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COMPLETE ISOMETRIES BETWEEN SUBSPACES OF NONCOMMUTATIVE L_p-SPACES

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ABSTRACT. We prove some noncommutative analogues of a theorem proved by Plotkin and Rudin about isometries between subspaces of L_p-spaces.

Let $0 < p < \infty$, p not an even integer. The main result of this paper states that in the category of unital subspaces of noncommutative probability L_p-spaces, under some boundedness condition, the unital completely isometric maps come from $*$-isomorphisms of the underlying von Neumann algebras.

Some applications are given, including to noncommutative H^p-spaces.

KEYWORDS: *Non commutative probability, complete isometries between non commutative L_p spaces, von Neumann algebra.*

MSC (2000): 46L51, 46L52, 46L07.

INTRODUCTION

The study of isometries between Banach spaces has been an active area of research in the theory of Banach spaces for a long time, see for example the survey [5]. The isometries between L_p-spaces were first described by Banach, with a final proof given by Lamperti. The study of isometries between subspaces of L_p-spaces, goes back at least to the 1960's with Forelli's work [6], but the most general result is due independently to Plotkin in a series of articles in the 1970′s [15] and to Rudin in [16]; see also Hardin [8]. The reader is refered the Chapter 2 of [10] for a survey.

The study of isometries between whole noncommutative L_p-spaces has already interested a few mathematicians, and the final characterization (in the tracial case) was given by Yeadon in [18]. Recent results were also obtained for non semifinite von Neumann algebras ([17] and [9]). The study of complete isometries between noncommutative L_p has also been more recently studied by Junge, Ruan and Sherman in [9].

In this paper we will be interested in the study of complete isometries between subspaces of noncommutative L_p-spaces, and the main results are close analogues of the result for isometries in subspaces of classical L_p-spaces.

We first recall Plotkin's and Rudin's theorem:

THEOREM 0.1 (Plotkin, Rudin). *Let $0 < p < \infty$ and $p \neq 2, 4, 6, 8, \ldots$. Let μ and ν be two probability measures (on arbitrary measure spaces Ω and Ω'). Let finally n be a positive integer and $f_1, \ldots, f_n \in L_p(\mu)$, $g_1, \ldots, g_n \in L_p(\nu)$.*

Assume that for all complex numbers $z_1, \ldots, z_n \in \mathbb{C}$,

$$\int |1 + z_1 f_1 + \cdots + z_n f_n|^p \mathrm{d}\mu = \int |1 + z_1 g_1 + \cdots + z_n g_n|^p \mathrm{d}\nu. \tag{0.1}$$

Then $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ form two equimeasurable families. Probabilistically, this means that the $\mathbb{C}^n$-valued random variables $(f_1, \ldots, f_n)$ and $(g_1, \ldots, g_n)$ have the same distribution.

The following theorem was also proved by Rudin in his paper [16]. It had previously been proved in weaker forms by Forelli ([6] and [7]).

THEOREM 0.2 (Rudin). *Let μ and ν be as above, and $0 < p < \infty$, $p \neq 2$. Let $M \subset L_p(\mu)$ be a (complex) unital algebra (with respect to the point-wise product), and $A : M \to L_p(\nu)$ a unital linear isometry: $A(1) = 1$ and*

$$\int |f|^p \mathrm{d}\mu = \int |A(f)|^p \mathrm{d}\nu \quad \forall f \in M.$$

(i) *Then for all $f, g \in M$:*

$$A(fg) = A(f)A(g) \quad \forall f, g \in M \quad \textit{and} \quad \|A(f)\|_\infty = \|f\|_\infty.$$

(ii) *If moreover $M \subset L_\infty$ or $p \neq 4, 6, 8, \ldots$, then for all n and $f_1, \ldots, f_n \in M$, $(f_1, \ldots, f_n)$ and $(Af_1, \ldots, Af_n)$ are equimeasurable.*

In this paper similar results are proved in the noncommutative setting (with some additional boundedness conditions). The commutative L_p-spaces have to be replaced by noncommutative spaces $L_p(\mathcal{M}, \tau)$ associated to a von Neumann algebra $\mathcal{M}$ with a finite normalized trace τ, and isometries are replaced by complete isometries. Let us briefly introduce the vocabulary.

In the whole paper $(\mathcal{M}, \tau)$ and $(\mathcal{N}, \widetilde{\tau})$ are von Neumann algebras equipped with normal faithful finite (n.f.f.) traces. The units of $\mathcal{M}$ and $\mathcal{N}$ are denoted by $1_\mathcal{M}$ and $1_\mathcal{N}$ or simply by 1. The traces will always be assumed to be normalized: $\tau(1) = 1$.

When n is an integer, the set of $\mathcal{M}$-valued $n \times n$ matrices is denoted by $M_n(\mathcal{M})$, is identified with the tensor product $M_n \otimes \mathcal{M}$ and is provided with a normal faithful tracial state $\tau^{(n)} \stackrel{\text{def}}{=} \mathrm{tr}_n \otimes \tau$. Here tr_n denotes the normalized trace on M_n:

$$\mathrm{tr}_n(a) = \frac{1}{n}\mathrm{Tr}(a) = \frac{1}{n} \sum_{1 \leqslant j \leqslant n} a_{j,j}.$$

The unit of $M_n(\mathcal{M})$ is $1_n \otimes 1_{\mathcal{M}}$ and will be denoted simply by 1 when no confusion is possible.

Let $0 < p < \infty$. If $x \in \mathcal{M}$, the "p-norm" of x is denoted by $\|x\|_p$ and is equal to

$$\|x\|_p = \|x\|_{L_p(\tau)} \stackrel{\text{def}}{=} (\tau(|x|^p))^{1/p}.$$

In the same way, if $x \in M_n(\mathcal{M})$, $\|x\|_p$ denotes the quantity $\|x\|_{L_p(\tau^{(n)})}$. Remark that $\|\cdot\|_p$ is a norm only if $p \geqslant 1$. In this case, $L_p(\mathcal{M}, \tau)$ is defined as the completion of $\mathcal{M}$ with respect to the norm $\|\cdot\|_p$ (see the survey [14] for more details, see also Section 2.2). If $p = \infty$, $L_\infty(\mathcal{M}, \tau)$ is just $\mathcal{M}$ with the operator norm. The space $L_p(\mathcal{M}, \tau)$ will be denoted by $L_p(\mathcal{M})$ or $L_p(\tau)$ when no confusion is possible.

As usual, the main modification one has to bring in order to deal with the noncommutativity is the fact that one has to allow operator coefficients instead of scalar coefficients in (0.1).

In the whole paper, we will try to use the following notation: unless explicitly specified, small letters x or y will stand for elements of the von Neumann algebras $\mathcal{M}$ or $\mathcal{N}$, a, b will stand for finite complex-valued matrices viewed as matricial coefficients. Operators written with capital letters will be matrices with coefficients in $\mathcal{M}$ or $\mathcal{N}$. The letters z and λ (respectively s and t) will denote complex (respectively real) numbers. In a typical equation like

$$S = \sum_k z_k a_k \otimes x_k \in M_n(\mathcal{M}),$$

it should thus be clear to which set all the z_k, a_k and x_k belong.

