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CYCLIC VECTORS AND INVARIANT SUBSPACES FOR BERGMAN AND DIRICHLET SHIFTS

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ABSTRACT. It is shown that the invariant subspaces for the Bergman and Dirichlet shifts on the right half-plane correspond to the common invariant subspaces of the right shift operators on certain weighted Lebesgue spaces on the half-line. As a particular instance, the corresponding result for invariant subspaces of multipliers induced by weak-star generators of $\mathcal{H}^\infty(\mathbb{D})$ on weighted Bergman spaces of the unit disc is deduced. Finally, cyclic vectors for the Bergman and Dirichlet shifts are also studied.

KEYWORDS: *Invariant subspaces, shift operator, Bergman spaces, Dirichlet spaces, cyclic vectors.*

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1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{C}_+$ denote the open right complex half-plane. Recall that if $\alpha > -1$ the *weighted Bergman spaces* $\mathcal{A}^2_\alpha(\mathbb{C}_+)$ consist of those functions F analytic on $\mathbb{C}_+$ for which the norm

$$\|F\|_{\mathcal{A}^2_\alpha(\mathbb{C}_+)} = \Big\{ \frac{1}{\pi} \int\limits_{\mathbb{C}_+} |F(x+\mathrm{i}y)|^2 x^\alpha \,\mathrm{d}x\,\mathrm{d}y \Big\}^{1/2}$$

is finite. Note that $\mathcal{A}^2_\alpha(\mathbb{C}_+)$ is isometrically isomorphic in a natural way to the *weighted Bergman space* $\mathcal{A}^2_\alpha(\mathbb{D})$ on the unit disc $\mathbb{D}$ which consists of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{A}^2_\alpha(\mathbb{D})} = \Big\{ \int\limits_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\alpha \,\mathrm{d}\sigma(z) \Big\}^{1/2} < \infty.$$

Here $\mathrm{d}\sigma(z)$ denotes the area measure on $\mathbb{D}$ normalized so that $\sigma(\mathbb{D}) = 1$. Observe that $\alpha = 0$ corresponds to the classical Bergman space $\mathcal{A}^2(\mathbb{D})$.

Recall also that if $-1 < \alpha < 1$ the *weighted Dirichlet space* $\mathcal{D}^2_\alpha(\mathbb{C}_+)$ consists of those functions F analytic on $\mathbb{C}_+$, such that their derivatives F' belong to $\mathcal{A}^2_\alpha(\mathbb{C}_+)$. In a similar way, one can define the *weighted Dirichlet space* $\mathcal{D}^2_\alpha(\mathbb{D})$. Note that when $\alpha \neq 0$, the spaces $\mathcal{D}^2_\alpha(\mathbb{C}_+)$ and $\mathcal{D}^2_\alpha(\mathbb{D})$ are no longer isometrically isomorphic.

In this work, we focus our attention in the study of the invariant subspaces (i.e., closed linear manifolds) for the forward shift operator S

$$(Sg)(z) = zg(z) \quad (z \in \mathbb{D}),$$

acting on weighted Bergman and Dirichlet spaces on the unit disc.

Firstly, we would like to point out that the structure of the invariant subspaces for S in the weighted Dirichlet spaces $\mathcal{D}^2_\alpha(\mathbb{D})$ was completely described by Richter in [18] at the end of the eighties (see also Aleman's work [1]). Concerning the Bergman space case, such a description was carried out by Aleman, Richter and Sundberg in [3]. As in the Hardy space case (see Beurling's Theorem [5]), in both cases it holds that the invariant subspaces for S are in one-to-one correspondence with their wandering subspaces. Recall that a subspace X of a Hilbert space is called wandering for an operator T if X is orthogonal to $T^n(X)$ for $n = 1, 2, \ldots$ (see [11]).

The aim of this work is to take a bit further the study of the invariant subspaces for S. In this sense, as a preliminary result, we show that an analogue of Lax's extension of Beurling's theorem (cf. [14] and Section 3.1 of [16]) holds for $\mathcal{A}^2_\alpha(\mathbb{D})$. More precisely, it can be shown as established in [9] that the invariant subspaces for S in $\mathcal{A}^2_\alpha(\mathbb{D})$ are in a one-to-one correspondence with those for the convolution operator

$$(Tf)(x) = \int_0^x \mathrm{e}^{-(x-t)} f(t)\,\mathrm{d}t \quad (x > 0)$$

in $L^2(\mathbb{R}_+, \mathrm{d}t/t^{\alpha+1})$. In Section 2, we prove that a closed subspace X contained in $L^2(\mathbb{R}_+, \mathrm{d}t/t^{\alpha+1})$ is invariant under T if and only if it is invariant under every right shift operator acting on $L^2(\mathbb{R}_+, \mathrm{d}t/t^{\alpha+1})$. In particular, it follows that $X \subset \mathcal{A}^2_\alpha(\mathbb{D})$ is invariant under S if and only if it is invariant under multiplication by every bounded analytic function. In addition, we show that the same holds if we replace S by a multiplication operator induced by an $\mathcal{H}^\infty(\mathbb{D})$ weak-star generator. Here, $\mathcal{H}^\infty(\mathbb{D})$ denotes the space of bounded analytic functions on $\mathbb{D}$.

In Section 3, we consider the shift operator acting on weighted Dirichlet spaces of the right half-plane $\mathcal{D}^2_\alpha(\mathbb{C}_+)$. Our goal is to prove that the lattice of invariant subspaces correspond to the common invariant subspaces of the right shift operators on certain weighted Lebesgue spaces on the half-line. In this case, a more careful analysis is required since technical difficulties arise because the right shift operators in the corresponding Lebesgue spaces are no longer bounded. In addition, by means of the adjoint of the shift in $\mathcal{A}^2_\alpha(\mathbb{C}_+)$, we also show that a kind of Lax Theorem for the backward shift operator in $\mathcal{D}^2_\alpha(\mathbb{C}_+)$ holds.

