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UNIFORM SUBELLIPTICITY

A.F.M. TER ELST and DEREK W. ROBINSON

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ABSTRACT. We prove that uniform subellipticity of a positive symmetric second-order partial differential operator on $L_2(\mathbb{R}^d)$ is self-improving in the sense that it automatically extends to higher powers of the operator. The range of extension is governed by the degree of smoothness of the coefficients of the operator. Secondly, if the operator is of the form $\sum_{i=1}^{N} X_i^* X_i$, where the X_i are vector fields on $\mathbb{R}^d$ with coefficients in $C_b^\infty(\mathbb{R}^d)$ satisfying a uniform version of Hörmander's criterion for hypoellipticity, then we prove that it is uniformly subelliptic of order r^{-1}, where r is the rank of the set of vector fields.

KEYWORDS: *Subelliptic operator, Hörmander sums of squares, double commutators.*

MSC (2000): 47B47, 47B44, 58G03.

1. INTRODUCTION

Our aim is to derive two global subellipticity properties of second-order self-adjoint elliptic operators on $L_2(\mathbb{R}^d)$. Initially we consider operators of the form

$$H_0 = -\sum_{i,j=0}^{d} \partial_i \, c_{ij} \, \partial_j \tag{1.1}$$

with domain $D(H_0) = W^{\infty,2}(\mathbb{R}^d)$, where $\partial_0 = \mathrm{i}I$ and $\partial_j = \partial/\partial x_j$ if $j \in \{1, \ldots, d\}$. We assume throughout that the coefficients $c_{ij} \in W^{m+1,\infty}(\mathbb{R}^d)$, where $m \in \mathbb{N}$, are complex-valued and $C = (c_{ij})$ is a symmetric positive-definite matrix. In particular, the coefficients are always at least twice differentiable. Although we allow the c_{ij} to be complex one could use symmetry to re-express H_0 in the form (1.1) but with real-valued coefficients. Then, however, the corresponding c_{i0} and c_{0j} are not necessarily in $W^{m+1,\infty}(\mathbb{R}^d)$. Since $c_{ij} \in W^{2,\infty}(\mathbb{R}^d)$ it follows, however, that H_0 is essentially self-adjoint on $W^{\infty,2}(\mathbb{R}^d)$ (see, for example, Section 6 of [17], or Proposition 2.3 below) and we denote the self-adjoint closure by H.

If $\gamma \in \langle 0,1]$ then H_0 is defined to be *subelliptic of order* γ if there is a $c > 0$ such that

$$c\,(\varphi, (I + H_0)\varphi) \geqslant \|\Delta^{\gamma/2}\varphi\|_2^2 \tag{1.2}$$

for all $\varphi \in W^{\infty,2}(\mathbb{R}^d)$. Then the subellipticity condition extends to H and $c\,(I + H) \geqslant \Delta^\gamma$ in the sense of quadratic forms. A local version of condition (1.2) arose in Hörmander's work [10] and is significant as it implies hypoellipticity of H_0. The global version implies uniform boundedness of the semigroup kernel associated with H by an argument based on Nash inequalities.

Our first result establishes that the subellipticity condition is self-improving.

THEOREM 1.1. *Let H_0 be a positive, symmetric, subelliptic operator of order $\gamma \in \langle 0,1]$ with coefficients $c_{ij} \in W^{m+1,\infty}(\mathbb{R}^d)$, where $m \in \mathbb{N}$, and with self-adjoint closure H. Then $D(H^\alpha) \subseteq D(\Delta^{\alpha\gamma})$ for all $\alpha \in [0, 2^{-1}(m+1+\gamma^{-1})\rangle$ and there is a $c > 0$ such that, for all $\varphi \in D(H^\alpha)$,*

$$c\,\|(I+H)^\alpha\varphi\|_2 \geqslant \|\Delta^{\alpha\gamma}\varphi\|_2. \tag{1.3}$$

The theorem is a strengthened global version of a local result of Fefferman and Phong (see first part of Theorem 1 of [7]). Fefferman and Phong established the local version for $\alpha = 1$ by a double commutator estimate and the theory of pseudodifferential operators. The latter limits the result to operators with C^∞-coefficients. But if the coefficients are smooth then much more is true.

COROLLARY 1.2. *If H_0 is a subelliptic operator of order $\gamma \in \langle 0,1]$ with coefficients $c_{ij} \in C_b^\infty(\mathbb{R}^d)$ then $D(H^\alpha) \subseteq D(\Delta^{\alpha\gamma})$ and (1.3) is valid for all $\alpha \geqslant 0$.*

Our proof of Theorem 1.1 uses a double commutator estimate combined with techniques of functional analysis [3], [17].

Our second result deals with operators

$$H_0 = \sum_{i=1}^N X_i^*\, X_i \tag{1.4}$$

constructed from C_b^∞-vector fields $X_1, \ldots, X_N$, i.e. vector fields on $\mathbb{R}^d$ with coefficients in $C_b^\infty(\mathbb{R}^d)$, satisfying a uniform version of Hörmander's criterion for hypoellipticity. Specifically, if $r \in \mathbb{N}$ then the vector fields $X_1, \ldots, X_N$ are defined to satisfy the *uniform Hörmander condition of order r* if each C_b^∞-vector field X can be expressed as a linear combination

$$X = \sum_{\alpha\colon 1 \leqslant |\alpha| \leqslant r} \psi_\alpha\, X_{[\alpha]}$$

with $\psi_\alpha \in C_b^\infty(\mathbb{R}^d)$ where $\alpha = (i_1, \ldots, i_n)$ is a multi-index with $i_k \in \{1, \ldots, N\}$, $|\alpha| = n$, and

$$X_{[\alpha]} = [X_{i_1}, [X_{i_2}, \ldots, [X_{i_{n-1}}, X_{i_n}] \ldots]]$$

is the corresponding multi-commutator. This version of the Hörmander condition was introduced by Kusuoka and Stroock (see condition (H) on page 400 of

[12]). In Section 5 we present several different characterizations of the uniform Hörmander condition.

THEOREM 1.3. *Let H_0 be given by* (1.4) *where $X_1, \ldots, X_N$ are C_b^∞-vector fields on $\mathbb{R}^d$ satisfying the uniform Hörmander condition of order r. Further let H denote the closure of H_0. If $n \in \mathbb{N}$ then $D(H^n) \subseteq D(\Delta^{n/r})$ and there exists a $c > 0$ such that, for all $\varphi \in D(H^n)$,*

$$c\,\|(I+H)^n\varphi\|_2 \geqslant \|\Delta^{n/r}\varphi\|_2. \tag{1.5}$$

Theorem 1.3 follows from Theorem 1.1, or Corollary 1.2, once one establishes that the operator H_0 given by (1.4) satisfies the estimate (1.2) with $\gamma = r^{-1}$. The latter is a global version of Hörmander's key estimate ([10], Theorem 4.3). Hörmander proved a local version of (1.2) for all $\gamma \in \langle 0, r^{-1}\rangle$. Rothschild and Stein [18] subsequently established the local estimate for the optimal value $\gamma = r^{-1}$. The arguments of Rothschild and Stein, which also give optimal local versions of the estimates (1.5), are based on an application of their general lifting theory. Our arguments are completely independent of this technique and provide an alternative proof of the optimal local results.

2. IMPROVEMENT PROPERTIES

In this section we prove Theorem 1.1 by use of commutator estimates. Commutator theory was initially developed by Glimm and Jaffe [8] to derive self-adjointness and regularity properties of quantum fields. It has since developed into a useful tool for various applications in mathematical physics (see, for example, Section 19.4 of [9], Section X.5 of [16], Section II.12 of [6], or Section 4.1 of [2]). Most of these applications are based on single commutator estimates but the analysis of degenerate operators requires double commutator estimates [3], [17].

In the sequel we need to estimate double commutators such as $[\Delta, [\Delta, H_0]]$ or analogous commutators with powers and fractional powers of Δ. If the coefficients of H_0 are in $C_b^\infty(\mathbb{R}^d)$ then the commutators are defined as operators on $W^{\infty,2}(\mathbb{R}^d)$. If, however, the coefficients of H_0 are only twice differentiable then the double commutators have to be defined as sesquilinear forms on $W^{\infty,2}(\mathbb{R}^d) \times W^{\infty,2}(\mathbb{R}^d)$. In general, if A, B are two symmetric operators in a Hilbert space $\mathcal{H}$ and $\mathcal{D} \subset D(A) \cap D(B)$ is a subspace of $\mathcal{H}$ then the commutator $[B, A]$ is defined as a sesquilinear form, with form domain $\mathcal{D}$, by

$$(\psi, [B, A]\varphi) = (B\psi, A\varphi) - (A\psi, B\varphi).$$

Moreover, if A, B_1, B_2 are three symmetric operators in $\mathcal{H}$ and $\mathcal{D} \subset D(A) \cap D(B_2B_1) \cap D(AB_1) \cap D(AB_2)$ then the double commutator is defined as a sesquilinear form $[B_1, [B_2, A]]$, with form domain $\mathcal{D}$, by

$$(\psi, [B_1, [B_2, A]]\varphi) = (B_2B_1\psi, A\varphi) - (AB_1\psi, B_2\varphi) + (A\psi, B_2B_1\varphi) - (AB_2\psi, B_1\varphi).$$

Although this is a slight abuse of notation it should not cause any confusion. Subsequent calculations of commutators involving differential operators and multiplication operators have to be interpreted in this form sense. Such commutators simplify by use of the relations $[\partial_i, c\,]\varphi = (\partial_i c)\varphi$ where c is a differentiable function acting as a multiplication operator.

Double commutators enter estimates through the two identities

$$\mathrm{Re}(B_2\varphi, [B_1, A]\varphi) = 2^{-1}(\varphi, [B_2, [B_1, A]]\varphi), \tag{2.1}$$

$$\mathrm{Re}(A\varphi, B^2\varphi) = (B\varphi, AB\varphi) + 2^{-1}(\varphi, [B, [B, A]]\varphi) \tag{2.2}$$

for all $\varphi \in \mathcal{D}$. In particular if $A \geqslant 0$ the first term on the right of (2.2) is positive and the double commutator gives a lower bound.

Throughout the rest of this section we set $L = I + \Delta$ and let S_t denote the self-adjoint contraction semigroup generated by L. Further we let H_0 be the second-order positive operator in divergence form with coefficients c_{ij} given by (1.1) where the $c_{ij} \in W^{m+1,\infty}(\mathbb{R}^d)$ and $m \in \mathbb{N}$ are fixed.

LEMMA 2.1. *The following commutator estimates are valid.*

(i) *There is a $c > 0$ such that, for all $\varphi, \psi \in W^{\infty,2}(\mathbb{R}^d)$,*

$$\sum_{k=0}^{d} |(\psi, [\partial_k^m, [\partial_k^m, H_0]]\varphi)| \leqslant c\, \|L^{m/2}\psi\|_2\, \|L^{m/2}\varphi\|_2.$$

(ii) *There is a $c > 0$ such that, for all $\varphi, \psi \in W^{\infty,2}(\mathbb{R}^d)$,*

$$|(\psi, [L^m, [L^m, H_0]]\varphi)| \leqslant c \sum_{n=m}^{3m} \|L^{n/2}\psi\|_2\, \|L^{(4m-n)/2}\varphi\|_2.$$

(iii) *If, moreover, $c_{ij} \in W^{3,\infty}(\mathbb{R}^d)$ then there is a $c > 0$ such that, for all $\varphi, \psi \in W^{\infty,2}(\mathbb{R}^d)$,*

$$|(\psi, [L, [L, H_0]]\varphi)| \leqslant c\, \|L\psi\|_2\, \|L\varphi\|_2.$$

Proof. The proof is by straightforward calculation using the fact that the coefficients are $m + 1$ times differentiable. ∎

The lemma has an important corollary which is in two parts. The first was a key observation of [17]. The second will be used in our analysis of Hörmander operators in Section 4.

