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NIL POLYNOMIALS AND REDUCIBILITY OF OPERATOR SEMIGROUPS

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ABSTRACT. If $\mathcal{S}$ is a multiplicative semigroup of bounded operators on a Banach space, what is the effect of a polynomial identity on reducibility of $\mathcal{S}$, i.e., the existence of a closed invariant subspace for $\mathcal{S}$? More specifically, which noncommutative polynomials in two variables have the property that whenever $f(A,B) = 0$, or more generally, $f(A,B)$ is quasinilpotent for all A and B in $\mathcal{S}$, then $\mathcal{S}$ is reducible or possibly (simultaneously) triangularizable? A well-known example of such polynomials that works at least for semigroups of compact operators is $f(x,y) = xy - yx$. Extensions of this result are also known for certain classes of polynomials that yield reducibility and triangularizability. We study this question for arbitrary homogeneous polynomials and present fairly general reducing and triangularizing conditions. As a corollary, we obtain polynomial conditions under which every compact group is necessarily Abelian.

KEYWORDS: *Semigroups of operators, invariant subspaces, simultaneous triangularizability, polynomial conditions.*

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0. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{S}$ be a semigroup of operators on a complex Banach space $\mathcal{X}$ of finite or infinite dimension. Let f be a noncommutative polynomial, homogeneous in each of its two variables. If $f(A,B) = 0$ for all A and B in $\mathcal{S}$, or, more generally, if $f(A,B)$ is nilpotent (or quasinilpotent in the infinite-dimensional case), what can we say about invariant subspaces of $\mathcal{S}$?

All operators considered here are meant to be linear and bounded. By subspaces we mean closed subspaces, and by a semigroup we mean a set of operators closed under multiplication.

The simplest and best-known example of such polynomials is $f(x,y) = xy - yx$. The vanishing of this polynomial on any set of operators on a finite-dimensional $\mathcal{X}$, and more generally, on any set $\mathcal{S}$ of compact operators, is easily seen to imply (simultaneous) triangularizability for $\mathcal{S}$, i.e. the existence of a chain of invariant subspaces which is maximal as a subspace chain. A stronger assertion holds for this polynomial: one has to assume merely that $AB - BA$ is quasinilpotent for all A and B in a semigroup $\mathcal{S}$. (See [5] or [11].) More recently, this result has been further strengthened with approximate versions of the hypothesis, e.g., the assumption that the spectral radius of $AB - BA$ is small for A and B in $\mathcal{S}$ [2].

The situation is more complicated as soon as the polynomial f has degree two or more in either of the variables, as shown by examples at the end of this section. We should mention here that if rings or algebras are considered as opposed to semigroups, then it is of course much harder for a polynomial to be zero or quasinilpotent. This is the theme of much studied polynomial-identity rings and algebras. For a brief and informative introduction, see Formanek's monograph [4]. A standard reference is [13].

Semigroup identities on abstract semigroups (or groups) have been studied extensively: they correspond to a special case of polynomials in our case, namely $f(x,y) = w_1(x,y) - w_2(x,y)$, where w_1 and w_2 are fixed words. Every faithful representation of an irreducible group or semigroup on $\mathbb{C}^n$ satisfies all the identities of $\mathcal{M}_n(\mathbb{C})$ as a polynomial-identity ring [13]. Our question is: what other polynomials can it satisfy? For a recent treatment of topics in semigroup identities, see [7].

We remark, before proceeding further, that without restricting to compact operators, even the simplest noncommutative polynomial $xy - yx$ vanishing on a semigroup of operators does not necessarily imply reducibility, i.e., the existence of a single nontrivial invariant subspace for the semigroup. (Read [12] proves the existence of a singly generated algebra of quasinilpotent operators on $\mathcal{L}_1$ that is irreducible. For a Hilbert space, the problem is still unsettled, of course.)

We also remark that the interesting polynomials to be considered here are those with coefficients adding up to zero. For let the sum a of the coefficients be nonzero. It follows from the homogeneity of f that if $f(A,B)$ is quasinilpotent for all A and B in a semigroup $\mathcal{S}$ of compact operators, then for every A in $\mathcal{S}$,

$$af(A,A) = aA^{r+s}$$

is quasinilpotent (where r and s are the degrees of homogeneity of f in x and y respectively.) Hence $\mathcal{S}$ consists of quasinilpotent compact operators and is thus known to be triangularizable by Turovskii's result [14].

Why consider only homogeneous polynomials? Because if the given f is not homogeneous in either of the variables, say y, then by replacing y with a suitable power of x we obtain a nontrivial polynomial g in one variable x which will be zero (respectively , quasinilpotent) on a semigroup $\mathcal{S}$ if f is. This amounts to restricting the spectra of members of $\mathcal{S}$ to a finite set. The connections of this condition to the existence of invariant subspaces has been explored elsewhere, e.g. in [6].

Why only two variables? Of course more variables can be considered with possibly interesting results, but it should be noted that we can always reduce the number of variables to two. For instance, if $f(x, y, z)$ is a nontrivial noncommuting polynomial, then it is not hard to verify that the two-variable noncommutative polynomial $f(x, y, x^m y)$ is nontrivial for many values of m.

We now state our definitions formally.

DEFINITION 0.1. Let f be a noncommutative polynomial in two variables, homogeneous of degree r in x, and homogeneous of degree s in y. Let $\mathcal{S}$ be a semigroup of operators on a Banach space $\mathcal{X}$.

(i) f is said to *vanish* (or be *zero*) on $\mathcal{S}$ if $f(A, B) = 0$ for all A and B in $\mathcal{S}$.

(ii) f is said to be *nil* on $\mathcal{S}$ if $f(A, B)$ is quasinilpotent for every A and B in $\mathcal{S}$.

Observe that if a homogeneous polynomial is zero (or nil) on $\mathcal{S}$, then it is also zero (or nil) on its homogeneous closure,

$$\overline{\mathbb{C}\mathcal{S}} = \text{norm closure of } \{cS : c \in \mathbb{C}, S \in \mathcal{S}\}.$$

This fact is very useful in reducing many cases involving compact operators to those of finite-rank ones.

In [9] some polynomial conditions for reducibility and triangularizability of semigroups of compact operators were given, but they were mainly restricted to polynomials of the special form $f(xy, yx)$, a direct extension of the already-known case of $xy - yx$ mentioned above. In this paper, we consider the general case.

It turns out that the problem of verifying reducibility is hardest when $\mathcal{S}$ is a group (and thus the underlying space is of finite dimension n, although n cannot be assumed bounded). Hence we first study the case of matrix groups. For semigroups of compact operators on infinite-dimensional $\mathcal{X}$ and semigroups of noninvertible matrices, our problems reduce to two cases: subgroups of $\mathcal{GL}_n(\mathbb{C})$ and subsemigroups of $\mathcal{M}_2(\mathbb{C})$ consisting of singular matrices. The "small" irreducible groups below play an important role in these reductions.

