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COMMON CYCLIC VECTORS FOR UNITARY OPERATORS

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ABSTRACT. In this paper, we determine whether or not certain natural classes of unitary multiplication operators on $L^2(\mathrm{d}\theta)$ have common cyclic vectors. For some classes which have common cyclic vectors, we obtain a classification of these vectors.

KEYWORDS: *Unitary operators, multiplication operators, cyclic vectors, Szegö's theorem.*

MSC (2000): 47A16, 47C15, 47B38.

1. INTRODUCTION

Suppose $\mathcal{H}$ is a separable complex Hilbert space and $\mathcal{B}(\mathcal{H})$ is the algebra of bounded linear operators on $\mathcal{H}$. In this paper, we explore the common cyclic vectors for certain classes of unitary operators on $\mathcal{H}$. An operator $U \in \mathcal{B}(\mathcal{H})$ is *cyclic* if there is an $x_0 \in \mathcal{H}$ such that

$$\bigvee\{U^n x_0 : n \in \mathbb{N}_0\} = \mathcal{H}.$$

In the above equation, $\bigvee$ is the closed linear span in $\mathcal{H}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The vector x_0 is called a *cyclic vector* for U. If U is unitary and cyclic with $\sigma(U) = \mathbb{T}$, where $\mathbb{T} := \{\mathrm{e}^{\mathrm{i}\theta} : 0 \leqslant \theta < 2\pi\}$, the spectral theorem assigns to each $\psi \in C(\mathbb{T})$ (the complex-valued continuous functions on $\mathbb{T}$), an operator $\psi(U) \in \mathcal{C}^*(U)$ (the C^*-algebra generated by U). Furthermore, if μ is the scalar-valued spectral measure for U, the map $\psi(U) \mapsto (M_\psi, L^2(\mu))$ is a spatial isomorphism of $\mathcal{C}^*(U)$ onto $\{(M_\psi, L^2(\mu)) : \psi \in C(\mathbb{T})\}$. We will use the notation $(M_\psi, L^2(\mu))$ for the multiplication operator $M_\psi : L^2(\mu) \to L^2(\mu)$ defined by

$$M_\psi f = \psi f, \quad f \in L^2(\mu).$$

Recall that two sets of operators $\mathcal{S}_1 \subset \mathcal{B}(\mathcal{H}_1)$ and $\mathcal{S}_2 \subset \mathcal{B}(\mathcal{H}_2)$ are *spatially isomorphic* if there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\mathcal{S}_1 U^{-1} = \mathcal{S}_2$.

The restriction of the map $\psi(U) \mapsto (M_\psi, L^2(\mu))$ to the class of cyclic unitary operators in $\mathcal{C}^*(U)$ establishes a spatial isomorphism between this class and

$$\{(M_\psi, L^2(\mu)) : \psi \in H(\mathbb{T})\}, \tag{1.1}$$

where $H(\mathbb{T})$ are the homeomorphisms of $\mathbb{T}$. We refer the reader to [5] for the details of the above discussion.

In this paper we ask whether the class of multiplication operators in equation (1.1) has a common cyclic vector. In other words, is there one function $f \in L^2(\mu)$ that is a cyclic vector for *every* $M_\psi, \psi \in H(\mathbb{T})$? The answer, a combination of Proposition 2.3 in [11], and Theorem 4.1, is contained in the following theorem.

THEOREM 1.1. *Let μ be a finite positive Borel measure on $\mathbb{T}$ whose support is $\mathbb{T}$. If μ is discrete, then the class in equation* (1.1) *has a common cyclic vector while if μ is continuous, the class in equation* (1.1) *does not have a common cyclic vector.*

When the measure μ is discrete, the existence of a common cyclic vector for the class in equation (1.1) was shown by Sibilev [11] (see also [9]). When μ is continuous, the current authors in [9] proved that the larger class of multiplication operators on $L^2(\mu)$ whose symbols are univalent μ-almost everywhere does not have a common cyclic vector. In fact, the papers [9], [11] deal with the more general problem of determining if the cyclic operators in $\mathcal{W}^*(N)$, the von Neumann algebra generated by a cyclic normal operator $N \in \mathcal{B}(\mathcal{H})$ have a common cyclic vector. When the scalar-valued spectral measure for N is continuous, the current authors in [9] show that the cyclic operators in $\mathcal{W}^*(N)$ do not have a common cyclic vector. They do this by first proving the result when N is the operator $N_0 := (M_z, L^2(m))$, where m is normalized Lebesgue measure on $\mathbb{T}$. Then, by means of Lebesgue spaces [6], they prove the result for $\mathcal{W}^*(N)$ by establishing a spatial isomorphism between the cyclic operators in $\mathcal{W}^*(N)$ and the cyclic operators in $\mathcal{W}^*(N_0)$.

Using standard techniques (see Proposition 2.3 and the final remark at the end of the paper), we reduce our common cyclic vector problem to the case where $\mu = m$. In this case (see Theorem 4.1, Theorem 5.1, and Theorem 5.2 below), we have the following theorem.

THEOREM 1.2. *Let B denote the homeomorphisms h of $[0, 2\pi]$ such that h is absolutely continuous with $h' \neq 0$ almost everywhere and B_1 denote the $h \in B$ such that h^{-1} is Lipschitz in some interval I_h.*

(i) *The class $\{(M_{e^{ih}}, L^2(m)) : h \in B\}$ does not have a common cyclic vector.*

(ii) *The class $\{(M_{e^{ih}}, L^2(m)) : h \in B_1\}$ does have a common cyclic vector and in fact, $\phi \in L^2(m)$ is a common cyclic vector for this class if and only if $|\phi| > 0$ almost everywhere and* $\log|\phi|$ *is not integrable on any arc of $\mathbb{T}$.*

As an aside, we point out that $\{M_{e^{ih}} : h \in B\}$ are precisely the cyclic unitary operators in $\mathcal{C}^*(N_0)$ which are unitarily equivalent to N_0.