Finally, in Section 4, we study cyclic vectors for the Bergman and Dirichlet shifts which, of course, correspond to cyclic vectors for shifts on weighted L^2 spaces on $\mathbb{R}_+$. We end the section giving also results about cyclic vectors for the backward shifts on the Dirichlet spaces.

2. INVARIANT SUBSPACES FOR THE BERGMAN SHIFT

In this section we deal with the invariant subspaces of the shift operator acting on weighted Bergman spaces $\mathcal{A}^2_\alpha(\mathbb{D})$. As stated in the introduction, the *weighted Bergman space* $\mathcal{A}^2_\alpha(\mathbb{C}_+)$ is isometrically isomorphic to the weighted Bergman space $\mathcal{A}^2_\alpha(\mathbb{D})$. In fact, an isometry $J : \mathcal{A}^2_\alpha(\mathbb{D}) \to \mathcal{A}^2_\alpha(\mathbb{C}_+)$ is given by $F = Jf$, where $f \in \mathcal{A}^2_\alpha(\mathbb{D})$ and

$$F(s) = \frac{2^{\alpha+1}}{(1+s)^{\alpha+2}} f\Big(\frac{1-s}{1+s}\Big), \quad (s \in \mathbb{C}_+). \tag{2.1}$$

In order to establish a further isometric isomorphism between $\mathcal{A}^2_\alpha(\mathbb{C}_+)$ and a space of functions on the positive real axis $\mathbb{R}_+$, we consider the weighted Lebesgue measure μ_α for $\alpha > -1$ defined by

$$\mathrm{d}\mu_\alpha = \frac{\Gamma(\alpha+1)}{2^\alpha t^{\alpha+1}}\,\mathrm{d}t, \quad (t > 0),$$

where Γ denotes the Euler Gamma function. By means of the Laplace transform $\mathcal{L}$, there is a further isometric isomorphism $\mathcal{L} : L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha) \to \mathcal{A}^2_\alpha(\mathbb{C}_+)$. That is,

$$\|f\|_{L^2(\mathbb{R}_+,\mathrm{d}\mu_\alpha)} = \|F\|_{\mathcal{A}^2_\alpha(\mathbb{C}_+)}, \quad \text{where } F(s) = (\mathcal{L}f)(s) := \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t \quad (s \in \mathbb{C}_+).$$

These isometric isomorphisms yield that the forward shift operator S is unitarily equivalent to $\widetilde{S}$ on $\mathcal{A}^2_\alpha(\mathbb{C}_+)$ defined by

$$(\widetilde{S}F)(s) = \Big(\frac{1-s}{1+s}\Big)F(s) \quad (s \in \mathbb{C}_+), \tag{2.2}$$

which, in turn, is unitarily equivalent to $I - 2T$, where $T : L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha) \to L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$ is the convolution operator defined by:

$$(Tf)(x) = \int_0^x \mathrm{e}^{-(x-t)} f(t)\,\mathrm{d}t \quad (x > 0). \tag{2.3}$$

This operator is bounded since the convolution kernel is in $L^1(\mathbb{R}_+)$. Therefore, the invariant subspaces of S in $\mathcal{A}^2_\alpha(\mathbb{D})$ are in one-to-one correspondence with those of T (see [9]) .

For $\tau > 0$, let S_τ be the bounded operator on $L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$ defined by

$$(S_\tau f)(t) = \begin{cases} f(t-\tau) & \text{if } t > \tau, \\ 0 & \text{if } t < \tau. \end{cases}$$

The following result holds.

THEOREM 2.1. *Let $\alpha > -1$. A closed subspace of $L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$ is invariant under every right shift S_τ, $\tau > 0$, if and only if it is invariant under the convolution operator T defined above.*

Proof. Suppose that $X \subseteq L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$ is invariant under T. We show that it is invariant under every S_τ by transforming the problem to the Bergman space of the unit disc. In detail, the operator $\mathcal{L}S_\tau\mathcal{L}^{-1}$ is the operator of multiplication by the function $s \mapsto \mathrm{e}^{-s\tau}$, and the operator $J^{-1}\mathcal{L}S_\tau\mathcal{L}^{-1}J$ is the operator on $\mathcal{A}^2_\alpha(\mathbb{D})$ of multiplication by the function $\phi : z \mapsto \exp(-\tau(1-z)/(1+z))$, which lies in $\mathcal{H}^\infty(\mathbb{D})$. Now $J^{-1}\mathcal{L}X$ is invariant under S, by hypothesis, so it is invariant under multiplication by every polynomial. Therefore, we can find a uniformly bounded sequence (p_n) of polynomials in $\mathcal{H}^\infty(\mathbb{D})$ such that $p_n f \to \phi f$ pointwise for every $f \in \mathcal{A}^2_\alpha(\mathbb{D})$: to do this, take (p_n) to be the Fejér sums of the Fourier series of ϕ, i.e.,

$$p_n(z) = \frac{1}{n+1} \sum_{k=0}^{n} \phi_k z^k,$$

where ϕ_k is the kth Taylor coefficient of ϕ. Now $p_n(z)f(z) \to \phi(z)f(z)$ as $n \to \infty$ for each $z \in \mathbb{D}$, i.e., $\langle p_n f, k_z\rangle \to \langle \phi f, k_z\rangle$ for each reproducing kernel function $k_z \in \mathcal{A}^2_\alpha(\mathbb{D})$. Since the closed linear span of these kernels is the whole space and the sequence (p_n) is bounded, we conclude that $p_n f \to \phi f$ weakly, and hence ϕf lies in $J^{-1}\mathcal{L}X$. This implies that $S_\tau X \subseteq X$.

