exists x \in [1, \infty), y \in [1, \infty), \text{ such that } (x, y, z) \in H\}.
\end{aligned}$$

Then $H = I_\alpha \times I_\beta \times J_\gamma$, where

$$I_\alpha = \begin{cases} [1, \alpha] & \text{if } \alpha < \infty \text{ and } \exists y \in [1, \infty), z \in [0, \infty) \text{ with } (\alpha, y, z) \in H; \\ [1, \alpha) & \text{if } \alpha < \infty, \text{but } (\alpha, y, z) \notin H \text{ for any } y \in [1, \infty), z \in [0, \infty); \\ [1, \infty) & \text{if } \alpha = \infty. \end{cases}$$

Similarly, define I_β and J_γ. For simplicity, let us put:

$$\begin{aligned}
I_\alpha^{(1)} &= [1, \alpha] \quad (1 \leqslant \alpha < \infty), & I_\alpha^{(2)} &= [1, \alpha) \quad (1 < \alpha < \infty), \\
J_\alpha^{(1)} &= [0, \alpha] \quad (0 \leqslant \alpha < \infty), & J_\alpha^{(2)} &= [0, \alpha) \quad (0 < \alpha < \infty), \\
I_\infty &= I_\infty^{(1)} = I_\infty^{(2)} = [1, \infty), & J_\infty &= J_\infty^{(1)} = J_\infty^{(2)} = [0, \infty).
\end{aligned}$$

Then

$$\begin{aligned}
\mathbf{\Omega}' &= \{I_\alpha \times I_\beta \times J_\gamma : 1 \leqslant \alpha, \beta \leqslant \infty, 0 \leqslant \gamma \leqslant \infty\} \\
&= \{I_\alpha^{(i)} \times I_\beta^{(j)} \times J_\gamma^{(k)} : 1 \leqslant \alpha, \beta \leqslant \infty, 0 \leqslant \gamma \leqslant \infty,\ i, j, k \in \{1, 2\}\}.
\end{aligned}$$

A topology on $\mathbf{\Omega}'$ can be induced by the bijection $\wedge$ defined by (1.4) so that $\mathbf{\Omega}'$ becomes a compact Hausdorff space, and a net $\{A_\alpha\}$ in $\mathbf{\Omega}'$ converges to A if and only if $\chi_{A_\alpha}(x) \to \chi_A(x)$ for any $x \in [1, \infty) \times [1, \infty) \times [0, \infty)$. As shown in Example 2.5, it is easy to verify that for any $i, j, k \in \{1, 2\}$,

(1) $\overline{\{I_\alpha^{(i)} \times I_\beta^{(j)} \times J_\gamma^{(k)} : \alpha, \beta, \gamma < \infty\}} = \mathbf{\Omega}'$;

(2) $\overline{\{I_\infty \times I_\beta^{(i)} \times J_\gamma^{(j)} : \beta, \gamma < \infty\}} = \{I_\infty \times I_\beta \times J_\gamma : \beta, \gamma \leqslant \infty\}$;

(3) $\overline{\{I_\alpha^{(i)}\times I_\infty\times J_\gamma^{(j)}:\alpha,\gamma<\infty\}}=\{I_\alpha\times I_\infty\times J_\gamma:\alpha,\gamma\leqslant\infty\}$;
(4) $\overline{\{I_\alpha^{(i)}\times I_\beta^{(j)}\times J_\infty:\alpha,\beta<\infty\}}=\{I_\alpha\times I_\beta\times J_\infty:\alpha,\beta\leqslant\infty\}$;
(5) $\overline{\{I_\infty\times I_\infty\times J_\gamma^{(i)}:\gamma<\infty\}}=\{I_\infty\times I_\infty\times J_\gamma:\gamma\leqslant\infty\}$;
(6) $\overline{\{I_\infty\times I_\beta^{(i)}\times J_\infty:\beta<\infty\}}=\{I_\infty\times I_\beta\times J_\infty:\beta\leqslant\infty\}$;
(7) $\overline{\{I_\alpha^{(i)}\times I_\infty\times J_\infty:\alpha<\infty\}}=\{I_\alpha\times I_\infty\times J_\infty:\alpha\leqslant\infty\}$;
(8) $\overline{\{I_\infty\times I_\infty\times J_\infty\}}=\{I_\infty\times I_\infty\times J_\infty\}$.

Let $a,b,c\in\mathbb{R}$ with $1\leqslant a, 1\leqslant b$ and $0<c$, put

$$H=\wedge^{-1}([1,a]\times[1,b]\times[0,c))=\Big\{(x_{ij})\in G_+g: x_{11}\leqslant a, x_{22}\leqslant b, \frac{x_{12}}{x_{22}}<c\Big\}.$$

Take the above H for example, let us study the detailed structure of $S(H)$.

Step 1. For any $x=(x_{ij})\in H$, we prove that

$$\theta_x^{-1}(H)=\wedge^{-1}\Big(\Big[1,\frac{a}{x_{11}}\Big]\times\Big[1,\frac{b}{x_{22}}\Big]\times\Big[0,\frac{x_{22}c-x_{12}}{x_{11}}\Big)\Big). \tag{2.7}$$

In fact, for any $y=(y_{ij})\in H$, we have $x^{-1}(x\vee y)=(z_{ij})$, where

$$z_{11}=\frac{x_{11}\vee y_{11}}{x_{11}},\quad z_{22}=\frac{x_{22}\vee y_{22}}{x_{22}},\quad z_{12}=\frac{x_{22}\vee y_{22}}{x_{11}}\Big(\frac{x_{12}}{x_{22}}\vee\frac{y_{12}}{y_{22}}\Big)-\frac{(x_{22}\vee y_{22})x_{12}}{x_{11}x_{22}}.$$

Since $x_{11}\vee y_{11}\leqslant a, x_{22}\vee y_{22}\leqslant b$ and $\frac{x_{12}}{x_{22}}\vee\frac{y_{12}}{y_{22}}<c$, we know that

$$z_{11}\leqslant\frac{a}{x_{11}},\quad z_{22}\leqslant\frac{b}{x_{22}},\quad \frac{z_{12}}{z_{22}}<\frac{x_{22}c-x_{12}}{x_{11}},$$

so

$$\theta_x^{-1}(H)\subseteq\wedge^{-1}\Big(\Big[1,\frac{a}{x_{11}}\Big]\times\Big[1,\frac{b}{x_{22}}\Big]\times[0,\frac{x_{22}c-x_{12}}{x_{11}})\Big).$$

On the other hand, for any r with $0<r<\frac{x_{22}c-x_{12}}{x_{11}}$, choose a natural number n_0 such that

$$c-\frac{1}{n_0}>\frac{x_{12}}{x_{22}},\quad \frac{x_{22}(c-\frac{1}{n_0})-x_{12}}{x_{11}}>r.$$

Let $y=(y_{ij})\in H$ with $y_{11}=a, y_{22}=b, y_{12}=\big(c-\frac{1}{n_0}\big)b$, denote $x^{-1}(x\vee y)=(z_{ij})$, then

$$z_{11}=\frac{a}{x_{11}},\quad z_{22}=\frac{b}{x_{22}},\quad z_{12}=\frac{(c-\frac{1}{n_0})b}{x_{11}}-\frac{x_{12}b}{x_{11}x_{22}}.$$

Therefore $\frac{z_{12}}{z_{22}}=\frac{x_{22}\left(c-\frac{1}{n_0}\right)-x_{12}}{x_{11}}>r$.

Now for any $t=(t_{ij})\in\wedge^{-1}g\big(\big[1,\frac{a}{x_{11}}\big]\times\big[1,\frac{b}{x_{22}}\big]\times\big[0,\frac{x_{22}c-x_{12}}{x_{11}}\big)g\big)$, let $r=\frac{t_{12}}{t_{22}}$, then by the above discussion, we know there exists $z=(z_{ij})\in\theta_x^{-1}(H)$ such that

$$z_{11}=\frac{a}{x_{11}},\quad z_{22}=\frac{b}{x_{22}},\quad \frac{z_{12}}{z_{22}}>r,$$

so $t \leqslant z \in \theta_x^{-1}(H)$, and thus $t \in \theta_x^{-1}(H)$. By the arbitrariness of t, we know that

$$\wedge^{-1}\Big(\Big[1, \frac{a}{x_{11}}\Big] \times \Big[1, \frac{b}{x_{22}}\Big] \times \Big[0, \frac{x_{22}c - x_{12}}{x_{11}}\Big)\Big) \subseteq \theta_x^{-1}(H).$$

Step 2. Similarly, we can prove that for any $x = (x_{ij}) \in G_+$,

$$\theta_x(H) = \wedge^{-1}\Big([1, ax_{11}] \times [1, bx_{22}] \times \Big[0, \frac{x_{11}c + x_{12}}{x_{22}}\Big)\Big). \tag{2.8}$$

Step 3. We prove that for any $a, b, c \in \mathbb{R}$ with $1 \leqslant a, 1 \leqslant b$ and $0 < c$, there exist $x, y \in G_+$ such that

$$\wedge^{-1}([1,a] \times [1,b] \times [0,c)) = \theta_y \circ \theta_x^{-1}(\wedge^{-1}(\{1\} \times \{1\} \times [0,1))). \tag{2.9}$$

This can be verified by choosing a positive number s with $0 < s < 1, 1 - s$ is small enough so that $bc - a(1-s) > 0$, and putting

$$x = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} a & bc - a(1-s) \\ 0 & b \end{pmatrix}.$$

Step 4. Let $H_0 = \wedge^{-1}(\{1\} \times \{1\} \times [0,1))$, then

$$\{\wedge^{-1}([1,a] \times [1,b] \times [0,c)) : a, b, c < \infty\} = \{\theta_y \circ \theta_x^{-1}(H_0) : y \in G_+, x \in H_0\}.$$

Therefore for any $H = \wedge^{-1}([1,a] \times [1,b] \times [0,c)) \in \mathbf{\Omega}$,

$$\begin{aligned} S(H) = S(H_0) &= \overline{\{\wedge^{-1}([1,r] \times [1,s] \times [0,t)) : r,s,t < \infty\}} \\ &= \wedge^{-1}(\overline{\{[1,r] \times [1,s] \times [0,t) : r,s,t < \infty\}}) = \wedge^{-1}(\mathbf{\Omega}') = \mathbf{\Omega}. \end{aligned}$$

REMARK 2.8. Let (G, G_+) be as in Example 1.4. For any $b, c \in \mathbb{R}$ with $1 < b, 0 < c$, set $H_1 = \wedge^{-1}([1,\infty) \times [1,b) \times [0,c)) \in \mathbf{\Omega}$, then it can be proved that

$$\{\wedge^{-1}([1,\infty) \times [1,s) \times [0,t)) : s,t < \infty\} = \{\theta_y \circ \theta_x^{-1}(H_1) : y \in G_+, x \in H_1\}.$$

It follows that

$$\begin{aligned} S(H_1) &= \overline{\{\wedge^{-1}([1,\infty) \times [1,s) \times [0,t)) : s,t < \infty\}} \\ &= \wedge^{-1}(\overline{\{[1,\infty) \times [1,s) \times [0,t) : s,t < \infty\}}) \\ &= \wedge^{-1}(\{I_\infty \times I_\beta \times J_\gamma : \beta, \gamma \leqslant \infty\}). \end{aligned}$$

3. THE LARGEST CLOSED θ-INVARIANT PROPER SUBSET OF $\mathbf{\Omega}$

As before, throughout this section, (G, G_+) is a quasi-lattice ordered group, and $\mathbf{\Omega}$ is the collection of hereditary and directed subsets of G_+. Let $\mathbb{F}$ be a closed θ-invariant *proper* subset of $\mathbf{\Omega}$, we say that $\mathbb{F}$ is the largest if $\mathbb{K} \subseteq \mathbb{F}$ holds for any other closed θ-invariant proper subset $\mathbb{K}$ of $\mathbf{\Omega}$. In this section, we will investigate conditions under which $S(H) \neq \mathbf{\Omega}$ for $H \in \mathbf{\Omega}$. In the peculiar case when (G, G_+) is an ordered group, we will prove that in $\mathbf{\Omega}$ exists a largest closed θ-invariant proper subset if and only if there exists a smallest semigroup of G strictly containing G_+ (see Theorem 3.6).