We also address the following question. For a given set A of $\phi \in L^2(m)$ such that $|\phi| > 0$ almost everywhere, what are the multiplication operators $M_\psi, \psi \in H(\mathbb{T})$, which have A as common cyclic vectors? Reversing this question, we ask, for a given set $\mathcal{A} \subset \{M_\psi : \psi \in H(\mathbb{T})\}$, what are the common cyclic vectors for $\mathcal{A}$?

The study of common cyclic vectors is not new. For example, the papers [1], [12] study common cyclic vectors for adjoints of multiplication operators on various Banach spaces of analytic functions.

2. BASIC FACTS ABOUT CYCLIC OPERATORS

Let us mention a few facts about cyclic multiplication operators and then establish a spatial isomorphism between the classes

$$\{(M_\psi, L^2(\mu)) : \psi \in H(\mathbb{T})\} \quad \text{and} \quad \{(M_\psi, L^2(m)) : \psi \in H(\mathbb{T})\}$$

when μ is a continuous probability measure on $\mathbb{T}$ whose support is all of $\mathbb{T}$. We start with two results for general positive, finite, compactly supported, Borel measures μ on the plane. The first result is trivial.

PROPOSITION 2.1. *If $\psi \in L^\infty(\mu)$ and $\phi \in L^2(\mu)$ is a cyclic vector for $(M_\psi, L^2(\mu))$, then $|\phi| > 0$ μ-a.e.*

This next result is a bit more complicated.

PROPOSITION 2.2. *The operator $(M_\psi, L^2(\mu))$ is cyclic if and only if ψ is injective on a set of full μ-measure.*

Proof. When ψ is injective on a set of full μ-measure, the change of variables formula on p. 180 in [7] shows that the operator

$$V : L^2(\mu) \to L^2(\mu \circ \psi^{-1}), \quad Vf := f \circ \psi^{-1}, \tag{2.1}$$

is unitary and

$$V(M_\psi, L^2(\mu)) = (M_z, L^2(\mu \circ \psi^{-1}))V.$$

Thus to show that $(M_\psi, L^2(\mu))$ is cyclic, it is enough to show that the operator $(M_z, L^2(\mu \circ \psi^{-1}))$ is cyclic. This last fact is a beautiful result of Bram ([2], Theorem 6). The other direction of the proof is a bit more delicate ([10], Lemma 3.1). ∎

We now show that we can focus our attention on the class

$$\{(M_\psi, L^2(m)) : \psi \in H(\mathbb{T})\}.$$

PROPOSITION 2.3. *For a continuous probability measure μ on $\mathbb{T}$ whose support is all of $\mathbb{T}$, the C^*-algebras $C^*(M_z, L^2(\mu))$ and $C^*(M_z, L^2(m))$ are spatially isomorphic. Moreover, the classes*

$$\{(M_\psi, L^2(\mu)) : \psi \in H(\mathbb{T})\} \quad \textit{and} \quad \{(M_\psi, L^2(m)) : \psi \in H(\mathbb{T})\}$$

are spatially isomorphic.

Proof. The measure μ_1 on $[0,2\pi]$ defined by

$$\mu_1(E) := \mu(\{e^{i\theta} : \theta \in E\}), \quad E \subset [0,2\pi],$$

is a continuous probability measure on $[0,2\pi]$ whose support is all of $[0,2\pi]$. Consequently, the distribution function

$$h(\theta) := 2\pi\mu_1([0,\theta]), \quad \theta \in [0,2\pi],$$

is continuous and strictly increasing with $h(0) = 0$ and $h(2\pi) = 2\pi$. That is to say, h is a homeomorphism of $[0,2\pi]$ and hence

$$\psi(e^{i\theta}) := e^{ih(\theta)} \in H(\mathbb{T}).$$

Since

$$\mu_1([a,b]) = \frac{1}{2\pi}(h(b) - h(a)),$$

the pull-back measure $\mu_1 \circ h^{-1}$ is $m_1/2\pi$, where m_1 is Lebesgue measure on $[0,2\pi]$. Thus $\mu \circ \psi^{-1} = m$. Hence the operator $V : L^2(\mu) \to L^2(m)$ from equation (2.1) is unitary and, for $\varphi \in H(\mathbb{T})$,

$$V(M_\varphi, L^2(\mu)) = (M_{\varphi\circ\psi^{-1}}, L^2(m))V.$$

It follows from the spectral theorem that the map

$$(M_\varphi, L^2(\mu)) \mapsto (M_{\varphi\circ\psi^{-1}}, L^2(m))$$

is the desired spatial isomorphism. ∎

Thus, in order to study cyclic vector questions for

$$\{(M_\psi, L^2(\mu)) : \psi \in H(\mathbb{T})\},$$

it suffices to consider the class

$$\{(M_\psi, L^2(m)) : \psi \in H(\mathbb{T})\}.$$

REMARK 2.4. The notation $(M_\psi, L^2(m))$ is somewhat cumbersome and is unnecessary beyond this point since, for the rest of the paper, we will be restricting our discussion to multiplication operators on $L^2(m)$. Thus we will use M_ψ to denote $(M_\psi, L^2(m))$.

This next result is known. However, for the benefit of the reader, we include a short proof.

PROPOSITION 2.5. *Let $\psi \in H(\mathbb{T})$. A function $\phi \in L^2(m)$ is cyclic for M_ψ if and only if the following two conditions are satisfied:*

$$|\phi| > 0 \text{ almost everywhere;} \tag{2.2}$$

$$\inf_p \int_{\mathbb{T}} |p(\psi)\phi - \overline{\psi}\phi|^2 dm = 0. \tag{2.3}$$

REMARK 2.6. In equation (2.3) and throughout this paper, the expression

$$\inf_p$$

will mean the infimum over all analytic polynomials p with complex coefficients.