Conversely, suppose that $S_\tau X \subseteq X$ for each $\tau > 0$. We show that X is invariant under T, or, equivalently, that $\mathcal{L}X$ is invariant under the operator $\widetilde{S}$ in (2.2). Since $(1-s)/(1+s) = -1 + 2/(1+s)$, it is sufficient to consider the operator $V : \mathcal{A}^2_\alpha(\mathbb{C}_+) \to \mathcal{A}^2_\alpha(\mathbb{C}_+)$ where

$$(Vf)(s) = \frac{f(s)}{1+s} = f(s) \int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{-t}\,\mathrm{d}t.$$

Now if we define

$$(V_n f)(s) = f(s) \int_0^n \mathrm{e}^{-st}\mathrm{e}^{-t}\,\mathrm{d}t = \frac{f(s)(1-\mathrm{e}^{-n(s+1)})}{s+1},$$

then, since

$$\int_n^\infty \mathrm{e}^{-st}\mathrm{e}^{-t}\,\mathrm{d}t \to 0$$

uniformly on $\mathbb{C}_+$, it is clear that $V_n f$ tends to Vf in norm as $n \to \infty$. It remains to show that $\mathcal{L}X$ is invariant under each V_n. However, this follows since the integral defining V_n can be approximated by Riemann sums, that is,

$$\frac{n}{N}\sum_{k=1}^{N} \mathrm{e}^{-(s+1)kn/N} \to \int_0^n \mathrm{e}^{-(s+1)t}\,\mathrm{d}t,$$

pointwise on $\mathbb{C}_+$ while the functions on the left-hand side above are uniformly bounded in $\mathcal{H}^\infty(\mathbb{D})$ norm by $\frac{n}{N}\sum\limits_{k=1}^{N} \mathrm{e}^{-kn/N} \leqslant n$. Thus by the dominated convergence theorem the operators $W_N^{(n)}$ defined by

$$(W_N^{(n)} f)(s) = \frac{n}{N}\sum_{k=1}^{N} \mathrm{e}^{-kn/N}\mathrm{e}^{-skn/N} f(s)$$

converge strongly to V_n for each n, as $N \to \infty$; hence $\mathcal{L}X$ is invariant under each V_n, and the result follows. ∎

We remark that one can give an alternative proof of Theorem 2.1 using the Nagy–Foiaş functional calculus along with some facts concerning strongly continuous semigroups of contractions.

In addition, we point out that a vector-valued version of Theorem 2.1 also holds, with a completely similar proof to that one we presented. Namely, if we consider the vector-valued Bergman spaces and Lebesgue spaces above, or, equivalently, a finite orthogonal direct sum of the corresponding scalar spaces, then we have a similar equivalence of closed invariant subspaces for shifts (i.e., operators such as $S \oplus \cdots \oplus S$ on $\mathcal{A}^2_\alpha(\mathbb{D}) \oplus \cdots \oplus \mathcal{A}^2_\alpha(\mathbb{D})$).

This enables us to make use of a technique apparently originating in [10] to study closed, possibly unbounded, operators commuting with shifts (see also [13], [16]).

Suppose that $T : \mathcal{D}(T) \to L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$ is a not necessarily bounded operator with $\mathcal{D}(T) \subseteq L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$ and closed graph

$$\mathcal{G}(T) := \{(f, Tf) : f \in \mathcal{D}(T)\} \subseteq L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha) \oplus L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha).$$

Then we say that T is *shift-invariant* if $(S_\tau \oplus S_\tau)\mathcal{G}(T) \subseteq \mathcal{G}(T)$ for each $\tau > 0$. If $\mathcal{D}(T) = L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$, then this is the same as saying that $S_\tau T = TS_\tau$ for each $\tau > 0$.

COROLLARY 2.2. *Let $\alpha > -1$. A closed operator $T : \mathcal{D}(T) \to L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$ with $\mathcal{D}(T) \subseteq L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$ is shift-invariant if and only if the graph of the operator $J^{-1}\mathcal{L}T\mathcal{L}^{-1}J$ (defined on a subdomain of $\mathcal{A}^2_\alpha(\mathbb{D})$) is closed in $\mathcal{A}^2_\alpha(\mathbb{D}) \oplus \mathcal{A}^2_\alpha(\mathbb{D})$, and invariant under $S \oplus S$. If T is bounded and shift-invariant, then the operator $J^{-1}\mathcal{L}T\mathcal{L}^{-1}J$ has the form $g \mapsto w \cdot g$ for some $w \in \mathcal{H}^\infty(\mathbb{D})$ with $\|T\| = \|w\|_\infty$; likewise, $\mathcal{L}T\mathcal{L}^{-1}$ is a multiplication operator by a function in $\mathcal{H}^\infty(\mathbb{C}_+)$.*

Proof. Most of this follows from the unitary equivalences above, combined with Theorem 2.1. The fact that bounded operators W on $\mathcal{A}^2_\alpha(\mathbb{D})$ that commute with S are given by multiplication by $\mathcal{H}^\infty(\mathbb{D})$ functions is easily derived, since if $We = w$, say, where $e(z) \equiv 1$, then $Wp = wp$ for each polynomial p, and by continuity W is the operator of multiplication by w on $\mathcal{A}^2_\alpha(\mathbb{D})$, easily seen to have norm $\|w\|_\infty$. ∎

Thus, if we could characterize all closed subspaces of $\mathcal{A}^2_\alpha(\mathbb{D}) \oplus \mathcal{A}^2_\alpha(\mathbb{D})$ invariant under $S \oplus S$, we would have a description of all the shift-invariant closed operators on $\mathcal{A}^2_\alpha(\mathbb{D})$, $\mathcal{A}^2_\alpha(\mathbb{C}_+)$ and $L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$. The link between shift-invariant operators on some function spaces on $\mathbb{R}_+$ and multiplication operators on the space of Laplace transforms is also explored in [17], [25].

2.1. MULTIPLIERS INDUCED BY WEAK-STAR GENERATORS. Let $\mathcal{H}$ be a Hilbert space of holomorphic functions on a domain Ω. Recall that a complex valued function ϕ in Ω is called a *multiplier* if the pointwise product ϕf belongs to $\mathcal{H}$ whenever $f \in \mathcal{H}$. Clearly, every multiplier of $\mathcal{H}$ induces a linear multiplication operator

$$M_\phi f = \phi f \quad (f \in \mathcal{H}).$$

Observe that the Closed Graph Theorem implies that any multiplier induces a bounded multiplication operator on the weighted Bergman spaces $\mathcal{A}^2_\alpha(\mathbb{D})$.