COROLLARY 2.2. *The following commutator estimates are valid.*

(i) *There is a $c > 0$ such that*

$$|(\psi, [S_t, [S_t, H_0]]\varphi)| \leqslant c\, \|\psi\|_2\, \|\varphi\|_2$$

uniformly for all $\varphi, \psi \in W^{\infty,2}(\mathbb{R}^d)$ and $t > 0$.

(ii) *If, moreover, $c_{ij} \in W^{3,\infty}(\mathbb{R}^d)$ then there is a $c > 0$ such that*

$$|(\psi, [S_t, [S_t, H_0]]\varphi)| \leqslant c\, \|(I - S_t)\psi\|_2\, \|(I - S_t)\varphi\|_2$$

uniformly for all $\varphi, \psi \in W^{\infty,2}(\mathbb{R}^d)$ *and* $t > 0$.

Proof. The proof of both statements is based on the identity

$$(\psi, [S_t, [S_t, H_0]]\varphi) = \int_0^t du \int_0^t dv\, (S_{u+v}\psi, [L, [L, H_0]]S_{2t-u-v}\varphi).$$

If $c > 0$ is as in Lemma 2.1(ii) applied with $m = 1$, then

$$\begin{aligned}|(\psi, [S_t, [S_t, H_0]]\varphi)| &\leqslant c \sum_{n=1}^{3} \int_0^t du \int_0^t dv\, \|L^{n/2} S_{u+v}\psi\|_2\, \|L^{(4-n)/2} S_{2t-u-v}\varphi\|_2 \\ &\leqslant c \sum_{n=1}^{3} \Big(\int_0^1 du\, u^{-n/4}\, (1-u)^{-(4-n)/4} \Big)^2 \|\psi\|_2\, \|\varphi\|_2^2\end{aligned}$$

for all $\varphi, \psi \in W^{\infty,2}(\mathbb{R}^d)$ and $t > 0$ and the first statement follows.

The second statement follows from Lemma 2.1(iii) and the Cauchy–Schwarz inequality, to obtain the bounds

$$|(\psi, [S_t, [S_t, H_0]]\varphi)| \leqslant c \Big(\int_0^t du \int_0^t dv \|LS_{u+v}\psi\|_2^2 \Big)^{1/2} \cdot \Big(\int_0^t du \int_0^t dv \|LS_{2t-u-v}\varphi\|_2^2 \Big)^{1/2}.$$

Then, however, one has, with a similar estimate for the φ-factor,

$$\int_0^t du \int_0^t dv\, \|LS_{u+v}\psi\|_2^2 \leqslant \int_0^t du \int_0^t dv\, \|LS_{(u+v)/2}\psi\|_2^2 = \|(I - S_t)\psi\|_2^2. \qquad ∎$$

The foregoing commutator estimates allow one to extend the argument used to prove Theorem 2.10 in [17] and to conclude the essential self-adjointness of H_0. Further if $c_{ij} \in C_b^\infty(\mathbb{R}^d)$ one can deduce from Theorems 2.16 and 2.17 of [17] that the closure H of H_0 generates a semigroup T which leaves the Sobolev spaces $W^{\sigma,2}(\mathbb{R}^d) = D(L^{\sigma/2})$ invariant. We will give shorter self-contained proofs of these results and establish a key invariance property of T for coefficients $c_{ij} \in W^{m+1,\infty}(\mathbb{R}^d)$.

First, note that H_0 maps $W^{\infty,2}(\mathbb{R}^d)$ into $W^{m,2}(\mathbb{R}^d)$ since $c_{ij} \in W^{m+1,\infty}(\mathbb{R}^d)$. Secondly, note that for all $\sigma \geqslant 0$ the space $W^{\sigma,2}(\mathbb{R}^d)$ is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle_\sigma$ given by $\langle \psi, \varphi \rangle_\sigma = (L^{\sigma/2}\psi, L^{\sigma/2}\varphi)$. Moreover, if $n \in \mathbb{N}_0$ then the inner product $\langle \cdot, \cdot \rangle'_n$ on $W^{n,2}(\mathbb{R}^d)$ defined by

$$\langle \psi, \varphi \rangle'_n = \sum_{k=0}^{d} (\partial_k^n \psi, \partial_k^n \varphi)$$

is norm-equivalent to $\langle \cdot, \cdot \rangle_n$.

PROPOSITION 2.3. *Let* H_0 *be a positive, symmetric, second-order, divergence form operator with coefficients* $c_{ij} \in W^{m+1,\infty}(\mathbb{R}^d)$ *where* $m \in \mathbb{N}$. *Then the operator* H_0

is essentially self-adjoint on $W^{\infty,2}(\mathbb{R}^d)$. *If* T *is the self-adjoint contraction semigroup generated by the closure* H *of* H_0 *then* T *leaves the Sobolev spaces* $W^{\sigma,2}(\mathbb{R}^d)$ *invariant for all* $\sigma \in [0,m]$. *Moreover, the restriction of* T *to* $W^{\sigma,2}(\mathbb{R}^d)$ *is a continuous semigroup on* $W^{\sigma,2}(\mathbb{R}^d)$ *and* $W^{\infty,2}(\mathbb{R}^d)$ *is a core for its generator* $H_{(\sigma)}$.

The proof consists of verifying the criteria of the Lumer–Phillips theorem on the Hilbert spaces $L_2(\mathbb{R}^d)$ and $W^{m,2}(\mathbb{R}^d)$ with inner product $\langle\,\cdot\,,\,\cdot\,\rangle'_m$.

LEMMA 2.4. *There is an* $\omega \geqslant 0$ *such that, for all* $\varphi \in W^{\infty,2}(\mathbb{R}^d)$,

$$\operatorname{Re}\langle\varphi,(H_0+\omega I)\varphi\rangle'_m \geqslant 0. \tag{2.3}$$

Proof. Since $(i\partial_k)^m$ is symmetric one deduces from (2.2) that

$$\begin{aligned}\operatorname{Re}\langle\varphi,H_0\varphi\rangle'_m &= \sum_{k=0}^d(\partial_k^m\varphi,\partial_k^m H_0\varphi) = \sum_{k=0}^d(\partial_k^m\varphi,H_0\partial_k^m\varphi)+(-1)^m 2^{-1}(\varphi,[\partial_k^m,[\partial_k^m,H_0]]\varphi)\\ &\geqslant -2^{-1}c\,\|L^{m/2}\varphi\|_2^2 = -2^{-1}c\,\langle\varphi,\varphi\rangle_m\end{aligned}$$

for all $\varphi \in W^{\infty,2}(\mathbb{R}^d)$, where c is the constant in Lemma 2.1(i). Then (2.3) is valid because the norms associated with the inner products $\langle\,\cdot\,,\,\cdot\,\rangle_m$ and $\langle\,\cdot\,,\,\cdot\,\rangle'_m$ are equivalent. ∎

LEMMA 2.5. *There exists an* $\varepsilon > 0$ *such that* $(I+\varepsilon H_0)W^{\infty,2}(\mathbb{R}^d)$ *is dense in* $L_2(\mathbb{R}^d)$ *and* $W^{m,2}(\mathbb{R}^d)$.

Proof. Let $n \in \{0,m\}$. We establish below that there exists a $c > 0$ such that

$$-\operatorname{Re}\langle\varphi,H_0S_{2t}\varphi\rangle'_n \leqslant c\,\|L^{n/2}\varphi\|_2^2 \tag{2.4}$$

uniformly for all $t > 0$ and $\varphi \in W^{\infty,2}(\mathbb{R}^d)$. It then follows by continuity that (2.4) is valid uniformly for all $t > 0$ and $\varphi \in D(L^{n/2})$. Moreover, there exists a $c_1 > 0$ such that

$$\|L^{n/2}\varphi\|_2^2 \leqslant c_1\sum_{k=0}^d\|\partial_k^n\varphi\|_2^2$$

for all $\varphi \in W^{\infty,2}(\mathbb{R}^d)$. Now set $\varepsilon = (2cc_1)^{-1}$. Let $\varphi \in D(L^{n/2})$ and suppose that the inner product $\langle\varphi,(I+\varepsilon H_0)\psi\rangle'_n = 0$ for all $\psi \in W^{\infty,2}(\mathbb{R}^d)$. Then $S_{2t}\varphi \in W^{\infty,2}(\mathbb{R}^d)$ and

$$\begin{aligned}c_1^{-1}\|S_tL^{n/2}\varphi\|_2^2 \leqslant \sum_{k=0}^d\|S_t\partial_k^n\varphi\|_2^2 &= \langle\varphi,S_{2t}\varphi\rangle'_n = -\varepsilon\operatorname{Re}\langle\varphi,H_0S_{2t}\varphi\rangle'_n\\ &\leqslant c\,\varepsilon\|L^{n/2}\varphi\|_2^2 = (2c_1)^{-1}\|L^{n/2}\varphi\|_2^2\end{aligned}$$

for all $t > 0$. So $\|L^{n/2}\varphi\|_2^2 = \lim_{t\downarrow 0}\|S_tL^{n/2}\varphi\|_2^2 \leqslant 2^{-1}\|L^{n/2}\varphi\|_2^2$ and $\varphi = 0$. Therefore it remains to prove (2.4).

Let $t > 0$ and $\varphi \in W^{\infty,2}(\mathbb{R}^d)$. The starting point is the identity

$$\mathrm{Re}\langle\varphi, H_0 S_{2t}\varphi\rangle'_n = \sum_{k=0}^{d} \mathrm{Re}(\partial_k^n\varphi, H_0 S_{2t}\partial_k^n\varphi) + \sum_{k=0}^{d} \mathrm{Re}(\partial_k^n\varphi, [\partial_k^n, H_0]S_{2t}\varphi). \tag{2.5}$$

We will bound the two terms separately.

The first term satisfies the identity

$$\sum_{k=0}^{d} \mathrm{Re}(\partial_k^n\varphi, H_0 S_{2t}\partial_k^n\varphi) = \sum_{k=0}^{d} (S_t\partial_k^n\varphi, H_0 S_t\partial_k^n\varphi) + \sum_{k=0}^{d} 2^{-1}(\partial_k^n\varphi, [S_t,[S_t,H_0]]\partial_k^n\varphi)$$

where we have again used (2.2). Therefore if $c > 0$ is as in Corollary 2.2(i) then

$$\sum_{k=0}^{d} \mathrm{Re}\,(\partial_k^n\varphi, H_0 S_{2t}\partial_k^n\varphi) \geqslant -2^{-1}c\,\|L^{n/2}\varphi\|_2^2.$$

Note that c is independent of t and φ. If $n = 0$ this completes the proof since the second term in (2.5) is identically zero.