Proof. If ϕ is cyclic for M_ψ, then, using Proposition 2.1, the two conditions hold. Now suppose that the conditions in equation (2.2) and equation (2.3) hold. Then there is a sequence of polynomials $(p_n)_{n\geqslant 1}$ such that $p_n(\psi)\phi \to \overline{\psi}\phi$ in the norm of $L^2(m)$ as $n \to \infty$. For any analytic polynomial q, notice that $q(\psi)p_n(\psi)\phi \to q(\psi)\overline{\psi}\phi$ in $L^2(m)$. From here it follows that

$$\bigvee\{\psi^n\overline{\psi}^m\phi : m,n \in \mathbb{N}_0\} \subset \bigvee\{\psi^n\phi : n \in \mathbb{N}_0\}.$$

Since ψ is a homeomorphism of $\mathbb{T}$, the Stone–Weierstrass theorem says that the complex, uniformly closed, $*$-algebra generated by ψ is equal to $C(\mathbb{T})$. This, together with the fact that $|\phi| > 0$ almost everywhere, will show that

$$L^2(m) = \bigvee\{\psi^n\overline{\psi}^m\phi : m,n \in \mathbb{N}_0\} \subset \bigvee\{\psi^n\phi : n \in \mathbb{N}_0\}.$$

This proves that ϕ is cyclic for M_ψ. ∎

A famous theorem of Szegö gives a formula for the infimum on the left-hand side of equation (2.3) when $\psi(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{e}^{\mathrm{i}\theta}$ and thus gives a complete characterization of the cyclic vectors for $M_{\mathrm{e}^{\mathrm{i}\theta}}$. The general form of Szegö's theorem is the following ([5], p. 143).

THEOREM 2.7 (Szegö). *For a positive finite Borel measure μ on $\mathbb{T}$,*

$$\inf_p \int_{\mathbb{T}} |p - \overline{z}|^2 \mathrm{d}\mu = \exp\Big(\int_{\mathbb{T}} \log\Big(\frac{\mathrm{d}\mu}{\mathrm{d}m}\Big)\mathrm{d}m\Big). \tag{2.4}$$

COROLLARY 2.8. *A function $\phi \in L^2(m)$ is cyclic for $M_{\mathrm{e}^{\mathrm{i}\theta}}$ if and only if $|\phi| > 0$ almost everywhere and* $\log|\phi|$ *is not integrable on $\mathbb{T}$.*

Proof. Apply Proposition 2.5 and Szegö's theorem to the measure $\mathrm{d}\mu = |\phi|^2\mathrm{d}m$. ∎

3. AN EXTENSION OF SZEGÖ'S FORMULA

To discuss cyclic and common cyclic vectors, we need the following two extensions of Szegö's formula. These two extensions are basically two changes of variables and a use of the classical Szegö's theorem. Let us recall the following version of the change of variable formula ([7], p. 344).

PROPOSITION 3.1. *Suppose τ is an absolutely continuous homeomorphism of $[0,2\pi]$ and $f \in L^1[0,2\pi]$. Then we have $(f \circ \tau)|\tau'| \in L^1[0,2\pi]$ and*

$$\int_0^{2\pi} f(t)\mathrm{d}t = \int_0^{2\pi} f \circ \tau(x)|\tau'(x)|\mathrm{d}x.$$

If $\psi \in H(\mathbb{T})$, then $\psi = \mathrm{e}^{\mathrm{i}c}\mathrm{e}^{\mathrm{i}h}$, where h is a homeomorphism of $[0,2\pi]$ and c is a real constant. Since multiplication by the complex constant $\mathrm{e}^{\mathrm{i}c}$ preserves cyclicity, we assume henceforth that $c = 0$. Note that h and h^{-1} are strictly monotone and hence are of bounded variation. Thus their derivatives exist and are finite almost everywhere, they can be written uniquely (up to additive constants) as the sum of an absolutely continuous and a singular function, etc. (see [7], [8]).

THEOREM 3.2. *Suppose $\psi = \mathrm{e}^{\mathrm{i}h} \in H(\mathbb{T})$. Then for $\phi \in L^2(m)$ we have*

$$\inf_p \int_{\mathbb{T}} |p(\psi)\phi - \overline{\psi}\phi|^2 \mathrm{d}m = \exp\Big(\int_0^{2\pi} \log(|\phi(\mathrm{e}^{\mathrm{i}h^{-1}(\theta)})|^2|(h^{-1})'(\theta)|)\frac{\mathrm{d}\theta}{2\pi}\Big).$$

Proof. Without loss of generality, assume that h^{-1} is increasing. By the Lebesgue decomposition theorem,

$$h^{-1} = (h^{-1})_{\mathrm{a}} + (h^{-1})_{\mathrm{s}}$$

where $(h^{-1})_{\mathrm{a}}$ and $(h^{-1})_{\mathrm{s}}$ are the absolutely continuous and singular parts of h^{-1}. Notice also that

$$\frac{\mathrm{d}(m_1 \circ h^{-1})}{\mathrm{d}m_1} = ((h^{-1})_{\mathrm{a}})' = (h^{-1})' \quad \text{a.e.} \tag{3.1}$$

Then we have

$$\begin{aligned}
&\inf_p \int_{\mathbb{T}} |p(\psi)\phi - \overline{\psi}\phi|^2 \mathrm{d}m \\
&= \inf_p \int_{\mathbb{T}} |p(\psi) - \overline{\psi}|^2 |\phi|^2 \mathrm{d}m \\
&= \inf_p \int_0^{2\pi} |p(\mathrm{e}^{\mathrm{i}h(\theta)}) - \mathrm{e}^{-\mathrm{i}h(\theta)}|^2 |\phi(\mathrm{e}^{\mathrm{i}\theta})|^2 \frac{\mathrm{d}\theta}{2\pi} \\
&= \inf_p \int_0^{2\pi} |p(\mathrm{e}^{\mathrm{i}\theta}) - \mathrm{e}^{-\mathrm{i}\theta}|^2 |\phi(\mathrm{e}^{\mathrm{i}h^{-1}(\theta)})|^2 \mathrm{d}\Big(\frac{m_1}{2\pi} \circ h^{-1}\Big) \\
&= \exp\Big(\int_0^{2\pi} \log\Big(|\phi(\mathrm{e}^{\mathrm{i}h^{-1}(\theta)})|^2 \frac{\mathrm{d}(m_1 \circ h^{-1})}{\mathrm{d}m_1}\Big)\frac{\mathrm{d}\theta}{2\pi}\Big) \quad \text{(Theorem 2.7)}
\end{aligned}$$

$$= \exp\Big(\int_0^{2\pi} \log(|\phi(e^{ih^{-1}(\theta)})|^2 (h^{-1})'(\theta)) \frac{d\theta}{2\pi}\Big) \quad \text{(equation (3.1))}. \qquad \blacksquare$$

Recall from the introduction that B denotes the class of absolutely continuous homeomorphisms h of $[0, 2\pi]$ such that $h' \neq 0$ almost everywhere.