We point out that it is possible to get an analogue of Theorem 2.1 for M_φ on $\mathcal{A}^2_\alpha(\mathbb{D})$ in the case where $\varphi \in \mathcal{H}^\infty(\mathbb{D})$ is a weak-star generator. Recall that $\varphi \in \mathcal{H}^\infty(\mathbb{D})$ is a weak-star generator of $\mathcal{H}^\infty(\mathbb{D})$ if the polynomials in φ are weak-star dense in $\mathcal{H}^\infty(\mathbb{D})$. We refer to Sarason's work [22] for a complete characterization of weak-star generators.

If $\varphi \in \mathcal{H}^\infty(\mathbb{D})$ is a weak-star generator, then the lattice of invariant subspaces for M_φ in the Hardy space $\mathcal{H}^2(\mathbb{D})$ coincides with the one of the forward shift S, that is, $\mathrm{Lat}(S) = \mathrm{Lat}(M_\varphi)$ on $\mathcal{H}^2(\mathbb{D})$. Bourdon proved ([6], Proposition 2.1) that the same holds on the Bergman space and a completely similar argument also works for the weighted Bergman spaces $\mathcal{A}^2_\alpha(\mathbb{D})$. This along with Theorem 2.1 yields the following

THEOREM 2.3. *Let $\alpha > -1$ and $\varphi \in \mathcal{H}^\infty(\mathbb{D})$ a weak-star generator. A closed subspace $\mathcal{M} \subset \mathcal{A}^2_\alpha(\mathbb{D})$ is invariant under the multiplier φ if and only if $\mathcal{L}J\mathcal{M} \subset L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$ is invariant under every right shift S_τ with $\tau > 0$.*

3. INVARIANT SUBSPACES FOR THE DIRICHLET SHIFT

Recall from the introduction that if $-1 < \alpha < 1$ the *weighted Dirichlet space* $\mathcal{D}^2_\alpha(\mathbb{C}_+)$ on the right half-plane $\mathbb{C}_+$ consists of those functions F analytic on $\mathbb{C}_+$ such that $F' \in \mathcal{A}^2_\alpha(\mathbb{C}_+)$. Upon identifying functions which differ by a constant,

$\mathcal{D}^2_\alpha(\mathbb{C}_+)$ turns out to be a Hilbert space endowed with the norm

$$\|F\|_{\mathcal{D}^2_\alpha(\mathbb{C}_+)} = \Big\{\frac{1}{\pi}\int\limits_{\mathbb{C}_+} |F'(x+\mathrm{i}y)|^2 x^\alpha \,\mathrm{d}x\,\mathrm{d}y\Big\}^{1/2}.$$

Note that just for $\alpha = 0$, the space $\mathcal{D}^2_0(\mathbb{C}_+)$ is isometrically isomorphic to the classical Dirichlet space of the unit disc $\mathcal{D}^2(\mathbb{D})$ modulo constant functions.

Note that since $F \in \mathcal{D}^2_\alpha(\mathbb{C}_+)$ if and only if $F' \in \mathcal{A}^2_\alpha(\mathbb{C}_+)$, it follows that $\mathcal{D}^2_\alpha(\mathbb{C}_+)$ is isometrically isomorphic, via the Laplace transform, to the function space $L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$, where

$$\mathrm{d}\nu_\alpha = \frac{\Gamma(\alpha+1)}{2^\alpha t^{\alpha-1}}\,\mathrm{d}t, \quad (t>0).$$

Observe that if $\alpha = 0$, the shift operator $(Sg)(z) = zg(z)$ on the classical Dirichlet space $\mathcal{D}^2(\mathbb{D})$ modulo constant functions is unitarily equivalent to the operator $\widetilde{S}$ on $\mathcal{D}^2_0(\mathbb{C}_+)$ defined by

$$(\widetilde{S}F)(s) = \Big(\frac{1-s}{1+s}\Big)F(s) \quad (s \in \mathbb{C}_+).$$

Nevertheless, if $\alpha \neq 0$, this is no longer true. Of course, for any $-1 < \alpha < 1$, the shift operator $\widetilde{S}$ on $\mathcal{D}^2_\alpha(\mathbb{C}_+)$ is unitarily equivalent to $I - 2T$ on $L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$, where T is the convolution operator in (2.3).

In order to relate the invariant subspaces of T to the invariant subspaces of the right shift S_τ on $L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$ for any $-1 < \alpha < 1$ a technical difficulty arises. The point is that the right shift operators S_τ are no longer bounded in $L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$. The following preliminary observation will be helpful.

LEMMA 3.1. *Suppose that $-1 < \alpha < 1$, that $f \in L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$ and that $\tau > 0$. Then $S_\tau f \in L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$ if and only if $f \in L^2(\mathbb{R}_+)$.*

Proof. If $f, S_\tau f \in L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$, then

$$\int\limits_0^1 |f(t)|^2 \mathrm{d}t \leqslant \int\limits_\tau^{\tau+1} |f(t-\tau)|^2 \Big(\frac{t}{\tau}\Big)^{1-\alpha} \mathrm{d}t < \infty \quad \text{and} \quad \int\limits_1^\infty |f(t)|^2 \mathrm{d}t \leqslant \int\limits_1^\infty |f(t)|^2 t^{1-\alpha} \mathrm{d}t < \infty,$$

so $f \in L^2(\mathbb{R}_+)$. Conversely, if $f \in L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha) \cap L^2(\mathbb{R}_+)$, then

$$\int\limits_\tau^{\tau+1} |f(t-\tau)|^2 t^{1-\alpha} \mathrm{d}t \leqslant (\tau+1)^{1-\alpha} \int\limits_0^1 |f(t)|^2 \mathrm{d}t < \infty,$$

$$\int\limits_{\tau+1}^\infty |f(t-\tau)|^2 t^{1-\alpha} \mathrm{d}t \leqslant (1+\tau)^{1-\alpha} \int\limits_1^\infty |f(t)|^2 t^{1-\alpha} \mathrm{d}t < \infty,$$

so $S_\tau f \in L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$. ∎

The following result is the analogue of Theorem 2.1 in the context of Dirichlet spaces.