LEMMA 3.1. *Let (G, G_+) be a quasi-lattice ordered group, and $\mathbf{\Omega}$ be the collection of hereditary and directed subsets of G_+. Let*

$$M(\mathbf{\Omega}) = \{H \in \mathbf{\Omega} : S(H) \neq \mathbf{\Omega}\}, \tag{3.1}$$

then $\mathbf{\Omega}$ contains a largest closed θ-invariant proper subset if and only if $\emptyset \neq M(\mathbf{\Omega}) \subseteq \overline{M(\mathbf{\Omega})} \neq \mathbf{\Omega}$. And if this happens, $M(\mathbf{\Omega})$ is the largest closed θ-invariant proper subset of $\mathbf{\Omega}$.

Proof. For a proof, see Proposition 5.8 of [8]. ∎

PROPOSITION 3.2. *Let (G, G_+) be a quasi-lattice ordered group, and $\mathbf{\Omega}$ be the collection of hereditary and directed subsets of G_+ and $M(\mathbf{\Omega})$ be defined by (3.1). Then for any $H \in \mathbf{\Omega}$, the following conditions are equivalent:*

(i) *$\exists F = \{a_1, a_2, \ldots, a_n\} \subseteq G_+ \setminus \{e\}$, such that for any $s \in G_+, t \in H$, there exists $a_{i_0} \in F$ which is contained in $\theta_s \circ \theta_t^{-1}(H)$;*

(ii) *$[e, e] = \{e\} \notin S(H)$;*

(iii) *$S(H) \neq \mathbf{\Omega}$, i.e., $H \in M(\mathbf{\Omega})$.*

Proof. Since $S(\{e\}) = S([e, e]) = \mathbf{\Omega}$, conditions (ii) and (iii) are equivalent. For any $A \in \mathbf{\Omega}$ and any non-empty finite subset F of G_+, let

$$N(A; F) = \{B \in \mathbf{\Omega} : \chi_B(x) = \chi_A(x), \forall x \in F\} = \{B \in \mathbf{\Omega} : B \cap F = A \cap F\}. \tag{3.2}$$

Then $\{N(A; F) : \emptyset \neq F \subseteq G_+, F \text{ is finite}\}$ is a local base at A.

Now for any $H \in \mathbf{\Omega}$, since $S(H) = \overline{\{\theta_s \circ \theta_t^{-1}(H) : s \in G_+, t \in H\}}$, we know $[e, e] \notin S(H) \iff \exists F = \{a_1, a_2, \ldots, a_n\} \subseteq G_+ \setminus \{e\}$, such that $\forall s \in G_+, \forall t \in H$,

$$N([e, e]; F) \cap \{\theta_s \circ \theta_t^{-1}(H) : s \in G_+, t \in H\} = \emptyset,$$

or equivalently, there exists $a_{i_0} \in F$ which is contained in $\theta_s \circ \theta_t^{-1}(H)$, this completes the proof of the equivalence of conditions (i) and (ii). ∎

When applied to ordered groups, Proposition 3.2 has a much simpler version, which can be stated as follows:

COROLLARY 3.3. *Let (G, G_+) be an ordered group, $\mathbf{\Omega}$ be the collection of hereditary subsets of G_+, then for any $H \in \mathbf{\Omega}$, $S(H) \neq \mathbf{\Omega}$ if and only if there exists some $a \in G_+ \setminus \{e\}$, such that:*

(i) *for any $t \in H$, $ta \in H$;*

(ii) *for any $s \in G_+$ with $s^{-1}a \in G_+$, $s^{-1}a \in H$.*

Proof. Since (G, G_+) is an ordered group, we know for any $x \in G_+$ and $y \in H$, either $xy^{-1} \in G_+$ or $yx^{-1} \in G_+$, which implies

$$\theta_x \circ \theta_y^{-1}(H) = \begin{cases} \theta_{xy^{-1}}(H) & \text{if } xy^{-1} \in G_+, \\ \theta_{yx^{-1}}^{-1}(H) & \text{if } yx^{-1} \in G_+. \end{cases}$$

Therefore, $S(H) = \overline{\{\theta_s(H), \theta_t^{-1}(H) : s \in G_+, t \in H\}}$. By Proposition 3.2 we know $S(H) \neq \mathbf{\Omega} \iff [e,e] \notin S(H) \iff \exists F = \{a_1, a_2, \ldots, a_n\} \subseteq G_+ \setminus \{e\}$, such that

$$N([e,e];F) \cap \{\theta_s(H), \theta_t^{-1}(H) : s \in G_+, t \in H\} = \emptyset. \tag{3.3}$$

Let a be the smallest element of F (that is, $a \in F$ and $a \leqslant a_i, \forall a_i \in F$), then condition (3.3) can be simplified as

$$\forall s \in G_+, \forall t \in H \Longrightarrow a \in \theta_s(H) \text{ and } a \in \theta_t^{-1}(H). \tag{3.4}$$

Note that for any $s \in G_+$, we have

$$s^{-1}(a \vee s) = \begin{cases} s^{-1}a & \text{if } s \leqslant a, \\ e & \text{if } a \leqslant s. \end{cases}$$

It follows that condition (3.4) is satisfied if and only if the asserted conditions (i) and (ii) hold. ∎

PROPOSITION 3.4. *Let (G, G_+) be an ordered group, $\mathbf{\Omega}$ be the collection of hereditary subsets of G_+. For any $H \in M(\mathbf{\Omega})$, let*

$$E = \{a \in G_+ : \forall t \in H \Longrightarrow ta \in H; \forall s \in G_+, s^{-1}a \in G_+ \Longrightarrow s^{-1}a \in H\}. \tag{3.5}$$

Then $E \in \mathbf{\Omega}$ and $G_E \stackrel{\text{def}}{=} G_+ \cdot E^{-1}$ is a semigroup of G with

$$G_E^0 \stackrel{\text{def}}{=} G_E \cap G_E^{-1} = E \cup E^{-1} \neq \{e\}.$$

Proof. Clearly, $e \in E \subseteq H$ (since $e \in H$), and by Corollary 3.3 we know that the set E defined as (3.5) is not equal to $\{e\}$.

Step 1. Suppose that $e \leqslant x \leqslant y \in E$. Then for any $t \in H$, we have $e \leqslant tx \leqslant ty \in H$, therefore $tx \in H$ since H is hereditary; $\forall s \in G_+$, if $s^{-1}x \in G_+$, then $s^{-1}y = (s^{-1}x)(x^{-1}y) \in G_+ \cdot G_+ = G_+$, hence $s^{-1}y \in H$, which implies that $s^{-1}x \in H$ since $e \leqslant s^{-1}x \leqslant s^{-1}y \in H$ and H is hereditary, this completes the proof of $x \in E$. Obviously, E is also directed since $G = G_+ \cup G_+^{-1}$, therefore $E \in \mathbf{\Omega}$.

Step 2. Let $x, y \in E$, then for any $t \in H$, we have $t(xy) = (tx)y \in Hy \subseteq H$; let $s \in G_+$ such that $s^{-1}(xy) \in G_+$, we prove $s^{-1}(xy) \in H$ so that E will be a semigroup of G_+.

Case 1. $s^{-1}x \in G_+$. In this case, $s^{-1}x \in H$ since $x \in E$, so $s^{-1}(xy) = (s^{-1}x)y \in Hy \subseteq H$.

Case 2. $s^{-1}x \in G_+^{-1}$. In this case, $s^{-1}x = g^{-1}$ for some $g \in G_+$. Then $g^{-1}y = s^{-1}(xy) \in G_+ \Longrightarrow g^{-1}y \in H$, i.e., $s^{-1}(xy) \in H$.

Step 3. For any $x, y \in E$, we prove that $x^{-1}y \in E \cup E^{-1}$. Without loss of generality, we may suppose that $x \leqslant y$. Let $s \in G_+$ with $s^{-1}(x^{-1}y) \in G_+$, then $(xs)^{-1}y \in G_+ \Longrightarrow s^{-1}(x^{-1}y) = (xs)^{-1}y \in H$; let $t \in H$, we will prove that $t(x^{-1}y) \in H$, therefore $x^{-1}y \in E$.

Case 1. $tx^{-1} \in G_+^{-1}$. In this case, $tx^{-1} = g^{-1}$ for some $g \in G_+$, hence $t(x^{-1}y) = g^{-1}y \in G_+ \Longrightarrow g^{-1}y \in H$, i.e., $t(x^{-1}y) \in H$.

Case 2. $tx^{-1} \in G_+$. Let $s = tx^{-1} \in G_+$, then $e \leqslant s \leqslant t \in H \Longrightarrow s \in H \Longrightarrow sy \in H$, i.e., $t(x^{-1}y) \in H$.

Step 4. We prove $(xy^{-1})(st^{-1}) \in G_+ \cdot E^{-1}$ for any $x, s \in G_+$ and $y, t \in E$, therefore $G_+ \cdot E^{-1}$ is a semigroup of G.

Case 1. $y^{-1}s \in G_+$. In this case, $(xy^{-1})(st^{-1}) = x(y^{-1}s)t^{-1} \in G_+ \cdot G_+ \cdot E^{-1} = G_+ \cdot E^{-1}$.

Case 2. $y^{-1}s \in G_+^{-1}$. In this case, $e \leqslant s \leqslant y \in E \Longrightarrow s \in E \Longrightarrow s^{-1}y \in E$. Therefore, $(xy^{-1})(st^{-1}) = x(t(s^{-1}y))^{-1} \in G_+ \cdot (E \cdot E)^{-1} = G_+ \cdot E^{-1}$.

Step 5. Let $G_E = G_+ \cdot E^{-1}$ and denote by $G_E^0 = G_E \cap G_E^{-1}$. Then G_E^0 is a subgroup of G which contains both E and E^{-1}. On the other hand, given any $x \in G_E^0$, if $x \in G_+$, then $x^{-1} = st^{-1}$ for some $s \in G_+$ and $t \in E$, hence $e \leqslant x \leqslant t \in E \Longrightarrow x \in E$; otherwise, $x \in G_+^{-1} \Longrightarrow x^{-1} \in G_+ \cap G_E^0 \Longrightarrow x^{-1} \in E \Longrightarrow x \in E^{-1}$. So in any case, we have $x \in E \cup E^{-1}$, this completes the proof of $G_E^0 = E \cup E^{-1}$. ∎

DEFINITION 3.5. Let (G, G_+) be an ordered group. For any $g \in G_+ \setminus \{e\}$, let G_g be the semigroup of G generated by G_+ and g^{-1}. Denote by

$$F(G) = \bigcap_{g \in G_+ \setminus \{e\}} G_g, \quad F(G_+) = \{t \in G_+ : t^{-1} \in F(G)\}. \tag{3.6}$$

As in the abelian case [9], elements of $F(G)$ are called *finite elements of* G.

The semigroup $F(G)$ defined as above may equal G_+; in the case when $F(G) \neq G_+$, $F(G)$ will be the smallest semigroup of G strictly containing G_+. Note also $F(G_+)$ is hereditary so that $F(G_+) \in \mathbf{\Omega}$. For, if $e \leqslant x \leqslant y \in F(G_+)$, then for any $g \in G_+$, we have

$$x^{-1} = (x^{-1}y)y^{-1} \in G_+ \cdot G_g = G_g. \tag{3.7}$$

THEOREM 3.6. *Let (G, G_+) be an ordered group, $\mathbf{\Omega}$ be the collection of hereditary subsets of G_+. Then $\mathbf{\Omega}$ contains a largest closed θ-invariant proper subset if and only if there exists a smallest semigroup of G strictly containing G_+; if and only if $F(G_+) \neq \{e\}$. And if this happens, $S(F(G_+))$ is the largest closed θ-invariant proper subset of $\mathbf{\Omega}$.*

Proof. Step 1. Suppose $F(G_+) \neq \{e\}$, we prove that $S(F(G_+))$ is the largest closed θ-invariant proper subset of $\mathbf{\Omega}$.

Step 1.1. Choose any $a \in F(G_+) \setminus \{e\}$, since $F(G_+)$ is a semigroup of G_+, we know

$$ta \in F(G_+) \cdot F(G_+) = F(G_+) \quad \text{for any } t \in F(G_+);$$

let $s \in G_+$ and suppose $s^{-1}a \in G_+$, then for any $g \in G_+ \setminus \{e\}$,

$$(s^{-1}a)^{-1} = a^{-1}s \in G_g \cdot s \subseteq G_g \cdot G_+ = G_g, \quad \text{so} \quad s^{-1}a \in F(G_+).$$

By Corollary 3.3 we know $S(F(G_+)) \neq \mathbf{\Omega}$.