DEFINITION 0.2. Let p and q be primes, not necessarily distinct. Let A be a nonscalar, diagonal, $p \times p$ matrix satisfying $A^q = I$ and let B be the $p \times p$ cycle

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

The notation $\mathcal{G}(p, q, A)$ will stand for the group generated by A and B.

Note for later reference, that $\mathcal{G}(p, q, A)$ is always a solvable group. It is a nilpotent group if and only if $p = q$.

We shall need the following two results.

LEMMA 0.3 (J. Bernik, R. Guralnick, and M. Mastnak [1]). *A linear algebraic group over an algebraically closed field is triangularizable if and only if all its finite subgroups are triangularizable.*

LEMMA 0.4 ([9]). *A noncommutative polynomial, homogeneous in each of its two variables, is zero (respectively nil) on an irreducible finite subgroup of $\mathcal{GL}_n(\mathbb{C})$ for some $n > 1$ if and only if it is zero (respectively nil) on some $\mathcal{G}(p, q, A)$.*

Combining these two results, we record the following lemma for later use.

LEMMA 0.5. *Assume a noncommutative polynomial f, homogeneous in each of its two variables is not zero (respectively nil) on any $\mathcal{G}(p, q, A)$. If f is zero (respectively nil) on any group $\mathcal{G}$ of matrices, then $\mathcal{G}$ is triangularizable.*

Proof. If f is zero (respectively nil) on any semigroup S, then it is clearly also zero (respectively nil) on the Zariski closure of $\mathcal{S}$. Thus we can assume, without loss of generality, that the group $\mathcal{G}$ is closed (in the relative topology), so that Lemmas 0.3 and 0.4 are applicable. ∎

The idea now is to find workable sufficient conditions on a given polynomial (in terms of its form and its coefficients) which would make it a "triangularizing" polynomial.

We conclude this section with examples of polynomials that are identically zero on some irreducible semigroups. For any integer $m \geqslant 2$, the simple polynomial $x^m y - y x^m$ is zero on the group $\mathcal{G}(p, p, A)$ where p is an arbitrary prime dividing m, and $A = \operatorname{diag}(1, \omega, \ldots, \omega^{p-1})$ with ω a p-th primitive root of unity. (Just observe that $A^p = I$ for all A in this group.) In fact, any polynomial of the form $g(x^p, y)$ with coefficients summing to zero vanishes on this group. There are less trivial polynomials vanishing on $\mathcal{G}$. For example, as will be clear from the next result, the polynomial $y^2xyx^3 - yx^2yx^2y$, or more generally,

$$f(x,y) = a_1 y^2 x y x^3 + a_2 y x^3 y^2 x + a_3 y x^2 y x^2 y + a_4 x^2 y^3 x^2 + a_5 x y^2 x^3 y$$

with $a_1 + \cdots + a_5 = 0$ is zero on $\mathcal{G}(5, 5, A)$.

EXAMPLE 0.6. Consider the general polynomial f of homogeneous degrees r and s respectively in x and y, i.e.,

$$f(x,y) = \sum a_{n_0, n_1, \ldots, n_s} x^{n_0} y x^{n_1} y x^{n_2} y \cdots x^{n_s} y$$

with $n_i \geqslant 0$ and $\sum\limits_{i=0}^{s} n_i = r$. Assume the coefficients add up to zero. If, for the exponents of every term, $\sum\limits_{i=0}^{s} i n_i$ is a constant c modulo some prime p, then f is zero on the irreducible group $\mathcal{G}(p, p, A)$ as described above.

Proof. It is not hard to verify that the commutator subgroup of this p-group is just its centre, i.e., for every pair S and T in $\mathcal{G}$,

$$TS = \omega ST,$$

where ω is some p-th root of unity. It follows that $T^iS^j = w^{ij}S^jT^i$ for all i and j. Now, by a straight-forward calculation,

$$\begin{aligned} f(S,T) &= \sum a_{n_0,\dots,n_s} S^{n_0}TS^{n_1}T\cdots S^{n_s}T = \sum a_{n_0,\dots,n_s}\omega^{\sum_{i=0}^{s} in_i}S^rT^s \\ &= \Big(\sum a_{n_0,\dots,n_s}\Big)\omega^c S^rT^s = 0. \quad \blacksquare \end{aligned}$$

As motivation for the results in later sections, we also include the following, slightly more general, set of examples whose verification should be easy.

EXAMPLE 0.7. Let f be a polynomial of the general form given in Example 0.6. Assume, furthermore, that the set of index vectors $R = (n_0, n_1, \dots, n_s)$ is partitioned into subsets $\mathcal{R}_1, \dots, \mathcal{R}_k$ satisfying

$$\sum\{a_R : R \in \mathcal{R}_j\} = 0$$

for $j = 1, \dots, k$. If there is a prime p and constants c_j modulo p such that,

$$\sum\{in_i : (n_0, n_1, \dots, n_s) \in \mathcal{R}_j\} = c_j$$

for every j, then f vanishes on $\mathcal{G}(p, p, A)$.

In the first two sections of this paper, we consider the significant case of groups of operators on finite dimensions, first the case of a vanishing polynomial, and then the more complicated case of a nil polynomial. The results are then used in Section 3 to treat the general case of semigroups of operators on an infinite-dimensional space.

1. VANISHING OF A POLYNOMIAL ON A GROUP

We have to deal with a certain type of integer-valued matrix whose rank is the same modulo every prime. Thus it is convenient to define the following number associated with any rectangular matrix of integers.

DEFINITION 1.1. Let M be a rectangular matrix of integers with $m+1$ columns and constant row sums. Denote its rows by $R_1, \dots, R_l$. Form a matrix M_0 whose rows are $R_i - R_j$ with $i < j$. (Thus M_0 has zero row sums and its rank is at most m.) Let

$$\Delta = \{\det N : N\ m \times m \text{ submatrix of } M_0\}.$$

Define $\delta(M) = 0$ if $\Delta = \emptyset$ or $\Delta = \{0\}$. Otherwise, set

$$\delta(M) = \gcd\{\delta \in \Delta : \delta \neq 0\}.$$

REMARKS 1.2. (i) $\delta(M)$ is necessarily zero if $l < m$.

(ii) The condition $\delta(M) = 1$ is easily seen to be equivalent to the assertion that the rank of M_0 is one less than the number of its columns when it is viewed as a matrix over the field $\mathbb{Z}_p$ for any prime p.

(ii) Let $\{e_j\}_{j=1}^{m+1}$ be the standard column basis. Let $k \leqslant m$ and let T be the $(m+1) \times k$ matrix whose j-th column is

$$e_j + e_{j+k} + e_{j+2k} + \cdots$$

(the last term having index $j + tk$ with the largest t satisfying $j + tk \leqslant m + 1$). With M as in the definition, it can be verified that $\delta(M) = 1$ implies that $\delta(MT) = 1$.

We are now ready to associate a useful integer with every noncommutative homogeneous polynomial, defined in terms of the sequences of exponents in each of its monomial terms.