THEOREM 3.3. *Suppose $\psi = e^{ih}$, where $h \in B$. Then for any $\phi \in L^2(m)$, we have*

$$\inf_p \int_{\mathbb{T}} |p(\psi)\phi - \overline{\psi}\phi|^2 dm = \exp\Big(\int_0^{2\pi} \log\Big(\frac{|\phi(e^{i\theta})|^2}{|h'(\theta)|}\Big) |h'(\theta)| \frac{d\theta}{2\pi}\Big).$$

To prove this theorem, apply the change of variable $\theta = h(t)$ in Theorem 3.2 along with Proposition 3.1 and the following lemma.

LEMMA 3.4. *If $h \in B$, then for almost every $\theta \in [0, 2\pi]$,*

$$(h^{-1})'(h(\theta)) = \frac{1}{h'(\theta)}. \tag{3.2}$$

Proof. The definition of the derivative shows that whenever $\theta \in [0, 2\pi]$ and $h'(\theta)$ is finite and non-zero, then equation (3.2) holds.

The result now follows since $h \in B$ and so h' is both finite and non-zero almost everywhere. $\blacksquare$

REMARK 3.5. (i) There are homeomorphisms of $[0, 2\pi]$ whose derivative vanishes almost everywhere ([7], p. 278). There are also absolutely continuous homeomorphisms h of $[0, 2\pi]$ such that $h' = 0$ on a set of positive measure.

(ii) It is routine to check that $h \in B$ if and only if $h^{-1} \in B$.

The following triviality will be used several times in this paper.

LEMMA 3.6. *If $f \in L^1[0, 2\pi]$ and $|f| > 0$ almost everywhere, then*

$$\int_0^{2\pi} |f| \log |f| d\theta \in (-\infty, \infty].$$

4. LACK OF COMMON CYCLIC VECTORS

In a previous paper [9] the current authors showed that the class of multiplication operators on $L^2(m)$ with bounded almost injective symbols does not have a common cyclic vector, even though each operator individually is cyclic. The following says that a much smaller class does not have a common cyclic vector.

THEOREM 4.1. *The class $\{M_\psi : \psi = e^{ih}, h \in B\}$ does not have a common cyclic vector.*

This follows immediately from the following.

THEOREM 4.2. *Suppose $\phi \in L^2(m)$ and $|\phi| > 0$ almost everywhere. Then there is an $h \in B$ such that ϕ is not cyclic for $M_{e^{ih}}$.*

Proof. Without loss of generality, suppose that

$$\int_0^{2\pi} |\phi(e^{it})|^2 dt = 2\pi.$$

Define the function

$$h(\theta) := \int_0^{\theta} |\phi(e^{it})|^2 dt, \quad \theta \in [0, 2\pi].$$

Since $|\phi| > 0$ almost everywhere, h is a strictly increasing absolutely continuous function from the interval $[0, 2\pi]$ onto itself and moreover, from the Lebesgue differentiation theorem,

$$h'(\theta) = |\phi(e^{i\theta})|^2 > 0 \quad \text{a.e.} \tag{4.1}$$

Thus $h \in B$. We will now prove that ϕ is *not* a cyclic vector for $M_{e^{ih}}$. Indeed, by Theorem 3.3,

$$\begin{aligned} \inf_p \int_{\mathbb{T}} |p(\psi)\phi - \overline{\psi}\phi|^2 dm &= \exp\Big(\int_0^{2\pi} \log \Big(|\phi(e^{i\theta})|^2 \frac{1}{h'(\theta)} \Big) h'(\theta) \frac{d\theta}{2\pi} \Big) \\ &= \exp\Big(\log 1 \int_0^{2\pi} |\phi(e^{i\theta})|^2 \frac{d\theta}{2\pi} \Big) \quad \text{(equation (4.1))} = 1. \end{aligned}$$

Now apply Proposition 2.5. ∎

5. A POSITIVE RESULT

We know from Theorem 4.1 that the class $\{M_\psi : \psi = e^{ih}, h \in B\}$ does not have a common cyclic vector. However, a slightly smaller subclass does have a common cyclic vector.

THEOREM 5.1. *Let B_1 denote the subclass of B consisting of the $h \in B$ such that h^{-1} is Lipschitz in some interval I_h. Then, the following class of multiplication operators has a common cyclic vector:*

$$V := \{M_\psi : \psi = e^{ih}, h \in B_1\}.$$

Proof. Suppose $0 < |\phi| \leqslant 1$ almost everywhere and not log-integrable on any arc (see Remark 5.3 below for an example of such a function). We will now show that ϕ is a common cyclic vector for the class V.