Step 1.2. For any $H \in M(\mathbf{\Omega})$, define the set E as (3.5). Then $G_E = G_+ \cdot E^{-1}$ is a semigroup of G which strictly contains G_+, so $F(G_+)^{-1} \subseteq G_E$. It follows that for any $a \in F(G_+) \subseteq G_+$, $a \in G_E^0 \cap G_+ = E$, hence

$$\forall t \in H, \forall a \in F(G_+) \Longrightarrow ta \in H. \tag{3.8}$$

Let us prove that $H \in S(F(G_+))$ so that $S(F(G_+))$ will be the largest closed θ-invariant proper subset of $\mathbf{\Omega}$. It suffices to prove that for any open neighborhood U of H,

$$U \cap \{\theta_t(F(G_+)), \theta_s^{-1}(F(G_+)) : t \in G_+, s \in F(G_+)\} \neq \emptyset.$$

Since (G, G_+) is an ordered group, we might as well consider the following three cases:

Case 1. $U = N(H; \{x, y\}) = \{B \in \mathbf{\Omega} : x \in B, y \notin B\}$ for some $x, y \in G_+$ with $x \in H$ and $y \notin H$. In this case, $x < y$ (otherwise, $y < x \in H \Longrightarrow y \in H$, a contradiction). Clearly $x \in \theta_x(F(G_+))$ and $y \notin \theta_x(F(G_+))$. For, if $y \in \theta_x(F(G_+))$, then $x^{-1}y = x^{-1}(x \vee y) \in F(G_+)$. So by (3.8) we know $y = x(x^{-1}y) \in H \cdot F(G_+) \subseteq H$, which is a contradiction. Therefore,

$$\theta_x(F(G_+)) \in U \cap \{\theta_t(F(G_+)), \theta_s^{-1}(F(G_+)) : t \in G_+, s \in F(G_+)\}.$$

Case 2. $U = N(H; \{x\}) = \{B \in \mathbf{\Omega} : x \in B\}$ for some $x \in H$. In this case,

$$\theta_x(F(G_+)) \in U \cap \{\theta_t(F(G_+)), \theta_s^{-1}(F(G_+)) : t \in G_+, s \in F(G_+)\}.$$

Case 3. $U = N(H; \{y\}) = \{B \in \mathbf{\Omega} : y \notin B\}$ for some $y \in G_+ \setminus H$. By (3.8) we know $F(G_+) \subseteq H$, hence $y \notin F(G_+)$, so

$$F(G_+) \in U \cap \{\theta_t(F(G_+)), \theta_s^{-1}(F(G_+)) : t \in G_+, s \in F(G_+)\}.$$

Step 2. Suppose that $\mathbf{\Omega}$ contains a largest closed θ-invariant proper subset, then by Lemma 3.1 we know this largest closed θ-invariant proper subset of $\mathbf{\Omega}$ is equal to $M(\mathbf{\Omega})$. In particular, $M(\mathbf{\Omega})$ is compact since it is closed in the compact Hausdorff space $\mathbf{\Omega}$. As before, for any $g \in G_+ \setminus \{e\}$, let G_g be the semigroup of G generated by G_+ and g^{-1}, and put $H_g = \{t \in G_+ : t^{-1} \in G_g\}$. Then $H_g \in \mathbf{\Omega}$ demonstrated as (3.7). Obviously, $tg \in H_g$ for any $t \in H_g$; and if $s^{-1}g \in G_+$ for $s \in G_+$, then $(s^{-1})^{-1} = g^{-1}s \in G_g \cdot G_+ = G_g$, so $s^{-1}g \in H_g$. It follows by Corollary 3.3 that $H_g \in M(\mathbf{\Omega})$. Let

$$\mathbb{D} = \{\mathbb{A} \subseteq M(\mathbf{\Omega}) : \exists t \in G_+ \setminus \{e\}, \text{such that } \forall s, e < s \leqslant t \Longrightarrow H_s \in \mathbb{A}\}.$$

Then it is easy to verify that $\mathbb{D}$ is a filter on $M(\mathbf{\Omega})$. Since $M(\mathbf{\Omega})$ is compact, $\mathbb{D}$ has a cluster point H in $M(\mathbf{\Omega})$ ([5], Theorem 3.11). So for any open neighborhood U of H, any $\mathbb{A} \in \mathbb{D}$, the intersection of U and $\mathbb{A}$ is always non-empty. By Corollary 3.3, there exists an $a \in H \setminus \{e\}$, such that $\forall t \in H$ implies $ta \in H$.

Now for any $g \in G_+ \setminus \{e\}$, let

$$U = N(H; \{a\}) = \{B \in M(\mathbf{\Omega}) : a \in B\}, \quad \mathbb{A} = \{H_s : e < s \leqslant g\},$$

then there exists an s with $e < s \leqslant g$ such that $a \in H_s$. Note that $s^{-1} = (s^{-1})g^{-1} \in G_g$, so $H_s \subseteq H_g$ and hence $a \in H_g$, or equivalently $a^{-1} \in G_g$. By the arbitrariness of g, we know $a \in F(G_+)$ and thus $F(G_+) \neq \{e\}$. ∎

REMARK 3.7. (i) Let (G, G_+) be an ordered group. If G is abelian, then for any $g \in G_+$, $G_g = G_+ - \mathbb{Z}_+ g = \{t - ng : t \in G_+, n \in \mathbb{Z}_+\}$. So in this case,

$$F(G_+) = \{x \in G_+ : \forall y \in G_+ \setminus \{0\}, \exists n_0 \in \mathbb{N}, \text{ such that } x \leqslant n_0 y\}.$$

And the reader is referred to Example 1.7 for a non-abelian ordered group which satisfies the condition stated in Theorem 3.6.

(ii) Let (G, G_+) be a quasi-lattice ordered group, and $\mathbf{\Omega}$ be the collection of hereditary and directed subsets of G_+. An element A of $\mathbf{\Omega}$ is said to be *maximal* if $A \subseteq B \in \mathbf{\Omega}$ implies $A = B$. Let $\mathbf{\Omega}_\infty$ be the collection of the maximal elements of $\mathbf{\Omega}$. It was proved in [6] that for any $A \in \mathbf{\Omega}$, there exists some $B \in \mathbf{\Omega}_\infty$ such that $A \subseteq B$, and the closure of $\mathbf{\Omega}_\infty$ is the smallest closed θ-invariant subset of $\mathbf{\Omega}$. It follows that $S(H) = \overline{\mathbf{\Omega}}_\infty$ for any $H \in \mathbf{\Omega}_\infty$.

4. THE INDUCED IDEALS OF THE TOEPLITZ ALGEBRAS

Let (G, G_+) be a quasi-lattice ordered group, H be a hereditary and directed subset of G_+. Denote by $\mathcal{T}^{G_+}$ and $\mathcal{T}^{G_H}$ the two associated Toeplitz algebras (for the definitions, see below), where $G_H = G_+ \cdot H^{-1}$. Then by Theorem 2.12 of [8] we know there exists a natural C^*-morphism γ^{G_H, G_+} from $\mathcal{T}^{G_+}$ onto $\mathcal{T}^{G_H}$. In this section, we will show that the induced ideals of $\mathcal{T}^{G_+}$ studied by Laca and Nica et al. all come from the intersection of such kernels as Ker γ^{G_H, G_+}. As an application, conditions under which $\mathcal{T}^{G_+}$ becomes simple are given.

Let us first recall some definitions about Toeplitz algebras on discrete groups. Let G be a discrete group and $\{\delta_g : g \in G\}$ be the usual orthonormal basis for $\ell^2(G)$. For any $g \in G$, a unitary operator u_g on $\ell^2(G)$ is defined by $u_g(\delta_h) = \delta_{gh}$ for $h \in G$. For any subset E of G, let $\ell^2(E)$ be the closed subspace of $\ell^2(G)$ generated by $\{\delta_g : g \in E\}$; the projection from $\ell^2(G)$ onto $\ell^2(E)$ is denoted by p^E. The C^*-algebra generated by $\{T_g^E \stackrel{\text{def}}{=} p^E u_g p^E : g \in G\}$ is denoted by $\mathcal{T}^E$ and is called the *Toeplitz algebra* with respect to E.

By definition we know $(T_g^E)^* = T_{g^{-1}}^E$ for any $g \in G$, and for any $T \in \mathcal{T}^E$, let $\theta^E(T)$ be the associated diagonal operator acting on $\ell^2(G)$ which is defined by

$$\theta^E(T)\delta_g = \begin{cases} \langle T\delta_g \ \delta_g \rangle \delta_g & \text{if } g \in E, \\ 0 & \text{if } g \notin E. \end{cases}$$

By definition, we know for any $g, h \in G$,

$$\theta^E(T_g^E T_{h^{-1}}^E) = \begin{cases} T_g^E T_{g^{-1}}^E & \text{if } g = h, \\ 0 & \text{otherwise.} \end{cases} \tag{4.1}$$

The properties of $\mathcal{T}^E$ are generally closely related to the underlying pair (G, E). When (G, G_+) is a quasi-lattice ordered group, with the convention $x^{-1}\infty = \infty$ ($x \in G_+$) and $T_\infty^{G_+} = 0$, we know from [10] that the following proposition holds:

PROPOSITION 4.1. *Let (G, G_+) be a quasi-lattice ordered group. Then:*

(i) *For any $x \in G$,* $T_x^{G_+} = \begin{cases} T_{\sigma(x)}^{G_+} T_{\tau(x)^{-1}}^{G_+} & \text{if } x \in G_+ \cdot G_+^{-1}, \\ 0 & \text{otherwise.} \end{cases}$

(ii) *For any $x, y \in G_+$,* $(T_x^{G_+} T_{x^{-1}}^{G_+}) \cdot (T_y^{G_+} T_{y^{-1}}^{G_+}) = T_{x\vee y}^{G_+} \quad T_{(x\vee y)^{-1}}^{G_+}$.

(iii) *For any $x, y \in G_+$,* $T_{x^{-1}}^{G_+} T_y^{G_+} = T_{x^{-1}(x\vee y)}^{G_+} \; T_{(y^{-1}(x\vee y))^{-1}}^{G_+}$.

Let

$$\mathcal{T}^\infty(G_+) = \operatorname{span}\{T_g^{G_+} T_{h^{-1}}^{G_+} : g, h \in G_+\}, \quad \mathcal{D}^{G_+} = \operatorname{closp}\{T_g^{G_+} T_{g^{-1}}^{G_+} : g \in G_+\},$$

then $\mathcal{T}^\infty(G_+)$ is a dense $*$-subalgebra of $\mathcal{T}^{G_+}$, whereas $\mathcal{D}^{G_+}$ is a commutative C^*-subalgebra of $\mathcal{T}^{G_+}$, and θ^{G_+} is a faithful positive compress linear operator from $\mathcal{T}^{G_+}$ onto $\mathcal{D}^{G_+}$ (in other words, θ^{G_+} is a faithful conditional expectation). Since $\mathcal{T}^\infty(G_+)$ is dense in $\mathcal{T}^{G_+}$ and θ^{G_+} is continuous, by (4.1) we know the following proposition holds:

PROPOSITION 4.2 (cf. Lemma 4.1 of [6]). *Let (G, G_+) be a quasi-lattice ordered group. Then for any $g \in G_+$, $X \in \mathcal{T}^{G_+}$ and $Y \in \mathcal{D}^{G_+}$,*

(i) $\theta^{G_+}(T_g^{G_+} X T_{g^{-1}}^{G_+}) = T_g^{G_+} \theta^{G_+}(X) T_{g^{-1}}^{G_+}$;

(ii) $\theta^{G_+}(T_{g^{-1}}^{G_+} X T_g^{G_+}) = T_{g^{-1}}^{G_+} \theta^{G_+}(X) T_g^{G_+}$;

(iii) $\theta^{G_+}(XY) = \theta^{G_+}(X)Y$, $\theta^{G_+}(YX) = Y\theta^{G_+}(X)$.