If $n = m$ then, by (2.1), the second term in (2.5) satisfies the identity

$$\sum_{k=0}^{d} \mathrm{Re}\,(\partial_k^m\varphi, [\partial_k^m, H_0]S_{2t}\varphi)$$

$$= (-1)^m \sum_{k=0}^{d} 2^{-1}(S_t\varphi, [\partial_k^m,[\partial_k^m,H_0]]S_t\varphi) - \sum_{k=0}^{d} \mathrm{Re}\,(\partial_k^m\varphi, [S_t,[\partial_k^m,H_0]]S_t\varphi). \tag{2.6}$$

But the first term in (2.6) satisfies

$$\sum_{k=0}^{d} 2^{-1}|(S_t\varphi, [\partial_k^m,[\partial_k^m,H_0]]S_t\varphi)| \leqslant 2^{-1}c\,\|L^{m/2}S_t\varphi\|_2^2 \leqslant 2^{-1}c\,\|L^{m/2}\varphi\|_2^2$$

with c the constant in Lemma 2.1(i). Finally, if $k \in \{1,\ldots,d\}$ then we estimate the last term as follows. The Cauchy inequality gives

$$|(\partial_k^m\varphi, [S_t,[\partial_k^m,H_0]]S_t\varphi)| \leqslant \|L^{m/2}\varphi\|_2\,\|[S_t,[\partial_k^m,H_0]]S_t\varphi\|_2.$$

Moreover,

$$[\partial_k^m, H_0] = \sum_{i,j=0}^{d}\sum_{p=0}^{m-1}\binom{m}{p+1}\big((\partial_i\partial_k^{p+1}c_{ij})\partial_k^{m-p-1}\partial_j + (\partial_k^{p+1}c_{ij})\partial_i\partial_k^{m-p-1}\partial_j\big).$$

Therefore

$$\|[S_t,[\partial_k^m,H_0]]S_t\varphi\|_2$$
$$\leqslant \sum_{i,j=1}^{d}\sum_{p=0}^{m-1}\binom{m}{p+1}\big(\|[S_t,(\partial_i\partial_k^{p+1}c_{ij})]S_t\|_{2\to2} + \|[S_t,(\partial_k^{p+1}c_{ij})]\partial_i S_t\|_{2\to2}\big)\|L^{m/2}\varphi\|_2.$$

The first term on the right is clearly bounded by a multiple of $\|L^{m/2}\varphi\|_2$. But the second satisfies a similar bound since

$$\begin{aligned}
&\|[S_t,(\partial_k^{p+1}c_{ij})]\partial_i S_t\|_{2\to 2}\\
&\qquad\leqslant \sum_{l=1}^d \int_0^t du \|\partial_l S_u(\partial_l\partial_k^{p+1}c_{ij})\partial_i S_{2t-u} + S_u(\partial_l\partial_k^{p+1}c_{ij})\partial_l\partial_i S_{2t-u}\|_{2\to 2}\\
&\qquad\leqslant \sum_{l=1}^d \|\partial_l\partial_k^{p+1}c_{ij}\|_\infty \int_0^1 du(u^{-1/2}(2-u)^{-1/2}+(2-u)^{-1})
\end{aligned}$$

for all $p\in\{0,\ldots,m-1\}$. This completes the proof of (2.4) and the proof of the lemma. ∎

Proof of Proposition 2.3. The operator H_0 is positive and symmetric on the space $L_2(\mathbb{R}^d)$. It then follows from Lemma 2.5 that it is essentially self-adjoint in $L_2(\mathbb{R}^d)$. Therefore the self-adjoint closure H generates a self-adjoint contraction semigroup T on $L_2(\mathbb{R}^d)$. It follows from Lemmas 2.4, 2.5 and the Lumer–Phillips theorem ([14], Theorem 3.1) that the operator H_0 is closable on $W^{m,2}(\mathbb{R}^d)$ and that its closure generates a continuous quasi-contraction semigroup on $W^{m,2}(\mathbb{R}^d)$ if $W^{m,2}(\mathbb{R}^d)$ is equipped with the norm induced from the inner product $\langle\,\cdot\,,\,\cdot\,\rangle'_m$. But this quasi-contraction semigroup is automatically the restriction of T to the space $W^{m,2}(\mathbb{R}^d)$. Moreover, it is a continuous semigroup on $W^{m,2}(\mathbb{R}^d)$ equipped with the norm induced by $\langle\,\cdot\,,\,\cdot\,\rangle_m$. If $\sigma\in\langle 0,m\rangle$ it follows by interpolation that T leaves the Sobolev space $W^{\sigma,2}(\mathbb{R}^d)$ invariant and the restriction of T to $W^{\sigma,2}(\mathbb{R}^d)$ is a continuous semigroup on $W^{\sigma,2}(\mathbb{R}^d)$. Let $H_{(\sigma)}$ denotes the generator on $W^{\sigma,2}(\mathbb{R}^d)$. Then $(\lambda I+H_0)W^{\infty,2}(\mathbb{R}^d)$ is dense in $W^{m,2}(\mathbb{R}^d)$ for large $\lambda>0$ by Lemma 2.5, so $(\lambda I+H_{(\sigma)})W^{\infty,2}(\mathbb{R}^d)=(\lambda I+H_0)W^{\infty,2}(\mathbb{R}^d)$ is dense in $W^{\sigma,2}(\mathbb{R}^d)$. Therefore $W^{\infty,2}(\mathbb{R}^d)$ is a core for $H_{(\sigma)}$. ∎

Next we turn to the problem of improving order properties. Note that if A and B are self-adjoint operators and $A\geqslant B^2$ then it is not true in general that $A^2\geqslant B^4$ although it is true if A and B commute. The next lemma draws a similar conclusion from a double commutator bound.

LEMMA 2.6. *Let $\mathcal{D}$ be a subspace of a Hilbert space $\mathcal{H}$ and A,B a symmetric and self-adjoint operator on $\mathcal{H}$, respectively, such that $\mathcal{D}\subset D(A)\cap D(B)$ and $B\mathcal{D}\subset\mathcal{D}$. Assume*

(2.7) $$(\varphi,A\varphi)\geqslant\|B\varphi\|^2$$

for all $\varphi\in\mathcal{D}$. If there are $\varepsilon\in[0,1\rangle$ and $c>0$ such that

(2.8) $$|(\varphi,[B,[B,A]]\varphi)|\leqslant\varepsilon\,\|B^2\varphi\|^2+c\,\|\varphi\|^2$$

for all $\varphi\in\mathcal{D}$, then, for all $\varphi\in\mathcal{D}$,

$$\|A\varphi\|^2\geqslant(1-\varepsilon)\|B^2\varphi\|^2-c\,\|\varphi\|$$

and in particular,

$$\|A\varphi\| \geqslant (1-\varepsilon)\|B^2\varphi\| - c^{1/2}\,\|\varphi\|.$$

Proof. One estimates that

$$\begin{aligned}\|A\varphi\|^2 + \|B^2\varphi\|^2 &\geqslant 2\,\mathrm{Re}(A\varphi, B^2\varphi) = 2\,(B\varphi, AB\varphi) + (\varphi, [B,[B,A]]\varphi)\\ &\geqslant (2-\varepsilon)\,\|B^2\varphi\|^2 - c\,\|\varphi\|^2\end{aligned}$$

for all $\varphi \in \mathcal{D}$ where we have successively used (2.2), (2.7) and (2.8). The statement of the lemma follows immediately. ∎

The double commutator estimate (2.8) is a rather weak requirement for second-order differential operators. For example, if $B = L$ and $A = H_0$ then Lemma 2.1(iii) gives the much stronger bound

$$|(\varphi, [B,[B,A]]\varphi)| \leqslant c\,\|B\varphi\|^2.$$

But our proof of the improvement of subelliptic properties follows from application of Lemma 2.6 with B a fractional power of L and this leads to a slight "loss of derivatives". Recall that we assume $c_{ij} \in W^{m+1,\infty}(\mathbb{R}^d)$ with $m \in \mathbb{N}$.

LEMMA 2.7. *For all* $\rho \in [0, m\rangle$ *and* $\delta > 0 \vee (\rho - 2^{-1}m)$ *there is a* $c > 0$ *such that, for all* $\varphi \in W^{\infty,2}(\mathbb{R}^d)$,

$$|(\varphi, [L^\rho, [L^\rho, H_0]]\varphi)| \leqslant c\,\|L^{\rho+\delta}\varphi\|_2^2.$$

Proof. The case $\rho = 0$ is trivial, so we may assume that $\rho > 0$. Set $\tau = m^{-1}\rho \in \langle 0, 1\rangle$. Then

$$L^\rho = (L^m)^\tau = c_1 \int_0^\infty d\lambda\, \lambda^{-1+\tau} L^m (\lambda I + L^m)^{-1}$$

where $c_1 = \int_0^\infty d\lambda\, \lambda^{-1+\tau}(1+\lambda)^{-1}$. Let $c > 0$ be as in Lemma 2.1(ii).

Let $\varphi \in W^{\infty,2}(\mathbb{R}^d)$. Then

$$\begin{aligned}(\varphi, [L^\rho, [L^\rho, H_0]]\varphi) &= c_1^{-2} \int_0^\infty d\lambda \int_0^\infty d\mu\, (\lambda\mu)^\tau (R_\lambda R_\mu \varphi, [L^m, [L^m, H_0]] R_\lambda R_\mu \varphi)\\ &\leqslant c c_1^{-2} \sum_{n=m}^{3m} \int_0^\infty d\lambda \int_0^\infty d\mu (\lambda\mu)^\tau \|L^{n/2} R_\lambda R_\mu \varphi\|_2 \|L^{(4m-n)/2} R_\lambda R_\mu \varphi\|_2\end{aligned}$$

where $R_\lambda = (\lambda I + L^m)^{-1}$. It follows from spectral theory that $\|L^\alpha R_\lambda\|_{2\to 2} \leqslant (1+\lambda)^{-(m-\alpha)/m}$ for all $\lambda > 0$ and $\alpha \in [0, m]$.

Let $n \in \{m, \ldots, 3m\}$. Set $\eta_1 = (2^{-1}n) \wedge (\rho + \delta)$ and $\eta_2 = (2^{-1}(4m - n)) \wedge (\rho + \delta)$. Then

$$\|L^{n/2} R_\lambda R_\mu \varphi\|_2 \leqslant \|L^{(n-2\eta_1)/2} R_\lambda R_\mu\|_{2\to 2} \|L^{\eta_1}\varphi\|_2$$
$$\leqslant ((1+\lambda)(1+\mu))^{-(4m)^{-1}(4m-n+2\eta_1)} \|L^{\rho+\delta}\varphi\|_2$$

for all $\lambda, \mu > 0$. Similarly,

$$\|L^{(4m-n)/2} R_\lambda R_\mu \varphi\|_2 \leqslant ((1+\lambda)(1+\mu))^{-(4m)^{-1}(n+2\eta_2)} \|L^{\rho+\delta}\varphi\|_2.$$

Therefore

$$\int_0^\infty d\lambda \int_0^\infty d\mu \, (\lambda\mu)^\tau \|L^{n/2} R_\lambda R_\mu \varphi\|_2 \, \|L^{(4m-n)/2} R_\lambda R_\mu \varphi\|_2$$
$$\leqslant \Big(\int_0^\infty d\lambda \, \lambda^\tau \, (1+\lambda)^{-(2m)^{-1}(2m+\eta_1+\eta_2)} \Big)^2 \|L^{\rho+\delta}\varphi\|_2^2.$$

So it remains to verify that the integral is finite, i.e. to show that $\eta_1 + \eta_2 > 2m\tau = 2\rho$.

If $\rho + \delta \leqslant 2^{-1}n \wedge 2^{-1}(4m - n)$ then $\eta_1 + \eta_2 = 2(\rho + \delta) > 2\rho$. If $2^{-1}n \vee 2^{-1}(4m - n) \leqslant \rho + \delta$ then $\eta_1 + \eta_2 = 2m > 2\rho$. Finally, if $2^{-1}n \leqslant \rho + \delta \leqslant 2^{-1}(4m - n)$ then $\eta_1 + \eta_2 = 2^{-1}n + \rho + \delta \geqslant 2^{-1}m + \rho + \delta > 2\rho$ since $\delta > \rho - 2^{-1}m$. Similarly $\eta_1 + \eta_2 > 2\rho$ if $2^{-1}(4m - n) \leqslant \rho + \delta \leqslant 2^{-1}n$. This proves the lemma. ∎

The previous lemmas can be applied to establish an improvement of Theorem 1.1.