Fix $\psi = e^{ih} \in V$ and assume, without loss of generality, that $h' > 0$ almost everywhere. The hypothesis implies there is an interval $I \subset [0, 2\pi]$ (depending on h) and some $\delta > 0$ so that $h' \geqslant \delta$ almost everywhere on I. Theorem 3.3 says that

$$\inf_p \int_{\mathbb{T}} |p(\psi)\phi - \overline{\psi}\phi|^2 \mathrm{d}m = \exp\Big(\int_0^{2\pi} \log\Big(|\phi(e^{i\theta})|^2 \frac{1}{h'(\theta)}\Big) h'(\theta)\frac{\mathrm{d}\theta}{2\pi}\Big). \tag{5.1}$$

Notice that $\log|\phi|^2 \leqslant 0$ and so

$$\int_I h' \log|\phi|^2 \mathrm{d}\theta \leqslant \delta \int_I \log|\phi|^2 \mathrm{d}\theta = -\infty. \tag{5.2}$$

Again, using the inequality $\log|\phi|^2 \leqslant 0$, we have

$$\int_{[0,2\pi]\setminus I} h' \log|\phi|^2 \mathrm{d}\theta \in [-\infty, 0]. \tag{5.3}$$

Finally, observe from Lemma 3.6 that

$$\int_0^{2\pi} h' \log h' \mathrm{d}\theta \in (-\infty, \infty]. \tag{5.4}$$

From equation (5.2), equation (5.3), and equation (5.4), it follows that

$$\int_0^{2\pi} \log\Big(|\phi|^2 \frac{1}{h'}\Big) h' \mathrm{d}\theta = \int_I h' \log|\phi|^2 \mathrm{d}\theta + \int_{[0,2\pi]\setminus I} h' \log|\phi|^2 \mathrm{d}\theta - \int_0^{2\pi} h' \log h' \mathrm{d}\theta = -\infty.$$

Thus, from equation (5.1), Theorem 3.3, and Proposition 2.5, ϕ is a cyclic vector for M_ψ. ∎

The proof of Theorem 5.1 provides a collection of common cyclic vectors for V. We now generalize Theorem 5.1 by describing *all* common cyclic vectors for V.

THEOREM 5.2. *A function $\phi \in L^2(m)$ is a common cyclic vector for the class V if and only if $|\phi| > 0$ almost everywhere and* $\log|\phi|$ *is not integrable on any arc of* $\mathbb{T}$.

Proof. Let us first prove the result for *bounded* ϕ. Indeed, the proof of Theorem 5.1 shows that any bounded ϕ for which $|\phi| > 0$ almost everywhere and $\log|\phi|$ is not integrable on any arc of $\mathbb{T}$, is a cyclic vector for V.

Now suppose that ϕ is bounded and is a common cyclic vector for the class V. From Proposition 2.1 we know that $|\phi| > 0$ almost everywhere. Let I be an interval in $[0, 2\pi]$ and define

$$h(\theta) := c \int_0^\theta k(t)\mathrm{d}t, \quad \theta \in [0, 2\pi], \tag{5.5}$$

where

$$k(t) := \begin{cases} |\phi(e^{it})|^2 & t \notin I, \\ 1 & t \in I, \end{cases}$$

and the *positive* constant c guarantees that h maps $[0,2\pi]$ onto itself. One can check that $M_\psi := M_{e^{ih}} \in V$ (i.e., $h \in B_1$) and moreover, using the fact that ϕ is cyclic for M_ψ, along with Proposition 2.5 and Theorem 3.3, we get

$$0 = \inf_p \int_{\mathbb{T}} |p(\psi)\phi - \overline{\psi}\phi|^2 \mathrm{d}m = \exp\Big(\int_0^{2\pi} \log\Big(|\phi|^2 \frac{1}{h'}\Big) h' \frac{\mathrm{d}\theta}{2\pi}\Big). \tag{5.6}$$

We now examine the integral in the exponential in the above formula. Notice that

$$\int_I h' \log|\phi|^2 \mathrm{d}\theta = c \int_I \log|\phi|^2 \mathrm{d}\theta. \tag{5.7}$$

From the definition of h in equation (5.5),

$$\int_{[0,2\pi]\setminus I} h' \log|\phi|^2 \mathrm{d}\theta = \int_{[0,2\pi]\setminus I} c|\phi|^2 \log|\phi|^2 \mathrm{d}\theta = c_1 \in (-\infty, \infty), \tag{5.8}$$

since ϕ is bounded and Lemma 3.6. We also see, since h' is bounded, that

$$\int_0^{2\pi} h' \log h' \mathrm{d}\theta = c_2 \in (-\infty, \infty). \tag{5.9}$$

From equation (5.7), equation (5.8), and equation (5.9), it follows that

$$\begin{aligned}\int_0^{2\pi} \log\Big(|\phi|^2 \frac{1}{h'}\Big) h' \mathrm{d}\theta &= \int_I h' \log|\phi|^2 \mathrm{d}\theta + \int_{[0,2\pi]\setminus I} h' \log|\phi|^2 \mathrm{d}\theta - \int_0^{2\pi} h' \log h' \mathrm{d}\theta \\ &= c \int_I \log|\phi|^2 \mathrm{d}\theta + c_1 + c_2 \in [-\infty, \infty).\end{aligned}$$

Since, from equation (5.6),

$$0 = \exp\Big(c \int_I \log|\phi|^2 \mathrm{d}\theta + c_1 + c_2\Big),$$

and $c > 0$, this forces the condition

$$\int_I \log|\phi|^2 \mathrm{d}\theta = -\infty.$$

Hence $|\phi| > 0$ almost everywhere and $\log|\phi|$ is not log-integrable on any arc of $\mathbb{T}$.

Thus far, we have established the result for bounded ϕ. Let us now prove the result for general $\phi \in L^2(m)$.

Suppose that ϕ is a common cyclic vector for V. Let $\phi_1 = \min(1, |\phi|)$ and observe that for any $M_\psi \in V$, and any analytic polynomial p,

$$\int_{\mathbb{T}} |p(\psi)\phi - \overline{\psi}\phi|^2 \mathrm{d}m = \int_{\mathbb{T}} |p(\psi) - \overline{\psi}|^2 |\phi|^2 \mathrm{d}m \geqslant \int_{\mathbb{T}} |p(\psi)\phi_1 - \overline{\psi}\phi_1|^2 \mathrm{d}m.$$

It follows from Proposition 2.5 that ϕ_1 is a common cyclic vector for V. Since we already know the result is true for bounded functions, we see that $\phi_1 > 0$ almost everywhere and $\log \phi_1$ is not integrable on any arc of $\mathbb{T}$. Let I be any arc of $\mathbb{T}$ and note, since $\phi \in L^1(m)$, that

$$\int_{\mathbb{T}} \log^+ |\phi| \mathrm{d}m < \infty.$$

Thus

$$\int_I \log |\phi| \mathrm{d}m = \int_I \log^+ |\phi| \mathrm{d}m + \int_I \log \phi_1 \mathrm{d}m = -\infty. \tag{5.10}$$

Hence we have shown that if $\phi \in L^2(m)$ is a common cyclic vector for V, then $|\phi| > 0$ almost everywhere and $\log |\phi|$ is not integrable on any arc of $\mathbb{T}$.