At least as far as bounded operators are concerned, the fact that two families of noncommutative random variables (i.e. elements of the L_p-spaces) are equimeasurable can be expressed by requesting that their $*$-distributions are the same. Let us recall the definition of the distribution of noncommutative random variables. If $(x_i)_{i \in I} \subset \mathcal{M}$ is a family of operators in $\mathcal{M}$, its distribution with respect to τ is the linear form on the free algebra generated by elements indexed by I that maps a polynomial $P((X_i)_{i \in I})$ in noncommuting variables to $\tau(P((x_i)_{i \in I})$. Its $*$-distribution is the distribution of $(x_i, x_i^*)_{i \in I}$. The fact that two families of *bounded* operators have the same $*$-distributions is known to be equivalent to saying that they generate isomorphic tracial von Neumann algebras (Lemma 3.3).

The main result of this paper is the following theorem:

THEOREM 0.3. *Let $(\mathcal{M}, \tau)$ and $(\mathcal{N}, \widetilde{\tau})$ be von Neumann algebras equipped with faithful normal finite normalized traces. Let $E \subset L_p(\mathcal{M}, \tau)$ be a subspace of $L_p(\mathcal{M}, \tau)$, and let $u : E \to L_p(\mathcal{N}, \widetilde{\tau})$ be a linear map. Denote by* $\mathrm{id} \otimes u : M_n \otimes E \to M_n \otimes L_p(\widetilde{\tau})$ *the natural extension of u to $M_n(E)$. Fix $0 < p < \infty$ such that p is not an even integer.*

Assume that the following boundedness condition holds: $E \subset L_\infty(\mathcal{M}) (= \mathcal{M})$.

Assume that for all $n \in \mathbb{N}$ and all $X \in M_n(E)$, the following equality between the p-"norms" holds:

$$\|1_n \otimes 1_{\mathcal{M}} + X\|_p = \|1_n \otimes 1_{\mathcal{N}} + (\mathrm{id} \otimes u)(X)\|_p. \tag{0.2}$$

Let $VN(E)$ denote the von Neumann subalgebra generated by E in $\mathcal{M}$. Then $u(E) \subset L_\infty(\mathcal{N})$ and u extends to a von Neumann algebra isomorphism $u : VN(E) \to VN(u(E))$ that preserves the traces, and this extension is unique.

In particular, if E is an algebra, then u agrees with the multiplicative structure of E: if $x, y \in E$, then $u(xy) = u(x)u(y)$. Moreover, if $x \in E$ and $x^* \in E$, then $u(x^*) = u(x)^*$.

First some remarks: as in the commutative case, the condition $p \notin 2\mathbb{N}$ is crucial. Indeed in the simplest case when $p = 2n$ and $E = \mathbb{C}X$ is one-dimensional, with $X^* = X$ and $Y = u(X) = Y^*$, then condition (0.2) holds as soon as the distributions of X and Y coincide on every polynomial of degree less than $2n$, which does not imply that the distributions agree on every polynomial.

It is also easy to see that it is necessary to allow matrix coefficients to appear in (0.2), and that the theorem does not hold when (0.2) is assumed only for $x \in E$. A simple example is when $E = \mathcal{M} = \mathcal{N} = M_n$ equipped with its normalized trace tr_n and u is the transposition map $u : (a_{ij}) \to (a_{ji})$. Then u is isometric for every p-norm but is not a morphism of algebras.

However, it is unclear whether the theorem holds if (0.2) is only assumed for every $x \in M_n(E)$ for a *fixed* n (even for $n = 2$).

When $p = 2m$ is an even integer, the situation is different: it is possible to show that if (0.2) holds for $n = m$, then (0.2) holds for any n. See Theorem 3.6.

The techniques used in the proof of Theorem 0.3 do not allow to state the result when E is a general subspace of $L_p(\mathcal{M})$ (i.e. not necessarily made of bounded operators). Indeed the proof relies on Lemma 3.3 which says that the $*$-distribution of a family of bounded operators characterizes the von Neumann algebra they generate. This result is known to be false for unbounded operators even in the commutative case (it is the moment problem). Moreover the proof relies on the expansion in power series of operators of the form $|1 + x|^p$, which allows to compute the $*$-distribution of operators (Lemma 2.2). At first sight this seems to require that the operator x is bounded. However it is possible to get some results of this kind for unbounded operators using a noncommutative version of dominated convergence theorem from [4]: see Lemma 2.7. It is also immediate to see that Theorem 0.3 still holds if the boundedness condition is replaced by the assumption that $E \cap L_\infty(\mathcal{M})$ (or even $E \cap L_\infty + u^{-1}(u(E) \cap L_\infty)$ by Theorem 2.11) is dense in E.

In the case when E is self-adjoint and u is assumed to map a self-adjoint operator to a self-adjoint operator (which is *a posteriori* always true, see Lemma 4.2), Theorem 0.3 can be deduced from the commutative Theorem 0.1. Although it is contained in the general case, this special case is proved in the first section of this paper, since the proof uses the same idea as in the general case but with simpler computations.

In the second section of this paper the main technical results are proved. The first one establishes the link between the trace of products of operators and

p-norms of linear combinations of these operators (Lemma 2.2 for bounded operators and Lemma 2.7 for the general case). The second one (Theorem 2.11) proves that in the setting of Theorem 0.3, if $E \subset L_\infty(\mathcal{M})$ then $u(E) \subset L_\infty(\mathcal{N})$.

In Section 3 the main theorem (analogous to Theorem 0.1) is derived from Lemma 2.2 (Theorem 3.1 and Theorem 0.3) and also reformulated in the operator space setting (Corollary 3.5). We also derive an approximation result and we discuss the necessity of taking matrices of arbitrary size in (0.2) (but this question is mainly left open).

In a last part, some other consequences of the results of Section 2 are established, dealing with maps defined on subspaces of L_p which have an additional algebraic structure (e.g. self-adjoint, or stable by multiplication ...). In particular a noncommutative analogue of Rudin's Theorem 0.2 is derived. We end the paper with some comments and questions.

1. SELF-ADJOINT CASE

In this section we prove the special case explained in the introduction as a consequence of the commutative theorem.

Let $p \in \mathbb{R}^+ \setminus 2\mathbb{N}$, $E \subset \mathcal{M}$ and $u : E \to \mathcal{N}$ be as in Theorem 0.3. Assume furthermore that E is self-adjoint (if $x \in E$, $x^* \in E$) and that $u(x^*) = u(x)^*$ for $x \in E$.

Let us sketch the proof in this special case: for any self-adjoint operators $x_1, \dots, x_n$ in E, denote $y_k = u(x_k)$. Then for any self-adjoint matrices $a_1, \dots, a_n$, since $\sum\limits_k a_k \otimes x_k$ and $\sum\limits_k a_k \otimes y_k$ are self-adjoint, they generate commutative von Neumann algebras, and Theorem 0.1 can be applied to deduce that they have the same distribution. The conclusion thus follows from Lemma 3.3 and from the following linearization result (and the fact that E is spanned by self-adjoint operators):

LEMMA 1.1. *Let $x_1, \dots, x_n \in \mathcal{M}$ and $y_1, \dots, y_n \in \mathcal{N}$ be self-adjoint operators. Assume that for all m and all self-adjoint $m \times m$ matrices $a_1, \dots, a_n$, the operators $a_1 \otimes x_1 + \dots + a_n \otimes x_n$ and $a_1 \otimes y_1 + \dots + a_n \otimes y_n$ have the same distribution with respect to the traces $\mathrm{tr}_m \otimes \tau$ and $\mathrm{tr}_m \otimes \widetilde{\tau}$:*

$$\mathrm{dist}(a_1 \otimes x_1 + \dots + a_n \otimes x_n) = \mathrm{dist}(a_1 \otimes y_1 + \dots + a_n \otimes y_n). \tag{1.1}$$

Then $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ have the same distribution.