THEOREM 2.8. *Let H_0 be a subelliptic operator of order $\gamma \in \langle 0, 1]$ with coefficients $c_{ij} \in W^{m+1,\infty}(\mathbb{R}^d)$, where $m \in \mathbb{N}$, and with self-adjoint closure H. Further, let $\sigma \in [0, 2^{-1}m\rangle$. If $\varphi \in D(H)$ and $H\varphi \in W^{2\sigma,2}(\mathbb{R}^d)$ then $\varphi \in W^{2\sigma+2\gamma,2}(\mathbb{R}^d)$. Moreover, there exist $c, \omega_0 > 0$ such that*

$$c \, \|L^\sigma(\omega_0 I + H_{(2\sigma)})\varphi\|_2 \geqslant \|L^{\sigma+\gamma}\varphi\|_2 \tag{2.9}$$

for all $\varphi \in D(H_{(2\sigma)})$, where $H_{(2\sigma)}$ denotes the closure of H_0 on $W^{2\sigma,2}(\mathbb{R}^d)$.

Proof. Since the restriction of S to $W^{2\sigma,2}(\mathbb{R}^d)$ is a continuous semigroup there exists an $\omega_0 > 1$ such that

$$\|L^\sigma \varphi\|_2 \leqslant \|L^\sigma(\omega_0 I + H_0)\varphi\|_2$$

for all $\varphi \in W^{\infty,2}(\mathbb{R}^d)$. Set $\tau = 2^{-1}\gamma$. Since $\|L^\tau\varphi\|_2^2 \leqslant 4 \, \|\Delta^\tau\varphi\|_2^2 + \|\varphi\|_2^2$ there exists by subellipticity a $c > 0$ such that

$$c \, (\varphi, (\omega_0 I + H_0)\varphi) \geqslant \|L^\tau\varphi\|_2^2$$

for all $\varphi \in W^{\infty,2}(\mathbb{R}^d)$. Set $A = c \, L^\sigma(\omega_0 I + H_0)L^\sigma$ and $B = L^{\sigma+\tau}$. Then $(\varphi, A\varphi) \geqslant \|B\varphi\|_2^2$ for all $\varphi \in W^{\infty,2}(\mathbb{R}^d)$. So (2.7) in Lemma 2.6 is satisfied with $\mathcal{D} = W^{\infty,2}(\mathbb{R}^d)$.

Fix $\delta \in \langle(\sigma - 2^{-1}m + \tau) \vee 0, \tau\rangle$. Then it follows from Lemma 2.7 that there are $c', \omega > 0$ such that, for all $\varphi \in W^{\infty,2}(\mathbb{R}^d)$,

$$|(\varphi, [B, [B, A]]\varphi)| = c\,|(L^\sigma \varphi, [L^{\sigma+\tau}, [L^{\sigma+\tau}, H_0]]L^\sigma \varphi)| \leqslant c' \|L^{2\sigma+\tau+\delta}\varphi\|_2^2 \leqslant 2^{-1}\|L^{2\sigma+2\tau}\varphi\|_2^2 + \omega\,\|\varphi\|_2^2.$$

So by Lemma 2.6

$$c^2\|L^\sigma(\omega_0 I + H_0)L^\sigma\varphi\|_2^2 \geqslant 2^{-1}\|L^{2\sigma+2\tau}\varphi\|_2^2 - \omega\|\varphi\|_2^2 \geqslant 2^{-1}\|L^{2\sigma+2\tau}\varphi\|_2^2 - \omega\|L^{2\sigma}\varphi\|_2^2$$

for all $\varphi \in W^{\infty,2}(\mathbb{R}^d)$. Since L^σ is a bijection from $W^{\infty,2}(\mathbb{R}^d)$ onto $W^{\infty,2}(\mathbb{R}^d)$ one may replace $L^\sigma\varphi$ by φ and then (2.9) follows by rearrangement uniformly for all $\varphi \in W^{\infty,2}(\mathbb{R}^d)$. But $W^{\infty,2}(\mathbb{R}^d)$ is a core for $H_{(2\sigma)}$ by Proposition 2.3. So (2.9) is valid for all $\varphi \in D(H_{(2\sigma)})$.

The first statement follows immediately from the second. ∎

Proof of Theorem 1.1. This follows immediately from Theorem 2.8 by interpolation and a telescopic argument. ∎

We conclude this section with four remarks on Theorem 1.1.

First, Theorem 1.1 can be rephrased in terms of order relations. The estimate (2.9) with $\sigma = 0$ is equivalent to the quadratic form estimate

$$c^2\,(I + H)^2 \geqslant L^{2\gamma}.$$

Then since the order relation between positive self-adjoint operators is respected by taking fractional powers one has

$$c^{2\alpha}\,(I + H)^{2\alpha} \geqslant L^{2\alpha\gamma}$$

for all $\alpha \in \langle 0, 1]$. But (2.9) with $\sigma > 0$ is equivalent to the estimate

$$c^2\,(I + H)L^{2\sigma}(I + H) \geqslant L^{2\sigma+2\gamma}$$

and then the previous argument can be iterated to obtain the order relations covered by Theorem 1.1.

Secondly, if $c_{ij} \in W^{2,\infty}(\mathbb{R}^d)$ then Theorem 1.1 establishes that the subellipticity condition (1.2) implies the estimate (1.3) with $\alpha = 1$. But the foregoing observation on order properties establishes the converse. Thus (1.2) is equivalent to (1.3) with $\alpha = 1$. This is a global strengthening of the first statement in Theorem 1 of Fefferman and Phong [7].

Thirdly, the statement of Theorem 1.1 is partly redundant since the closed graph theorem implies that if the inclusion $D(H^\alpha) \subseteq D(\Delta^{\alpha\gamma})$ is valid, with $\alpha \geqslant 0$, then there exists a $c > 0$ such that (1.3) is valid.

Finally, if $\gamma = 1$, i.e. if H_0 is strongly elliptic, then the statement of the theorem is also valid for $\alpha = 2^{-1}(m + 1 + \gamma^{-1}) = 2^{-1}(m + 2)$, and this is the best estimate one could expect for operators with coefficients $c_{ij} \in W^{m+1,\infty}(\mathbb{R}^d)$. The extension is a consequence of the theorem, applied with $\alpha = 2^{-1}(m + 1)$, together with a simple commutator estimate. In fact for $\gamma = 1$ the domain inclusion in the theorem is an equality which is also valid on the L_p-spaces if $p \in \langle 1, \infty \rangle$ (see Theorem 1.5.II of [5]).

3. C_b^∞-FLOWS

In this section we prepare for the discussion of elliptic operators (1.4) of Hörmander type by recalling some basic properties of the flows corresponding to C_b^∞-vector fields. We also give several estimates for products and commutators of such flows. Local estimates of a similar nature are an important feature in the work of Hörmander [10] and Nagel, Stein and Wainger [15] but our emphasis is on estimates which are uniform over $\mathbb{R}^d$. The uniform Hörmander condition is not relevant in this section.

Let X be a C_b^∞-vector field on $\mathbb{R}^d$ with coefficients a_i. Then it follows from the theory of ordinary differential equations that there exists a unique C^∞-function $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ such that

$$\frac{\partial f_i}{\partial t}(t, x) = a_i(f(t, x)) \quad \text{and} \quad f(0, x) = x$$

for all $x \in \mathbb{R}^d$, $t \in \mathbb{R}$ and $i \in \{1, \ldots, d\}$. We adopt the conventional notation $\exp(tX)(x) = f(t, x)$. Then for all $\varphi \in C^\infty(\mathbb{R}^d)$ and $t \in \mathbb{R}$ we define $e^{tX}\varphi \in C^\infty(\mathbb{R}^d)$ by $(e^{tX}\varphi)(x) = \varphi(\exp(tX)(x))$. The relevant properties of these maps are summarized as follows.

LEMMA 3.1. *Let X be a C_b^∞-vector field on $\mathbb{R}^d$. Then one has the following:*

(i) $\exp(tX)(\exp(sX)(x)) = \exp((t+s)X)(x)$ *for all $x \in \mathbb{R}^d$ and $t, s \in \mathbb{R}$. Hence for each $t \in \mathbb{R}$ the map* $\exp(tX)$ *is a diffeomorphism of $\mathbb{R}^d$.*

(ii) *If $\varphi \in C^\infty(\mathbb{R}^d)$ and $t \in \mathbb{R}$ then, for all $x \in \mathbb{R}^d$, where $\sim$ denotes the Taylor series $($in $t)$ around* 0,

$$\varphi(\exp(tX)(x)) = (e^{tX}\varphi)(x) \sim \sum_{n=0}^{\infty} t^n\, n!^{-1} (X^n \varphi)(x).$$

We also need some quantitative estimates. It is convenient to introduce a multi-index notation. For all $N \in \mathbb{N}$ and $n \in \mathbb{N}_0$ set

$$J_n(N) = \bigoplus_{k=0}^{n} \{1, \ldots, N\}^k, \quad J(N) = \bigoplus_{k=0}^{\infty} \{1, \ldots, N\}^k$$

and let $J_n^+(N)$, $J^+(N)$ denote the corresponding sets with the restrictions $k \geqslant 1$.

One can prove the next lemma with the aid of Gronwall's lemma and induction.

LEMMA 3.2. *Let X be a C_b^∞-vector field.*

(i) *For all $k \in \mathbb{N}$ there exists an $M > 0$ such that*

$$|\partial_t^k \exp(tX)(x)| \leqslant M$$

uniformly for all $t \in \mathbb{R}$ *and* $x \in \mathbb{R}^d$.

(ii) *For all* $\alpha \in J(d)$ *and* $k \in \mathbb{N}_0$ *with* $|\alpha| + k \geqslant 1$ *there exist* $M, \omega > 0$ *such that*

$$|\partial_t^k \, \partial_x^\alpha \exp(tX)(x)| \leqslant M e^{\omega |t|}$$

uniformly for all $t \in \mathbb{R}$ *and* $x \in \mathbb{R}^d$.

(iii) *There are* $M, \omega > 0$ *such that, for all* $t \in \mathbb{R}$ *and* $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\|e^{tX}\varphi\|_2 \leqslant M e^{\omega|t|} \, \|\varphi\|_2.$$

Next we need several estimates which follow from the Campbell–Baker–Hausdorff formula. The first of these is an estimate for the product of two flows generated by C_b^∞-vector fields. The key observation is contained in the following lemma.

LEMMA 3.3. *Let* Y_1 *and* Y_2 *be* C_b^∞*-vector fields and let* $N \in \mathbb{N}\backslash\{1\}$. *Then there exist* $Z_2, \ldots, Z_N$ *with* $Z_j \in \operatorname{span}\{Y_{[\alpha]} : \alpha \in J(2),\ |\alpha| = j\}$ *for all* $j \in \{2, \ldots, N\}$, *such that, for all* $\varphi \in C^\infty(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ *and* $k \in \{1, \ldots, N\}$,

$$\Big(\frac{d}{dt}\Big)^k \varphi(\exp(t(Y_1+Y_2))\exp(-tY_1)\exp(-tY_2)\circ\exp(-t^2 Z_2)\cdots\exp(-t^N Z_N)(x))\Big|_{t=0} = 0.$$

Proof. This follows from the Campbell–Baker–Hausdorff formula as in the discussion preceding Lemma 4.5 of [10]. See in particular pp. 160–161 of [10]. ∎

As a direct consequence one has the following estimate which is uniform over $\mathbb{R}^d$.