Throughout the rest of this section, (G, G_+) is a quasi-lattice ordered group, and $\mathbf{\Omega}$ is the collection of hereditary and directed subsets of G_+. By an ideal, we always mean it is closed, two-sided, and proper. There is a correspondence between the induced ideals of $\mathcal{T}^{G_+}$, α-invariant ideals of $\mathcal{D}^{G_+}$ and the closed θ-invariant subsets of $\mathbf{\Omega}$ described as follows:

Step 1. For any $s \in G_+$, α_s and $\alpha_{s^{-1}}$ are two C^*-automorphisms of $\mathcal{D}^{G_+}$ defined by

$$\alpha_s(X) = T_s^{G_+} X T_{s^{-1}}^{G_+}, \quad \alpha_{s^{-1}}(X) = T_{s^{-1}}^{G_+} X T_s^{G_+} \quad \text{for } X \in \mathcal{D}^{G_+}. \tag{4.2}$$

An ideal $\mathcal{I}$ of $\mathcal{D}^{G_+}$ is said to be *α-invariant* if $\alpha_s(\mathcal{I}) \subseteq \mathcal{I}$ and $\alpha_{s^{-1}}(\mathcal{I}) \subseteq \mathcal{I}$ for every $s \in G_+$. Clearly any ideal $\mathcal{J}$ of $\mathcal{T}^{G_+}$ can induce an α-invariant ideal $\mathcal{I}$ of $\mathcal{D}^{G_+}$ by simply letting $\mathcal{I} = \mathcal{J} \cap \mathcal{D}^{G_+}$, and it is remarkable that $\mathcal{J} \cap \mathcal{D}^{G_+}$ is nonzero provided that $\mathcal{J}$ is nonzero (for the detail, see [7] or [8]). On the other hand, any α-invariant ideal $\mathcal{I}$ of $\mathcal{D}^{G_+}$ can induce an *induced ideal* Ind $\mathcal{I}$ of $\mathcal{T}^{G_+}$ which is defined as

$$\mathrm{Ind}\mathcal{I} = \{T \in \mathcal{T}^{G_+} : \theta^{G_+}(T^*T) \in \mathcal{I}\} \tag{4.3}$$

with the property that (Ind $\mathcal{I}) \cap \mathcal{D}^{G_+} = \mathcal{I}$. It follows that $\mathcal{T}^{G_+}$ is simple if and only if there is no nonzero α-invariant ideal of $\mathcal{D}^{G_+}$.

Step 2. Let $\widehat{\mathcal{D}^{G_+}}$ be the maximal ideal space of $\mathcal{D}^{G_+}$, and Γ be the Gelfand transform from $\mathcal{D}^{G_+}$ onto $C(\widehat{\mathcal{D}^{G_+}})$. For any $t \in G_+$, since $T_t^{G_+} T_{t^{-1}}^{G_+}$ is a projection,

$\gamma(T_t^{G_+}T_{t^{-1}}^{G_+}) \in \{0,1\}$ for any $\gamma \in \widehat{\mathcal{D}^{G_+}}$. Let

$$A_\gamma = \{t \in G_+ : \gamma(T_t^{G_+}T_{t^{-1}}^{G_+}) = 1\} \quad \text{for } \gamma \in \widehat{\mathcal{D}^{G_+}}.$$

Then $A_\gamma \in \mathbf{\Omega}$, and Nica showed in [10] (also demonstrated in the proof of Theorem 4.5 down below) that $\rho : \gamma \to A_\gamma$ is a homeomorphism from the compact Hausdorff space $\widehat{\mathcal{D}^{G_+}}$ onto $\mathbf{\Omega}$, and thus induces an isomorphism $\rho^* : C(\widehat{\mathcal{D}^{G_+}}) \to C(\mathbf{\Omega})$ defined as

$$\rho^*(f) = f \circ \rho^{-1} \quad \text{for } f \in C(\widehat{\mathcal{D}^{G_+}}).$$

Therefore, $\rho^* \circ \Gamma$ realizes an isomorphism from $\mathcal{D}^{G_+}$ onto $C(\mathbf{\Omega})$ with the property that

$$(\rho^* \circ \Gamma(T_t^{G_+}T_{t^{-1}}^{G_+}))(A) = \chi_A(t) \quad \text{for any } t \in G_+ \text{ and } A \in \mathbf{\Omega}. \tag{4.4}$$

Accordingly given any ideal $\mathcal{I}$ of $\mathcal{D}^{G_+}$, since $(\rho^* \circ \Gamma)(\mathcal{I})$ is an ideal of $C(\mathbf{\Omega})$, there exists uniquely a nonempty closed subset $\mathbb{K}$ of $\mathbf{\Omega}$ such that

$$\mathcal{I} = \mathcal{I}_{\mathbb{K}} \overset{\text{def}}{=} \{T \in \mathcal{D}^{G_+} : (\rho^* \circ \Gamma(T))(A) = 0 \text{ for any } A \in \mathbb{K}\}. \tag{4.5}$$

Laca showed that $\mathcal{I}_{\mathbb{K}}$ is α-invariant if and only if $\mathbb{K}$ is θ-invariant ([6], Proposition 3.2).

Now for any closed θ-invariant subset $\mathbb{K}$ of $\mathbf{\Omega}$, let

$$\mathcal{J}_{\mathbb{K}} \overset{\text{def}}{=} \text{Ind}\mathcal{I}_{\mathbb{K}} = \{x \in \mathcal{T}^{G_+} : \theta^{G_+}(x^*x) \in \mathcal{I}_{\mathbb{K}}\}. \tag{4.6}$$

Let $\mathbf{\Omega}_\infty$ be the collection of the maximal elements of $\mathbf{\Omega}$ and $\text{cl}(\mathbf{\Omega}_\infty)$ be the closure of $\mathbf{\Omega}_\infty$. Since $\text{cl}(\mathbf{\Omega}_\infty)$ is the smallest closed θ-invariant subset of $\mathbf{\Omega}$, we know $\mathcal{I}_{\text{cl}(\mathbf{\Omega}_\infty)}$ is the largest α-invariant ideal of $\mathcal{D}^{G_+}$, while $\mathcal{J}_{\text{cl}(\mathbf{\Omega}_\infty)}$ is the largest induced ideal of $\mathcal{T}^{G_+}$ (the reader should be aware that $\mathcal{I}_{\text{cl}(\mathbf{\Omega}_\infty)} = 0$ if $\text{cl}(\mathbf{\Omega}_\infty) = \mathbf{\Omega}$).

Now given $H \in \mathbf{\Omega}$, let $G_H = G_+ \cdot H^{-1}$. By Theorem 2.12 of [8] we know there exists a natural C^*-morphism γ^{G_H,G_+} from $\mathcal{T}^{G_+}$ onto $\mathcal{T}^{G_H}$ such that

$$\gamma^{G_H,G_+}(T_g^{G_+}) = T_g^{G_H} \quad \text{for any } g \in G.$$

It follows that

$$\mathcal{T}^{G_H} = \gamma^{G_H,G_+}(\mathcal{T}^{G_+}) = \text{closp}\{T_g^{G_H}T_{h^{-1}}^{G_H} : g,h \in G_+\}$$

and $\mathcal{D}^{G_H} = \text{closp}\{T_g^{G_H}T_{g^{-1}}^{G_H} : g \in G_+\}$ is a commutative C^*-subalgebra of $\mathcal{T}^{G_H}$ with a faithful conditional expectation $\theta^{G_H} : \mathcal{T}^{G_H} \to \mathcal{D}^{G_H}$ satisfying

$$\gamma^{G_H,G_+} \circ \theta^{G_+}(T) = \theta^{G_H} \circ \gamma^{G_H,G_+}(T) \quad \text{for any } T \in \mathcal{T}^{G_+}. \tag{4.7}$$

Upon replacing G_+ by G_H, the same propositions as Proposition 4.1 and Proposition 4.2 also hold.

PROPOSITION 4.3. *Let (G, G_+) be a quasi-lattice ordered group, H be a hereditary and directed subset of G_+ and $G_H = G_+ \cdot H^{-1}$. Then*

$$\text{Ind}((\text{Ker}\gamma^{G_H,G_+}) \cap \mathcal{D}^{G_+}) = \text{Ker}\gamma^{G_H,G_+}.$$

Proof. Since θ^{G_H} is faithful, we know for any $T \in \mathcal{T}^{G_+}$,

$$T \in \mathrm{Ind}((\mathrm{Ker}\gamma^{G_H,G_+}) \cap \mathcal{D}^{G_+}) \Longleftrightarrow \theta^{G_+}(T^*T) \in \mathrm{Ker}\gamma^{G_H,G_+}$$
$$\Longleftrightarrow \theta^{G_H}(\gamma^{G_H,G_+}(T^*T)) = 0 \Longleftrightarrow \gamma^{G_H,G_+}(T^*T) = 0$$
$$\Longleftrightarrow T^*T \in \mathrm{Ker}\gamma^{G_H,G_+} \Longleftrightarrow T \in \mathrm{Ker}\gamma^{G_H,G_+}. \quad \blacksquare$$

The following proposition was established in [8]. For a proof, see Lemma 2.8 and Lemma 2.9 of [8].

PROPOSITION 4.4. *Let (G, G_+) be a quasi-lattice ordered group, H be a hereditary and directed subset of G_+ and $G_H = G_+ \cdot H^{-1}$. Then for any $x, y \in G_+$ and $g \in G$,*
(i) *$g \in G_H \Longleftrightarrow g \in G_+ \cdot G_+^{-1}$ with $\tau(g) \in H$;*
(ii) *$x^{-1}y \in G_+ \cdot G_+^{-1} \Longleftrightarrow x \vee y \neq \infty$, and if this happens,*

$$\sigma(x^{-1}y) = x^{-1}(x \vee y) \quad \textit{and} \quad \tau(x^{-1}y) = y^{-1}(x \vee y).$$

The main result of this section is as follows:

THEOREM 4.5. *Let (G, G_+) be a quasi-lattice ordered group, $\mathbf{\Omega}$ be the collection of hereditary and directed subsets of G_+. Let $H \in \mathbf{\Omega}$, and denote by $S(H)$ the closed θ-invariant subset of $\mathbf{\Omega}$ generated by $\{H\}$. Then*

$$\mathcal{I}_{S(H)} = (\mathrm{Ker}\gamma^{G_H,G_+}) \cap \mathcal{D}^{G_+} \quad \textit{and} \quad \mathcal{J}_{S(H)} = \mathrm{Ker}\gamma^{G_H,G_+}. \tag{4.8}$$

Proof. Step 1. For any $\gamma \in \widehat{\mathcal{D}^{G_H}}$, as in Section 6.2 of [10] we define

$$A_\gamma = \{t \in G_+ : \gamma(T_t^{G_H} T_{t^{-1}}^{G_H}) = 1\}.$$

It is easy to check that $A_\gamma \in \mathbf{\Omega}$. For any $s \in G_+, t \in H$, let $\gamma_{s,t}$ be in $\widehat{\mathcal{D}^{G_H}}$ defined as

$$\gamma_{s,t}(\cdot) = \langle \cdot \delta_{st^{-1}}, \delta_{st^{-1}} \rangle.$$

We prove that

$$A_{\gamma_{s,t}(\cdot)} = \theta_s \circ \theta_t^{-1}(H). \tag{4.9}$$

To this end, let us first prove that for such $t \in H$,

$$G_+ \cdot H^{-1}t = G_+ \cdot (\theta_t^{-1}(H))^{-1}. \tag{4.10}$$

For one direction, $xh^{-1}t = (x \cdot (h^{-1}(h \vee t)))(t^{-1}(h \vee t))^{-1} \in G_+ \cdot (\theta_t^{-1}(H))^{-1}$ whenever $x \in G_+$ and $h \in H$. By (2.2) we know $ty \in H$ for $y \in \theta_t^{-1}(H)$. The reverse direction then follows as $xy^{-1} = x(ty)^{-1}t \in G_+ \cdot H^{-1}t$ for such y and $x \in G_+$. Hence for any $g \in G_+$,

$$g \in A_{\gamma_{s,t}(\cdot)} \Longleftrightarrow \gamma_{s,t}(T_g^{G_H} T_{g^{-1}}^{G_H}) = 1 \Longleftrightarrow g^{-1}st^{-1} \in G_+ \cdot H^{-1}$$
$$\Longleftrightarrow g^{-1}s \in G_+ \cdot H^{-1}t = G_+ \cdot (\theta_t^{-1}(H))^{-1} \Longleftrightarrow \tau(g^{-1}s) \in \theta_t^{-1}(H)$$
$$\Longleftrightarrow s^{-1}(g \vee s) \in \theta_t^{-1}(H) \Longleftrightarrow g \in \theta_s(\theta_t^{-1}(H)).$$

Step 2. Let ρ_H be a morphism from $\widehat{\mathcal{D}^{G_H}}$ to $\mathbf{\Omega}$ defined by $\rho_H(\gamma) = A_\gamma$ for $\gamma \in \widehat{\mathcal{D}^{G_H}}$. Then ρ_H is continuous and one-to-one, so the compactness of $\widehat{\mathcal{D}^{G_H}}$ implies that $\rho_H(\widehat{\mathcal{D}^{G_H}})$ is compact (and hence closed) in $\mathbf{\Omega}$. It follows that ρ_H is a homeomorphism from $\widehat{\mathcal{D}^{G_H}}$ onto $\rho_H(\widehat{\mathcal{D}^{G_H}})$.