Independently of our work, this lemma was obtained in [3] using random matrices, and was used to give a new formulation of Connes's embedding problem.

Here we provide a different and elementary proof that consists in exhibiting specific matrices $a_1, \dots, a_n$. The idea is the same as in the proof of the general case of Theorem 0.3, but here the computations are simpler.

In fact the result of Collins and Dykema is apparently slightly stronger than the one stated above in the sense that they only assume that (1.1) holds for any self-adjoint matrices a_i with a spectrum included in $[c, d]$ for some fixed real numbers $c < d$. But it is not hard to deduce their result from the one above. More precisely, $m \in \mathbb{N}$ and $c < d$ being fixed, if one only assumes that (1.1) holds for any self-adjoint matrices a_i of size m with a spectrum included in $[c, d]$, then it holds for any self-adjoint matrices $a_i \in M_m$ (without restriction on the spectrum). Indeed, if $c < \lambda < d$ and $a_i \in M_m$ are arbitrary, then for $t \in \mathbb{R}$ small enough, the matrices $\lambda 1_m + ta_i$ all have spectrum in $[c, d]$; and the distribution of $\sum(1_m + ta_i) \otimes x_i$ for infinitely many different values of t is enough to determine the distribution of $\sum a_i \otimes x_i$.

Proof of Lemma 1.1. Let m be an integer and take $(i_1, \ldots, i_m) \in \{1, 2, \ldots, n\}^m$. We want to prove that

$$\tau(x_{i_1} x_{i_2} \cdots x_{i_n}) = \widetilde{\tau}(y_{i_1} y_{i_2} \cdots y_{i_n}).$$

Relabeling and repeating if necessary the x_i's and y_i's, it is enough to prove it when $m = n$ and $i_k = k$ for all k. We are left to prove that

$$\tau(x_1 x_2 \cdots x_n) = \widetilde{\tau}(y_1 y_2 \cdots y_n). \tag{1.2}$$

Take n complex numbers $z_1, \ldots, z_n$ and consider the $n \times n$ self-adjoint matrices $a_k = z_k e_{k,k+1} + \overline{z_k} e_{k+1,k}$ if $k < n$ and $a_n = z_n e_{n,1} + \overline{z_n} e_{1,n}$; the expression $(\mathrm{tr}_n \otimes \tau)\Big(\Big(\sum_{k=1}^{n} a_k \otimes x_k\Big)^n\Big)$ can be viewed as a polynomial in the z_j's and the $\overline{z_j}$'s, and the coefficient in front of $z_1 z_2 \cdots z_n$ is equal to $\tau(x_1 x_2 \cdots x_n)$. This is not hard to check from the trace property of τ and from the fact that for a permutation σ on $\{1; 2; \ldots; n\}$, $\mathrm{tr}_n(e_{\sigma(1),\sigma(1)+1 \mod n} e_{\sigma(2),\sigma(2)+1 \mod n} \cdots e_{\sigma(n),\sigma(n)+1 \mod n})$ is nonzero if and only if σ is a circular permutation, in which case it is equal to $1/n$.

Thus (1.2) holds, and this concludes the proof. ∎

REMARK 1.2. The following property for n-tuples $a_1, \ldots, a_n$ of $m \times m$ matrices is the key combinatorial property used in the proof above and will later on be considered in this paper:

$$\mathrm{tr}_m(a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)}) = \begin{cases} 1 & \text{for a circular permutation } \sigma \text{ on } \{1; 2; \ldots; n\}, \\ 0 & \text{for another permutation } \sigma. \end{cases} \tag{1.3}$$

A permutation σ is said to be circular if there is an integer k such that $\sigma(j) = j + k \mod n$ for all $1 \leqslant j \leqslant n$.

As noted in the proof above (and it was the main combinatorial trick in the proof), the matrices $a_j = n^{1/n} e_{j,j+1 \mod n} \in M_n$ have the property (1.3). But in fact in Section 4, it will be interesting to find n matrices $a_1 \cdots a_n$ with the same property but with smaller size. And this is possible with matrices of size m for $m \geqslant n/2$:

If $n = 2m$, then the following choice of the $a_j \in M_m$ $(j = 1, \dots, n)$ works:

$a_{2j-1} = e_{j,j}$ for $j = 1, \dots, m$; $a_{2j} = e_{j,j+1}$ for $j = 1, \dots, m-1$; $a_{2m} = me_{m,1}$.

If $n = 2m-1$, then the following choice of the $a_j \in M_m$ $(j = 1, \dots, n)$ works:

$a_{2j-1} = e_{j,j}$ for $j = 1, \dots, m-1$; $a_{2j} = e_{j,j+1}$ for $j = 1, \dots, m-1$; $a_{2m-1} = me_{m,1}$.

2. EXPRESSION OF THE MOMENTS IN TERM OF THE p-NORMS

In this section, we prove that the trace of a product of finitely many operators or of their adjoints can be computed from the p-norm of the linear (matrix-valued) combinations of these operators. The main results are Lemma 2.2 for bounded operators and its refinement Lemma 2.7 for unbounded operators. We also prove that a map u as in Theorem 0.3 maps a bounded operator to a bounded operator (Theorem 2.11).

2.1. CASE OF BOUNDED OPERATORS. First suppose we are given $x_1, x_2, \dots, x_n$ elements of the von Neumann algebra $\mathcal{M}$ (here the x_i's are *bounded operators*), and $\varepsilon_1, \dots, \varepsilon_n \in \{1, *\}$. If x is an element of a von Neumann algebra and $\varepsilon \in \{1, *\}$, let x^ε denote x if $\varepsilon = 1$ and x^* if $\varepsilon = *$ (for a complex number z, $z^* = \overline{z}$).

For clearness, we adopt the following (classical) notation: for every $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $k = (k_1, \dots, k_n) \in \mathbb{N}^n$, we write $z^k = \prod_j z_j^{k_j}$ and $\overline{z}^k = \prod_j \overline{z}_j^{k_j}$. In the same way, one writes $z^\varepsilon = \prod_j z_j^{\varepsilon_j}$. If f is a formal series $f(z) = \sum_{k,l \in \mathbb{N}^n} a_{k,l} z^k \overline{z}^l$, we denote $f(z)[z^k \overline{z}^l] = a_{k,l}$. We will also denote by $\binom{\beta}{n}$ the generalized binomial coefficient defined, for $\beta \in \mathbb{C}$ and $n \in \mathbb{N}$, by:

$$\binom{\beta}{n} = \beta(\beta-1) \cdots \frac{\beta - n + 1}{n!}. \tag{2.1}$$

Pick n matrices $a_1, \dots, a_n$ with complex coefficients (say of size m). The a_j's will soon be assumed to satisfy (1.3). For all $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$, denote by $S_z \in M_m(\mathcal{M})$ the matrix

$$S_z = S(z_1, \dots, z_n) = 1 + \sum_{j=1}^{n} z_j a_j^{\varepsilon_j} \otimes x_j. \tag{2.2}$$

The following combinatorial lemma justifies the choice of the $a_j's$:

LEMMA 2.1. *Denote by $\alpha(\varepsilon)$ or simply α the number of indices $1 \leqslant j \leqslant n$ such that $\varepsilon_j = *$ and $\varepsilon_{j+1} = 1$ (again if $j = n$, $\varepsilon_{n+1} = \varepsilon_1$).*

If the a_j's satisfy (1.3) *and S_z is defined by* (2.2), *then for any integer k,*

$$\tau^{(m)}((S_z^* S_z - 1)^k [z^\varepsilon]) = \tau(x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}) k \binom{\alpha}{n-k} \tag{2.3}$$