DEFINITION 1.3. Let f be a noncommutative polynomial separately homogeneous of degrees r and s in x and y respectively. We express f uniquely as

$$\sum_R a_R x^{n_0} y x^{n_1} y x^{n_2} \cdots x^{n_s - 1} y x^{n_s},$$

with $R = (n_0, \ldots, n_s)$, $0 \leqslant n_j$, and $n_0 + \cdots + n_s = r$. Let M be the matrix whose rows consist of all R with $a_R \neq 0$. Define $\delta(f) = \delta(M)$. (Note that δ is unchanged if the rows of M are permuted.)

We pause here to make a point of clarification and apology to the reader: whenever we speak, admittedly loosely, of "a proper subset of nonzero coefficients" of a polynomial f with coefficients $\{a_R : R \in \mathcal{R}\}$ as above, we mean a subset $\{a_R : R \in \mathcal{R}_1\}$, where $\mathcal{R}_1$ is a proper subset of the index set $\mathcal{R}$.

THEOREM 1.4. *Let f be a noncommutative polynomial, separately homogeneous in each of its variables x and y, such that*

(i) $\delta(f) = 1$ *and*

(ii) *no proper subset of the nonzero coefficients of f sums to zero.*

If f vanishes on a group $\mathcal{G}$ of matrices, then $\mathcal{G}$ is triangularizable. If $\mathcal{G}$ is also compact, then it is diagonalizable (and thus Abelian).

Before presenting the proof, we find it convenient to give a definition and a lemma concerning commutative polynomials in several variables. These will be used more than once.

DEFINITION 1.5. Let g be a commutative polynomial in the variables z_j, $0 \leqslant j \leqslant m$. Assume g is jointly homogeneous of degree d, so that

$$g(z_0, \ldots, z_m) = \sum_R a_R z_0^{r_0} z_1^{r_1} \cdots z_m^{r_m},$$

where each $R = (r_0, \ldots, r_m) \in \mathbb{Z}^{m+1}$, $r_j \geqslant 0$ for all j , and $\sum r_j = d$. Let M be the matrix whose rows consist of those R for which $a_R \neq 0$. Define

$$\delta_0(g) = \delta(M).$$

REMARKS 1.6. If $f(x,y)$ is expressed as in Definition 1.3, we can introduce formal transformations $z_j = y^j x y^{-j}$, $j = 0, \dots, s$, and rewrite f uniquely as

$$g(z_0, \dots, z_s) y^s,$$

where

$$g(z_0, z_1, \dots, z_s) = \sum_R a_R z_0^{n_0} z_1^{n_1} \cdots z_s^{n_s}.$$

Treating g as a commutative polynomial, we make two observations.

(i) The correspondence between f and g is one-to-one. In other words, given g, a jointly homogeneous polynomial of degree r in $s+1$ commuting variables z_j, we arrange the variables in each monomial in ascending order of the indices j and form

$$f(x,y) = g(x, yxy^{-1}, \dots, y^s x y^{-s}) y^s.$$

Later, we will have occasion to consider a more involved commutative "transform" for f, which will not be one-to-one. We may call g the first transform of f.

(ii) It is easily seen that $\delta(f) = \delta_0(g)$.

Our technical lemma follows.

LEMMA 1.7. *Let g be a nonzero, jointly homogeneous, commutative polynomial in $m+1$ variables,*

$$g(z_0, \dots, z_m) = \sum_R a_R z_0^{r_0} z_1^{r_1} \cdots z_m^{r_m}.$$

Assume that no proper subset of nonzero coefficients of g sums to zero. If there exists a prime p and a sequence $\{k_0, \dots, k_m\}$ of integers that is not constant modulo p such that the polynomial $h(x) = g(x^{k_0}, \dots, x^{k_m})$ is divisible by $x^p - 1$, then $\delta_0(g) \neq 1$.

Proof. Let θ be a primitive p-th root of unity. It follows from $x^p - 1 | h(x)$ that $h(\theta^t) = 0$ for all integers t, so that

$$\sum_R a_R \theta^{(\sum_{i=0}^m r_i k_i) t} = 0.$$

Collecting terms, this yields $\sum_{j=0}^{p-1} \left(\sum \left\{ a_R : \sum_{i=0}^m r_i k_i = j \mod p \right\} \right) \theta^{jt} = 0$. Hence, for every j, the members of the set

$$\mathcal{E}_j = \left\{ a_R : \sum_{i=0}^m r_i k_i = j \mod p \right\}$$

have zero sums. Since no proper subset of the coefficients sums to zero, there is a unique j such that $\mathcal{E}_j$ contains all the coefficients a_R. In other words, $\sum_{i=0}^m r_i k_i = j \mod p$ for all R. It follows that the system

$$\sum_{i=0}^m (r_i - r_i') k_i = 0 \mod p,$$

where R and R' range over all distinct pairs of rows corresponding to a_R and $a_{R'}$, has a solution $\{k_0, \dots, k_m\}$, nonconstant modulo p. The system also has the constant solution $\{1, 1, \dots, 1\}$. This implies that the rank of the coefficient matrix A of the system is at most $m - 1$ over the field $\mathbb{Z}_p$. Thus the determinant of every possible $m \times m$ submatrix of A over $\mathbb{Z}_p$ is zero. Since the matrix M_0 of Definition 1.1 obtained from the matrix M of Definition 1.5 is a submatrix of A, we conclude that $\delta(M) \neq 1$. ∎

Proof of Theorem 1.4. By Lemma 0.5 it suffices to show that f does not vanish on any of the groups $\mathcal{G}(p, q, A)$ of Definition 0.2.

Suppose otherwise. Thus there are $p \times p$ matrices

$$A = \operatorname{diag}(\theta^{k_0}, \theta^{k_1}, \dots, \theta^{k_{p-1}}) \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where θ is a primitive q-th root of unity, A is nonscalar, and $f(S, T) = 0$ for all S and T in the group generated by A and B.

We first assume $s < p$. Replacing A with $B^j A B^{-j}$ for an appropriate j if necessary (and still keeping it diagonal) we make sure that the first s diagonal entries of A are not all the same. Noting that the set $\{A, BAB^{-1}, \dots, B^s A B^{-s}\}$ is commutative, we employ the notation given in the remarks following Definition 1.5 to deduce that

$$0 = f(A, B)B^{-s} = g(A, BAB^{-1}, B^2 A B^{-2}, \dots, B^s A B^{-s}).$$

We collect the first diagonal entries on the right hand side to have $g(\theta^{k_0}, \theta^{k_1}, \dots, \theta^{k_s}) = 0$. Similarly, the equation $0 = f(A^t, B)B^{-s}$ yields $g(\theta^{tk_0}, \theta^{tk_1}, \dots, \theta^{tk_s}) = 0$ for all integers t. This implies that the polynomial

$$g_0(x) = g(x^{k_0}, x^{k_1}, \dots, x^{k_s})$$

is divisible by $x^q - 1$. Since no proper subset of the coefficients has zero sum, Lemma 1.7 applies and

$$\delta(f) = \delta_0(g) \neq 1,$$

which is a contradiction.