PROPOSITION 3.4. *Let* Y_1 *and* Y_2 *be* C_b^∞*-vector fields and let* $N \in \mathbb{N}\backslash\{1\}$. *Then there exist* $c > 0$ *and* $Z_2, \ldots, Z_N$ *with* $Z_j \in \operatorname{span}\{Y_{[\alpha]} : \alpha \in J(2),\ |\alpha| = j\}$ *for all* $j \in \{2, \ldots, N\}$, *such that*

$$|\exp(t(Y_1+Y_2))\exp(-tY_1)\exp(-tY_2)\exp(-t^2 Z_2)\cdots\exp(-t^N Z_N)(x) - x| \leqslant c t^{N+1}$$

uniformly for all $x \in \mathbb{R}^d$ *and* $t \in [-1, 1]$.

Proof. Define $\Phi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ by

$$\Phi(x,t) = \exp(t(Y_1+Y_2))\exp(-tY_1)\exp(-tY_2)\exp(-t^2 Z_2)\cdots\exp(-t^N Z_N)(x).$$

If $\varphi \in C^\infty(\mathbb{R}^d)$ then it follows from Lemma 3.3 and the Taylor integral remainder formula that

$$|\varphi(\Phi(x,t)) - \varphi(x)| = |\varphi(\Phi(x,t)) - \varphi(\Phi(x,0))| = \Big| N!^{-1} \int_0^t ds\,(t-s)^N \, \partial_s^{N+1} \varphi(\Phi(x,s)) \Big|$$

for all $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$. Now apply the above to $\varphi = \pi_k$. It follows from Lemma 3.2(ii) that there exists an $M > 0$ such that $|\partial_s^{N+1} \pi_k(\Phi(x,s))| \leqslant M$ uniformly for all $x \in \mathbb{R}^d$ and $s \in [-1,1]$. Then

$$|\pi_k(\Phi(x,t)) - x_k| \leqslant N!^{-1} \int_0^t ds\, (t-s)^N M = M\,(N+1)!^{-1}\, t^{(N+1)}$$

for all $x \in \mathbb{R}^d$, $t \in [-1,1]$ and $k \in \{1,\ldots,d\}$. So $|\Phi(x,t) - x| \leqslant Md\, t^{N+1}$ for all $x \in \mathbb{R}^d$ and $t \in [-1,1]$. ∎

Finally we give an estimate comparing the flow generated by a combination of multi-commutators in terms of products of the elementary flows.

LEMMA 3.5. *Let $X_1, \ldots, X_N$ be C_b^∞-vector fields and $s \in \mathbb{N}$. Then there exists an $M > 0$ such that for all $b : J_s^+(N) \to [-1,1]$ there are $n \in \{1,\ldots,3(2d)^s\}$, $i_1, \ldots, i_n \in \{1,\ldots,N\}$ and $a_1, \ldots, a_n \in [-M, M]$ such that, for all $k \in \{1,\ldots,N\}$, $\varphi \in C^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,*

$$\Big(\frac{d}{dt}\Big)^k \varphi\Big(\exp\Big(\sum_{\alpha \in J_s^+(N)} b(\alpha)\, t^{|\alpha|} X_{[\alpha]}\Big) \exp(-a_1 t X_{i_1}) \cdots \exp(-a_n t X_{i_n})(x)\Big)\Big|_{t=0} = 0.$$

Proof. This follows from the arguments given in the proof of Lemma 2.22 in [15]. ∎

PROPOSITION 3.6. *Let $s \in \mathbb{N}$ and $X_1, \ldots, X_N$ be C_b^∞-vector fields. Then there exist $M, M' > 0$ such that for all $\delta \in \langle 0,1]$ and $b : J_s^+(N) \to [-1,1]$ with $|b(\alpha)| \leqslant \delta^{|\alpha|}$ for all $\alpha \in J_s^+(N)$ there are $n \in \{1,\ldots,3(2d)^s\}$, $i_1, \ldots, i_n \in \{1,\ldots,N\}$ and $a_1, \ldots, a_n \in [-M\delta, M\delta]$ such that*

$$\Big|\exp\Big(\sum_{\alpha \in J_s^+(N)} b(\alpha) t^\alpha X_{[\alpha]}\Big) \exp(-a_1 t X_{i_1}) \cdots \exp(-a_n t X_{i_n})(x) - x\Big| \leqslant M'(\delta\,|t|)^{s+1}$$

uniformly for all $x \in \mathbb{R}^d$ and $t \in [-1,1]$.

Proof. Let $M > 0$ be as in Lemma 3.5. Let $\delta \in \langle 0,1]$ and $b : J_s^+(N) \to [-1,1]$ with $|b(\alpha)| \leqslant \delta^{|\alpha|}$ for all $\alpha \in J_s^+(N)$. Then $\delta^{-|\alpha|}\,|b(\alpha)| \leqslant 1$ for all $\alpha \in J_s^+(N)$. By Lemma 3.5 there are $n \in \{1,\ldots,3(2d)^s\}$, $i_1, \ldots, i_n \in \{1,\ldots,N\}$ and $a_1, \ldots, a_n \in [-M, M]$ such that

$$\Big(\frac{d}{dt}\Big)^k \varphi\Big(\exp\Big(\sum_{\alpha \in J_s^+(N)} \delta^{-|\alpha|} b(\alpha)\, t^{|\alpha|} X_{[\alpha]}\Big) \exp(-a_1 t X_{i_1}) \cdots \exp(-a_n t X_{i_n})(x)\Big)\Big|_{t=0} = 0$$

for all $k \in \{1,\ldots,N\}$, $\varphi \in C^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Define $\Phi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ by

$$\Phi(x,t) = \exp\Big(\sum_{\alpha \in J_s^+(N)} \delta^{-|\alpha|} b(\alpha)\, t^{|\alpha|} X_{[\alpha]}\Big) \exp(-a_1 t X_{i_1}) \cdots \exp(-a_n t X_{i_n})(x).$$

Then it follows from Lemma 3.2(ii) as in the proof of Proposition 3.4 that there exists an $M' > 0$, depending only on the $X_{[\alpha]}$ with $\alpha \in J_s^+(N)$ and n, such that

$|\Phi(x,t) - x| \leqslant M' \, |t|^{s+1}$ uniformly for all $t \in [-1,1]$ and $x \in \mathbb{R}^d$. Replacing t by $\delta \, t$ and a_i by $a_i \delta^{-1}$ yields the proposition. ∎

4. SUBELLIPTICITY ESTIMATES

In this section we prove Theorem 1.3. The proof follows closely Hörmander's reasoning and the subsequent discussion should be read in conjunction with Section 4 of [10]. Throughout the section $X_1, \ldots, X_N$ are C_{b}^∞-vector fields but we do not require that they satisfy the uniform Hörmander condition until Proposition 4.7. Set $H_0 = \sum_{i=1}^N X_i^* \, X_i$ with domain $D(H_0) = W^{\infty,2}(\mathbb{R}^d)$. Then H_0 is essentially self-adjoint by Proposition 2.3. Let H denote the closure of H_0. Alternatively, define the quadratic form h on $L_2(\mathbb{R}^d)$ by

$$h(\varphi) = \sum_{i=1}^N \|X_i \varphi\|_2^2$$

and domain $D(h) = \bigcap_{i=1}^N D(X_i)$. Then the form h is closed. Let $\widetilde{H}$ be the positive self-adjoint operator associated with the closed quadratic form h. Then obviously $H_0 \subseteq \widetilde{H}$ and by uniqueness of self-adjoint extensions one has $H = \widetilde{H}$.

Hörmander's proofs are based on the extensive use of Hölder norms. Therefore we associate with each C_{b}^∞-vector field X a family of such norms. Specifically for all $\gamma \in \langle 0, 1]$ we define the Hölder norm $\| \cdot \|_{2;X,\gamma}$ by

$$\|\varphi\|_{2;X,\gamma} = \|\varphi\|_2 + \sup_{0<|t|\leqslant 1} |t|^{-\gamma} \|e^{tX}\varphi - \varphi\|_2$$

for all $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$. In addition we introduce the universal Hölder norms

$$\|\varphi\|_{2;\gamma} = \|\varphi\|_2 + \sup_{0<|x|\leqslant 1} |x|^{-\gamma} \|L(x)\varphi - \varphi\|_2$$

for all $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$, where $L(\,\cdot\,)$ denotes the left regular representation of $\mathbb{R}^d$ on $L_2(\mathbb{R}^d)$.

The next two lemmas are similar to Lemmas 4.1 and 4.2 in [10].

LEMMA 4.1. *Let X be a C_{b}^∞-vector field. Further let $\psi \in C_{\mathrm{b}}^\infty(\mathbb{R}^d)$ and $\gamma \in \langle 0, 1]$. Then there exists a $c > 0$ such that, for all $\varphi \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$,*

$$\|\varphi\|_{2;\psi X,\gamma} \leqslant c \, \|\varphi\|_{2;X,\gamma} \, .$$

Proof. Following Hörmander's proof of Lemma 4.1 we define $\tau : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ to be the solution for each $x \in \mathbb{R}^d$ of the initial value problem

$$\partial_t \tau(x,t) = \psi(\exp(\tau(x,t)X)(x)) \quad \text{and} \quad \tau(x,0) = 0.$$

Then it follows from Gronwall's lemma that there are $M, \omega > 0$ such that

$$|\partial_k \tau(x,t)| \leqslant M \, \mathrm{e}^{\omega |t|} \tag{4.1}$$

for all $k \in \{1, \ldots, d\}$, $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$. Consequently one calculates as in [10] that, for all $t \in [-1,1]\backslash\{0\}$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\|\mathrm{e}^{t\psi X}\varphi - \varphi\|_2$$
$$\leqslant 2|t|^{-1}\int \mathrm{d}x \int_{\{\sigma:|\sigma|\leqslant|t|\}} \mathrm{d}\sigma |\varphi(\exp(\tau(x,t)X)(x)) - \varphi(\exp(\sigma X)(x))|^2 + 2|t|^{2\gamma}\|\varphi\|^2_{2;X,\gamma} .$$

Fix $t \in [-1,1]\backslash\{0\}$. Introduce new variables $y = \exp(\sigma X)(x)$ and $w = \tau(x,t) - \sigma$. Then the Jacobian of the coordinate transformation is given by

$$J_{x,\sigma} = \left| \begin{array}{cc} \partial_x \exp(\sigma X)(x) & \partial_\sigma \exp(\sigma X)(x) \\ (\partial_x \tau)(x,t) & -1 \end{array} \right| .$$

Since $|\sigma| \leqslant |t| \leqslant 1$ it follows from (4.1) and Lemma 3.2(ii) that there exists an $M > 0$ such that $|J_{x,\sigma}| \leqslant M$ uniformly for all $x \in \mathbb{R}^d$ and $\sigma \in [-1,1]$. Moreover, $|\tau(x,t)| \leqslant \|\psi\|_\infty \, |t|$, so $|w| \leqslant (1 + \|\psi\|_\infty) \, |t|$. Hence

$$|t|^{-1} \int \mathrm{d}x \int_{\{\sigma:|\sigma|\leqslant|t|\}} \mathrm{d}\sigma \, |\varphi(\exp(\tau(x,t)X)(x)) - \varphi(\exp(\sigma X)(x))|^2$$
$$\leqslant M|t|^{-1}\int \mathrm{d}y \int_{\{w:|w|\leqslant(1+\|\psi\|_\infty)|t|\}} \mathrm{d}w |\varphi(\exp(wX)(y)) - \varphi(y)|^2 \leqslant M'|t|^{2\gamma}\|\varphi\|^2_{2;X,\gamma}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, where we have used Lemma 3.2(iii). The statement of the lemma follows immediately. ∎

LEMMA 4.2. *Let $\Phi : \mathbb{R}^d \times \langle -2,2 \rangle \to \mathbb{R}^d$ be a C^∞-function, $N' \in \mathbb{N}$ and $\gamma \in \langle 0,1]$. Suppose there exists an $M > 0$ such that*

$$|\Phi(x,t) - x| \leqslant M \, t^{N'} \quad \textit{and} \quad |\partial_k \Phi(x,t)| \leqslant M$$

uniformly for all $x \in \mathbb{R}^d$, $t \in [-1,1]$ and $k \in \{1, \ldots, d\}$. Then there exists a $c > 0$ such that, for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $t \in [-1,1]$,

$$\int_{\mathbb{R}^d} \mathrm{d}x \, |\varphi(\Phi(x,t)) - \varphi(x)|^2 \leqslant c \, |t|^{2N'\gamma} \|\varphi\|^2_{2;\gamma} .$$

Proof. The proof is similar to the proof of Lemma 4.2 in [10], with the same modifications as in the proof of Lemma 4.1. ∎

The conclusion of Lemma 4.2 can be immediately translated into a bound on the Hölder norm.