Proof. Recall that

$$S_z^*S_z - 1 = \sum_{j\leqslant n} z_j a_j^{\varepsilon_j} \otimes x_j + \sum_{j\leqslant n} \overline{z_j} a_j^{\varepsilon_j *} \otimes x_j^* + \sum_{i,j\leqslant n} \overline{z_i} z_j a_i^{\varepsilon_i *} a_j^{\varepsilon_j} \otimes x_i^* x_j.$$

For one of the terms of $\sum_{j\leqslant n} z_j a_j^{\varepsilon_j} \otimes x_j$ to bring a contribution to the coefficient of $\prod_j z_j^{\varepsilon_j}$ in $(S_z^*S_z-1)^k$, it is necessary that $\varepsilon_j = 1$, and then $z_j a_j^{\varepsilon_j} \otimes x_j = z_j^{\varepsilon_j} a_j \otimes x_j^{\varepsilon_j}$. In the same way, for one of the terms of $\sum_{j\leqslant n} \overline{z_j} a_j^{\varepsilon_j *} \otimes x_j^*$ to bring a contribution, it is necessary that $\varepsilon_j = *$ and then $\overline{z_j} a_j^{\varepsilon_j *} \otimes x_j^* = z_j^{\varepsilon_j} a_j \otimes x_j^{\varepsilon_j}$. Last, for one of the terms of $\sum_{i,j\leqslant n} \overline{z_i} z_j a_i^{\varepsilon_i *} a_j^{\varepsilon_j} \otimes x_i^* x_j$ to have a nonzero contribution, the values of ε_i and ε_j must be $\varepsilon_i = *$ and $\varepsilon_j = 1$, and then $\overline{z_i} z_j a_i^{\varepsilon_i *} a_j^{\varepsilon_j} \otimes x_i^* x_j = z_i^{\varepsilon_i} z_j^{\varepsilon_j} a_i a_j \otimes x_i^{\varepsilon_i} x_j^{\varepsilon_j}$. Thus if one denotes $y_j = x_j^{\varepsilon_j}$,

$$\tau^{(m)}((S_z^*S_z-1)^k)[z^\varepsilon] = \tau^{(m)}\Big(\Big(\sum_{1\leqslant j\leqslant n} z_j^{\varepsilon_j} a_j \otimes y_j + \sum_{i,j,\varepsilon_i = * \text{ and } \varepsilon_j = 1} z_i^{\varepsilon_i} z_j^{\varepsilon_j} a_i a_j \otimes y_i y_j\Big)^k\Big)[z^\varepsilon].$$

Developing and using the assumption (1.3) on the a_j's, one gets

$$\tau^{(m)}((S_z^*S_z - 1)^k)[z^\varepsilon] = \sum_{l=1}^n C_l \tau(y_l y_{l+1} \cdots y_{l-1}), \tag{2.4}$$

where the indices have to be understood modulo n and where C_l denotes the number of ways of writing formally the word $y_l y_{l+1} \cdots y_{l-1}$ (which is of length n) as a concatenation of k "elementary bricks" of the form y_j (for $1 \leqslant j \leqslant n$) or $y_j y_{j+1}$ with $\varepsilon_j = *$ and $\varepsilon_{j+1} = 1$. Each of these bricks has length 1 or 2. If α_l denotes the number of apparitions of the subsequence $*, 1$ in the sequence $\varepsilon_l, \varepsilon_{l+1 \mod n}, \dots, \varepsilon_{l-1 \mod n}$ (not cyclically this time!), then for C_l to be non zero it is necessary that $k \leqslant n \leqslant k + \alpha_l$. In that case C_l is equal to the number of ways of choosing the $n-k$ bricks of size 2 among the α_j possible, the other bricks being of size 1. Thus $C_l = \binom{\alpha_l}{n-k}$. The fact that τ is a trace then allows to write (2.4) as

$$\tau^{(m)}((S_z^*S_z - 1)^k)[z^\varepsilon] = \tau(y_1 y_2 \cdots y_n) \sum_l \binom{\alpha_l}{n-k} = \tau(x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}) \sum_l \binom{\alpha_l}{n-k}.$$

It remains to notice that $\alpha_l = \alpha - 1$ if $\varepsilon_{l-1} = *$ and $\varepsilon_l = 1$ (which is the case for α different values of l), and that $\alpha_l = \alpha$ otherwise (for the $n - \alpha$ remaining values of l). The preceding equation then becomes

$$\tau^{(m)}((S_z^*S_z - 1)^k)[z^\varepsilon] = \tau(x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}) \Big(\alpha \binom{\alpha - 1}{n-k} + (n-\alpha)\binom{\alpha}{n-k}\Big).$$

Equation (2.3) follows from the elementary equality

$$\alpha \binom{\alpha-1}{n-k} + (n-\alpha)\binom{\alpha}{n-k} = k \binom{\alpha}{n-k}. \qquad \blacksquare$$

Note that the above proof only uses combinatorial arguments, it therefore also holds with minor modifications when the assumption $x_j \in \mathcal{M}$ is replaced by $x_j \in L_n(\mathcal{M}, \tau)$ for all j.

The following lemma establishes the link between the p-norm of S_z and the trace of the product of the $x_j^{\varepsilon_j}$.

LEMMA 2.2. *Let $0 < p < \infty$. Let $a_1, \dots, a_n$ be matrices satisfying* (1.3), *and, remembering* (2.2), *define the function $\varphi : \mathbb{R}^n \to \mathbb{C}$ by*

$$\varphi(r_1, \dots, r_n) = \frac{1}{(2\pi)^n} \int\limits_{[0,2\pi]^n} \|S(r_1 \mathrm{e}^{\mathrm{i}\theta_1}, \dots, r_n \mathrm{e}^{\mathrm{i}\theta_n})\|_p^p \prod_j \exp(-\mathrm{i}\theta_j)^{\varepsilon_j} \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n.$$

Then φ is indefinitely differentiable on a neighborhood of 0, *and if α is defined as in Lemma* 2.1

$$\frac{\mathrm{d}^{(n)}}{\mathrm{d}r_1 \cdots \mathrm{d}r_n} \varphi(0, \dots, 0) = \lim_{r_1, \dots, r_n \to 0} \frac{1}{r_1 \cdots r_n} \varphi(r_1, \dots, r_n) = \tau(x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}) \sum_{k=0}^{\alpha} (n-k) \binom{p/2}{n-k} \binom{\alpha}{k}. \tag{2.5}$$

Proof. The idea of the proof is the following: S_z is a small perturbation of the unit, which allows to write $|S_z|^p$ as a converging series. Equation (2.5) follows from the identification of the term in front of $\prod_j z_j^{\varepsilon_j}$. First write:

$$S_z^* S_z = 1 + \sum_{j \leqslant n} z_j a_j^{\varepsilon_j} \otimes x_j + \sum_{j \leqslant n} \overline{z_j} a_j^{\varepsilon_j *} \otimes x_j^* + \sum_{i,j \leqslant n} \overline{z_i} z_j a_i^{\varepsilon_i *} a_j^{\varepsilon_j} \otimes x_i^* x_j = 1 + \sum_{1 \leqslant j \leqslant n^2 + 2n} C_j.$$

In the last line, we denoted by C_j the $n^2 + 2n$ terms that appear on the preceding line. Remark that if $\sup |z_j| = \delta \leqslant 1$, then $\|C_j\| \leqslant \delta K$ where $K = \max_j (\|a_j\| \|x_j\|, \|a_j\|^2 \|x_j\|^2)$.