Next we prove that the collection of all such $A_{\gamma_{s,t}}$ is dense in $\rho_H(\widehat{\mathcal{D}^{G_H}})$, so that

$$\rho_H(\widehat{\mathcal{D}^{G_H}}) = \overline{\{A_{\gamma_{s,t}} : s \in G_+, t \in H\}} = \overline{\{\theta_s \circ \theta_t^{-1}(H) : s \in G_+, t \in H\}} = S(H).$$

Toward this end, it suffices to prove that for any $\gamma \in \widehat{\mathcal{D}^{G_H}}$, any finite nonempty subset F of G_+, there exist $s \in G_+$ and $t \in H$ such that

$$x \in A\gamma \Longleftrightarrow x \in \theta_s(\theta_t^{-1}(H)) \quad \text{for any } x \in F. \tag{4.11}$$

Case 1. $F_1 \overset{\text{def}}{=} F \cap A_\gamma$ and $F_2 \overset{\text{def}}{=} F \cap (G_+ \setminus A_\gamma)$ both are non-empty. Let $F_1 = \{s_1, s_2, \ldots, s_n\}$ and $F_2 = \{t_1, t_2, \ldots, t_m\}$. Since A_γ is directed, the least common upper bound of s_i, $\sigma(s_1, s_2, \ldots, s_n) \overset{\text{def}}{=} s_0$ belongs to A_γ. Note if $s \in G_+$ such that $s \geqslant s_0$, then for any s_i in F_1, $s_i \leqslant s_0 \leqslant s \in \theta_s(A)$ for every $A \in \mathbf{\Omega}$, accordingly $F_1 \subseteq \theta_s(\theta_t^{-1}(H))$ for all such s and any $t \in H$. So it reduces to prove there exist $s \geqslant s_0$ and $t \in H$ such that $F_2 \subseteq G_+ \setminus \theta_s(\theta_t^{-1}(H))$, or equivalently, $T_{t_j}^{G_H} T_{t_j^{-1}}^{G_H} \delta_{st^{-1}} = 0$ for any $t_j \in F_2$ (see (4.9)). Suppose on the contrary that

$$\prod_{j=1}^{m} (1 - T_{t_j}^{G_H} T_{t_j^{-1}}^{G_H}) \delta_{st^{-1}} = 0 \quad \text{for any } t \in H, s \in G_+ \text{ with } s \geqslant s_0, \tag{4.12}$$

then we show

$$\Big(\prod_{j=1}^{m} (1 - T_{t_j}^{G_H} T_{t_j^{-1}}^{G_H}) \Big) \cdot T_{s_0}^{G_H} T_{s_0^{-1}}^{G_H} = 0. \tag{4.13}$$

In fact, for any $x \in G_+$ and $y \in H$, if $T_{s_0}^{G_H} T_{s_0^{-1}}^{G_H} \delta_{xy^{-1}} \neq 0$, then we will prove that $T_{s_0}^{G_H} T_{s_0^{-1}}^{G_H} \delta_{xy^{-1}} = \delta_{st^{-1}}$ for some $t \in H$ and $s \in G_+$ with $s \geqslant s_0$, the conclusion then follows by (4.12). The proof of the asserted property can be demonstrated as follows:

$$\begin{aligned}
& T_{s_0}^{G_H} T_{s_0^{-1}}^{G_H} \delta_{xy^{-1}} \neq 0 \Longleftrightarrow \langle T_{s_0}^{G_H} T_{s_0^{-1}}^{G_H} \delta_{xy^{-1}}, \, \delta_{xy^{-1}} \rangle = 1 \\
& \Longleftrightarrow s_0 \in \theta_x(\theta_y^{-1}(H)) \Longleftrightarrow x^{-1}(x \vee s_0) \in \theta_y^{-1}(H) \\
& \Longleftrightarrow yx^{-1}(x \vee s_0) \in H \Longleftrightarrow xy^{-1} = (x \vee s_0) t^{-1} \quad \text{for some } t \in H.
\end{aligned}$$

By the definition of A_γ, we know $\prod\limits_{j=1}^{m} (1 - \gamma(T_{t_j}^{G_H} T_{t_j^{-1}}^{G_H})) \cdot \gamma(T_{s_0}^{G_H} T_{s_0^{-1}}^{G_H}) = 1$. But by (4.13) we also have $\prod\limits_{j=1}^{m} (1 - \gamma(T_{t_j}^{G_H} T_{t_j^{-1}}^{G_H})) \cdot \gamma(T_{s_0}^{G_H} T_{s_0^{-1}}^{G_H}) = 0$, a contradiction.

Case 2. $F = F_1, F_2 = \emptyset$. Let s_0 be as in Case 1, and put $s = s_0, t = e$, then (4.11) holds.

Case 3. $F = F_2, F_1 = \emptyset$. In this case, $\gamma(T_{t_j}^{G_H} T_{t_j^{-1}}^{G_H}) = 0$ for all $t_j \in F$ ($1 \leqslant j \leqslant m$). It follows that $\prod_{j=1}^{m}(1 - \gamma(T_{t_j}^{G_H} T_{t_j^{-1}}^{G_H})) = 1$, and hence $\prod_{j=1}^{m}(1 - T_{t_j}^{G_H} T_{t_j^{-1}}^{G_H}) \neq 0$, so there exist some $s \in G_+$ and $t \in H$, such that

$$\prod_{j=1}^{m}(1 - T_{t_j}^{G_H} T_{t_j^{-1}}^{G_H})\delta_{st^{-1}} \neq 0.$$

It follows that $t_j \notin \theta_s \circ \theta_t^{-1}(H)$ for $1 \leqslant j \leqslant m$.

Step 3. As before, let $\rho_H^* : C(\widehat{\mathcal{D}^{G_H}}) \to C(S(H))$ be the induced morphism defined by

$$\rho_H^*(f) = f \circ \rho_H^{-1} \quad \text{for } f \in C(\widehat{\mathcal{D}^{G_H}}).$$

Also, let Γ be the Gelfand transformation from $\mathcal{D}^{G_H}$ onto $C(\widehat{\mathcal{D}^{G_H}})$. Then it is easy to verify that for any $A \in S(H)$ and $t \in G_+$,

$$(\rho_H^* \circ \Gamma(T_t^{G_H} T_{t^{-1}}^{G_H}))(A) = \chi_A(t). \tag{4.14}$$

Since the linear span of $\{T_t^{G_+} T_{t^{-1}}^{G_+} : g \in G_+\}$ is dense in $\mathcal{D}^{G_+}$, by (4.4) and (4.14) we know for any $T \in \mathcal{D}^{G_+}$ and $A \in S(H)$,

$$(\rho^* \circ \Gamma(T))(A) = (\rho_H^* \circ \Gamma(\gamma^{G_H, G_+}(T)))(A). \tag{4.15}$$

Thus we have the following C^*-algebras, all of which are isomorphic:

$$\mathcal{D}^{G_H} \cong C(S(H)) \cong C(\mathbf{\Omega})/\Delta_{S(H)} \cong \mathcal{D}^{G_+}/\mathcal{I}_{S(H)},$$

where

$$\begin{aligned} \Delta_{S(H)} &= \{f \in C(\mathbf{\Omega}) : f(A) = 0, \forall A \in S(H)\}, \\ \mathcal{I}_{S(H)} &= \{T \in \mathcal{D}^{G_+} : (\rho^* \circ \Gamma(T))(A) = 0, \forall A \in S(H)\}. \end{aligned}$$

We are now ready to prove that $\mathcal{I}_{S(H)} = (\mathrm{Ker}\gamma^{G_H, G_+}) \cap \mathcal{D}^{G_+}$. In fact, for any $x \in \mathcal{D}^{G_+}$,

$$\begin{aligned} & x \in \mathcal{I}_{S(H)} \iff [x] = 0 \text{ in } \mathcal{D}^{G_+}/\mathcal{I}_{S(H)} \iff [\rho^* \circ \Gamma(x)] = 0 \text{ in } C(\mathbf{\Omega})/\Delta_{S(H)} \\ & \iff \rho^* \circ \Gamma(x)|_{S(H)} = 0 \iff \forall A \in S(H),\ \rho^* \circ \Gamma(x)(A) = 0 \\ & \iff \forall A \in S(H),\ (\rho_H^* \circ \Gamma(\gamma^{G_H, G_+}(x)))(A) = 0 \iff \gamma^{G_H, G_+}(x) = 0. \end{aligned}$$

The assertion that $\mathcal{J}_{S(H)} = \mathrm{Ker}\gamma^{G_H, G_+}$ then follows from Proposition 4.3. ∎

COROLLARY 4.6. *Let (G, G_+) be a quasi-lattice ordered group, $\mathbf{\Omega}$ be the collection of hereditary and directed subsets of G_+. Let $\mathbb{K}$ be a closed θ-invariant subset of $\mathbf{\Omega}$, and define $\mathcal{I}_{\mathbb{K}}$ and $\mathcal{J}_{\mathbb{K}}$ by* (4.5) *and* (4.6) *respectively, then*

$$\mathcal{I}_{\mathbb{K}} = \bigcap_{H \in \mathbb{K}} ((\mathrm{Ker}\gamma^{G_H,G_+}) \cap \mathcal{D}^{G_+}) \quad \textit{and} \quad \mathcal{J}_{\mathbb{K}} = \bigcap_{H \in \mathbb{K}} \mathrm{Ker}\gamma^{G_H,G_+}.$$

Proof. Obviously $\mathbb{K} = \bigcup_{H \in \mathbb{K}} S(H)$, the conclusion then follows by the definitions of $\mathcal{I}_{\mathbb{K}}, \mathcal{J}_{\mathbb{K}}$ and the preceeding theorem. ∎

COROLLARY 4.7. *Let (G, G_+) be a quasi-lattice ordered group, $\mathbf{\Omega}$ be the collection of hereditary and directed subsets of G_+. Then the following conditions are all equivalent:*
(i) $\mathcal{T}^{G_+}$ *is simple;*
(ii) *for any $H \in \mathbf{\Omega}$, γ^{G_H,G_+} is a C^*-isomorphism;*
(iii) *for any $H \in \mathbf{\Omega}$, $S(H) = \mathbf{\Omega}$;*
(iv) $\mathrm{cl}(\mathbf{\Omega}_\infty) = \mathbf{\Omega}$;
(v) *the only closed θ-invariant subset of $\mathbf{\Omega}$ is $\mathbf{\Omega}$ itself;*
(vi) *there is no nonzero α-invariant ideal of $\mathcal{D}^{G_+}$;*
(vii) *for every finite subset F of $G_+ \setminus \{e\}$, there exists $z \in G_+$ such that $z \vee x = \infty$ for all $x \in F$;*
(viii) *for every nonzero element A in $\mathcal{T}^{G_+}$, there exist $B, C \in \mathcal{T}^{G_+}$ such that $BAC = 1$.*

Proof. Since $\mathrm{cl}(\mathbf{\Omega}\infty)$ is the smallest closed θ-invariant subset of $\mathbf{\Omega}$, $S(H) = \mathrm{cl}(\mathbf{\Omega}_\infty)$ for any $H \in \mathrm{cl}(\mathbf{\Omega}_\infty)$. The equivalence of conditions (i) through (vi) then follows from Theorem 4.5. For the rest, see Lemma 5.2 and Theorem 5.4 of [6]. ∎

5. THE MAXIMAL IDEALS OF THE TOEPLITZ ALGEBRAS

For $n \geqslant 2$, the Cuntz algebra $\mathcal{O}_n$ is the universal C^*-algebra generated by isometries $S_1, S_2, \ldots, S_n$ such that $\sum_{i=1}^{n} S_i S_i^* = 1$. It is known that for any non-zero element $A \in \mathcal{O}_n$, there exist B and C such that $BAC = 1$, which means $\mathcal{O}_n$ is simple, and thus if $T_1, T_2, \ldots, T_n$ are any n isometries such that $\sum_{i=1}^{n} T_i T_i^* = 1$, then the C^*-algebra generated by $T_1, T_2, \ldots, T_n$ is isomorphic to $\mathcal{O}_n$. Meanwhile, the C^*-algebra $\mathcal{O}_\infty$ is the universal C^*-algebra generated by countably many isometries S_k such that $\sum_{k=1}^{n} S_k S_k^* < 1$ for all $n \geqslant 1$. The C^*-algebra $\mathcal{O}_\infty$ is also simple. For the details, the reader is referred to [2] and [3]. The purpose of this section is to give a new look at these Cuntz algebras. We will show that they can be expressed as certain Toeplitz algebras, and the property of purely infiniteness can be proved in a unified way.