We must now consider the case $p \leqslant s$. Extend the vector $K = (k_0, k_1, \dots, k_{p-1})$ of exponents by periodicity to

$$\widehat{K} = (k_0, k_1, \dots, k_{p-1}; k_0, k_1, \dots, k_{p-1}; k_0, k_1, \dots)$$

with $s + 1$ components. Since $B^p = I$, it is easily seen that the equation $f(A, B)B^{-s} = 0$ still gives

$$g(\theta^{tk_0}, \dots, \theta^{tk_s}) = 0$$

for all t as before, yielding a contradiction. To complete the proof, we must only observe that if the group $\mathcal{G}$ is compact, then it is simultaneously similar to a unitary group, which is diagonalizable (and thus commutative) if it is triangularizable. ∎

A converse to Theorem 1.4 can be stated if f is of degree 1 in one of the variables, say y. (See also [8] and [9] for related results.)

THEOREM 1.8. *Let f be homogeneous of degree r in x and degree 1 in y. Assume that the coefficients of f add up to zero. If $\delta(f) \neq 1$, then there exists an irreducible group of matrices on which f vanishes.*

Proof. Assume $\delta(f) \neq 1$. We write

$$f(x,y) = a_0 yx^r + a_1 xyx^{r-1} + \cdots + a_r x^r y.$$

Doing away with the vacuous case in which f is a monomial, we see from the definition that $\delta(f) = \gcd\{j-k : j \neq k, a_j a_k \neq 0\}$. It follows from $\delta(f) \neq 1$ that there is a prime p that divides $\delta(f)$. Hence there is an integer $t \geqslant 0$ such that after discarding some of the zero coefficients,

$$f(x,y) = x^t(a_t yx^{pk} + a_{t+p}x^p yx^{p(k-1)} + \cdots + a_{t+kp}x^{kp}y)x^{r-t-kp}.$$

Now let $\mathcal{G} = \mathcal{G}(p,p,A)$ with $A = \operatorname{diag}(1, w, \ldots, w^{p-1})$. Since $S^p = I$ for all $S \in \mathcal{G}$, we conclude that for all S and T in $\mathcal{G}$,

$$f(S,T) = S^t\Big(\Big(\sum a_i\Big)T\Big)S^{r-t-kp} = 0. \quad ∎$$

EXAMPLE 1.9. Let f be homogeneous of degrees r in x and 2 in y. Thus

$$f(x,y) = \sum_R a_R x^{r_0} yx^{r_1} yx^{r_2}$$

with $R = (r_0, r_1, r_2)$, $r_i \geqslant 0$ and $r_0 + r_1 + r_2 = r$. Finding $\delta(f)$ then amounts to a calculation of determinants of the typical form

$$\det\begin{pmatrix} r_0 - r_0' & r_1 - r_1' \\ r_0 - r_1'' & r_1 - r_1'' \end{pmatrix}$$

for distinct vectors R, R', R''. More specifically, let us consider the case where r also equals 2 so that

$$f(x,y) = a_1(xy)^2 + a_2(yx)^2 + a_3 xy^2x + a_4 yx^2y + a_5 x^2y^2 + a_6 y^2x^2.$$

The calculation of $\delta(f)$ is easy. For instance, if $a_1a_2a_3 \neq 0$, then the corresponding rows R are $R_1 = (1,1,0)$, $R_2 = (0,1,1)$, and $R_3 = (1,0,1)$. Thus the matrix of the differences $R_i - R_j$ has a submatrix

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

so that $\delta(f) = 1$ by definition. For a very specific example, if

$$(xy)^2 + (yx)^2 - 2xy^2x$$

vanishes on a compact matrix group $\mathcal{G}$, then $\mathcal{G}$ is Abelian.

The following result is an application to the special case of nilpotent groups. In this case, a stronger result than our main theorem above can be proved by replacing the hypothesis $f(A, B) \equiv 0$ on $\mathcal{G}$ with $\det f(A, B) \equiv 0$ on $\mathcal{G}$.

THEOREM 1.10. *Let f be a noncommutative polynomial of homogeneous degree r in x and s in y such that*

(i) $\delta(f) = 1$*;*

(ii) *no proper subset of nonzero coefficients of f sums to zero.*

If $\mathcal{G}$ is a nilpotent subgroup of $\mathcal{M}_n(\mathbb{C})$ and if $\det f(S, T) = 0$ *for all S and T in $\mathcal{G}$, then $\mathcal{G}$ is triangularizable (and diagonalizable if $\mathcal{G}$ is also compact.)*

Proof. Suppose $\mathcal{G}$ is not triangularizable. Since the Zariski closure of $\mathcal{G}$ (in $\mathcal{GL}_n(\mathbb{C})$) is also nilpotent, we can assume $\mathcal{G}$ is Zariski-closed. By [1], $\mathcal{G}$ has a finite nontriangularizable subgroup $\mathcal{G}_0$. Let $\mathcal{G}_{00}$ be a minimal non Abelian subgroup of $\mathcal{G}_0$, which is nilpotent as is $\mathcal{G}_0$. Then by Lemma 4.2.9 of [10] together with the fact that we can assume $\mathcal{G} = \mathbb{C}\mathcal{G}$, there exists an invariant subspace $\mathcal{M}$ for $\mathcal{G}_{00}$ such that $\mathcal{G}_{00}|\mathcal{M}$ is of the form $\mathcal{G}(p, q, A)$. But $p = q$ since $\mathcal{G}_{00}|\mathcal{M}$ is nilpotent.

Now it is not hard to verify that $\mathcal{G}(p, p, A) = \mathcal{G}(p, p, A')$, where A' has the pleasant feature of a geometric progression on its diagonal. So we replace A with A' and assume that

$$A = \operatorname{diag}(1, \theta, \ldots, \theta^{p-1}),$$

where θ is a primitive p-th root of unity.

Observe that in the proof of Theorem 1.4 we collected the first diagonal entries of

$$g(A, BAB^{-1}, B^2AB^{-2}, \ldots) = f(A, B)B^{-s} = 0$$

and set it equal to zero to get a contradiction. In the current case, some diagonal entry of $f(A, B)B^{-s}$, not necessarily the first, must be zero, say the j-th. But by the perfect regularity of the form of A, we can see that the j-th entry is

$$\begin{aligned} 0 = g(\theta^{j-1}, \theta^j, \ldots, \theta^{j+s-1}) &= \sum_R a_R \theta^{\sum_{i=0}^s (i+j-1)r_i} \\ &= \sum_R a_R \theta^{\sum_{i=0}^s ir_i + (j-1)\sum_{i=0}^s r_i} = \theta^{(j-1)r} \sum_R a_R \theta^{\sum_{i=0}^s ir_i}. \end{aligned}$$

Hence the first diagonal entry of $f(A, B)B^{-s}$ is zero as well. A contradiction is then obtained as in Theorem 1.4. ∎

As we have seen in the introduction, one cannot expect affirmative results without the condition (i) in Theorems 1.4 and 1.10. For example, the simple polynomial $x^m y - yx^m$, mentioned in the introduction, has $\delta = m$ and satisfies the condition (ii) of the theorems above. But if $m \neq 1$, then for any p dividing m, the polynomial vanishes on $\mathcal{G}(p,p,A)$. The following example shows that the condition (i) is not sufficient either.