By the functional calculus for bounded operators, for z small enough (i.e. $\|1 - S_z^* S_z\| < 1$), one has:

$$(S_z^* S_z)^{p/2} = \sum_{k \geqslant 0} \binom{p/2}{k} (S_z^* S_z - 1)^k = \sum_{k \geqslant 0} \binom{p/2}{k} \sum_{1 \leqslant j_1, \dots, j_k \leqslant n^2 + 2n} C_{j_1} \cdots C_{j_k}. \tag{2.6}$$

The series above converges absolutely and uniformly when $\delta = \sup |z_j|$ is small enough, i.e. $K(n^2 + 2n)\delta < 1$. Indeed, $\|C_{j_1} \cdots C_{j_k}\| \leqslant \delta^k K^k$, and since $\binom{p/2}{k}$ tends to 0 as $k \to \infty$, one has

$$\sum_{k \geqslant 0} \sum_{1 \leqslant j_1, \dots, j_k \leqslant n^2 + 2n} \sup_{|z_j| \leqslant \delta \forall j} \left\| \binom{p/2}{k} C_{j_1} \cdots C_{j_k} \right\| \leqslant \sum_{k \geqslant 0} (n^2 + 2n)^k \left| \binom{p/2}{k} \right| \delta^k K^k < \infty.$$

We can thus reorder the terms of the sum (2.6) along powers of z_j and $\overline{z}_j$:

$$|S_z|^p = \sum_{k,l \in \mathbb{N}^n} z_1^{k_1} \cdots z_n^{k_n} \overline{z}_1^{l_1} \cdots \overline{z}_n^{l_n} D_{k,l}, \tag{2.7}$$

where $D_{k,l}$ are some operators in $M_m \otimes \mathcal{M}$, which are in fact some polynomials in $a_1^{\varepsilon_1} \otimes x_1 \cdots a_n^{\varepsilon_n} \otimes x_n$ and their adjoints. Taking the trace $\tau^{(m)}$ on both sides of (2.7) , one gets

$$\|S_z\|_p^p = \sum_{k,l \in \mathbb{N}^n} \lambda_{k,l}\, z_1^{k_1} \cdots z_n^{k_n} \overline{z}_1^{\,l_1} \cdots \overline{z}_n^{\,l_n}. \tag{2.8}$$

In this sum, we wrote $k = (k_1, \ldots, k_n)$ and $l = (l_1, \ldots l_n)$. The coefficient $\lambda_{k,l}$ is equal to

$$\lambda_{k,l} = \sum_{j \leqslant |l|+|k|} \binom{p/2}{j} \tau^{(m)}((S_z^* S_z - 1)^j)[z^k \overline{z}^l].$$

If E is defined as the set of indices $(k,l) \in \mathbb{N}^n \times \mathbb{N}^n$ such that $k_j - l_j = 1$ if $\varepsilon_j = 1$ and $k_j - l_j = -1$ if $\varepsilon_j = *$, then for $r_1, \ldots, r_n$ small enough, we are allowed to exchange the series and the integral in the definition of $\varphi(r_1, \ldots, r_n)$ and we get the following expression of φ as a converging power series:

$$\varphi(r_1, \ldots, r_n) = \sum_{(k,l) \in E} \lambda_{k,l}\, r_1^{k_1+l_1} \cdots r_n^{k_n+l_n}.$$

The two left-hand sides of (2.5) are thus equal to λ_{k^0,l^0} where $k_j^0 = 1$ if $\varepsilon_j = 1$, $k_j^0 = 0$ else, and $l_j^0 = 1 - k_j^0$. In other words, λ_{k^0,l^0} is the coefficient of $\prod_j z_j^{\varepsilon_j}$ in (2.8):

$$\frac{\mathrm{d}^{(n)}}{\mathrm{d}r_1 \cdots \mathrm{d}r_n} \varphi(0, \ldots, 0) = \lim_{r_1, \ldots, r_n \to 0} \frac{1}{r_1 \cdots r_n} \varphi(r_1, \ldots, r_n) = \lambda_{k^0,l^0}, \tag{2.9}$$

with

$$\lambda_{k^0,l^0} = \sum_{j \in \mathbb{N}} \binom{p/2}{j} \underbrace{\tau^{(m)}((S_z^* S_z - 1)^j)[z^\varepsilon]}_{\stackrel{\text{def}}{=} \gamma_j}. \tag{2.10}$$

But from Lemma 2.1,

$$\gamma_j = \tau(x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n})\, j \binom{\alpha}{n-j}.$$

Putting this equation together with (2.9) and (2.10), we finally get (2.5), which proves the lemma. ∎

REMARK 2.3. In the case when $p/2$ is an integer (i.e. p is an even integer), the same result holds in a more general setting, when the x_i's are not bounded but are in the noncommutative L_p-space associated to $(\mathcal{M}, \tau)$. Indeed, then the sum on the right-hand side of (2.6) makes sense as a finite sum of elements which all are in $L_1(\mathcal{M}, \tau)$. Indeed, from Hölder's inequality, a product of k elements of L_p is in $L_{p/k}$. This allows to take the trace in (2.6) and to follow the rest of the proof.

Of course when p is different from an even integer, the proof does not apply for unbounded operators: it is indeed unclear what sense should be given to the series (2.6), and more importantly taking the trace to get (2.8) makes no sense. However, using a noncommutative dominated convergence theorem from [4], it is possible to modify the proof and get similar results with unbounded operators.

2.2. UNBOUNDED CASE. The reader is refered to [4] for all facts and definitions on measure topology and generalized s-numbers. Just recall that if $(\mathcal{M}, \tau)$ is a von Neumann algebra with a n.f.f. normalized trace, the t-th singular number of a closed densely defined (possibly unbounded) operator Y affiliated with $\mathcal{M}$ is defined as

$$\mu_t(Y) = \inf\{\|YE\|, E \text{ is a projection in } \mathcal{M} \text{ with } \tau(1-E) \leqslant t\}.$$

The map $t \to \mu_t(Y)$ is non-increasing and vanishes on $t \geqslant 1$, and $\mu_t(Y) < \infty$ if $t > 0$.

Moreover the measure topology makes the set of τ-measurable operators affiliated with $\mathcal{M}$ a topological $*$-algebra in which a sequence (Y_n) converges to Y if and only if $\mu_t(Y - Y_n) \to 0$ for all $t > 0$. More precisely, the following inequalities hold for any positive real numbers $s, t > 0$ and any (closed densely defined) operators T and S affiliated with $\mathcal{M}$ (Lemma 2.5 in [4]):

$$\mu_t(\lambda T) = |\lambda| \mu_t(T) \quad \text{for any } \lambda \in \mathbb{C}, \tag{2.11}$$

$$\mu_t(T) = \mu_t(|T|) = \mu_t(T^*), \tag{2.12}$$

$$\mu_{t+s}(T+S) \leqslant \mu_t(T) + \mu_s(S), \tag{2.13}$$

$$\mu_{t+s}(TS) \leqslant \mu_t(T)\mu_s(S). \tag{2.14}$$

Another property from Lemma 2.5 of [4] is the fact that $\mu_s(f(T)) = f(\mu_s(T))$ for any operator $T \geqslant 0$ and any continuous increasing function on $\mathbb{R}$ with $f(0) = 0$. As a consequence, for any continuous function f on $\mathbb{R}$ with $f(0) = 0$ and any self-adjoint T affiliated with $\mathcal{M}$,

$$\mu_t(f(T)) \leqslant \sup_{|u| \leqslant \mu_t(T)} |f(u)|. \tag{2.15}$$

For any $0 < p \leqslant \infty$, the noncommutative L_p-space $L_p(\mathcal{M}, \tau)$ is identified with the set of closed densely defined operators Y affiliated with $\mathcal{M}$ such that the function $t \mapsto \mu_t(Y)$ is in $L_p([0,1], \mathrm{d}t)$. Moreover, the p-norm of this function is equal to $\|Y\|_p$.