THEOREM 3.2. *Suppose that* $-1 < \alpha < 1$, *that* X *is a closed* T*-invariant subspace of* $L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$ *and that* $f \in X$ *satisfies* $S_\tau f \in L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$ *for some* $\tau > 0$. *Then* $S_\tau f \in X$. *Conversely, if* X *is a closed subspace of* $L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$ *and* $f \in X$ *satisfies* $S_\tau f \in X$ *for every* $\tau > 0$, *then* $Tf \in X$.

Before proving Theorem 3.2, we would like to point out that when $\alpha = 0$, the first half of our result could be seen as a particular instance of a Richter and Shields result for the Dirichlet space of the unit disc $\mathcal{D}(\mathbb{D})$ which states that if $g \in \mathcal{H}^\infty(\mathbb{D})$, $f \in \mathcal{D}(\mathbb{D})$ and $f\,g \in \mathcal{D}(\mathbb{D})$, then $f\,g$ belongs to the shift invariant subspace generated by f (see [19]). Nevertheless, though Richter and Shields' result also holds for the weighted Dirichlet spaces $\mathcal{D}_\alpha(\mathbb{D})$ (see Aleman's paper [1]), it is no longer true that Theorem 3.2 could be inferred from Aleman's work.

Proof. Suppose that $g \in L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$ and that g is orthogonal to $\{T^n f : n \geqslant 0\}$. Thus

$$\begin{aligned}\langle T^n f, g\rangle &= \tfrac{\Gamma(\alpha+1)}{2^\alpha} \int_{x=0}^{\infty} \overline{g(x)} \int_{t=0}^{x} f(t)\mathrm{e}^{-(x-t)}(x-t)^n \,\mathrm{d}t\, x^{1-\alpha}\,\mathrm{d}x \\ &= \tfrac{\Gamma(\alpha+1)}{2^\alpha} \int_{x=0}^{\infty} \overline{g(x)} \int_{t=0}^{x} f(x-t)\mathrm{e}^{-t}t^n \,\mathrm{d}t\, x^{1-\alpha}\,\mathrm{d}x = 0\end{aligned}$$

for $n = 0, 1, 2, \ldots$. Hence, using Fubini's Theorem, we see that

$$\int_{t=0}^{\infty}\int_{x=t}^{\infty} f(x-t)\overline{g(x)}x^{1-\alpha}\,\mathrm{d}x\, t^n \mathrm{e}^{-t}\,\mathrm{d}t = 0 \tag{3.1}$$

for each n. Consider the function F defined by

$$F(z) = \int_{t=0}^{\infty}\int_{x=t}^{\infty} f(x-t)\overline{g(x)}x^{1-\alpha}\,\mathrm{d}x\, \mathrm{e}^{-tz}\,\mathrm{d}t. \tag{3.2}$$

It is easily seen that this function is analytic for $z \in \mathbb{C}_+$, since, for $f \in L^2(\mathbb{R}_+) \cap L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$, the quantity

$$\int_{x=t}^{\infty} f(x-t)\overline{g(x)}x^{1-\alpha}\,\mathrm{d}x$$

cannot grow faster than polynomially in t, as is seen by considering the integrals

$$\int_0^1 |f(u)|^2(t+u)^{1-\alpha}\,\mathrm{d}u \quad \text{and} \quad \int_1^\infty |f(u)|^2\Big(\frac{t+u}{u}\Big)^{1-\alpha} u^{1-\alpha}\,\mathrm{d}u.$$

Moreover the orthogonality relation (3.1) implies that $F^{(n)}(1) = 0$ for each n and thus F is identically 0. Hence

$$\int_{x=t}^{\infty} f(x-t)\overline{g(x)}x^{1-\alpha}\,\mathrm{d}x = 0$$

for all $t \geqslant 0$, and taking $t = \tau$ shows that $\langle S_\tau f, g\rangle = 0$. Since this is true for each $g \in X^\perp$, we see that $S_\tau f \in X$.

Conversely, if $f \in X$ and $g \in L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$ is orthogonal to each $S_\tau f$, then

$$\int_{x=\tau}^{\infty} f(x-\tau)\overline{g(x)}x^{1-\alpha}\,\mathrm{d}x = 0$$

for each $\tau \geqslant 0$, and we can form the analytic function $F(z)$ defined in (3.2), and note that it is the Laplace transform of the zero function, and hence vanishes on $\mathbb{C}_+$. Calculating its derivative at $z = 1$ shows once more that $\langle Tf, g\rangle = 0$; since this is true for each such g, it follows that $Tf \in X$. ∎

DIRICHLET BACKWARD SHIFTS. Note that the space $L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$ can naturally be identified as the dual space of $L^2(\mathbb{R}_+, \mathrm{d}\mu_\beta)$ with $\beta = -\alpha$ by means of the bilinear dual pairing

$$\langle f, g\rangle = \int_0^\infty f(t)g(t)\,\mathrm{d}t, \quad (f \in L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha), g \in L^2(\mathbb{R}_+, \mathrm{d}\mu_\beta)), \tag{3.3}$$

giving (up to a constant factor of $\Gamma(1+\alpha)\Gamma(1-\alpha)$) an isometric isomorphism. The backward shifts on $L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$ are therefore the adjoints of the forward shifts on $L^2(\mathbb{R}_+, \mathrm{d}\mu_\beta)$, and thus the invariant subspaces of the backward shifts on $L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$ can be identified as the annihilators of the invariant subspaces of the forward shifts on $L^2(\mathbb{R}_+, \mathrm{d}\mu_\beta)$.