EXAMPLE 1.11. Theorems 1.4 and 1.10 do not hold if we assume (i), but not (ii). To see this consider the noncommutative polynomial

$$f(x,y) = (xy - yx)(xy - \omega yx) \cdots (xy - \omega^{p-1} yx),$$

where ω is a primitive p-th root and p is a prime. In the p-group $\mathcal{G}(p,p,A)$ with $A = \operatorname{diag}(1,\omega,\ldots,\omega^{p-1})$, any two members S and T satisfy the relation $ST = \omega^j TS$ for some j, so that $f(S,T)$ is identically zero on $\mathcal{G}(p,p,A)$.

It is not hard to see that f does not satisfy the condition (ii). To verify that it does satisfy (i), it is convenient to consider the commutative transform $g(x_0,\ldots,x_p)$ of f (as constructed after Definition 1.5), which is

$$(x_0 - x_1)(x_1 - \omega x_2) \cdots (x_{p-1} - \omega^{p-1} x_p).$$

Now we observe that every one of the $p+1$ terms $\prod_{i \neq j} x_i$, $j = 0,\ldots,p$, is present in g (with some nonzero coefficient). Thus the rows $R_0,\ldots,R_p$ of the $(p+1)\times(p+1)$ matrix

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 0 \\ 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & \cdots & 0 & 1 \end{pmatrix}$$

are some of those of M, the matrix of exponent vectors of g as in Definition 1.5. The matrix with rows $R_0 - R_j$, $j > 0$, has a submatrix equal to the $p \times p$ identity. Hence $\delta_0(g) = 1$ or $\delta(f) = 1$.

REMARKS 1.12. (i) The results of this section can also be viewed as statements about irreducible representations. For example, Theorem 1.4 says that if G is an abstract group, then it has no nontrivial irreducible representation $\mathcal{G}$ satisfying $f(A,B) \equiv 0$ if f is any polynomial with properties (i) and (ii). Theorem 1.10 says that if G is an abstract nilpotent group, then it has no such representation even with $\det f(A,B) \equiv 0$.

(ii) If f can be factored from left or right, i.e., if

$$f(x,y) = w_1(x,y) f_1(x,y) w_2(x,y),$$

where w_1 and w_2 are words in x and y, then the vanishing of f on a group is clearly equivalent to that of f, so that $\delta(f_1)$ could more profitably used in applying the results above.

(iii) The reader may have noticed that the definition of δ is not symmetric relative to x and y. In fact, if we interchange x and y and define $g(x,y)$ as $f(y,x)$, then $\delta(g)$, which may well be different from $\delta(f)$, could, clearly, also be used in every one of the theorems above. At the cost of introducing yet another piece of notation, we could strengthen Theorems 1.4, 1.10, and their corollaries: just define

$$\widehat{\delta}(f) = \delta(f) + \delta(g) - \delta(f)\delta(g),$$

with g as above, so that $\widehat{\delta}(f) = 1$ if and only if one of the two numbers $\delta(f)$ and $\delta(g)$ equals 1. Then $\widehat{\delta}$ treats the two variables of f symmetrically and can replace δ in all our statements.

2. NILPOTENCY OF A POLYNOMIAL ON A GROUP

The effect of a polynomial f being nil on a group $\mathcal{G}$ is of course the same as that of a power of f vanishing on $\mathcal{G}$, but the main result of the preceding section does not apply to a power of f, because of the condition (ii) in Theorem 1.4. For example, many proper subsets of the nonzero coefficients of $(xy - yx)^n$ add up to zero if $n \geqslant 2$. Even if we fix that condition, finding the number δ would be cumbersome for large n. So our goal here is to keep the condition (ii) but define another integer δ_1 associated with f which would work with the nilpotency hypothesis. We shall restrict our attention to homogeneous f of the same degree m in x as well as in y. We remark again that the interesting case is the one in which the coefficients of f sum to zero. (Otherwise, nilpotency of $f(S, S)$ for all S in $\mathcal{G}$ implies that of S.)

It is not surprising that $\delta(f)$ will not be adequate for dealing with f when we merely assume nilpotency of f on $\mathcal{G}$, as opposed to the stronger condition of its vanishing on $\mathcal{G}$.

We construct a new commutative "transform" h for f as follows. As before, express $f(x, y)$ as a linear combination of monomials,

$$\sum_R a_R w_R(x, y),$$

where $R = (n_0, n_1, \ldots, n_m)$ with $n_i \geqslant 0$, and

$$w_R(x, y) = x^{n_0} y x^{n_1} y x^{n_2} \cdots x^{n_{m-1}} y x^{n_m}.$$

It is not hard to verify that each group word $w_R(y^{-m} x y^{m-1}, y)$ can be expressed uniquely as a product of m of the $2m$ terms

$$z_0 = x, \quad z_1 = y^{-1} x y, \quad z_2 = y^{-2} x y^2, \quad \ldots, \quad z_{2m-1} = y^{-(2m-1)} x y^{2m-1}.$$

Thus, given f, homogeneous of degree m in each variable, we can form

$$h(z_0, \ldots, z_{2m-1}) = \sum a_R w_R(y^{-m} x y^{m-1}, y)$$

unambiguously. We now view h as a commutative polynomial in $2m$ variables. (The function h does not uniquely determine f, unlike the case of the function g used in the preceding section, but we shall not need injectivity.)

DEFINITION 2.1. Let f be a noncommutative polynomial of the same homogeneous degree m in x and in y. Let h be the corresponding commutative polynomial obtained as in the paragraph above. Define

$$\delta_1(f) = \delta_0(h).$$

The main result of this section follows.

THEOREM 2.2. *Let f be a polynomial of homogeneous degree m in each of its two variables. Assume that*

(i) $\delta_1(f) = 1$ *and*

(ii) *no proper subset of the nonzero coefficients of f sums to zero.*

If f is nil on a group $\mathcal{G}$ of matrices, then $\mathcal{G}$ is triangularizable. In particular, $\mathcal{G}$ is Abelian if it is compact.

Proof. Let $f(x,y) = \sum\limits_R a_R w_R(x,y)$, and assume, without loss of generality, that $\sum\limits_R a_R = 0$. Let h be the commutative polynomial constructed from f as above. Observe that each monomial in h may come from several monomials in f, so that each of its coefficients is of the form

$$\sum(a_R \in \mathcal{R}_i),$$

where $\mathcal{R} = \bigcup \mathcal{R}_i$ is a partition of the original index set $\mathcal{R}$. This implies that the coefficients of h, like those of f, have the property that no proper subset of them sums to zero.