COROLLARY 4.3. *Let X be a C_b^∞-vector field and let $\gamma \in \langle 0,1]$. Then there exists a $c > 0$ such that $\|\varphi\|_{2;X,\gamma} \leqslant c \, \|\varphi\|_{2;\gamma}$ for all $\varphi \in C_c^\infty(\mathbb{R}^d)$.*

Proof. It follows from the Duhamel formula that

$$|\pi_k(\exp(tX))(x) - \pi_k(x)| \leqslant \int_0^t ds\, |(X\pi_k)(\exp(sX))(x)| \leqslant \|X\pi_k\|_\infty\, |t|$$

for all $k \in \{1, \ldots, d\}$. Therefore

$$|(\exp(tX)(x)) - x| \leqslant |t| \sum_{k=1}^{d} \|X\pi_k\|_\infty$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$. Then the corollary follows from Lemma 3.2(ii) and Lemma 4.2 applied with $\Phi(x,t) = \exp(tX)(x)$ and $N' = 1$. ∎

Proposition 3.4 immediately yields a global version of Lemma 4.5 in [10].

LEMMA 4.4. *Let Y_1 and Y_2 be C_b^∞-vector fields, $\gamma \in \langle 0,1]$ and $N \in \mathbb{N} \setminus \{1\}$. Let $Z_2, \ldots, Z_N$ be as in Proposition* 3.4. *Then there exists a $c > 0$ such that, for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $t \in [-1,1]$,*

$$\|e^{t(Y_1+Y_2)}\varphi - \varphi\|_2 \leqslant c\Big(\|e^{tY_1}\varphi - \varphi\|_2 + \|e^{tY_2}\varphi - \varphi\|_2 + \sum_{j=2}^{N} \|e^{t^j Z_j}\varphi - \varphi\|_2 + |t|^{\gamma(N+1)}\|\varphi\|_{2;\gamma}\Big).$$

Proof. Define $\Phi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ by

$$\Phi(x,t) = \exp(t(Y_1+Y_2))\exp(-tY_1)\exp(-tY_2)\exp(-t^2 Z_2)\cdots\exp(-t^N Z_N)(x).$$

If $c > 0$ is as in Proposition 3.4 then it follows that $|\Phi(x,t) - x| \leqslant c\,|t|^{N+1}$ for all $x \in \mathbb{R}^d$ and $t \in [-1,1]$. Secondly,

$$\sup_{k\in\{1,\ldots,d\}} \sup_{x\in\mathbb{R}^d} \sup_{t\in[-1,1]} |(\partial_k\Phi)(x,t)| < \infty$$

again by Lemma 3.2(ii). Hence by Lemma 4.2 it follows that there is a $c_1 > 0$ such that, for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $t \in [-1,1]$,

$$\int dx\, |\varphi(\Phi(x,t)) - \varphi(x)|^2 \leqslant c_1^2\, |t|^{2(N+1)\gamma}\|\varphi\|_{2;\gamma}^2 .$$

Next, for all $t \in \mathbb{R}$ define $H_t : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ by $(H_t\varphi)(x) = \varphi(\Phi(x,t))$. Then $\|H_t\varphi - \varphi\|_2 \leqslant c_1\, t^{(N+1)\gamma}\|\varphi\|_{2;\gamma}$ for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $t \in [-1,1]$. But

$$H_t = e^{-t^N Z_N}\cdots e^{-t^2 Z_2} e^{-tY_2} e^{-tY_1} e^{t(Y_1+Y_2)}$$

and

$$e^{t(Y_1+Y_2)} = e^{tY_1} e^{tY_2} e^{t^2 Z_2}\cdots e^{t^N Z_N} H_t$$

for all $t \in \mathbb{R}$. Then the lemma is a consequence of a concertina formula and Lemma 3.2(iii). ∎

We emphasize that in the next two lemmas it is not necessary for the vector fields to satisfy the uniform Hörmander condition.

LEMMA 4.5. *Let* $X_1, \ldots, X_N$ *be* C_b^∞*-vector fields and let* $\gamma, \delta \in \langle 0, 1]$. *Then for all* $\alpha \in J^+(N)$ *there exists* $c_1, c_2 > 0$ *such that, for all* $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\|\varphi\|_{2;X_{[\alpha]},\gamma|\alpha|^{-1}} \leqslant c_1 \sum_{j=1}^N \|\varphi\|_{2;X_j,\gamma} + c_2 \|\varphi\|_{2;\delta}.$$

Proof. Let $\alpha \in J^+(N)$. Fix $s \in \mathbb{N}$ with $s \geqslant |\alpha| \vee \gamma\,\delta^{-1}$. By Proposition 3.6 applied with $\delta = 1$ there exist $n \in \{1, \ldots, 3(2d)^s\}$, $i_1, \ldots, i_n \in \{1, \ldots, N\}$, $M, M' > 0$ and $a_1, \ldots, a_n \in [-M, M]$ such that if $\Phi : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is given by

$$\Phi(x,t) = \exp(t^{|\alpha|} X_{[\alpha]}) \exp(-a_1 t X_{i_1}) \cdots \exp(-a_n t X_{i_n})(x)$$

then $|\Phi(x,t) - x| \leqslant M' \, |t|^{s+1}$ uniformly for all $t \in [-1,1]$ and $x \in \mathbb{R}^d$. Moreover, by Lemma 3.2(ii), there exists an $M'' > 0$ such that $|\partial_k \Phi(x,t)| \leqslant M''$ uniformly for all $x \in \mathbb{R}^d$, $t \in [-1,1]$ and $k \in \{1, \ldots, d\}$. For all $t \in \mathbb{R}$ define $H_t : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ by $(H_t\varphi)(x) = \varphi(\Phi(x,t))$. Then by Lemma 4.2 there is a $c > 0$ such that

$$\|H_t\varphi - \varphi\|_2 \leqslant c\,|t|^{(s+1)\delta} \|\varphi\|_{2;\delta} \leqslant c\,|t|^{\gamma} \|\varphi\|_{2;\delta}$$

for all $t \in [-1,1]$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$. But

$$e^{t^{|\alpha|} X_{[\alpha]}} = e^{a_1 t X_{j_1}} \cdots e^{a_n t X_{i_n}} H_t.$$

Therefore Lemma 3.2(iii) implies that there is a $c' > 0$ such that

$$\begin{aligned}\|e^{t^{|\alpha|} X_{[\alpha]}} \varphi - \varphi\|_2 &\leqslant c' \Big(\sum_{l=1}^n \|e^{a_l t X_{i_l}} \varphi - \varphi\|_2 + \|H_t\varphi - \varphi\|_2 \Big) \\ &\leqslant c' \Big(|t|^\gamma \sum_{l=1}^n \|\varphi\|_{2;a_l X_{i_l},\gamma} + c\,|t|^\gamma \|\varphi\|_{2;\delta} \Big)\end{aligned}$$

for all $t \in [-1,1]$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$. Then the lemma follows from Lemma 4.1. ∎

LEMMA 4.6. *Let* $X_1, \ldots, X_N$ *be* C_b^∞*-vector fields. For all* $k \in \mathbb{N}$ *set*

$$\mathcal{D}^{(k)} = \operatorname{span}\{\psi X_{[\alpha]} : \psi \in C_b^\infty(\mathbb{R}^d),\ \alpha \in J_k^+(N)\}.$$

Then for all $\delta, \gamma \in \langle 0, 1]$, $k \in \mathbb{N}$ *and* $X \in \mathcal{D}^{(k)}$ *there exists a* $c > 0$ *such that, for all* $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\|\varphi\|_{2;X,\gamma k^{-1}} \leqslant c \Big(\sum_{j=1}^N \|\varphi\|_{2;X_j,\gamma} + \|\varphi\|_{2;\delta} \Big). \tag{4.2}$$

Proof. Fix $\delta \in \langle 0, 1]$. If $\gamma \leqslant \delta k$ then (4.2) follows from Corollary 4.3. For all $n \in \mathbb{N}$ let $P(n)$ be the following hypothesis:

> For all $k \in \mathbb{N}$, $\gamma \in \langle 0, \delta k 2^{n-2} \wedge 1]$ and $X \in \mathcal{D}^{(k)}$ there exists a $c > 0$ such that, for all $\varphi \in C_c^\infty(\mathbb{R}^d)$,
>
> $$\|\varphi\|_{2;X,\gamma k^{-1}} \leqslant c \Big(\sum_{j=1}^N \|\varphi\|_{2;X_j,\gamma} + \|\varphi\|_{2;\delta} \Big).$$

Then $P(1)$ is valid. Let $n \in \mathbb{N}$ and suppose that $P(n)$ is valid. Let $k \in \mathbb{N}$ and $\gamma \in \langle 0, \delta k 2^{n-1} \wedge 1]$. Consider

$$V = \Big\{ X \in \mathcal{D}^{(k)} : \text{there exists a } c > 0 \text{ such that}$$
$$\|\varphi\|_{X,\gamma k^{-1}} \leqslant c \Big(\sum_{j=1}^{N} \|\varphi\|_{2;X_j,\gamma} + \|\varphi\|_{2;\delta} \Big) \text{ for all } \varphi \in C_c^\infty(\mathbb{R}^d) \Big\}.$$

If $\alpha \in J_k^+(N)$ then $X_{[\alpha]} \in V$ by Lemma 4.5. Moreover, if in addition $\psi \in C_b^\infty(\mathbb{R}^d)$ then $\psi X_{[\alpha]} \in V$ by Lemma 4.1. So it remains to show that V is a vector space. But that follows from Lemma 4.4 and the induction hypothesis. ∎

The next proposition is the first application of the uniform Hörmander condition.