In fact, an easy computation shows that the adjoint of the convolution operator T using the dual pair (3.3) is given by

$$T^* f(x) = \int_x^\infty f(t)\mathrm{e}^{x-t}\,\mathrm{d}t, \quad (x > 0).$$

Hence, if S_τ^*, $\tau > 0$, denotes the left shift in $L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$, i.e.,

$$(S_\tau^* f)(t) = f(t+\tau) \quad (t > 0),$$

we may state the following

THEOREM 3.3. *A closed subspace of $L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$ is invariant under every left shift S_τ^*, $\tau > 0$, if and only if it is invariant under the convolution operator T^* defined above.*

Note that it is not possible to treat the backward shift on Bergman spaces by these means, since the forward shifts on $L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$ are unbounded operators (indeed their common domain is the zero function), and thus the duality methods cannot be used.

4. CYCLIC VECTORS

In this section, we focus on the study of cyclic vectors for the Bergman and Dirichlet shifts. Recall that if T is a linear bounded operator acting on a Hilbert space $\mathcal{H}$, a vector $f \in \mathcal{H}$ is called cyclic for T if the linear span generated by the orbit of f, i.e.,

$$\operatorname{span}\{T^n f : n \geqslant 0\}$$

is dense in $\mathcal{H}$. Here T^0 denotes the identity operator.

As a immediate consequence of the isometric isomorphism between $\mathcal{A}^2_\alpha(\mathbb{D})$ and $L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$, cyclic vectors in $\mathcal{A}^2_\alpha(\mathbb{D})$ for the Bergman shift operator S correspond to cyclic vectors in $L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$ for the convolution operator

$$(Tf)(x) = \int_0^x f(t)\mathrm{e}^{-(x-t)}\,\mathrm{d}t \quad (x > 0).$$

Of course, the same holds for the Dirichlet shift operator $\widetilde{S}$ in $\mathcal{D}^2_\alpha(\mathbb{C}_+)$. In this sense, the case $\alpha = 0$ is particularly important, since we get that the cyclic vectors for the Dirichlet shift S on the classical Dirichlet space $\mathcal{D}(\mathbb{D})$ are in correspondence with the cyclic vectors for T on $L^2(\mathbb{R}_+, t\,\mathrm{d}t)$. Note that any cyclic vector for S in $\mathcal{D}(\mathbb{D})$ is a cyclic vector for S in the Hardy space; hence it is an outer function. Nevertheless, Shields' question of characterizing cyclic vectors for S in $\mathcal{D}(\mathbb{D})$ remains still open.

Our goal in this section is to provide some necessary and some sufficient conditions for a vector f to be cyclic for T in $L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$ and $L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$, respectively.

First, a little computation shows that the iterates of T are given by

$$(T^n f)(x) = \frac{1}{\Gamma(n)} \int_0^x f(t)\mathrm{e}^{-(x-t)}(x-t)^{n-1}\,\mathrm{d}t \quad (n \geqslant 1).$$

Note that if f vanishes in a neighbourhood of 0, then so does any function in its orbit. Therefore, if we denote by $\operatorname{supp} f$ the support of f, that is,

$$\operatorname{supp} f = \{x \in \mathbb{R}_+ : \ \mathrm{m}(\{y \in (x-\varepsilon, x+\varepsilon) \cap \mathbb{R}_+ : \ f(y) \neq 0\}) > 0 \ \text{ for any } \varepsilon > 0\},$$

where m denotes the Lebesgue measure in $\mathbb{R}_+$, a straightforward necessary condition for a vector f being cyclic is the following

PROPOSITION 4.1. *Let $\alpha > -1$. If $f \in L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$ is cyclic for T, then $0 \in \operatorname{supp} f$. The same holds for cyclic vectors $f \in L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$ for $-1 < \alpha < 1$.*

One might ask if the converse of Proposition 4.1 also holds. In fact, it does when we are dealing with cyclic vectors for the classical Volterra operator in

$L^2[0,1]$. Recall that the Volterra operator in $L^2[0,1]$ is defined by

$$Vf(x) = \int_0^x f(t)\,\mathrm{d}t, \quad (0 \leqslant x \leqslant 1),$$

and f is cyclic for V if and only if $0 \in \operatorname{supp} f$. For more about the cyclic behaviour of the Volterra operator we refer to the recent survey [15].

The following example shows that the situation is different for T. Before presenting it, we state a result on zeroes of complex functions, which is taken from the theory of differential-difference equations. In the language of linear systems theory, the functions described below correspond to *delay systems of advanced type*.

PROPOSITION 4.2 (See [4] and 6.1 of [16]). *Let $f : \mathbb{C} \to \mathbb{C}$ be the analytic function given by $f(s) = q(s) + \mathrm{e}^{-sh} r(s)$, where $h > 0$ and q and r are nonzero polynomials. If $\deg r > \deg q$, then f has infinitely many zeroes (s_n) in $\mathbb{C}_+$.*

EXAMPLE 4.3. Let $f : (0, \infty) \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} t & \text{if } 0 < t < 1, \\ 0 & \text{if } t \geqslant 1. \end{cases}$$

This function lies in $L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$ for each α with $-1 < \alpha < 4$, in particular for $\alpha = 0$, which corresponds to the unweighted Bergman space. Moreover, it lies in $L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$ for each α with $-1 < \alpha < 1$. Its Laplace transform is given by

$$(\mathcal{L}f)(s) = \frac{1 - (s+1)\mathrm{e}^{-s}}{s^2},$$

which has infinitely many zeroes in $\mathbb{C}_+$ by Proposition 4.2. Thus the vector f is not cyclic for the convolution operator T nor the family of shifts $\{S_\tau : \tau > 0\}$, since, if s_0 is a zero of $\mathcal{L}f$, then all functions g in the cyclic subspace generated by f will also satisfy $(\mathcal{L}g)(s_0) = 0$.