Throughout this section, (G, G_+) is a quasi-lattice ordered group such that G_+ itself is not directed, which means there exist some $x, y \in G_+$ with $x \vee y = \infty$. Since $\mathrm{cl}(\mathbf{\Omega}_\infty)$ is the smallest closed θ-invariant subset of $\mathbf{\Omega}$, $\mathcal{J}_{\mathrm{cl}(\mathbf{\Omega}_\infty)}$ is the largest *induced ideal* of $\mathcal{T}^{G_+}$, which may however fail to be the largest ideal of $\mathcal{T}^{G_+}$ as shown in the next section. So we turn to investigate conditions under which $\mathcal{J}_{\mathrm{cl}(\mathbf{\Omega}_\infty)}$ becomes a *maximal ideal* of $\mathcal{T}^{G_+}$ in the sense that, for any ideal $\mathcal{J}$ of $\mathcal{T}^{G_+}$, $\mathcal{J}_{\mathrm{cl}(\mathbf{\Omega}_\infty)} \subseteq \mathcal{J} \implies \mathcal{J}_{\mathrm{cl}(\mathbf{\Omega}_\infty)} = \mathcal{J}$. Note for any $H \in \mathrm{cl}(\mathbf{\Omega}_\infty)$, $S(H) = \mathrm{cl}(\mathbf{\Omega}_\infty)$, so by Theorem 4.5 we know that $\mathcal{J}_{\mathrm{cl}(\mathbf{\Omega}_\infty)}$ is maximal if and only if the Toeplitz algebras $\mathcal{T}^{G_H}$ $(H \in \mathrm{cl}(\mathbf{\Omega}_\infty))$ are simple. Following the same lines as [2] and [6], in this section we study the purely infinite simpleness of the Toeplitz algebras $\mathcal{T}^{G_H}$ $(H \in \mathrm{cl}(\mathbf{\Omega}_\infty))$.

LEMMA 5.1 (cf. Lemma 3.9 of [10]). *Let (G, G_+) be a quasi-lattice ordered group, and $\{L(t) : t \in G_+\}$ be a family of projections of a unital C^*-algebra $\mathcal{B}$ such that $L(e) = 1$ and $L(s)L(t) = L(s \vee t)$ (with the convention that $L(\infty) = 0$). Then for any finite subset F of G_+, any $\lambda_x \in \mathbf{C}$, we have*

$$\Big\| \sum_{x\in F} \lambda_x L(x) \Big\| = \max \Big\{ \Big| \sum_{x\in A} \lambda_x \Big| : \emptyset \neq A \subseteq F, \prod_{x\in A} L(x) \cdot \prod_{y\notin A} (1 - L(y)) \neq 0 \Big\}.$$

(Note if $A = F$, then the product above should be understood as $\prod_{x\in F} L(x)$.)

LEMMA 5.2 (cf. Lemma 5.1 of [6]). *Let (G, G_+) be a quasi-lattice ordered group such that G_+ itself is not directed. Let H be in $\mathbf{\Omega}_\infty$ and F a nonempty finite subset of G_+ with $F \cap H = \emptyset$. If $a \in H$ satisfies $a < x$ for any $x \in F$, then there exists $y \in H$ with $a < y$ such that $x \vee y = \infty$ for all $x \in F$.*

THEOREM 5.3. *Let (G, G_+) be a quasi-lattice ordered group such that G_+ itself is not directed. Suppose for any $x, y \in G_+$ with $x \neq y$, there exists $g \in G_+$, such that*

$$T^{G_+}_{g^{-1}} T^{G_+}_{x} T^{G_+}_{y^{-1}} T^{G_+}_{g} = 0. \tag{5.1}$$

Then for any $H \in \mathrm{cl}(\mathbf{\Omega}_\infty)$, the Toeplitz algebra $\mathcal{T}^{G_H}$ is purely infinite simple.

Proof. We need to prove that for any $X \in \mathcal{T}^{G_H}$ with $X \neq 0$, there exist $B, C \in \mathcal{T}^{G_H}$ such that $BXC = 1$, which is further reduced to find some B_1 and C_1 such that $B_1 X C_1$ is invertible.

Step 1. Let $Y \in \mathcal{T}^\infty(G_H) \overset{\mathrm{def}}{=} \mathrm{span}\{T^{G_H}_g T^{G_H}_{h^{-1}} : g, h \in G_+\}$ with $Y \neq 0$. For any $g \in G_+$, denote $T^{G_H}_g T^{G_H}_{g^{-1}}$ simply by $L(g)$. Then there exists a finite subset F of G_+, and $\lambda_x \in \mathbf{C}$ for each $x \in F$, such that

$$\theta^{G_H}(Y^*Y) = \sum_{x\in F} \lambda_x L(x). \tag{5.2}$$

By Lemma 5.1, there exists $A \subseteq F$ such that

$$\prod_{x\in A} L(x) \cdot \prod_{y\notin A} (1 - L(y)) \neq 0, \quad \text{and} \quad \Big\| \sum_{x\in F} \lambda_x L(x) \Big\| = \Big| \sum_{x\in A} \lambda_x \Big|. \tag{5.3}$$

Case 1. $\emptyset \neq A \subseteq F, A \neq F$. Let a be the least common upper bound of elements in A. By (5.3) and Proposition 4.1 we know $a \in G_+$. Note for any $x \in A$

and $y \in F \setminus A$,

$$(L(a) - L(a \vee y))L(x) = L(a \vee x) - L((a \vee y) \vee x) = L(a) - L(a \vee y),$$
$$(L(a) - L(a \vee y))L(y) = L(a \vee y) - L(a \vee y) = 0.$$

Therefore,

$$Q \cdot \Big(\sum_{x \in F} \lambda_x L(x)\Big) \cdot Q = \Big(\sum_{x \in A} \lambda_x\Big) Q, \tag{5.4}$$

where

$$Q \overset{\text{def}}{=} \prod_{y \notin A} (L(a) - L(a \vee y)) = \prod_{x \in A} L(x) \cdot \prod_{y \notin A} (1 - L(y)) \neq 0.$$

Note $\theta^{G_H}(Y^*Y)$ is positive, by (5.2)–(5.4) we know that $\Big|\sum\limits_{x \in A} \lambda_x\Big| = \sum\limits_{x \in A} \lambda_x$. Therefore,

$$Q \cdot \theta^{G_H}(Y^*Y) \cdot Q = \|\theta^{G_H}(Y^*Y)\| Q. \tag{5.5}$$

Since $Q \neq 0$ and $H \in \mathrm{cl}(\mathbf{\Omega}_\infty)$, we know $\rho_H^* \cdot \Gamma(Q) \neq 0$ in $C(S(H)) = C(\mathrm{cl}(\mathbf{\Omega}_\infty))$. By the density of $\mathbf{\Omega}_\infty$ in $\mathrm{cl}(\mathbf{\Omega}_\infty)$, there exists $H_1 \in \mathbf{\Omega}_\infty$ such that

$$\prod_{y \notin A} (\chi_{H_1}(a) - \chi_{H_1}(a \vee y)) = (\rho_H^* \circ \Gamma(Q))(H_1) \neq 0,$$

which means that $a \in H_1$ and $a \vee y \notin H_1$ for all $y \notin A$. By Lemma 5.2 there exists $z \in G_+$ with $z \geqslant a$ and $z \vee (a \vee y) = \infty$ for all $y \in F \setminus A$, so that $L(z)Q = L(z)$. Hence,

$$L(z)Q \cdot \theta^{G_H}(Y^*Y) \cdot QL(z) = \|\theta^{G_H}(Y^*Y)\| L(z).$$

It follows that

$$T_{z^{-1}}^{G_H} Q \cdot \theta^{G_H}(Y^*Y) \cdot Q T_z^{G_H} = \|\theta^{G_H}(Y^*Y)\|. \tag{5.6}$$

Case 2. $\emptyset \neq A = F$. In this case, let a be the least common upper bound of elements in F, then $L(a)L(x) = L(x)L(a) = L(a)$ for any $x \in F$. Let $z = a$ and $Q = L(a)$, then (5.6) also holds.

Step 2. Suppose now that condition (5.1) is satisfied. Let $x_i, y_i \in G_+$ with $x_i \neq y_i$ for $i = 1, 2$. By assumption there exists $g_1 \in G_+$ such that

$$T_{g_1^{-1}}^{G_H} T_{x_1}^{G_H} T_{y_1^{-1}}^{G_H} T_{g_1}^{G_H} = \gamma^{G_H, G_+}(T_{g_1^{-1}}^{G_+} T_{x_1}^{G_+} T_{y_1^{-1}}^{G_+} T_{g_1}^{G_+}) = 0.$$

If the operator $T_{g_1^{-1}}^{G_H} T_{x_2}^{G_H} T_{y_2^{-1}}^{G_H} T_{g_1}^{G_H} \neq 0$, then it is equal to $T_u^{G_H} T_{v^{-1}}^{G_H}$ for some $u, v \in G_+$ with $u \neq v$ (see Proposition 4.1). Once again there exists $g_2 \in G_+$ such that $T_{g_2^{-1}}^{G_H} T_u^{G_H} T_{v^{-1}}^{G_H} T_{g_2}^{G_H} = 0$. Let $g = g_1 g_2$, then $T_g^{G_H} = T_{g_1}^{G_H} T_{g_2}^{G_H}$, so for any $\lambda_1, \lambda_2 \in \mathbf{C}$,

$$T_{g^{-1}}^{G_H}(\lambda_1 T_{x_1}^{G_H} T_{y_1^{-1}}^{G_H} + \lambda_2 T_{x_2}^{G_H} T_{y_2^{-1}}^{G_H}) T_g^{G_H} = 0.$$

The above process indicates for any $T \in \mathcal{T}^\infty(G_H)$, there exists $g \in G_+$ such that

$$T_{g^{-1}}^{G_H}(T - \theta^{G_H}(T)) T_g^{G_H} = 0. \tag{5.7}$$

Step 3. Let $X \in \mathcal{T}^{G_H}$ with $X \neq 0$. Since θ^{G_H} is faithful (in the sense that $\theta^{G_H}(S^*S) = 0 \iff S = 0$ for any $S \in \mathcal{T}^{G_H}$), we know $\theta^{G_H}(X^*X) \neq 0$. Since $\mathcal{T}^\infty(G_H)$ is dense in $\mathcal{T}^{G_H}$, we can choose a sequence $\{Y_n\}$ in $\mathcal{T}^\infty(G_H)$ such that $Y_n \to X$ in $\mathcal{T}^{G_H}$. Then $Y_n^*Y_n \to X^*X$ and $\|\theta^{G_H}(Y_n^*Y_n)\| \to \|\theta^{G_H}(X^*X)\| > 0$. It follows that there exists some $Y \in \mathcal{T}^\infty(G_H)$ such that

$$\theta^{G_H}(Y^*Y) \neq 0 \quad \text{and} \quad \frac{\|X^*X - Y^*Y\|}{\|\theta^{G_H}(Y^*Y)\|} < 1. \tag{5.8}$$

Let Q and z be as in Step 1 such that (5.6) holds. By Proposition 4.2, we know

$$\theta^{G_H}(T^{G_H}_{z^{-1}}Q(Y^*Y - \theta^{G_H}(Y^*Y))QT^{G_H}_z) = 0,$$

therefore by (5.6) and (5.7) we know there exists $g \in G_+$ such that

$$T^{G_H}_{g^{-1}}T^{G_H}_{z^{-1}}Q \cdot (Y^*Y) \cdot QT^{G_H}_z T^{G_H}_g = \|\theta^{G_H}(Y^*Y)\|. \tag{5.9}$$

Then

$$\begin{aligned}
&\| \|\theta^{G_H}(Y^*Y)\|^{-1} T^{G_H}_{g^{-1}}T^{G_H}_{z^{-1}}Q \cdot (X^*X) \cdot QT^{G_H}_z T^{G_H}_g - 1\| \\
&\quad = \|\theta^{G_H}(Y^*Y)\|^{-1} \cdot \|T^{G_H}_{g^{-1}}T^{G_H}_{z^{-1}}Q \cdot (X^*X - Y^*Y) \cdot QT^{G_H}_z T^{G_H}_g\| \\
&\quad \leqslant \|\theta^{G_H}(Y^*Y)\|^{-1} \cdot \|X^*X - Y^*Y\| < 1.
\end{aligned}$$

Let

$$B_1 = \|\theta^{G_H}(Y^*Y)\|^{-1} T^{G_H}_{g^{-1}}T^{G_H}_{z^{-1}}Q \cdot X^*, \quad \text{and} \quad C_1 = QT^{G_H}_z T^{G_H}_g.$$

Then $\|B_1XC_1 - 1\| < 1$, therefore B_1XC_1 is invertible. ∎

By Proposition 4.1, we know $T^{G_+}_{g^{-1}}T^{G_+}_x = 0$ for any $g, x \in G_+$ with $x \vee g = \infty$. So compared with the condition (5.1) given in the preceding theorem, a stronger condition can be stated as follows:

DEFINITION 5.4. A quasi-lattice ordered group (G, G_+) is said to be *extremely incomparable* if for any $x \in G_+ \setminus \{e\}$, there exists $g \in G_+$ such that $x \vee g = \infty$.