We will now prove the other direction. Assume $\phi \in L^2(m)$, $|\phi| > 0$ almost everywhere, and $\log |\phi|$ is not integrable on any arc of $\mathbb{T}$. Since ϕ and ϕ_1 share the same zeros, we see that $\phi_1 > 0$ almost everywhere. From equation (5.10) we conclude that $\log \phi_1$ is not integrable on any arc of $\mathbb{T}$.

By Theorem 3.3, it suffices to show, assuming without loss of generality that $h' > 0$ almost everywhere, that for fixed $M_{\mathrm{e}^{\mathrm{i}h}} \in V$,

$$\int_0^{2\pi} (h' \log |\phi|^2 - h' \log h') \mathrm{d}\theta = -\infty.$$

The above integral appears as the quantity in the exp in Theorem 3.3 and so it assumes a value in $[-\infty, \infty)$. Thus the above integral over any subset $E \subset [0, 2\pi]$ also has its value in $[-\infty, \infty)$. Thus to finish, we will show there is some $E \subset [0, 2\pi]$ of positive measure such that

$$\int_E (h' \log |\phi|^2 - h' \log h') \mathrm{d}\theta = -\infty. \tag{5.11}$$

By our assumptions on h, we know there is some interval $I \subset [0, 2\pi]$ and some $\delta > 0$ such that $h' \geqslant \delta$ on I. Since $\log |\phi|$ is not integrable on I, we see that the set

$$E := I \cap \{|\phi| \leqslant 1\}$$

has positive measure. Observe that

$$-\int_E h' \log h' \mathrm{d}\theta \in [-\infty, \infty) \tag{5.12}$$

since

$$\int_E h' \log h' \mathrm{d}\theta \geqslant \log \delta \int_E h' \mathrm{d}\theta$$

and h' is integrable. Finally, since $\log|\phi|^2 \leqslant 0$ on E, we see that

$$\int_E h' \log|\phi|^2 \mathrm{d}\theta \leqslant \delta \int_E \log|\phi|^2 \mathrm{d}\theta = \delta \int_E \log \phi_1^2 \mathrm{d}\theta.$$

But this last integral is equal to

$$\int_I \log \phi_1^2 \mathrm{d}\theta$$

since $\phi \geqslant 1$ on $I \setminus E$ and so $\phi_1 = 1$ on $I \setminus E$. However, $\log \phi_1$ is not integrable on I and so, combining this with equation (5.12), we have shown equation (5.11) and the proof is now complete. ∎

REMARK 5.3. (i) An example of a bounded ϕ such that $\phi > 0$ almost everywhere and such that $\log|\phi|$ is not integrable on any arc of $\mathbb{T}$ is

$$\phi(\mathrm{e}^{\mathrm{i}\theta}) := \exp\Big(-\sum_{n=1}^{\infty} \frac{2^{-n}}{|\theta - a_n|}\Big),$$

where $(a_n)_{n\geqslant 1}$ is a dense sequence in $[0, 2\pi]$. It is routine to see that $\log|\phi|$ is not integrable on any arc of $\mathbb{T}$. To see that $\phi > 0$ almost everywhere, it suffices to show that the quantity

$$\sum_{n=1}^{\infty} \frac{2^{-n}}{|\theta - a_n|}$$

is finite almost everywhere. Let

$$g(\theta) := \sum_{n=1}^{\infty} \frac{2^{-n/2}}{|\theta - a_n|^{1/2}}$$

and note that a simple integral computation shows that $g \in L^1[0, 2\pi]$. Thus $g(\theta)$ and hence $g(\theta)^2$ is finite almost everywhere. Moreover,

$$\sum_{n=1}^{\infty} \frac{2^{-n}}{|\theta - a_n|} \leqslant g(\theta)^2.$$

(ii) As a consequence of Szegö's theorem, we know that $\phi \in L^2(m)$ is a cyclic vector for $M_{\mathrm{e}^{\mathrm{i}\theta}}$ if and only if its truncation ϕ_1 is cyclic for $M_{\mathrm{e}^{\mathrm{i}\theta}}$. The proof of Theorem 5.2 shows that ϕ is a cyclic for $M_\psi, \psi \in V$ if and only if ϕ_1 is cyclic for M_ψ. The same is no longer true when V is replaced by the larger class $\{M_\psi : \psi = \mathrm{e}^{\mathrm{i}h}, h \in B\}$. For example, let $h \in B$ be defined by

$$h'(\theta) = \begin{cases} \frac{1}{\theta(\log\theta)^2} & \theta \in [0, \frac{1}{2}], \\ 1 & \theta \in [0, 2\pi] \setminus [0, \frac{1}{2}]. \end{cases}$$

Notice that $h' \in L^1[0,2\pi]$ and $h' \geqslant 1$ but $h' \log h' \notin L^1[0,2\pi]$. Let $\phi(e^{i\theta}) = \sqrt{h'(\theta)}$ and observe that $\phi \geqslant 1$ and so $\phi_1 \equiv 1$. Thus

$$\int_0^{2\pi} (h' \log \phi_1^2 - h' \log h')d\theta = -\int_0^{2\pi} h' \log h' = -\infty$$

and so $\phi_1 \equiv 1$ is cyclic for $M_{e^{ih}}$ (Theorem 3.3). But

$$\int_0^{2\pi} (h' \log \phi^2 - h' \log h')d\theta = 0$$

and so ϕ is not cyclic for $M_{e^{ih}}$.