The above example is a particular case of a more general phenomenon.

PROPOSITION 4.4. *Let $h > 0$ and let p be a polynomial such that $p(0) = 0$ and $p(h) \neq 0$ and suppose that $\lambda \in \mathbb{R}$. Then the function $g : t \mapsto p(t)\mathrm{e}^{-\lambda t}\chi_{[0,h]}(t)$ is not a cyclic vector for the family $\{S_\tau : \tau > 0\}$ acting on $L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$ or $L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$ for $-1 < \alpha < 1$.*

Proof. Let p have degree $N \geqslant 1$. Then the Laplace transform of g is given by

$$(\mathcal{L}g)(s) = \int_0^h p(t)\mathrm{e}^{-(s+\lambda)t}\,\mathrm{d}t = -p(h)\frac{\mathrm{e}^{-(s+\lambda)h}}{s+\lambda} + \frac{1}{s+\lambda}\int_0^h p'(t)\mathrm{e}^{-(s+\lambda)t}\,\mathrm{d}t,$$

and hence it is easily seen to take the form

$$(\mathcal{L}g)(s) = \frac{q(s) + r(s)\mathrm{e}^{-sh}}{(s+\lambda)^{N+1}},$$

where q and r are polynomials such that $\deg r = N$ and $\deg q < N$. By Proposition 4.2, $\mathcal{L}g$ has infinitely many zeroes in $\mathbb{C}_+$. Thus, as in Example 4.3, g is not a cyclic vector for T, nor for the family of shifts S_τ. ∎

On the other hand, since $\mathbb{R}_+$ seems to play an important role, one might think that there are no cyclic vectors with compact support for T. Nevertheless, that is not the case as the next example shows.

EXAMPLE 4.5. Let $f : (0, \infty) \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} e^{-t} & \text{if } 0 \leqslant t < 1, \\ 0 & \text{if } t \geqslant 1. \end{cases}$$

This function lies in the space $L^2(\mathbb{R}_+, d\mu_\alpha)$ for every $-1 < \alpha < 0$ and in the space $L^2(\mathbb{R}_+, d\nu_\alpha)$ for each α with $-1 < \alpha < 1$. Note that we have

$$e^{-t} = \sum_{k=0}^{\infty} e^{-k} S_k f(t),$$

where as usual S_k denotes the right shift by k. Note that the sum converges in norm. Since the function $t \mapsto e^{-t}$ is easily seen to be cyclic, it follows that f is also cyclic.

Actually, modifying the previous example, we may even find a continuous cyclic vector f with compact support and $f(0) = 0$.

EXAMPLE 4.6. Let $f : (0, \infty) \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} te^{-t} & \text{if } 0 \leqslant t < 1, \\ (2-t)e^{-t} & \text{if } 1 \leqslant t < 2, \\ 0 & \text{if } t \geqslant 2. \end{cases}$$

This function lies in the space $L^2(\mathbb{R}_+, d\mu_\alpha)$ for every $-1 < \alpha < 2$ and in the space $L^2(\mathbb{R}_+, d\nu_\alpha)$ for each α with $-1 < \alpha < 1$. Then

$$\sum_{k=0}^{\infty} e^{-k} S_k f(t) = g(t),$$

where

$$g(t) = \begin{cases} te^{-t} & \text{if } 0 \leqslant t < 1, \\ e^{-t} & \text{if } t \geqslant 1, \end{cases}$$

and finally

$$\sum_{k=0}^{\infty} e^{-k} S_k g(t) = te^{-t},$$

with again all sums converging in norm. Hence the smallest invariant subspace containing f contains a cyclic vector. Note that similar but more complicated examples can also be constructed in $L^2(\mathbb{R}_+, d\mu_\alpha)$ for $\alpha \geqslant 1$.

On the other hand, Example 4.5 illustrates a more general phenomenon.

THEOREM 4.7. *Suppose that $f \in L^2(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ is a convex monotonically decreasing function on $\mathbb{R}$ with $f(0) > 0$. Then f is a cyclic vector for the family $\{S_\tau : \tau > 0\}$ acting on $X = L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$ for each $-1 < \alpha < 0$.*

Before proceeding to prove Theorem 4.7, just recall that Beurling's Theorem [5] states that the outer functions are precisely the cyclic vectors for the forward shift S in the Hardy spaces $\mathcal{H}^p(\mathbb{D})$, when $1 \leqslant p < \infty$ (see [12], for instance).

Proof of Theorem 4.7. On one hand, since f is a monotonically decreasing function, we can write $f(t) = \int_t^\infty \mathrm{d}\mu(x)$ almost everywhere, for some positive finite Borel measure μ on $(0, \infty)$. Taking Laplace transforms, we see that

$$\mathcal{L}f(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\,\mathrm{d}t = \int_{x=0}^\infty \int_{t=0}^x \mathrm{e}^{-st}\,\mathrm{d}t\,\mathrm{d}\mu(x) = \int_0^\infty \frac{1-\mathrm{e}^{-sx}}{s}\,\mathrm{d}\mu(x).$$

Since $|\mathrm{e}^{-sx}| < 1$ for $x > 0$ and $s \in \mathbb{C}_+$, this implies that $\operatorname{Re} s(\mathcal{L}f)(s) \geqslant 0$ for $s \in \mathbb{C}_+$ and hence the function $s \mapsto s\mathcal{L}f(s)$ is an outer function in $\mathcal{H}^\infty(\mathbb{C}_+)$ (this follows easily from the characterizations of outer functions given in [12]). Set

$$g(z) = \frac{2}{(1+z)^{\alpha+2}}\,\mathcal{L}f\Big(\frac{1-z}{1+z}\Big), \quad (z \in \mathbb{D}). \tag{4.1}$$

The isometry J in (2.1) Section 2, along with the fact that $\mathcal{L}f(s) \in \mathcal{H}^2(\mathbb{C}_+)$, yields that

$$G(z) = (1+z)^{\alpha+1}g(z) \in \mathcal{H}^2(\mathbb{D}). \tag{4.2}$$

In addition, since $s \mapsto s\mathcal{L}f(s)$ is an outer function in $\mathcal{H}^\infty(\mathbb{C}_+)$, it follows that the function

$$(1-z)\,G(z) = (1-z)(1+z)^{\alpha+1}g(z) \tag{4.3}$$

is outer and belongs to $\mathcal{H}^\infty(\mathbb{D})$. From (4.2) and (4.3), we get that g is an outer function in $\mathcal{H}^1(\mathbb{D})$. In order to ensure that $g \in \mathcal{A}^2_\alpha(\mathbb{D})$ for $-1 < \alpha < 0$, we need a little more work.