Typical examples of extremely incomparable quasi-lattice ordered groups are the free groups with finite or countably infinite generators presented below:

EXAMPLE 5.5. Let F_n be the free group with $n (n \geqslant 2)$ generators $a_1, \ldots, a_n$, and F_n^+ be the semigroup of F_n generated by $a_1, \ldots, a_n$. Then (F_n, F_n^+) is a quasi-lattice ordered group with a property that

$$x \vee y \neq \infty \iff x \leqslant y \quad \text{or} \quad y \leqslant x \quad \text{for any } x, y \in F_n^+.$$

Specifically, for any $t \in F_n^+ \setminus \{e\}$, there exists one and only one generator a_{i_0} satisfies $a_{i_0} \leqslant t$, which means $t \vee a_i = \infty$ for any $i \neq i_0$. Let $H = \{e\} \cup \{a_1^m : m \in \mathbb{N}\}$ and denote $F_n^+ \cdot H^{-1}$ simply by G_H. Then $H \in \mathbf{\Omega}_\infty$, so by Theorem 5.3 we know $\mathcal{T}^{G_H}$ is purely infinite simple. Clearly, the Toeplitz algebra $\mathcal{T}^{G_H}$ is generated by

$\{T^{G_H}_{a_i} : i = 1, 2, \ldots, n\}$ with $\sum_{i=1}^{n} T^{G_H}_{a_i} T^{G_H}_{a_i^{-1}} = 1$, and hence $\mathcal{T}^{G_H} \cong \mathcal{O}_n$ by the uniqueness of $\mathcal{O}_n$.

EXAMPLE 5.6. Let G be the free group with countably infinite generators $\{a_n : n \in \mathbb{N}\}$, and G_+ be the semigroup of G generated by $\{a_n : n \in \mathbb{N}\}$. Let $H = \{e\} \cup \{a_1^m : m \in \mathbb{N}\}$, then the Toeplitz algebra $\mathcal{T}^{G_H}$ is purely infinite simple, which is isomorphic to $\mathcal{O}_\infty$.

COROLLARY 5.7. *Let (G_1, G_1^+) and (G_2, G_2^+) be two extremely incomparable quasi-latticed ordered groups. Denote by $\mathbf{\Omega_1}$ and $\mathbf{\Omega_2}$ the collection of hereditary and directed subsets of G_1^+ and G_2^+ respectively. Let $\mathbf{\Omega}_1^\infty$ and $\mathbf{\Omega}_2^\infty$ be the collection of the maximal elements of $\mathbf{\Omega}_1$ and $\mathbf{\Omega}_2$ respectively. Then for any $H_1 \in \mathbf{\Omega}_1^\infty$ and $H_2 \in \mathbf{\Omega}_2^\infty$, the spatial tensor product of C^*-algebras $\mathcal{T}^{G_{H_1}} \otimes \mathcal{T}^{G_{H_2}}$ is purely infinite simple.*

Proof. Let $G = G_1 \times G_2$, $G_+ = G_1^+ \times G_2^+$ and $H = H_1 \times H_2$, then (G, G_+) is also an extremely incomparable quasi-lattice ordered group and $H \in \mathbf{\Omega}_\infty$ (see Proposition 2.4). Let U be the natural unitary operator from $\ell^2(G)$ onto $\ell^2(G_1) \otimes \ell^2(G_2)$ which satisfies $U\delta_{(x,y)} = \delta_x \otimes \delta_y$ for $x \in G_1$ and $y \in G_2$. It is easy to verify that

$$U \cdot T^{G_H}_{(x,y)} \cdot U^* = T^{G_{H_1}}_x \otimes T^{G_{H_2}}_y \quad \text{for any } x \in G_1, y \in G_2.$$

Therefore, the spatial tensor product of C^*-algebras $\mathcal{T}^{G_{H_1}} \otimes \mathcal{T}^{G_{H_2}}$ is unitarily equivalent to $\mathcal{T}^{G_H}$. The conclusion then follows from Theorem 5.3. ∎

EXAMPLE 5.8. For $n, m \geqslant 2$, the (spatial) tensor product of the Cuntz algebras $\mathcal{O}_n \otimes \mathcal{O}_m$ is purely infinite simple.

6. THE LARGEST IDEALS OF THE TOEPLITZ ALGEBRAS

In this section, we will study the largest ideals of the Toeplitz algebras. Let (G, G_+), $\mathbf{\Omega}$ and $\mathbf{\Omega}_\infty$ be as in Section 5 except that G_+ itself might be directed. We will prove, under a certain assumption, that if the condition (5.1) given in Theorem 5.3 is not satisfied, then the largest induced ideal $\mathcal{J}_{\mathrm{cl}(\mathbf{\Omega}_\infty)}$ does fail to be the largest ideal of $\mathcal{T}^{G_+}$ (see Theorem 6.5).

DEFINITION 6.1. Let $\mathcal{B}$ be a unital C^*-algebra. A map V from G_+ to $\mathcal{B}$ is said to be an *isometric representation of* G_+ if it satisfies

$$V(e) = 1; \quad V(g)^*V(g) = 1; \quad V(g)V(h) = V(gh) \quad \text{for any } g, h \in G_+.$$

Moreover, V is said to be *covariant* if for any $s, t \in G_+$,

$$V(s)V(s)^* \cdot V(t)V(t)^* = \begin{cases} V(s \vee t)V(s \vee t)^* & \text{if } s \vee t \in G_+, \\ 0 & \text{if } s \vee t = \infty. \end{cases}$$

DEFINITION 6.2. The pair (G, G_+) is said to be *amenable* if every covariant isometric representation $V : G_+ \to \mathcal{B}$ can be lifted as a C^*-morphism $\pi_V : \mathcal{T}^{G_+} \to \mathcal{B}$ such that $\pi_V(T_g^{G_+}) = V(g)$ for any $g \in G_+$.

By Theorem 4.7 of [4] or Section 4 of [10] we know (G, G_+) is amenable provided that G is amenable. Furthermore, as in the case of the free group F_n ($n \geqslant 2$), the condition of the amenability of G can be replaced by a weaker one, which concerns certain approximation property. For the details, see Theorem 4.6 of [4] or Proposition 2 in Section 4 of [10].

PROPOSITION 6.3. *Let (G, G_+) be a quasi-lattice ordered group, $\mathbf{\Omega}$ be the collection of hereditary and directed subsets of G_+, and $\mathbf{\Omega}_\infty$ be the collection of the maximal elements of $\mathbf{\Omega}$. Denote by $H_\Delta = \bigcap\limits_{H \in \mathbf{\Omega}_\infty} H$, then $H_\Delta \cdot H_\Delta^{-1}$ is a subgroup of G with*

$$H_\Delta = \{x \in G_+ : \forall y \in G_+,\ x \vee y \neq \infty\}. \tag{6.1}$$

Proof. Step 1. Suppose that $x \in H_\Delta$. For any $y \in G_+$, we can choose some $H_y \in \mathbf{\Omega}_\infty$ such that $[e, y] \subseteq H_y$, where $[e, y] = \{s \in G_+ : s \leqslant y\}$. Since $x, y \in H_y$ and H_y is directed, we know $x \vee y \in H_y \subseteq G_+$. It follows that $H_\Delta \subseteq \{x \in G_+ : \forall y \in G_+,\ x \vee y \neq \infty\}$. On the other hand, suppose $x \in G_+$ with $x \vee y \neq \infty$ for any $y \in G_+$, then for any $H \in \mathbf{\Omega}_\infty$, we let

$$H_1 = \bigcup_{t \in H} [e, x \vee t].$$

Clearly $x \in H_1 \in \mathbf{\Omega}$ and $H \subseteq H_1$, hence $H = H_1$ by the maximality of H. Accordingly $x \in H$, and therefore equation (6.1) holds.

Step 2. Let $x \in H_\Delta$. By the definition of H_Δ, we know that

$$(\rho^* \circ \Gamma(T_x^{G_+} T_{x^{-1}}^{G_+}))(A) = \chi_A(x) = 1 \quad \text{for every } A \in \mathbf{\Omega}_\infty.$$

By the density of $\mathbf{\Omega}_\infty$ in $\mathrm{cl}(\mathbf{\Omega}_\infty)$, we know $\rho^* \circ \Gamma(1 - T_x^{G_+} T_{x^{-1}}^{G_+}) \equiv 0$ on $\mathrm{cl}(\mathbf{\Omega}_\infty) = S(H)$ for any $H \in \mathrm{cl}(\mathbf{\Omega}_\infty)$. By (4.14) and (4.15) we know

$$1 - T_x^{G_H} T_{x^{-1}}^{G_H} = 0 \quad \text{for any } H \in \mathrm{cl}(\mathbf{\Omega}_\infty). \tag{6.2}$$

In particular for $h \in H \in \mathbf{\Omega}_\infty$, $T_x^{G_H} T_{x^{-1}}^{G_H} \delta_{h^{-1}} = \delta_{h^{-1}}$ and thus $(hx)^{-1} \in G_H \iff hx \in H$. We have proved that $H \cdot x \subseteq H$ for any $x \in H_\Delta$ and $H \in \Omega_\infty$, which means that H_Δ is a semigroup of G_+.

Now let $x, y \in H_\Delta$, by (6.2) we have $T_x^{G_H} T_{x^{-1}}^{G_H} \delta_y = \delta_y$ for any $H \in \mathbf{\Omega}_\infty$. So $x^{-1}y \in G_H \iff \tau(x^{-1}y) \in H$, and hence $\tau(x^{-1}y) \in H_\Delta$ by the arbitrariness of H. Exchanging x with y, we know $\sigma(x^{-1}y) = \tau(y^{-1}x) \in H_\Delta$. It follows that $x^{-1}y = \sigma(x^{-1}y)\tau(x^{-1}y)^{-1} \subseteq H_\Delta \cdot H_\Delta^{-1}$, which implies that $H_\Delta \cdot H_\Delta^{-1}$ is a subgroup since we have already shown that H_Δ is a semigroup of G_+. ∎

Let

$$G_\Delta = \{s \in H_\Delta \cdot H_\Delta^{-1} : g^{-1}sg \in H_\Delta \cdot H_\Delta^{-1} \text{ for any } g \in G_+\}. \tag{6.3}$$

Then the set G_Δ is also a (possibly non-normal) subgroup of G. Since G_+ is a semigroup, by the definition of G_Δ we know that $g^{-1}\cdot G_\Delta\cdot g\subseteq G_\Delta$ for any $g\in G_+$.