By Lemma 0.5 it suffices to prove that f cannot be nil on any group $\mathcal{G}(p,q,A)$. Assume otherwise. Letting A and B be the diagonal and the cycle in $\mathcal{G}(p,q,A)$ as before, we deduce that $f(B^{-m}AB^{m-1}, B)$ is nilpotent. Since all the matrices $B^{-j}AB^j$, for $j = 0, \ldots, 2m-1$ are diagonal (and thus commute with each other), this implies that the diagonal matrix

$$h(A, B^{-1}AB, \ldots, B^{-(2m-1)}AB^{2m-1})$$

is nilpotent, and hence equal to zero. Assuming first, $2m-1 < p$, and collecting the first entries of diagonal terms, we obtain $h(\theta^{k_0}, \theta^{k_1}, \ldots, \theta^{k_{2m-1}}) = 0$. Proceeding as in the proof of Theorem 1.4, we replace A with A^t to obtain further $h(\theta^{tk_0}, \theta^{tk_1}, \ldots, \theta^{tk_{2m-1}}) = 0$ for all integers t. Thus

$$h_0(x) = h(x^{k_0}, \ldots, x^{k_{2m-1}})$$

is divisible by $x^q - 1$. This implies by Lemma 1.7 that $\delta_1(f) = \delta_0(h) \neq 1$, which is a contradiction. The case $2m-1 \geqslant p$ is treated exactly as it was in the proof of Theorem 1.4. ∎

EXAMPLE 2.3. Let f be the general homogeneous polynomial of degree 2 in each of the variables x and y:

$$f(x,y) = a_1(xy)^2 + a_2(yx)^2 + a_3xy^2x + a_4yx^2y + a_5x^2y^2 + a_6y^2x^2.$$

To find the corresponding commutative h, we replace x with $y^{-2}xy$ and obtain

$$\begin{aligned} f(y^{-2}xy, y) &= a_1 y^{-2}xy^2y^{-2}xy^2 + a_2 y^{-1}xyy^{-1}xy + a_3 y^{-2}xy^2y^{-1}xy \\ &\quad + a_4 y^{-1}xyy^{-2}xy^2 + a_5 y^{-2}xy^2y^{-3}xy^3 + a_6 xy^{-1}xy \\ &= a_1 z_2^2 + a_2 z_1^2 + a_3 z_2 z_1 + a_4 z_1 z_2 + a_5 z_2 z_3 + a_6 z_0 z_1. \end{aligned}$$

So the polynomial h is

$$h(z_0, z_1, z_2, z_3) = a_6 z_0 z_1 + a_2 z_1^2 + (a_3 + a_4) z_1 z_2 + a_1 z_2^2 + a_5 z_2 z_3.$$

The matrix M of the exponent vectors is a submatrix of

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

consisting of those rows for which the coefficient of the corresponding monomial is nonzero. If, for example, $a_1(a_3 + a_4)a_5 a_6 \neq 0$, then $\delta(M) = 1$. For a more specific example, let

$$f(x, y) = a(xy)^2 + bxy^2x + cx^2y^2 + dy^2x^2$$

with $abcd \neq 0$ and no proper subset of $\{a, b, c, d\}$ adding up to zero. If f is nil on a compact group $\mathcal{G}$ of matrices, then $\mathcal{G}$ is Abelian. (If $\mathcal{G}$ is not compact, then we can only deduce triangularizability.)

Paragraphs (i) and (iii) of the remarks at the end of the preceding section clearly apply to Theorem 2.2 as well.

3. EXTENSIONS TO SEMIGROUPS OF OPERATORS

The results of the preceding sections on groups are also valid for semigroups of invertible matrices (not necessarily closed under taking inverses). For if $\mathcal{S}$ is such a semigroup in $\mathcal{M}_n(\mathbb{C})$ and $f(x, y)$ is zero (or nil) on $\mathcal{S}$, then it is also zero (or nil) on the Zariski closure $\mathcal{G}$ of $\mathcal{S}$ in $\mathcal{GL}_n(\mathbb{C})$. But $\mathcal{G}$ is a group.

We now consider the general case of semigroups of compact operators on a (complex) Banach space $\mathcal{X}$ (which includes semigroups of arbitrary operators if $\mathcal{X}$ is finite-dimensional). The following useful lemma is a rewording of Lemma 1.1 and Corollary 1.5 of [9].

LEMMA 3.1. *Let $\mathcal{S}$ be an irreducible semigroup of compact operators on a Banach space $\mathcal{X}$ with $\mathcal{S} = \overline{\mathbb{R}^+\mathcal{S}}$. If $\mathcal{X}$ is finite-dimensional, also assume that $\mathcal{S}$ has a nonzero singular member. Then $\mathcal{S}$ contains a semigroup $\mathcal{S}_0$ with the following property: $\mathcal{S}_0$ has invariant subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ with $\mathcal{M}_1$ a subspace of codimension 2 in $\mathcal{M}_2$ such that* *the semigroup induced by $\mathcal{S}_0$ on $\mathcal{M}_2/\mathcal{M}_1$ is, up to simultaneous similarity, generated by a pair*

$$A = \begin{pmatrix} \alpha & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 0 & \beta \end{pmatrix}$$

in $\mathcal{M}_2(\mathbb{C})$ with $\alpha\beta \neq 1$.

REMARK 3.2. It is easily checked that the condition $\alpha\beta \neq 1$ is equivalent to the irreducibility of the pair $\{A, B\}$. Depending on whether neither, one, or both of the numbers α and β equal zero, one gets three distinct semigroups in $\mathcal{M}_2(\mathbb{C})$. It turns out that we shall not need to distinguish these cases in our considerations and the single parameter $\alpha\beta$ will do.

We extend a definition from [9] to more general polynomials.

DEFINITION 3.3. A noncommutative polynomial $f(x, y)$ is said to be *rigid* if it is not nil on any 2×2 semigroup generated by

$$\begin{pmatrix} \alpha & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 0 & \beta \end{pmatrix}$$

with $\alpha\beta \neq 1$. We denote this semigroup by $\mathcal{S}(\alpha, \beta)$.

REMARK 3.4. Checking a given polynomial for rigidity is not hard. Rather than rely on the long and detailed recipe given in [9], which distinguished among the three different cases of pairs, we present a simpler method, which is more suitable for the polynomials considered here. Let f be of the same homogeneous degree r in each of the variables x and y. It is convenient to represent every monomial in f with positive exponents only, i.e.,

$$x^{m_1}y^{n_1}x^{m_2}y^{n_2}\cdots \quad \text{or} \quad y^{n_1}x^{m_1}y^{n_2}x^{m_2}\cdots,$$

$n_i \geqslant 1$, $m_i \geqslant 1$. Replace each x^{m_i} with $\alpha^{m_i-1}x$ and each y^{n_i} with $\beta^{n_i-1}y$, where α and β are scalar variables and then collapse every xyx to X and every yxy to Y. A typical monomial starting and ending with x is of the form $x^{m_1}y^{n_1}x^{m_2}y^{n_2}\cdots y^{n_j-1}x^{m_j}$, which becomes

$$\alpha^{m_1+\cdots+m_j-j}\beta^{n_1+\cdots+n_{j-1}-(j-1)}X = \alpha^{r-j}\beta^{r-j+1}X = \beta(\alpha\beta)^{r-j}X.$$

Similarly, the transforms of monomials $x(\cdots)y$, $y(\cdots)x$, and $y(\cdots)y$ are, respectively, of the following general forms:

$$(\alpha\beta)^k XY, \quad (\alpha\beta)^k YX, \quad \text{and} \quad \alpha(\alpha\beta)^k Y.$$

Thus we obtain a simple polynomial, uniquely from f, of the form

$$f_{11}(\alpha\beta)\beta X + f_{12}(\alpha\beta)XY + f_{21}(\alpha\beta)YX + f_{22}(\alpha\beta)\alpha Y,$$

where each f_{ij} is a polynomial in one variable.