We now fix $0 < p < \infty$.

The first fact we prove is the following lemma, which basically says that when the x_j's are unbounded operators affiliated with $\mathcal{M}$, the development in power series of $|S_z|^p$ (2.7) still holds, but in the measure topology instead of the norm topology.

LEMMA 2.4. *Let X be a closed densely defined operator affiliated with a von Neumann algebra $(\mathcal{M},\tau)$. For $r>0$, denote by Y_r the operator*

$$Y_r = (1+rX)^*(1+rX) - 1 = rX + rX^* + r^2 X^* X.$$

Then as $r \to 0$, the following convergence holds in the measure topology:

$$\frac{1}{r^n}\Big(|1+rX|^p - \sum_{j=0}^{n} \binom{p/2}{j} Y_r^j\Big) \to 0. \tag{2.16}$$

Proof. We first claim that for all $t>0$, $\sup_{r<1} \mu_t(Y_r/r) < \infty$. Indeed, from (2.13), we have:

$$\mu_{2t}(Y_r/r) \leqslant \mu_t(X+X^*) + \mu_t(rX^*X) = \mu_t(X+X^*) + r\mu_t(X^*X).$$

The claim follows from the fact that $\mu_t(x) < \infty$ for all closed densely defined operator x.

Fix now $t>0$ and take $M = \sup_{r<1} \mu_t(Y_r/r)$. Then (Proposition 2.2 in [4]) if $E = E_{[-M,M]}(Y_r/r)$ and $r<1$, we have $\tau(1-E) \leqslant t$ and by the functional calculus, since Y and E commute and are self-adjoint,

$$(1+Y_r)^{p/2}E = (E+Y_rE)^{p/2}E = \sum_{j\geqslant 0} \binom{p/2}{j} (Y_rE)^j E = \sum_{j\geqslant 0} \binom{p/2}{j} Y_r^j E.$$

The previous series converges in the operator norm topology if $rM<1$, since in that case, $\|Y_rE\| \leqslant rM < 1$. Then

$$\begin{aligned}\Big\|\frac{1}{r^n}\Big(|S_{rz}|^p - \sum_{j=0}^{n}\binom{p/2}{j}Y_r^j\Big)E\Big\| &= \Big\|\frac{1}{r^n}\Big((1+Y_r)^{p/2} - \sum_{j=0}^{n}\binom{p/2}{j}Y_r^j\Big)E\Big\| \\ &= \Big\|\sum_{j\geqslant n+1} r^{-n}\binom{p/2}{j}(Y_rE)^j\Big\| \leqslant \sum_{j\geqslant n+1}\Big|\binom{p/2}{j}\Big| r^{-n}(Mr)^j \to 0.\end{aligned}$$

This proves that $\mu_t\Big(r^{-n}\Big(|1+rX|^p - \sum_{j=0}^{n}\binom{p/2}{j}Y_r^j\Big)\Big)$ tends to zero as $r\to 0$ for every $t>0$. This concludes the proof. ∎

Let us denote by Q_n the linear projection from the space of complex polynomial $\mathbb{C}[r]$ to the subspace $\mathbb{C}_n[r]$ of the polynomials of degree at most n given by: $Q_n(r^k) = r^k$ if $r \leqslant n$ and $Q_n(r^k) = 0$ if $k>n$. If V is any vector space over the field of complex numbers, this projection naturally extends to the space of polynomials with coefficients in V (this extension if simply the tensor product map $\mathrm{id} \otimes Q_n$ if one identifies the space of polynomials with coefficients in V with the tensor product $V \otimes \mathbb{C}[r]$). For simplicity this extension will still be denoted by Q_n. The following result follows from Lemma 2.4:

COROLLARY 2.5. *Let X and Y_r be as above (for any $r > 0$). Then as $r \to 0$,*

$$\frac{1}{r^n}\Big(|1+rX|^p - Q_n\Big(\sum_{j=0}^{n}\binom{p/2}{j}Y_r^j\Big)\Big) \to 0.$$

Proof. It follows immediately from Lemma 2.4 and from the fact that if T is affiliated with $(\mathcal{M}, \tau)$ and $k > n$, then

$$\frac{1}{r^n} r^k T \to 0 \text{ in the measure topology as } r \to 0. \quad \blacksquare$$

The next step is to get a domination result necessary to apply Fack and Kosaki's dominated convergence theorem. More precisely, we prove:

LEMMA 2.6. *With the same notation as above, there are constants C and K depending only on p and n such that for all $r < 1$ and all $0 < t \leqslant 1$,*

$$\mu_t\Big(\frac{1}{r^n}\Big(|1+rX|^p - Q_n\Big(\sum_{j=0}^{n}\binom{p/2}{j}Y_r^j\Big)\Big)\Big) \leqslant C(\mu_{t/K}(X)^n + \mu_{t/K}(X)^p). \tag{2.17}$$

Proof. Denote by $m_{t,r}$ the left-hand side of (2.17):

$$m_{t,r} \overset{\text{def}}{=} \mu_t\Big(\frac{1}{r^n}\Big(|1+rX|^p - Q_n\Big(\sum_{j=0}^{n}\binom{p/2}{j}Y_r^j\Big)\Big)\Big).$$

Fix an integer K such that $K \geqslant 2n2^{2n+1}$ and $K \geqslant 3 \times 2^{2n+1}$. Define a real number s by $s = t/K$. To prove that $m_{t,r} \leqslant C(\mu_s(X)^n + \mu_s(X)^p)$, we consider two cases, depending on the value of $r\mu_s(X)$.

First assume that $r\mu_s(X) \geqslant 1$.

Note that there are some real numbers $\lambda_{k,\varepsilon}$ indexed by the integers $k \geqslant 0$ and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_k) \in \{1, *\}^k$ such that for any $r > 0$ (and any n),

$$Q_n\Big(\sum_{j=0}^{n}\binom{p/2}{j}Y_r^j\Big) = \sum_{k=0}^{n}\sum_{\varepsilon\in\{1,*\}^k}\lambda_{k,\varepsilon}r^k X^{\varepsilon_1}X^{\varepsilon_2}\cdots X^{\varepsilon_k}.$$

Thus, using (2.13) 2^{n+1} times, one gets

$$r^n m_{t,r} \leqslant \mu_{t/2^{n+1}}(|1+rX|^p) + \sum_{k=0}^{n}\sum_{\varepsilon\in\{1,*\}^k}|\lambda_{k,\varepsilon}|r^k\mu_{t/2^{n+1}}(X^{\varepsilon_1}X^{\varepsilon_2}\cdots X^{\varepsilon_k}).$$

Since $t/2^{n+1} \geqslant t/K = s$, we have that

$$\mu_{t/2^{n+1}}(|1+rX|^p) \leqslant \mu_s(|1+rX|^p) = \mu_s(|1+rX|)^p = \mu_s(1+rX)^p \leqslant (1+r\mu_s(X))^p$$
$$\leqslant 2^p(r\mu_s(X))^p \leqslant \begin{cases} r^n 2^p \mu_s(X)^p & \text{if } p \geqslant n,\\ r^n 2^p \mu_s(X)^n & \text{if } p \leqslant n.\end{cases}$$

In these computations, the fact that $\mu_s(f(T)) = f(\mu_s(T))$ for any operator $T \geqslant 0$ and any continuous increasing function on $\mathbb{R}$ with $f(0) = 0$ was used, together with the assumption $1 \leqslant r\mu_s(X)$.