Define an equivalence relationship on G by $x\sim y\Longleftrightarrow x^{-1}y\in G_\Delta$. For any $g\in G$, let $[g]=\{s\in G: s\sim g\}$. Put $[G]=\{[g]: g\in G\}$ and $[G_+]=\{[g_+]: g_+\in G_+\}$. Then a left action of $[G]$ on $[G_+]$ can be defined unambiguously as

$$[g][g_+]\overset{\text{def}}{=}[gg_+]\quad\text{for } g\in G, g_+\in G_+.$$

So for any $[g]\in[G]$, a (generalized) Toeplitz operator $T_{[g]}^{[G_+]}$ on $\ell^2([G_+])\overset{\text{def}}{=}\text{closp}\{\delta_{[g_+]}: g_+\in G_+\}$ can also be defined as

$$T_{[g]}^{[G_+]}\delta_{[g_+]}=\begin{cases}\delta_{[gg_+]} & \text{if } [gg_+]\in[G_+],\\ 0 & \text{otherwise.}\end{cases}$$

By definition, we know $(T_{[g]}^{[G_+]})^*=T_{[g^{-1}]}^{[G_+]}$ for any $g\in G$.

PROPOSITION 6.4. *Let $\mathbb{B}(\ell^2([G_+]))$ be the set of all bounded linear operators on $\ell^2([G_+])$. If $G_\Delta=H_\Delta\cdot H_\Delta^{-1}$, then the isometric representation $V: G_+\to\mathbb{B}(\ell^2([G_+]))$ defined as $V(g_+)=T_{[g_+]}^{[G_+]}$ $(g_+\in G_+)$ is covariant.*

Proof. For any $x,g_+\in G_+$, it is easy to check that

$$V(x)V(x)^*\delta_{[g_+]}\neq 0\Longleftrightarrow x\in g_+\cdot G_\Delta\cdot G_+^{-1}\Longleftrightarrow x\leqslant g_+a\quad\text{for some } a\in G_\Delta.$$

It follows that V is covariant if and only if for any $x,y,g_+\in G_+$ and $a_1,a_2\in G_\Delta$,

$$x\leqslant g_+a_1, y\leqslant g_+a_2\Longrightarrow x\vee y\leqslant g_+a\quad\text{for some } a\in G_\Delta. \tag{6.4}$$

If $G_\Delta=H_\Delta\cdot H_\Delta^{-1}$, then $a_i\leqslant\sigma(a_1)\vee\sigma(a_2)\overset{\text{def}}{=}a\in H_\Delta\subseteq G_\Delta$, hence (6.4) holds. ∎

THEOREM 6.5. *Let (G,G_+) be an amenable quasi-lattice ordered group. Suppose that $G_\Delta=H_\Delta\cdot H_\Delta^{-1}$. If the condition* (5.1) *stated in Theorem* 5.3 *is not satisfied, then $\mathcal{J}_{\text{cl}(\mathbf{\Omega}_\infty)}$ fails to be the largest ideal of $\mathcal{T}^{G_+}$.*

Proof. Choose any $H_0\in\text{cl}(\mathbf{\Omega}_\infty)$ so that $\text{Ker}\,\gamma^{G_{H_0},G_+}=\mathcal{J}_{\text{cl}(\mathbf{\Omega}_\infty)}$. Let us first prove that $\mathcal{J}_{\text{cl}(\mathbf{\Omega}_\infty)}$ is the largest ideal of $\mathcal{T}^{G_+}$ if and only if $1\notin\overline{\theta^{G_+}(\mathcal{J})}$ for any ideal $\mathcal{J}$ of $\mathcal{T}^{G_+}$.

"$\Longrightarrow$": Suppose that Ker $\gamma^{G_{H_0},G_+}$ is the largest ideal of $\mathcal{T}^{G_+}$. Then for any ideal $\mathcal{J}$ of $\mathcal{T}^{G_+}$, $\theta^{G_+}(\mathcal{J})\subseteq\theta^{G_+}(\text{Ker}\gamma^{G_{H_0},G_+})\subseteq\text{Ker}\gamma^{G_{H_0},G_+}$. Hence $\overline{\theta^{G_+}(\mathcal{J})}\subseteq\text{Ker}\gamma^{G_{H_0},G_+}$, therefore $1\notin\overline{\theta^{G_+}(\mathcal{J})}$.

"$\Longleftarrow$": Suppose that $1\notin\overline{\theta^{G_+}(\mathcal{J})}$ for any ideal $\mathcal{J}$ of $\mathcal{T}^{G_+}$. Then for any ideal $\mathcal{J}$ of $\mathcal{T}^{G_+}$, by Proposition 4.2 we know $\overline{\theta^{G_+}(\mathcal{J})}$ is actually an α-invariant ideal of $\mathcal{D}^{G_+}$, and is thus contained in $(\text{Ker}\gamma^{G_{H_0},G_+})\cap\mathcal{D}^{G_+}$. So, $x\in\mathcal{J}\Longrightarrow\theta^{G_+}(x^*x)\in(\text{Ker}\gamma^{G_{H_0},G_+})\cap\mathcal{D}^{G_+}\Longrightarrow x\in\text{Ker}\gamma^{G_{H_0},G_+}$. It follows that $\text{Ker}\gamma^{G_{H_0},G_+}$ is the largest ideal of $\mathcal{T}^{G_+}$.

Since $G_\Delta = H_\Delta \cdot H_\Delta^{-1}$ and (G, G_+) is amenable, the covariant isometric representation $V : G_+ \to \mathbb{B}(\ell^2([G_+]))$ induces a C^*-morphism $\pi_V : \mathcal{T}^{G_+} \to \mathbb{B}(\ell^2([G_+]))$ such that $\pi_V(T_{g_+}^{G_+}) = T_{[g_+]}^{[G_+]}$ for any $g \in G_+$. Given any $x \in H_\Delta$ and $g_+ \in G_+$, $g_+^{-1}x^{-1}g_+ \in g_+^{-1} \cdot H_\Delta^{-1} \cdot g_+ \subseteq g_+^{-1} \cdot G_\Delta \cdot g_+ = G_\Delta$. So $x^{-1}g_+ \in G_+ \cdot G_\Delta$, or equivalently $[x^{-1}g_+] \in [G_+]$, and thus

$$T_{[x^{-1}]}^{[G_+]}\delta_{[g_+]} = \delta_{[x^{-1}g_+]} \quad \text{for } x \in H_\Delta \text{ and } g_+ \in G_+. \tag{6.5}$$

We claim that

$$\pi_V(T_t^{G_+}) = 1 \quad \text{for any } t \in G_\Delta. \tag{6.6}$$

In fact, for any $t \in G_\Delta$, we have $\sigma(t) \in G_+$ and $\tau(t) \in H_\Delta$, so by (6.5) we know for any $g_+ \in G_+$,

$$\pi_V(T_t^{G_+})\delta_{[g_+]} = T_{[\sigma(t)]}^{[G_+]} T_{[\tau(t)^{-1}]}^{[G_+]}\delta_{[g_+]} = \delta_{[tg_+]} = \delta_{[g_+]}.$$

Let

$$\mathcal{M} = \{(x, y) : x, y \in G_+, x \neq y, T_{g^{-1}}^{G_+} T_x^{G_+} T_{y^{-1}}^{G_+} T_g^{G_+} \neq 0, \forall g \in G_+\}.$$

By assumption, $\mathcal{M}$ is nonempty, so we can choose some $(x_0, y_0) \in \mathcal{M}$. Then

$$T_{g^{-1}}^{G_+} T_{x_0}^{G_+} \neq 0 \quad \text{and} \quad T_{g^{-1}}^{G_+} T_{y_0}^{G_+} \neq 0 \quad \text{for any } g \in G_+,$$

so $x_0, y_0 \in H_\Delta$ by (6.1), and by assumption $x_0 y_0^{-1} \in (H_\Delta \cdot H_\Delta^{-1}) \setminus \{e\} = G_\Delta \setminus \{e\}$.

Now let $\mathcal{J} = \mathrm{Ker}\pi_V$. By (6.6) we have $1 - T_{x_0 y_0^{-1}}^{G_+} \in \mathcal{J}$. But clearly, $1 = \theta^{G_+}(1 - T_{x_0 y_0^{-1}}^{G_+}) \in \theta^{G_+}(\mathcal{J})$. The first part of the proof indicates that $\mathcal{J}_{\mathrm{cl}(\boldsymbol{\Omega}_\infty)}$ fails to be the largest ideal of $\mathcal{T}^{G_+}$. ∎

REMARK 6.6. (i) Suppose that (G, G_+) is an abelian quasi-lattice ordered group. Replacing G by $G_+ - G_+$, we may assume further that (G, G_+) is a partially ordered group. In this case, $\mathrm{cl}(\boldsymbol{\Omega}_\infty) = \{G_+\}$, so $\mathcal{J}_{\mathrm{cl}(\boldsymbol{\Omega}_\infty)} = \mathrm{Ker}\gamma^{G,G_+}$. By Proposition 1.2 of [11] we know $\mathrm{Ker}\gamma^{G,G_+}$ is the commutator ideal of $\mathcal{T}^{G_+}$, which definitely cannot be the largest ideal of $\mathcal{T}^{G_+}$. Meanwhile condition (5.1) is also definitely not satisfied.

(ii) Suppose that (G, G_+) is an ordered group, H is a subset of G_+, then H belongs to $\boldsymbol{\Omega}$ if and only if it is hereditary. Let $H \in \boldsymbol{\Omega}$ such that $G_H = G_+ \cdot H^{-1}$ is a subgroup of G, then $G_H^0 \overset{\mathrm{def}}{=} G_H \cap G_H^{-1} = H \cup H^{-1}$ is an *order ideal* of G. Suppose further that G is abelian, then $T_x^{G_H} T_y^{G_H} = T_y^{G_H} T_x^{G_H}$ for any $x \in G_H^0, y \in G$, which means that $T_x^{G_H}$ $(x \in G_H^0)$ belongs to the commutant of $\mathcal{T}^{G_H}$. Therefore for any irreducible representation (π, H) of $\mathcal{T}^{G_H}$, $\pi(T_x^{G_H})$ $(x \in G_H^0)$ is the scalar multiple of I_H. Using this fact, one can give a shorter proof of Corollary 3.4 in [1], and the reader is referred to [12] for the details. In the non-abelian case, things may become much more complicated.

Acknowledgements. The author would like to thank the referee for some helpful comments and suggestions.

This research was partially supported by the Leading Academic Discipline Project of Shanghai Normal University (no. DZL803), the National Natural Science Foundation of China (no. 10371051), Shanghai Natural Science Foundation (no. 05ZR14094) and Shanghai Municipal Education Commission (no. 05DZ04 and no. 09YZ147).

REFERENCES

[1] S. ADJI, I. RAEBURN, The ideal structure of Toeplitz algebras, *Integral Equations Operator Theory* **48**(2004), 281–293.

[2] J. CUNTZ, Simple C^*-algebras generated by isometries, *Comm. Math. Phys.* **57**(1977), 173–185.

[3] K. DAVIDSON, *C^*-Algebras by Example*, Fields Inst. Monogr., Amer. Math. Soc., vol. 6, Providence, RI 1996.

[4] R. EXEL, Amenability for Fell bundles, *J. Reine Angew. Math.* **492**(1997), 41–73.

[5] K.D. JOSHI, *Introduction to General Topology*, A Halsted Press Book, John Wiley and Sons, Inc., New York 1983.

[6] M. LACA, Purely infinite simple Toeplitz algebras, *J. Operator Theory* **41**(1999), 421–435.

[7] M. LACA, I. RAEBURN, Semigroup crossed products and the Toeplitz algebras of nonabelian groups, *J. Funct. Anal.* **139**(1996), 415–440.

[8] J. LORCH, Q. XU, Quasi-lattice ordered groups and Toeplitz algebras, *J. Operator Theory* **50**(2003), 221–247.

[9] G. MURPHY, An index theorem for Toeplitz operators, *J. Operator Theory* **29**(1993), 97–114.

[10] A. NICA, C^*-algebras generated by isometries and Wiener–Hopf operators, *J. Operator Theory* **27**(1992), 17–52.

[11] Q. XU, X. CHEN, A note on Toeplitz operators on discrete groups, *Proc. Amer. Math. Soc.* **126**(1998), 3625–3631.

[12] Q. XU, Q. XIAO, The primitive ideals of Toeplitz algebras on discrete abelian ordered groups, *Bull. Adv. Math.* **58**(2006), 1–3.

QINGXIANG XU, DEPARTMENT OF MATHEMATICS, SHANGHAI NORMAL UNIVERSITY, SHANGHAI 200234, P.R. CHINA

E-mail address: qxxu@shnu.edu.cn

Received October 15, 2006.