DEFINITION 3.5. The four polynomials f_{ij} constructed above will be called the *reduced components* of f.

LEMMA 3.6. *Let f be of the same homogeneous degree r in x and y, and assume that its coefficients sum to zero. Let f_{ij} be its reduced components. Then f is rigid if and only if the following two equations have exactly one common solution $z = 1$:*

$$[f_{11}(z) + f_{22}(z)]z + f_{21}(z) + f_{12}(z) = 0,$$
$$f_{11}(z)f_{22}(z)z^2 - [f_{11}(z)f_{22}(z) + f_{12}(z)f_{21}(z)]z + f_{21}(z)f_{12}(z) = 0.$$

Proof. Assume f is not rigid. Then there exist α and β with $\alpha\beta \neq 1$ such that f is nil on the semigroup $\mathcal{S}(\alpha, \beta)$ of Definition 3.3. In particular, $f(A, B)$ is nilpotent with

$$A = \begin{pmatrix} \alpha & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & \beta \end{pmatrix}.$$

Since $ABA = A$, $BAB = B$, $A^m = \alpha^{m-1}A$, and $B^m = \beta^{m-1}B$ for all $m \geqslant 1$, it follows from the construction of the components f_{ij} that the matrix

$$C = f_{11}(\alpha, \beta)\beta A + f_{12}(\alpha, \beta)AB + f_{21}(\alpha, \beta)BA + f_{22}(\alpha, \beta)\alpha B$$

is nilpotent. Thus $\det C = \operatorname{tr} C = 0$. But

$$C = \begin{pmatrix} \alpha\beta f_{11}(\alpha, \beta) + f_{21}(\alpha, \beta) & [f_{12}(\alpha, \beta) + f_{22}(\alpha, \beta)]\alpha \\ [f_{12}(\alpha, \beta) + f_{22}(\alpha, \beta)]\beta & f_{12}(\alpha, \beta + f_{22}(\alpha, \beta) \end{pmatrix}.$$

This easily implies that the two equations are satisfied by $z = \alpha\beta \neq 1$.

Conversely, assume that the system has a solution $z_0 \neq 1$. We shall show that f is nil on some semigroup $\mathcal{S} = \mathcal{S}(\alpha, \beta)$ of the form given above with $\alpha\beta = z_0$. Since the argument just given for traces and determinants is reversible, we immediately deduce that $f(A, B)$ is nilpotent at least for the generators of $\mathcal{S}$. But we must prove that $f(R, S)$ is nilpotent for arbitrary R and S in $\mathcal{S}$.

First note that, since the coefficients of f are assumed to have zero sum, $f(T_1, T_2)$ is automatically nilpotent whenever T_1 and T_2 are simultaneously triangularizable (because the diagonal of $f(T_1, T_2)$ must be zero). Thus we can restrict ourselves to non-triangularizable pairs $\{R, S\}$ in $\mathcal{S}$.

Now if $z_0 = 0$, just take $\alpha = 0$, $\beta = 0$, and $\mathcal{S} = \mathcal{S}(0, 0)$. Then $\mathcal{S}$ has only four nonzero members and the only non-triangularizable ordered pairs in $\mathcal{S}$ are $\{A, B\}$ and $\{B, A\}$. Since a permutation of the basis shows that $\{A, B\}$ is simultaneously similar to $\{B, A\}$, and since $f(A, B)$ is nilpotent, so is $f(B, A)$, and we are done.

To complete the proof, assume $z_0 \neq 0$. Take $\alpha = \beta$ with $\alpha^2 = z_0$. Then every member of $\mathcal{S}(\alpha, \alpha)$ is a scalar multiple of one of the matrices

$$A = \begin{pmatrix} \alpha & 0 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & \alpha \end{pmatrix}, \quad C = \begin{pmatrix} 1/\alpha & 0 \\ 1 & 0 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} 0 & 1 \\ 0 & 1/\alpha \end{pmatrix}.$$

Of all the unordered pairs, $\{A, C\}$ and $\{B, D\}$ are obviously triangularizable. So are $\{A, D\}$ and $\{B, C\}$. (Just note that A and D have a common eigenvector, and so do B and C.) As in the preceding paragraph, $\{A, B\}$ is simultaneously similar to $\{B, A\}$ as is $\{C, D\}$ to $\{D, C\}$. So the only ordered pair on which we have to show that f is nil is $\{C, D\}$. But this is just $\{A, B\}$ with $1/\alpha$ in place of α. ∎

EXAMPLE 3.7. The only interesting polynomial of homogeneous degree $r = 1$ both in x and in y is $xy - yx$, which is of course rigid. The case $r = 2$ is also very easy to check: let f be the general polynomial

$$f(x,y) = a_1(xy)^2 + a_2(yx)^2 + a_3xy^2x + a_4yx^2y + a_5x^2y^2 + a_6y^2x^2$$

with $a_1 + \cdots + a_6 = 0$ (and of course, not all a_i equal to zero). If no subset of the nonzero a_i sums to zero, then f is rigid. To see this, just observe that the reduced components $f_{ij}(z)$ are obtained from

$$a_3\beta X + (a_1 + a_5\alpha\beta)XY + (a_2 + a_6\alpha\beta)YX + a_4\alpha Y$$

as $f_{11} = a_3$, $f_{12} = a_1 + a_5z$, $f_{21} = a_2 + a_6z$ and $f_{22} = a_4$. Hence the first equations in the test system of the lemma above is

$$(a_3 + a_4 + a_5 + a_6)z + a_1 + a_2 = 0$$

whose only solution is $z = 1$. Hence the lemma applies.

We can now state our most general theorem on triangularizability of semigroups of compact operators. We should point out that the mere assumption of rigidity for a polynomial f can yield reducibility results for a semigroup $\mathcal{S}$ of compact operators on which f is nil. It was proved in [9], for example, that under this hypothesis

(i) $\mathcal{S}$ has a block-triangularization in which every diagonal (irreducible) block is an essentially finite group of finite dimension, i.e., it is contained in $\mathbb{C}\mathcal{G}$ where $\mathcal{G}$ is a finite subgroup of $\mathcal{GL}_n(\mathbb{C})$ for some n, and

(ii) in particular, if every member of $\mathcal{S}$ has rank at most one, then it is triangular.

Note that if the underlying space $\mathcal{X}$ is infinite-dimensional or $\mathcal{S}$ contains nonzero singular elements, then the above hypothesis implies reducibility. We now consider the general case for triangularization.