From (2.14) and (2.12), we get, for $0 \leqslant k \leqslant n$,

$$\mu_{t/2^{n+1}}(X^{\varepsilon_1}X^{\varepsilon_2}\cdots X^{\varepsilon_k}) \leqslant \mu_{t/(k2^{n+1})}(X)^k.$$

Since $t/(k2^{n+1}) \geqslant t/K = s$, we have that

$$\frac{1}{r^n} r^k \mu_{t/2^{n+1}}(X^{\varepsilon_1}X^{\varepsilon_2}\cdots X^{\varepsilon_k}) \leqslant r^{k-n}\mu_s(X)^k \leqslant \mu_s(X)^n.$$

This concludes the proof that $m_{t,r} \leqslant C(\mu_s(X)^p + \mu_s(X)^n)$ for some C, in the case when $r\mu_s(X) \geqslant 1$.

Let us now assume that $r\mu_s(X) < 1$. We want to prove that in that case, there is a constant C not depending on r and t such that

$$r^n m_{r,t} \leqslant Cr^n \mu_s(X)^n. \tag{2.18}$$

In the same way as above, write

$$|1+rX|^p - Q_n\Big(\sum_{j=0}^n \binom{p/2}{j} Y_r^j\Big) = |1+rX|^p - \sum_{j=0}^n \binom{p/2}{j} Y_r^j + \sum_{k=n+1}^{2n} \sum_{\varepsilon\in\{1,*\}^k} \widetilde{\lambda}_{k,\varepsilon} r^k X^{\varepsilon_1}X^{\varepsilon_2}\cdots X^{\varepsilon_k},$$

for some real numbers $\widetilde{\lambda}_{k,\varepsilon}$ depending neither on r nor on t.

Again, using (2.13), one gets

$$r^n m_{r,t} \leqslant \mu_{t/2^{2n+1}}\Big(|1+rX|^p - \sum_{j=0}^n \binom{p/2}{j} Y_r^j\Big) + \sum_{k=n+1}^{2n} \sum_{\varepsilon\in\{1,*\}^k} \widetilde{\lambda}_{k,\varepsilon} r^k \mu_{t/2^{2n+1}}(X^{\varepsilon_1}X^{\varepsilon_2}\cdots X^{\varepsilon_k}).$$

The second term is easy to dominate using that $r\mu_s(X) < 1$ and that if $n < k \leqslant 2n$, then $t/k2^{2n+1} \geqslant t/K = s$:

$$\mu_{t/2^{2n+1}}(X^{\varepsilon_1}X^{\varepsilon_2}\cdots X^{\varepsilon_k}) \leqslant \mu_{t/k2^{2n+1}}(X)^k \leqslant \mu_s(X)^k \leqslant r^{n-k}\mu_s(X)^n.$$

For the first term, we use (2.15) for $T = Y_r$ and $f(x) = (1+x)^{p/2} - \sum_{k=0}^n \binom{p/2}{k} x^k$ (if $x \geqslant -1$, and say $f(x) = f(-1)$ else). Indeed, we have that $f(x) = o(x^n)$ as $x \to 0$, in particular there is a constant C_1 such that $|f(x)| \leqslant C_1|x|^n$ if $|x| \leqslant 3$. If one proves that $\mu_{t/2^{2n+1}}(Y_r) \leqslant 3r\mu_s(X)$, we thus have that

$$\mu_{t/2^{2n+1}}\Big(|1+rX|^p - \sum_{j=0}^n \binom{p/2}{j} Y_r^j\Big) \leqslant C_1 3^n r^n \mu_s(X)^n,$$

which would complete the proof of (2.18).

We are left to prove that $\mu_{t/2^{2n+1}}(Y_r) \leqslant 3r\mu_s(X)$. But since $t/2^{2n+1} \geqslant 3t/K$, we have that $\mu_{t/2^{2n+1}}(Y_r) \leqslant \mu_{3s}(Y_r)$, and thus using (2.13), one gets

$$\begin{aligned}\mu_{t/2^{2n+1}}(Y_r) \leqslant \mu_{3s}(r^2X^*X + rX + rX^*) &\leqslant r^2\mu_s(X^*X) + r\mu_s(X) + r\mu_s(X^*)\\ &= (r\mu_r(X))^2 + 2r\mu_s(X) \leqslant 3r\mu_s(X). \quad \blacksquare\end{aligned}$$

It is now possible to use Fack and Kosaki's dominated convergence theorem Theorem 3.6 of [4] to prove the main result of this part, which is the unbounded version of Lemma 2.2:

LEMMA 2.7. *Let $0 < p < \infty$. Assume that $x_1 \cdots x_n \in L_p(\mathcal{M}, \tau)$ and take $\varepsilon_1, \dots, \varepsilon_n \in \{1, *\}$. If $x_j \in L_n(\mathcal{M}, \tau)$ for all j, then the trace $\tau(x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n})$ "can be computed from the p-norms" of linear combinations of the x_i's with coefficients in M_m for some m.*

More precisely, let $m \in \mathbb{N}$ and $a_1, \dots, a_n \in M_m$ satisfying (1.3) *(as explained in Remark* 1.2*, such a choice of the a_j's can be achieved for any $m \geqslant n/2$). Define α as in Lemma* 2.1 *and denote $\forall z \in \mathbb{C}^n$*

$$S_z = 1 + \sum_{j=1}^{n} z_j a_j^{\varepsilon_j} \otimes x_j.$$

Let $Z = (Z_1, \dots, Z_n)$ be a (classical) $\mathbb{C}^n$-valued random variable where the Z_j's are uniformly distributed in $\{\exp(2ik\pi/3), k = 1, 2, 3\}$ and independent. Denote by $\mathbb{E}$ the expected value with respect to Z. Then

$$\frac{1}{r^n}\mathbb{E}\Big[\|S_{rZ}\|_p^p \prod_{j=1}^{n} \overline{Z_j}^{\varepsilon_j}\Big] \xrightarrow{r\to 0} \tau(x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}) \sum_{k=0}^{\alpha} (n-k) \binom{p/2}{n-k} \binom{\alpha}{k}. \tag{2.19}$$

Proof. The first step of the proof consists in using Corollary 2.5 and Lemma 2.6 in the von Neumann algebra $(M_m(\mathcal{M}), \tau^{(m)})$ in order to apply Theorem 3.6 of [4]. Fix $z \in \mathbb{C}^n$, and denote $Y_r = S_{rz}^* S_{rz} - 1$ (the dependence of Y_r on z is implicit). If $T_r = (1/r^n)\Big(|S_{rz}|^p - Q_n\Big(\sum_{j=0}^{n} \binom{p/2}{j} Y_r^j\Big)\Big)$, then from Corollary 2.5, T_r converges to 0 in measure, and from Lemma 2.6, T_r is dominated in the following way: there are positive constants C and K such that for any $0 < t \leqslant 1$ and any $0 < r < 1$,

$$\mu_t(T_r) \leqslant C(\mu_{t/K}(X)^p + \mu_{t/K}(X)^n), \tag{2.20}$$

where $X = \sum_{j=1}^{n} z_j a_j^{\varepsilon_j} \otimes x_j$. In particular, $X \in L_p(M_m(\mathcal{M}), \tau^{(m)})$ and $X \in L_n(M_m(\mathcal{M}), \tau^{(m)})$. To deduce that

$$\frac{1}{r^n} \tau^{(m)}\Big(|S_{rz}|^p - \sum_{j=0}^{n} \binom{p/2}{j} Y_r^j\Big) \to 0, \tag{2.21}$$

it is thus sufficient to prove that the domination term $C(\mu_{t/K}(X)^p + \mu_{t/K}(X)^n)$ is (as a function of t), in $L_1(\mathbb{R}_+, \mathrm{d}t)$ (see Theorem 3.6 of [4]). But this follows from the fact that, since $X \in L_p(M_m(\mathcal{M}), \tau^{(m)})$ (respectively $X \in L_n(M_m(\mathcal{M}), \tau^{(m)})$), the function $t \mapsto \mu_t(X)$ is in $L_p(\mathbb{R}_+, \mathrm{d}t)$ (respectively $L_n(\mathbb{R}_+, \mathrm{d}t)$. This proves (2.21).