By assumption, f is a convex function in $\mathbb{R}_+$, so we may write $f'(t) = -\int_t^\infty \mathrm{d}\nu(x)$ almost everywhere, for some positive finite Borel measure ν on $(0, \infty)$. Taking Laplace transforms, we get that

$$\mathcal{L}f(s) = \frac{f(0)}{s} - \int_0^\infty \frac{1-\mathrm{e}^{-sx}}{s^2}\,\mathrm{d}\nu(x) = \frac{f(0)}{s} - \frac{H(s)}{s^2}, \quad (s \in \mathbb{C}_+) \tag{4.4}$$

where $H(s) = \int_0^\infty (1-\mathrm{e}^{-sx})\,\mathrm{d}\nu(x)$ is in $\mathcal{H}^\infty(\mathbb{C}_+)$. Changing variables in (4.4), we have

$$\mathcal{L}f\Big(\frac{1-z}{1+z}\Big) = \frac{1+z}{1-z}f(0) - H\Big(\frac{1-z}{1+z}\Big)\Big(\frac{1+z}{1-z}\Big)^2, \quad (z \in \mathbb{D}).$$

Substituting the above expression in (4.1), it follows that the function G in (4.2) equals

$$G(z) = \frac{2f(0)}{1-z} - 2H\Big(\frac{1-z}{1+z}\Big)\,\frac{1+z}{(1-z)^2} \quad (z \in \mathbb{D}).$$

Note that $(1-z)^2 G(z)$ is an outer function in $\mathcal{H}^\infty(\mathbb{D})$. This along with (4.2) yields that g is an outer function belonging to $\mathcal{H}^2(\mathbb{D})$. Since $\mathcal{H}^2(\mathbb{D}) \subset \mathcal{A}^2_\alpha(\mathbb{D})$, it follows that g is also cyclic for the operator S on $\mathcal{A}^2_\alpha(\mathbb{D})$, and therefore, by means of the isometric isomorphism between $\mathcal{A}^2_\alpha(\mathbb{D})$ and $L^2(\mathbb{R}_+, \mathrm{d}\mu_\alpha)$, the function f given is cyclic for the family $\{S_\tau : \tau > 0\}$. ∎

Finally, we remark that there exist smooth functions with $\operatorname{supp} f = \mathbb{R}_+$ which are not cyclic for T. In fact, if f is the function in the Example 4.3, an easy computation yields:

$$Tf(t) = \begin{cases} t - 1 + \mathrm{e}^{-t} & \text{if } 0 < t \leqslant 1, \\ \mathrm{e}^{-t} & \text{if } t \geqslant 1, \end{cases}$$

which shows that $\operatorname{supp} Tf = \mathbb{R}_+$. Nevertheless, neither Tf, and of course, nor $T^n f$, $n > 1$, are cyclic vectors for T (otherwise f would be cyclic).

DIRICHLET BACKWARD SHIFTS. Cyclic vectors for the backward shift in the classical Dirichlet space of the unit disc $\mathcal{D}^2(\mathbb{D})$ were extensively studied in [2], showing that the situation is completely different from the Hardy space case. For more about the subject we refer the reader to [20] and [21]. Here we focus on the cyclic vectors in the Dirichlet space $\mathcal{D}^2_\alpha(\mathbb{C}_+)$ for the backward shift.

By using the duality between Bergman and Dirichlet spaces on the line, as given in (3.3), and their equivalent formulation in terms of shifts on the spaces $\mathcal{D}^2_\alpha(\mathbb{C}_+)$ and $\mathcal{A}^2_\alpha(\mathbb{C}_+)$, we show that there is no function $f \in L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$ of compact support such that $\mathcal{L}f$ is cyclic in $\mathcal{D}^2_\alpha(\mathbb{C}_+)$.

In order to show that, we recall that an entire function g is said to be of *exponential type* $A > 0$ if there is a constant $C > 0$ such that $|g(s)| \leqslant C\mathrm{e}^{A|s|}$ for $s \in \mathbb{C}$. If in addition g is square integrable on the imaginary axis, then g can be written as

$$g(s) = \int_{-A}^{A} f(t)\mathrm{e}^{-st}\,\mathrm{d}t$$

for some $f \in L^2(-A, A)$. We refer to Chapter 19 of [23] for further details.

THEOREM 4.8. *For $-1 < \alpha < 1$, suppose that $g \in \mathcal{D}^2_\alpha(\mathbb{C}_+)$ extends to a function of exponential type A on $\mathbb{C}$. Then g is not a cyclic vector for the backward shift on $\mathcal{D}^2_\alpha(\mathbb{C}_+)$.*

Proof. The conditions on f imply that it is the (bilateral) Laplace transform of a function f supported on $[-A, A]$ which is simultaneously in $L^2(\mathbb{R}_+, \mathrm{d}\nu_\alpha)$, and hence supported on $[0, A]$. Take any nonzero function $h \in L^2(\mathbb{R}_+, \mathrm{d}\mu_\beta)$ for

$\beta = -\alpha$ with h supported on $[A, \infty)$, and note that $\langle f, S_\tau h \rangle = 0$ for each $\tau > 0$. Hence $\langle f, T^n h \rangle = 0$ for each n, by Theorem 2.1. ∎

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