THEOREM 3.8. *Let f be a rigid polynomial of the same homogeneous degree r in x and in y. Assume that $\delta_1(f) = 1$ and that no proper subset of the coefficients of f sums to zero. If f is nil on any semigroup of compact operators on a Banach space $\mathcal{X}$, then $\mathcal{S}$ is triangularizable.*

Proof. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are any invariant subspaces for $\mathcal{S}$ with $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then f is nil on the semigroup induced by $\mathcal{S}$ on the quotient space $\mathcal{M}_2/\mathcal{M}_1$. Thus it suffices to show that $\mathcal{S}$ is reducible. (See the Triangularization Lemma in p. 155 of [10].)

Suppose $\mathcal{S}$ is irreducible. Theorem 2.2 together with the opening remarks of the present section allow us to assume that if $\mathcal{X}$ is finite-dimensional, then $\mathcal{S}$ has a singular element. Since f is nil on $\overline{\mathbb{R}^+\mathcal{S}}$ by homogeneity of f and continuity of spectral radius on compact operators, Lemma 3.1 implies that f is nil on some $\mathcal{S}(\alpha, \beta)$, which contradicts the rigidity hypothesis. ∎

The following corollary is immediate.

COROLLARY 3.9. *If in addition to the hypotheses of Theorem* 3.8, $\mathcal{X}$ *is assumed to be a Hilbert space and* $\mathcal{S}$ *to be self-adjoint, then* $\mathcal{S}$ *is commutative (and consists of normal operators.)*

COROLLARY 3.10. *Let* f *be as in Theorem* 3.8, *and let* $\mathcal{S}$ *be a semigroup in* $\mathcal{B}(\mathcal{X})$, *merely assumed to contain a nonzero compact operator. If* f *is nil on* $\mathcal{S}$, *then* $\mathcal{S}$ *is reducible.*

Proof. The semigroup ideal of $\mathcal{S}$ consisting of compact members of $\mathcal{S}$ satisfies the hypotheses of the theorem and is thus triangularizable. This implies that $\mathcal{S}$ is reducible. (See e.g., p. 200 of [10].) ∎

EXAMPLE 3.11. Consider the general f, quadratic in x and in y, given in Example 3.7 with $a_1 + \cdots + a_6 = 0$ and no proper subset of nonzero a_i summing to zero, so that f is rigid. If $\delta_1(f)$=1 and if f is nil on a semigroup $\mathcal{S}$ of compact operators, then $\mathcal{S}$ is triangularizable. If, for example,

$$f(x,y) = a(xy)^2 + bxy^2x + cx^2y^2 + dy^2x^2$$

with $abcd \neq 0$ and no proper subset of $\{a,b,c,d\}$ summing to zero, then $\delta_1(f) = 1$, as shown in Example 2.3.

4. ON THE CASE OF NONCOMPACT OPERATORS

As noted in the introduction, even the simplest of polynomials of the form considered in this paper, i.e., $xy - yx$ can vanish on an irreducible $\mathcal{S}$, where $\mathcal{S}$ is not just a semigroup, but a singly generated algebra. Compactness of the operators in $\mathcal{S}$ was used in the preceeding section to obtain affirmative results.

If nonzero compact operators are present in $\mathcal{S}$, certain corollaries of the results above can be obtained by making use of the fact that semigroup ideals of $\mathcal{S}$ are irreducible if $\mathcal{S}$ itself is irreducible. Corollary 3.10 above is a sample. Furthermore, if the compact operator present in $\mathcal{S}$ is also "substantial," e.g., injective and diagonalizable with distinct eigenvalues, then $\mathcal{S}$ is triangularizable. Proofs of these corollaries are similar to those given in [9] and will not be repeated here.

We shall now consider one case in which $\mathcal{S}$ is not assumed to contain any compact operators other than zero. Instead, we assume that $\mathcal{S}$ is a strongly compact group of invertible operators on a Banach space $\mathcal{X}$. The main tool here is the following lemma, which is implicit in most proofs of the Peter–Weyl Theorem. For a proof see [11].

LEMMA 4.1. *Let* $\mathcal{G}$ *be a group of bounded operators on a Banach space* $\mathcal{X}$ *and assume that* $\mathcal{G}$ *is compact in the strong operator topology. Then the following set is dense in* $\mathcal{X}$:

$$\mathcal{X}_0 = \{x \in \mathcal{X} : \mathcal{G}x \textit{ has finite dimensional span}\}.$$

THEOREM 4.2. *Let f be a noncommutative polynomial, separately homogeneous in each of its two variables, such that*
(i) $\delta(f) = 1$ *and*
(ii) *no proper subset of nonzero coefficients of f sums to zero.*
If f vanishes on a strongly compact group $\mathcal{G}$ in $\mathcal{B}(\mathcal{X})$, then $\mathcal{G}$ is Abelian.

Proof. Let $x \in \mathcal{X}_0$, so that the linear span $\mathcal{M}$ of $\mathcal{G}x$ is a finite-dimensional subspace of $\mathcal{X}$ invariant under $\mathcal{G}$. Then $\mathcal{G}_0 = \mathcal{G}|\mathcal{M}$ is a group of invertible operators on $\mathcal{M}$ which is compact, since $\mathcal{G}$ is strongly compact. Thus $\mathcal{G}_0$ is Abelian by Theorem 1.4. It follows that $ABx = BAx$ for all A and B in $\mathcal{G}$. Since this is true for all x in $\mathcal{X}_0$, and $\mathcal{X}_0$ is dense, we deduce that $AB = BA$ for all A and B in $\mathcal{G}$. ∎

COROLLARY 4.3. *Let f be a polynomial satisfying the conditions of Theorem* 4.2. *If $\mathcal{G}$ is a compact group of invertible elements in a unital Banach algebra $\mathcal{A}$, and if f vanishes on $\mathcal{G}$, then $\mathcal{G}$ is Abelian.*

Proof. Following [11], we treat $\mathcal{G}$ as a group of left-multiplications. This makes Theorem 4.2 immediately applicable. ∎

The following results can be proved in a similar way, using Theorem 2.2 instead of Theorem 1.4.

THEOREM 4.4. *Let f be a polynomial of homogeneous degree m in each of its two variables. Assume that*
(i) $\delta_1(f) = 1$ *and*
(ii) *no proper subset of nonzero coefficients of f adds up to zero.*
If f is nil on a strongly compact group $\mathcal{G}$ of operators on a Banach space, then $\mathcal{G}$ is Abelian.

COROLLARY 4.5. *Let f be as in the preceeding theorem. A compact group $\mathcal{G}$ in a unital Banach algebra is Abelian if f is nil on $\mathcal{G}$.*

One family of noncompact operators for which the polynomial conditions in this paper may prove useful is that of algebraic operators. As pointed out in [3], there exists an irreducible semigroup of unipotent operators, i.e. operators of the form $1 + N$, where N is nilpotent. However, assuming bounded index and using Zelmanov's result [15], it was shown in [3] that every semigroup of unipotent operators whose nilpotent parts have bounded index is triangularizable. Thus the following general questions seems reasonable to ask.

QUESTION 4.6. What polynomial conditions can be imposed on semigroups of algebraic operators on a Banach space to guarantee reducibility, triangularizability, or commutativity?

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