(iii) Using Theorem 3.3 we also observe that $\phi \equiv 1$ is a common cyclic vector for the class of $M_{e^{ih}}$, $h \in B$, for which

$$\int_0^{2\pi} |h'| \log |h'| d\theta = \infty.$$

Note from Lemma 3.6 that the above integral cannot evaluate to $-\infty$.

6. CLASSES OF CYCLIC OPERATORS AND COMMON CYCLIC VECTORS

Let

$$Q := \{\phi \in L^2(m) : |\phi| > 0 \text{ a.e.}\}.$$

We know from Proposition 2.5 that every cyclic vector for some M_ψ, $\psi \in H(\mathbb{T})$, must belong to Q. For certain symbols, this is the only requirement for cyclicity (see below). Other symbols, $\psi(e^{i\theta}) = e^{i\theta}$ for example, require more than the condition $\phi \in Q$ to be cyclic for M_ψ (see Corollary 2.8).

On the other hand, let us consider the class of operators

$$R := \{M_{e^{ih}} : e^{ih} \in H(\mathbb{T}), m_1(\{(h^{-1})' = 0\}) > 0\}.$$

An application of Proposition 2.5 along with Theorem 3.2 and the hypothesis that $(h^{-1})' = 0$ on some set of positive measure, shows that every $\phi \in Q$ is a common cyclic vector for R.

For a set

$$\mathcal{A} \subset \{M_\psi : \psi \in H(\mathbb{T})\},$$

consider the set $\mathcal{C}(\mathcal{A})$ of common cyclic vectors for $\mathcal{A}$. By Proposition 2.1, we know that

$$\mathcal{C}(\mathcal{A}) \subset Q. \tag{6.1}$$

Given a non-empty set $A \subset Q$, consider the set

$$\mathcal{L}(A) := \{M_\psi, \psi \in H(\mathbb{T}) : \text{every } \phi \in A \text{ is a cyclic vector for } M_\psi.\}.$$

Since every $\phi \in Q$ is a common cyclic vector for R, we see that

$$R \subset \mathcal{L}(A).$$

Using this terminology, we collect our previous results in the following proposition.

PROPOSITION 6.1. (i) (*Theorem* 3.2) *For each homeomorphism h of* $[0, 2\pi]$,

$$\mathcal{C}(\{M_{e^{ih}}\}) = \Big\{\phi \in Q : \int_0^{2\pi} \log(|\phi(e^{ih^{-1}(\theta)})|^2 |(h^{-1})'(\theta)|) d\theta = -\infty\Big\}.$$

(ii) (*Theorem* 4.1) $\mathcal{C}(\{M_{e^{ih}} : h \in B\}) = \emptyset$.

(iii) (*Theorem* 5.2) $\mathcal{C}(V) = \{\phi \in Q : \log|\phi|$ *is not integrable on any arc of* $\mathbb{T}\}$.

(iv) (*Theorem* 3.2 *and equation* (6.1)) *For each* $M_{e^{ih}} \in R$,

$$\mathcal{C}(\{M_{e^{ih}}\}) = Q. \tag{6.2}$$

Thus $\mathcal{C}(R) = Q$.

(v) (*Theorem* 3.2) *For each* $\phi \in Q$,

$$\mathcal{L}(\{\phi\}) = \Big\{M_{e^{ih}} : \int_0^{2\pi} \log(|\phi(e^{ih^{-1}(\theta)})|^2 |(h^{-1})'(\theta)|) d\theta = -\infty\Big\}.$$

THEOREM 6.2. $\mathcal{L}(Q) = R$.

Proof. Suppose $M_{e^{ih}} \in R$. An application of Proposition 2.5 along with Theorem 3.2 and the hypothesis that $(h^{-1})' = 0$ on some set of positive measure, shows that every $\phi \in Q$ is a cyclic vector for $M_{e^{ih}}$. Hence $M_{e^{ih}} \in \mathcal{L}(Q)$.

Now suppose that $M_{e^{ih}} \notin R$. Without loss of generality, let us assume that $(h^{-1})' > 0$ almost everywhere. This says that $h' \geqslant 0$ almost everywhere. The proof of Lemma 3.4 shows that whenever $(h^{-1})'(\theta)$ is both finite and non-zero, the formula

$$h'(h^{-1}(\theta))(h^{-1})'(\theta) = 1 \tag{6.3}$$

holds. Since $(h^{-1})'(\theta)$ is both finite and non-zero for almost all θ, equation (6.3) holds for almost all θ.

Let

$$\phi(e^{i\theta}) := \begin{cases} \sqrt{h'(\theta)} & \text{if } h'(\theta) > 0, \\ 1 & \text{if } h'(\theta) = 0. \end{cases}$$

Then $\phi \in Q$ and from Theorem 3.2 we have

$$\inf_p \int_{\mathbb{T}} |p(\psi)\phi - \overline{\psi}\phi|^2 dm = \exp\Big(\int_0^{2\pi} \log(|\phi(e^{ih^{-1}(\theta)})|^2 (h^{-1})'(\theta)) \frac{d\theta}{2\pi}\Big).$$

Observe that

$$\log(|\phi(e^{ih^{-1}(\theta)})|^2 (h^{-1})'(\theta)) = \log(h'(h^{-1}(\theta))(h^{-1})'(\theta)) = 0$$

whenever $h'(h^{-1}(\theta)) \neq 0$. But by equation (6.3), this holds for almost every θ. Thus

$$\inf_p \int_{\mathbb{T}} |p(\psi)\phi - \overline{\psi}\phi|^2 \mathrm{d}m > 0$$

and, by Proposition 2.5, ϕ is non-cyclic for $M_{e^{ih}}$. Hence $M_{e^{ih}} \notin \mathcal{L}(Q)$. ∎

Next, we wish to explore the sets

$$\mathcal{C}(\mathcal{L}(A)) \quad \text{and} \quad \mathcal{L}(\mathcal{C}(\mathcal{A})).$$

Clearly we have the containments,

$$A \subset \mathcal{C}(\mathcal{L}(A)) \quad \text{and} \quad \mathcal{A} \subset \mathcal{L}(\mathcal{C}(\mathcal{A})).$$

Let us say that $\mathcal{C}(\mathcal{L}(A))$ is the *cyclic closure* of A and $\mathcal{L}(\mathcal{C}(\mathcal{A}))$ is the *cyclic closure* of $\mathcal{A}$. Note that the cyclic closure is actually cyclic closed. Indeed, if $A_1 = \mathcal{C}(\mathcal{L}(A))$, then $\mathcal{C}(\mathcal{L}(A_1)) = A_1$. Similarly, if $\mathcal{A}_1 = \mathcal{L}(\mathcal{C}(\mathcal{A}))$, then $\mathcal{L}(\mathcal{C}(\mathcal{A}_1)) = \mathcal{A}_1$. These definitions set up a certain "duality" between subsets of Q and subsets of $\{M_\psi : \psi \in H(\mathbb{T})\}$.