PROPOSITION 4.7. *Let $X_1, \ldots, X_N$ be C_b^∞-vector fields satisfying the uniform Hörmander condition of order r on $\mathbb{R}^d$. Then for all $\gamma \in \langle 0, 1]$ there exists a $c > 0$ such that, for all $\varphi \in C_c^\infty(\mathbb{R}^d)$,*

$$\|\varphi\|_{2;\gamma r^{-1}} \leqslant c \Big(\sum_{j=1}^{N} \|\varphi\|_{2;X_j,\gamma} + \|\varphi\|_2 \Big).$$

Proof. It follows from Lemma 4.6 that for all $X \in \mathcal{D}^{(r)}$ there exists a $c > 0$ such that

$$\|\varphi\|_{2;\gamma r^{-1}} \leqslant c \Big(\sum_{j=1}^{N} \|\varphi\|_{2;X_j,\gamma} + \|\varphi\|_{2;\gamma(2r)^{-1}} \Big)$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$, where $\mathcal{D}^{(r)}$ is as in Lemma 4.6. By the uniform Hörmander condition one has $\partial_i \in \mathcal{D}^{(r)}$ for all $i \in \{1, \ldots, d\}$. Hence there is a $c > 0$ such that

$$\|\varphi\|_{2;\gamma r^{-1}} \leqslant d \sum_{i=1}^{d} \|\varphi\|_{2;\partial_i,\gamma r^{-1}} \leqslant c \Big(\sum_{j=1}^{N} \|\varphi\|_{2;X_j,\gamma} + \|\varphi\|_{2;\gamma(2r)^{-1}} \Big)$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$. But there is a $c_1 > 0$ such that

$$\|\varphi\|_{2;\gamma(2r)^{-1}} \leqslant \varepsilon \, \|\varphi\|_{2;\gamma r^{-1}} + c_1 \, \varepsilon^{-1} \|\varphi\|_2$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $\varepsilon > 0$. Choosing $\varepsilon = (2c)^{-1}$ one deduces that, for all $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\|\varphi\|_{2;\gamma r^{-1}} \leqslant 2c \Big(\sum_{j=1}^{N} \|\varphi\|_{2;X_j,\gamma} + 2c \, c_1 \, \|\varphi\|_2 \Big). \quad ∎$$

In order to prove Theorem 1.3 we need some additional interpolation spaces. If $r = 1$ then H_0 is strongly elliptic and the theorem is well known. So we may assume that $r \geqslant 2$.

If L is the generator of a continuous semigroup S on $L_2(\mathbb{R}^d)$, $p \in [1,\infty]$ and $\gamma \in \langle 0,1]$ define the functions $\|\cdot\|_{\gamma,p,S}$, $\|\cdot\|'_{\gamma,p,S}$, $\|\cdot\|_{\gamma,p,L} : L_2(\mathbb{R}^d) \to [0,\infty]$ by

$$\|\varphi\|_{\gamma,p,S} = \|\varphi\|_2 + \Big(\int_0^1 dt\, t^{-1} |t^{-\gamma} \|(I - S_t)\varphi\|_2|^p \Big)^{1/p},$$

$$\|\varphi\|'_{\gamma,p,S} = \|\varphi\|_2 + \Big(\int_0^1 dt\, t^{-1} |t^{-\gamma} \|(I - S_t)^2\varphi\|_2|^p \Big)^{1/p},$$

$$\|\varphi\|_{\gamma,p,L} = \|\varphi\|_2 + \Big(\int_0^1 dt\, t^{-1} |t^{-\gamma} \kappa_t(\varphi)|^p \Big)^{1/p},$$

if $p < \infty$, where

$$\kappa_t(\varphi) = \inf\{\|\varphi - \varphi_1\|_2 + t\,\|L\varphi_1\|_2 : \varphi_1 \in D(L)\}$$

with obvious modifications if $p = \infty$. Define the interpolation spaces

$$\mathcal{X}_{\gamma,p,S} = \{\varphi \in L_2(\mathbb{R}^d) : \|\varphi\|_{\gamma,p,S} < \infty\}, \quad \mathcal{X}'_{\gamma,p,S} = \{\varphi \in L_2(\mathbb{R}^d) : \|\varphi\|'_{\gamma,p,S} < \infty\},$$

with norms $\|\cdot\|_{\gamma,p,S}$ and $\|\cdot\|'_{\gamma,p,S}$. If $\mathcal{X}$ and $\mathcal{Y}$ are two Banach spaces which are embedded in a locally convex Hausdorff space denote by $(\mathcal{X},\mathcal{Y})_{\gamma,p,K}$ the interpolation space with respect to the K-method. Then

$$(L_2(\mathbb{R}^d), D(L))_{\gamma,p,K} = \{\varphi \in L_2(\mathbb{R}^d) : \|\varphi\|_{\gamma,p,L} < \infty\}$$

and the norm is equivalent to $\|\cdot\|_{\gamma,p,L}$.

If S is a continuous semigroup then it follows from Theorem 3.4.2 and Corollary 3.4.9 of [1], that the spaces $\mathcal{X}_{\gamma,p,S}$, $\mathcal{X}'_{\gamma,p,S}$ and $(L_2(\mathbb{R}^d), D(L))_{\gamma,p,K}$ are equal with equivalent norms if $\gamma < 1$. Moreover, if S is merely continuous, $p = \infty$ and $\gamma = 1$ then $D(L) \subset \mathcal{X}_{1,\infty,S}$ and the embedding is continuous. If, however, L is a positive self-adjoint operator and $p = 2$ then a much better result is valid: $D(L^\gamma) = \mathcal{X}_{\gamma,2,S}$ and the norms are equivalent (see Lemma 7.1 of [4]).

As in Section 2 we set $L = I + \Delta$ and let S be the semigroup generated by L.

LEMMA 4.8. *$D(H) \subset (L_2(\mathbb{R}^d), D(L))_{r^{-1},\infty,K}$ and the embedding is continuous.*

Proof. It follows from Theorem 3.2 of [4], that the two norms $\|\cdot\|_{2,\delta}$ and $\|\cdot\|_{2^{-1}\delta,\infty,S}$ are equivalent for all $\delta \in \langle 0,1\rangle$. Moreover, $D(X_i) \subset \mathcal{X}_{1,\infty,X_i}$ and the embedding is continuous. Hence it follows from Proposition 4.7, applied with $\gamma = 1$, that there is a $c_1 > 0$ such that

$$\sup_{0<t\leqslant 1} t^{-1/r} \|(I - S_t)\varphi\|_2^2 \leqslant c_1(\|\varphi\|_2^2 + (\varphi, H\varphi)) \tag{4.3}$$

for all $\varphi \in C_c^\infty(\mathbb{R}^d)$. Then by density (4.3) is valid for all $\varphi \in W^{2,2}(\mathbb{R}^d)$. Next let $c > 0$ be as in Corollary 2.2(ii). Set $\tau = (2r)^{-1}$ and let $t \in \langle 0,1]$. Choosing

$A = c_1\,(I + H_0)$ and $B = t^{-\tau}(I - S_t)$ it follows from (4.3) that $(\varphi, A\varphi) \geqslant \|B\varphi\|^2$ for all $\varphi \in W^{2,2}(\mathbb{R}^d)$. Moreover,

$$|(\varphi,[B,[B,A]]\varphi)| = c_1 t^{-2\tau}|(\varphi,[S_t,[S_t,H_0]]\varphi)| \leqslant c c_1 \|B\varphi\|_2^2 \leqslant 2^{-1}\|B^2\varphi\|_2^2 + c^2 c_1^2 \|\varphi\|_2^2$$

for all $\varphi \in W^{\infty,2}(\mathbb{R}^d)$ by Corollary 2.2. Therefore the assumptions of Lemma 2.6 are valid with $\mathcal{D} = W^{\infty,2}(\mathbb{R}^d)$ uniformly for all $t \in \langle 0, 1]$. Hence

$$c\|(I+H_0)\varphi\|_2 = \|A\varphi\|_2 \geqslant 2^{-1}\|B^2\varphi\|_2 - cc_1\|\varphi\|_2 = 2^{-1}t^{-2\tau}\|(I-S_t)^2\varphi\|_2 - cc_1\|\varphi\|_2$$

uniformly for all $\varphi \in W^{\infty,2}(\mathbb{R}^d)$ and $t \in \langle 0, 1]$. Therefore

$$c\,\|(I + H_0)\varphi\|_2 \geqslant 2^{-1}\|\varphi\|'_{2\tau,\infty,S} - (c\,c_1 + 1)\,\|\varphi\|_2$$

for all $\varphi \in W^{\infty,2}(\mathbb{R}^d)$. Since $W^{\infty,2}(\mathbb{R}^d)$ is a core of H_0 and $\mathcal{X}'_{2\tau,\infty,S}$ is complete it follows that $D(H) \subset \mathcal{X}'_{2\tau,\infty,S}$ and the embedding is continuous. Finally, the lemma follows because $\mathcal{X}'_{2\tau,\infty,S} = \mathcal{X}_{2\tau,\infty,L}$, with equivalent norms. ∎

Poof of Theorem 1.3. It follows from Lemma 4.8 that

$$D(H) \subset (L_2(\mathbb{R}^d), D(L))_{r^{-1},\infty,K}$$

and the embedding is continuous. Hence

$$(L_2(\mathbb{R}^d), D(H))_{2^{-1},2,K} \subset (L_2(\mathbb{R}^d), (L_2(\mathbb{R}^d), D(L))_{r^{-1},\infty,K})_{2^{-1},2,K}.$$

But by the reiteration theorem ([1], Theorem 3.2.20) one has

$$(L_2(\mathbb{R}^d), (L_2(\mathbb{R}^d), D(L))_{r^{-1},\infty,K})_{2^{-1},2,K} = (L_2(\mathbb{R}^d), D(L))_{\tau,2,K}$$

with equivalent norms, where again $\tau = (2r)^{-1}$. Moreover,

$$(L_2(\mathbb{R}^d), D(L))_{\tau,2,K} = D(L^\tau) \quad \text{and} \quad (L_2(\mathbb{R}^d), D(H))_{2^{-1},2,K} = D(H^{1/2})$$

with equivalent norms. So $D(H^{1/2}) \subset D(L^\tau)$ and the embedding is continuous. Therefore there exists a $c > 0$ such that

$$\|\Delta^\tau\varphi\|_2^2 \leqslant \|L^\tau\varphi\|_2^2 \leqslant c(\|\varphi\|_2^2 + \|H^{1/2}\varphi\|_2^2) = c(\|\varphi\|_2^2 + (\varphi, H\varphi))$$

for all $\varphi \in D(H)$. Then Theorem 1.3 is a corollary of Theorem 1.1. ∎

5. THE UNIFORM HÖRMANDER CONDITION

Let $X_1, \ldots, X_N$ be C_b^∞-vector fields on $\mathbb{R}^d$. We conclude by deriving several characterizations of the uniform version of the Hörmander condition.

Each vector field X_i can be expressed as a partial differential operator $X_i = \sum_{k=1}^{d} a_{ik}\,\partial_k$ with coefficients $a_{ik} = X_i\pi_k \in C_b^\infty(\mathbb{R}^d)$, where π_k denotes the projection on the k-th coordinate. The multi-commutator $X_{[\alpha]}$ is also a C_b^∞-vector field with coefficients $a_{\alpha\, k} = X_{[\alpha]} \pi_k \in C_{\rm b}^\infty(\mathbb{R}^d)$. Explicitly $X_{[\alpha]} = \sum_{k=1}^d a_{\alpha\, k}\, \partial_k$. Then for all $r \in \mathbb{N}$ and all $i, j \in \{1, \ldots, d\}$ define

$$c^{(r)}_{ij} = \sum_{\alpha \in J_r^+(N)} a_{\alpha\, i}\, a_{\alpha\, j}$$

and set $C^{(r)} = (c^{(r)}_{ij})$. The matrix $C^{(r)}$ is real symmetric and positive semidefinite. In particular the operator H_0 given by (1.4) is a second-order operator in divergence form with the matrix of coefficients $C^{(1)} = \sum_{k=1}^d a_{ki}\, a_{kj}$.