This "duality" extends to more general settings. For example, for any subalgebra $\mathfrak{B}$ of $\mathcal{B}(\mathcal{H})$, let $\mathcal{K}$ denote the cyclic operators in $\mathfrak{B}$ and let $\mathcal{Q}$ be the $x \in \mathcal{H}$ such that x is cyclic for some $T \in \mathcal{K}$. As before, given $\mathcal{A} \subset \mathcal{K}$ and $A \subset \mathcal{Q}$, we form

$$\mathcal{C}(\mathcal{A}) := \{x \in \mathcal{Q} : x \text{ is cyclic for every } T \in \mathcal{A}\},$$
$$\mathcal{L}(A) := \{T \in \mathcal{K} : \text{every } x \in A \text{ is cyclic for } T\}.$$

For example, let $\mathfrak{B}$ be the algebra of co-analytic Toeplitz operators on the Hardy space H^2 and $\mathcal{A}$ be the set of co-analytic Toeplitz operators with non-constant symbols. The main result of [12] says that $\mathcal{C}(\mathcal{A}) \neq \varnothing$. It then follows that $\mathcal{A}$ is cyclic closed. The above duality can be compared with the familiar duality between subspace lattices and operator algebras ([5], Chapter 4).

We now return to the case of multiplication operators considered in this paper. First note that Proposition 6.1 says that the sets of functions: $\varnothing$, $\{\phi \in Q : \log|\phi| \text{ is not integrable on any arc of } \mathbb{T}\}$, and Q are cyclic closed. (When $A = \varnothing$, we have $\mathcal{L}(A) = \{M_\psi : \psi \in H(\mathbb{T})\}$ by "default".) Secondly, Theorem 6.2 shows that R is a cyclic closed family of operators, and of course $\{M_\psi : \psi \in H(\mathbb{T})\}$ is cyclic closed.

Finally, it may be of interest to compute the cyclic closures, $\mathcal{L}(\mathcal{C}(\mathcal{A}))$ and $\mathcal{C}(\mathcal{L}(A))$, for other choices of $\mathcal{A} \subset \{M_\psi : \psi \in H(\mathbb{T})\}$ and $A \subset Q$. We conclude with two sample open questions.

QUESTION 6.3. If $\psi \in H(\mathbb{T})$, what is the cyclic closure of $\{M_\psi\}$? In other words, what is $\mathcal{L}(\mathcal{C}(\{M_\psi\}))$?

We can answer this question in a very special case.

COROLLARY 6.4. *If* $M_{e^{ih}} \in R$, *then* $\mathcal{L}(\mathcal{C}(\{M_{e^{ih}}\})) = R$.

Proof. Combine equation (6.2) with Theorem 6.2. ∎

The corollary actually shows that R is the cyclic closure of each nonempty subset of R.

QUESTION 6.5. If $\phi \in Q$, what is the cyclic closure of $\{\phi\}$? In other words, what is $\mathcal{C}(\mathcal{L}(\{\phi\}))$?

Routine arguments show that

$$\mathcal{C}(\mathcal{L}(\{\phi\})) \supset \{\rho\phi : \rho \in L^\infty(m) \cap Q\}.$$

We believe that the above inclusion is proper for all choices of $\phi \in Q$. For example, we can show that the inclusion is proper when $\phi \equiv 1$ by explicitly exhibiting an unbounded function in $\mathcal{C}(\mathcal{L}(\{1\}))$.

7. FINAL REMARK

The alert reader might wonder why we only consider continuous measures whose support is all of $\mathbb{T}$. If μ is any measure whose support is a proper subset K of $\mathbb{T}$, then a classical theorem of Lavrentiev ([3], p. 232) says that the the sup-norm closure on K of the analytic polynomials is $C(K)$ and it follows easily that for every homeomorphism ψ of K, the set of cyclic vectors for M_ψ is the set of all $f \in L^2(\mu)$ such that $|f| > 0$ μ-a.e. So now suppose that

$$\mu = \mu_{\mathrm{c}} + \mu_{\mathrm{d}},$$

where μ_{c} is a continuous measure with support $\mathbb{T}$ and μ_{d} is a discrete measure on $\mathbb{T}$. Then the unitary operator $(M_z, L^2(\mu))$ can be written as

$$(M_z, L^2(\mu)) = (M_z, L^2(\mu_{\mathrm{c}})) \oplus (M_z, L^2(\mu_{\mathrm{d}})).$$

If $\mathrm{Lat}(A)$ denotes the lattice of invariant subspaces of an operator A, then Corollary 2.2 of [4] says that

$$\mathrm{Lat}((M_z, L^2(\mu))) = \mathrm{Lat}((M_z, L^2(\mu_{\mathrm{c}}))) \oplus \mathrm{Lat}((M_z, L^2(\mu_{\mathrm{d}}))).$$

If follows that $f \in L^2(\mu)$ is cyclic for $(M_z, L^2(\mu))$ if and only if $f = f_{\mathrm{c}} + f_{\mathrm{d}}$ where f_{c} is cyclic for $(M_z, L^2(\mu_{\mathrm{c}}))$ and f_{d} is cyclic for $(M_z, L^2(\mu_{\mathrm{d}}))$. Since the existence of a common cyclic vector in the discrete case has been settled by Sibilev, we focus on continuous measures.

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