PROPOSITION 5.1. *Let $X_1, \ldots, X_N$ be $C_{\rm b}^\infty$-vector fields. For all $\alpha \in J^+(N)$ and $x \in \mathbb{R}^d$ let $a_\alpha(x) \in \mathbb{R}^d$ be such that $X_{[\alpha]} = \sum_{k=1}^d a_{\alpha\, k}\, \partial_k$. Moreover, fix $r \in \mathbb{N}$. Then the following statements are equivalent:*

(i) *The vector fields $X_1, \ldots, X_N$ satisfy the uniform Hörmander condition of order r on $\mathbb{R}^d$.*

(ii) *There exists a $\sigma > 0$ such that $C^{(r)}(x) \geqslant \sigma I$ uniformly for all $x \in \mathbb{R}^d$.*

(iii) *There exists an $M > 0$ such that for all $x \in \mathbb{R}^d$, $i \in \{1, \ldots, d\}$ and $\alpha \in J_r^+(d')$ there exists a $\lambda_\alpha \in [-M, M]$ such that*

$$e_i = \sum_{\alpha \in J_r^+(d')} \lambda_\alpha\, a_\alpha(x),$$

where e_i is the unit vector in the i-th direction.

(iv) *There exists a $\sigma > 0$ such that*

$$\mathrm{Vol}\Big\{ \sum_{\alpha \in J_r^+(N)} \lambda_\alpha\, a_\alpha(x) : |\lambda_\alpha| \leqslant 1 \textit{ for all } \alpha \in J_r^+(N) \Big\} \geqslant \sigma$$

uniformly for all $x \in \mathbb{R}^d$.

(v) *There exists a $\sigma > 0$ such that for all $x \in \mathbb{R}^d$ there are multi-indices $\alpha_1, \ldots, \alpha_d \in J_r^+(N)$ such that*

$$|\det((X_{[\alpha_i]} \pi_j)(x))| = |\det(a_{\alpha_1}(x), \ldots, a_{\alpha_d}(x))| \geqslant \sigma.$$

Proof. (i)$\Rightarrow$(iii). It follows from statement (i) that for all $i \in \{1, \ldots, d\}$ and $\alpha \in J_r^+(N)$ there are $\psi_{i\,\alpha} \in C_{\rm b}^\infty(\mathbb{R}^d)$ such that

$$\partial_i = \sum_{\alpha \in J_r^+(N)} \psi_{i\,\alpha} X_{[\alpha]}$$

for all $i \in \{1, \ldots, d\}$. Then $e_i = \sum_{\alpha \in J_r^+(N)} \psi_{i\,\alpha}(x)\, a_\alpha(x)$ for all $x \in \mathbb{R}^d$ and statement (iii) follows with $M = \max_{i \in \{1,\ldots,d\}} \max_{\alpha \in J_r^+(N)} \|\psi_{i\,\alpha}\|_\infty$.

(iii)⇒(iv). Let $M > 0$ be as in statement (iii). Then

$$\Big\{ \sum_{i=1}^{d} \lambda_i\, e_i : 0 \leqslant \lambda_i \leqslant (dM)^{-1} \text{ for all } i \Big\} \subseteq \Big\{ \sum_{\alpha \in J_r^+(N)} \lambda_\alpha\, a_\alpha(x) : |\lambda_\alpha| \leqslant 1 \text{ for all } \alpha \Big\}$$

for all $x \in \mathbb{R}^d$. Therefore

$$\mathrm{Vol}\Big\{ \sum_{\alpha \in J_r^+(N)} \lambda_\alpha\, a_\alpha(x) : |\lambda_\alpha| \leqslant 1 \text{ for all } \alpha \Big\} \geqslant \mathrm{Vol}\Big\{ \sum_{i=1}^{d} \lambda_i\, e_i : 0 \leqslant \lambda_i \leqslant (dM)^{-1} \text{ for all } i \Big\} = (dM)^{-d}$$

for all $x \in \mathbb{R}^d$ and statement (iv) follows.

(iv)⇒(v). Fix $x \in \mathbb{R}^d$. By Lemma 3.1.2 in [19] there are $\alpha_1, \ldots, \alpha_d \in J_r^+(N)$ and for all $\alpha \in J_r^+(N)$ and $k \in \{1, \ldots, d\}$ there are $\lambda_{\alpha\, k} \in \mathbb{R}$ with $|\lambda_{\alpha\, k}| \leqslant 2^{L-d}$ such that

$$a_\alpha(x) = \sum_{k=1}^{d} \lambda_{\alpha\, k}\, a_{\alpha_k}(x)$$

where $L = \mathrm{card} J_r^+(N)$. Then

$$\Big\{ \sum_{\alpha \in J_r^+(N)} \lambda_\alpha a_\alpha(x) : |\lambda_\alpha| \leqslant 1 \text{ for all } \alpha \Big\} \subseteq 2^{L-d} L \Big\{ \sum_{k=1}^{d} \lambda_k a_{\alpha_k}(x) : |\lambda_k| \leqslant 1 \text{ for all } k \Big\}.$$

Therefore

$$|\det(a_{\alpha_1}(x), \ldots, a_{\alpha_d}(x))| = 2^{-d} \mathrm{Vol}\Big\{ \sum_{k=1}^{d} \lambda_k\, a_{\alpha_k}(x) : |\lambda_k| \leqslant 1 \text{ for all } k \Big\}$$
$$\geqslant 2^{-L}\, L^{-1} \mathrm{Vol}\Big\{ \sum_{\alpha \in J_r^+(N)} \lambda_\alpha\, a_\alpha(x) : |\lambda_\alpha| \leqslant 1 \text{ for all } \alpha \Big\}.$$

Thus statement (iv) implies statement (v).

(v)⇒(ii). Let $\sigma > 0$ be as in statement (v). Fix $x \in \mathbb{R}^d$. Then there are $\alpha_1, \ldots, \alpha_d \in J_r^+(N)$ such that

$$|\det(a_{\alpha_1}(x), \ldots, a_{\alpha_d}(x))| \geqslant \sigma.$$

For all $k, l \in \{1, \ldots, d\}$ set $d_{kl} = \sum_{j=1}^{d} a_{\alpha_j\, k}(x)\, a_{\alpha_j\, l}(x)$ and $D = (d_{kl})$. Then

$$\det D = (\det(a_{\alpha_1}(x), \ldots, a_{\alpha_d}(x)))^2 \geqslant \sigma^2.$$

Moreover,

$$C^{(r)} \geqslant D \geqslant \|D\|^{-(d-1)}\, (\det D)\, I \geqslant \|D\|^{-(d-1)}\, \sigma^2\, I$$

where $\|D\|$ is the norm of the matrix D. Since the coefficients a_α are uniformly bounded statement (ii) follows.

(ii)⇒(i). If $X = \sum_{k=1}^{d} a_k \,\partial_k$ then since $C^{(r)}$ is invertible one computes that

$$X = \sum_{\alpha \in J_r^+(N)} ((C^{(r)})^{-1} a, a_\alpha)\, X_{[\alpha]} .$$

But the condition $C^{(r)} \geqslant \sigma I$ implies that the coefficients of the matrix $(C^{(r)})^{-1}$ are in $C_{\rm b}^\infty(\mathbb{R}^d)$. ∎

Statement (ii) of Proposition 5.1 is the formulation of the uniform Hörmander condition used by Kusuoka and Stroock in Section 3 et seq. of [12] and again in their analysis of long time behaviour in [13] (see Theorems 3.20 and 3.24). The determinant identified in statement (v) of Proposition 5.1 plays a ubiquitous role in the analysis of Nagel, Stein and Wainger [15] and was also identified by Jerison as an important parameter in the Poincaré inequality (see condition (2.3c) on page 505 of [11]).

Finally we note that for operators H_0 with C^∞-coefficients Fefferman and Phong have shown that the subellipticity condition (1.2) is locally equivalent to a property of the geometry associated with H_0. Nagel, Stein and Wainger [15] have then analyzed in detail the local geometry for operators (1.4) constructed from vector fields satisfying the local Hörmander condition. One could expect that there are global analogues of these results. In a separate paper we will indeed extend the conclusions of Nagel, Stein and Wainger and obtain uniform properties of the geometry, properties such as volume doubling, if the vector fields satisfy the uniform Hörmander condition.

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REFERENCES

[1] P.L. BUTZER, H. BERENS, *Semi-groups of Operators and Approximation*, Grundlehren Math. Wiss., vol. 145, Springer-Verlag, Berlin 1967.

[2] H.L. CYCON, R.G. FROESE, W. KIRSCH, B. SIMON, *Schrödinger Operators with Application to Quantum Mechanics and Global Geometry*, Springer-Verlag, Berlin 1987.

[3] W. DRIESSLER, S.J. SUMMERS, On commutators and selfadjointness, *Lett. Math. Phys.* **7**(1983), 319–326.

[4] A.F.M. TER ELST, D.W. ROBINSON, Subelliptic operators on Lie groups: regularity, *J. Austr. Math. Soc. Ser. A* **57**(1994), 179–229.

[5] A.F.M. TER ELST, D.W. ROBINSON, Second-order strongly elliptic operators on Lie groups with Hölder continuous coefficients, *J. Austr. Math. Soc. Ser. A* **63**(1997), 297–363.

[6] W.G. FARIS, *Self-adjoint Operators*, Lect. Notes in Math., vol. 433, Springer-Verlag, Berlin 1975.

[7] C. FEFFERMAN, D.H. PHONG, Subelliptic eigenvalue problems, in *Conference on Harmonic Analysis in Honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981)*, Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1983, pp. 590–606.

[8] J. GLIMM, A. JAFFE, The $\lambda\phi_2^4$ quantum field theory without cutoffs. IV. Perturbations of the Hamiltonian, *J. Math. Phys.* **13**(1972), 1568–1584.

[9] J. GLIMM, A. JAFFE, *Quantum Physics. A Functional Integral Point of View*, Springer-Verlag, New York 1981.

[10] L. HÖRMANDER, Hypoelliptic second order differential equations, *Acta Math.* **119**(1967), 147–171.

[11] D. JERISON, The Poincaré inequality for vector fields satisfying Hörmander's condition, *Duke Math. J.* **53**(1986), 503–523.

[12] S. KUSUOKA, D. STROOCK, Applications of the Malliavin calculus. III, *J. Fac. Sci. Univ. Tokyo Sect. IA Math.* **34**(1987), 391–442.

[13] S. KUSUOKA, D. STROOCK, Long time estimates for the heat kernel associated with a uniformly subelliptic symmetric second order operator, *Ann. Math.* **127**(1988), 165–189.

[14] G. LUMER, R.S. PHILLIPS, Dissipative operators in a Banach space, *Pacific J. Math.* **11**(1961), 679–698.

[15] A. NAGEL, E.M. STEIN, S. WAINGER, Balls and metrics defined by vector fields. I: basic properties, *Acta Math.* **155**(1985), 103–147.

[16] M. REED, B. SIMON, *Methods of Modern Mathematical Physics.* II *Fourier Analysis, Self-adjointness*, Academic Press, New York 1975.

[17] D.W. ROBINSON, Commutator theory on Hilbert space, *Canad. J. Math.* **34**(1987), 1235–1280.

[18] L.P. ROTHSCHILD, E.M. STEIN, Hypoelliptic differential operators and nilpotent groups, *Acta Math.* **137**(1976), 247–320.

[19] C.M.P.A. SMULDERS, *Reduced Heat Kernels on Homogeneous Spaces*, Ph.D. Dissertation, Eindhoven Univ. of Technology, The Netherlands 2000; see http://alexandria.tue.nl/extra2/200000153.pdf.

A.F.M. TER ELST, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF AUCKLAND, AUCKLAND, NEW ZEALAND

E-mail address: terelst@math.auckland.ac.nz

DEREK W. ROBINSON, CENTRE FOR MATHEMATICS AND ITS APPLICATIONS, AUSTRALIAN NATIONAL UNIVERSITY, CANBERRA, ACT 0200, AUSTRALIA

E-mail address: Derek.Robinson@anu.edu.au

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