Now replace z in (2.21) by the random variable Z defined above, multiply by $\prod_{j=1}^{n} \overline{Z_j}^{\varepsilon_j}$ and take the expected value. Since z is no longer fixed, Y_r is denoted by $Y_r(z)$ to remember that Y_r depends on z. Since Z only takes a finite number of values, equation (2.21) then becomes:

$$\frac{1}{r^n}\mathbb{E}\Big[\tau^{(m)}\Big(\prod_{j=1}^n \overline{Z_j}^{\varepsilon_j}|S_{rZ}|^p - Q_n\Big(\sum_{j=0}^n \binom{p/2}{j}\prod_{j=1}^n \overline{Z_j}^{\varepsilon_j} Y_r(Z)^j\Big)\Big)\Big] \xrightarrow{r\to 0} 0.$$

Note that $Q_n\Big(\sum_{j=0}^n \binom{p/2}{j} Y_r(z)^j\Big)$ is, as a function of $z=(z_1,\dots,z_n)$, a polynomial in the $2n$ variables z_i and $\overline{z_j}$ with coefficients in $L_1(M_m\otimes\mathcal{M},\tau^{(m)})$ (this follows from Hölder's inequality and from the fact that $x_j\in L_n(\mathcal{M},\tau)$). Moreover, if $P(z_1,\dots,z_n)$ is such a polynomial, i.e. $P(z)=\sum_{k,l\in\mathbb{N}^n,|k|+|l|\leqslant n} X_{k,l}z^k\overline{z}^l$, then

$$\mathbb{E}\Big[\prod_{j=1}^n \overline{Z_j}^{\varepsilon_j} P(Z)\Big] = X_{k^0,l^0},$$

where $k^0\in\mathbb{N}^n$ and $l^0\in\mathbb{N}^n$ are again defined by $k^0_j=1$ if $\varepsilon_j=1$, $k^0_j=0$ else, and $l^0_j=1-k^0_j$. If D_{k^0,l^0} denotes the coefficient in front of $z^{k^0}\overline{z}^{l^0}$ in $\sum_{j=0}^n\binom{p/2}{j}Y_1(z)^j$, then one has:

$$\mathbb{E}\Big[Q_n\Big(\sum_{j=0}^n\binom{p/2}{j}\prod_{j=1}^n\overline{Z_j}^{\varepsilon_j}Y_r(Z)^j\Big)\Big]=\mathbb{E}\Big[Q_n\Big(\sum_{j=0}^n\binom{p/2}{j}\prod_{j=1}^n\overline{Z_j}^{\varepsilon_j}Y_1(rZ)\Big)\Big]=r^nD_{k^0,l^0}.$$

Taking the trace $\tau^{(m)}$, dividing by r^n and taking the limit as $r\to 0$ in (2.21), one gets

$$\frac{1}{r^n}\mathbb{E}\Big[\prod_{j=1}^n\overline{Z_j}^{\varepsilon_j}\|S_{rZ}\|_p^p\Big]\xrightarrow{r\to 0}\tau^{(m)}(D_{k^0,l^0}).$$

This shows (2.19) since from Lemma 2.1,

$$\tau^{(m)}(D_{k^0,l^0})=\tau(x_1^{\varepsilon_1}x_2^{\varepsilon_2}\cdots x_n^{\varepsilon_n})\sum_{k=0}^{\alpha}(n-k)\binom{p/2}{n-k}\binom{\alpha}{k}. \quad ∎$$

2.3. BOUNDEDNESS ON $E\cap L_2$ OF ISOMETRIES ON $E\subset L_p$. In this subsection and in the next one, we study how isometric properties for one p-norm imply boundedness (and isometric) properties for the q-norms for $q\neq p$.

Here we first show that a unital map which is isometric between subspaces of noncommutative L_p-spaces for $1\leqslant p<\infty$ is also isometric for the 2-norm. This is a noncommutative analogue of Proposition 1 of [6], where the author proves that a unital isometry between subspaces of commutative probability L_p-spaces is also an isometry for the L_2-norm, and our proof is inspired by Forelli's proof.

Then in Theorem 2.11, we will prove that any unital and 2-isometric map between subspaces of noncommutative L_p-spaces for $0<p<\infty$, $p\notin 2\mathbb{N}$ is also isometric for the $2n$-norm for any $n\in\mathbb{N}\cup\{\infty\}$.

THEOREM 2.8. *Let $(\mathcal{M},\tau)$ and $(\mathcal{N},\widetilde{\tau})$ be as in the introduction, and $1 \leqslant p < \infty$. Let $x \in L_p(\mathcal{M},\tau)$ and $y \in L_p(\mathcal{N},\widetilde{\tau})$ such that for any $z \in \mathbb{C}$,*

$$\|1+zx\|_p = \|1+zy\|_p. \tag{2.22}$$

Then $\|x\|_2 < \infty$ if and only if $\|y\|_2 < \infty$, and $\|x\|_2 = \|y\|_2$.

The following lemma will be used; its proof was communicated to me by Pisier.

LEMMA 2.9. *Let A be a bounded operator on a Hilbert space H, and $p \geqslant 1$. Then*

$$|1+A|^p + |1-A|^p + |1+A^*|^p + |1-A^*|^p \geqslant 4. \tag{2.23}$$

Proof. By the operator convexity of the function $t \to t^r$ for $1 \leqslant r \leqslant 2$ and by an induction argument, it is enough to prove (2.23) for $p = 1$. For convenience we denote by $C = |1+A| + |1-A| + |1+A^*| + |1-A^*|$.

By Corollary 1.3.7 of [1], for any operator B on H, the following operator on $H \oplus H$ is positive:

$$\begin{pmatrix} |B| & B^* \\ B & |B^*| \end{pmatrix}.$$

Replacing B respectively by $1+A$, $1-A$, $1+A^*$ and $1-A^*$ and adding the four resulting positive operators, we get that the following operator is also positive:

$$\begin{pmatrix} C & 4 \\ 4 & C \end{pmatrix}.$$

It is classical that this implies that $C \geqslant 4$ (see for example Theorem 1.3.3 of [1]). ∎

REMARK 2.10. The lemma is stated for bounded operators, but by approximation it also applies to closed densely defined unbounded operators.

The inequality (2.23) does not hold for $0 < p < 1$ (take $A = 1$). But if one could find a finitely supported probability measure ν on $\mathbb{C} \setminus \{0\}$ such that

$$\int (|1+zA|^p + |1+zA^*|^p)\mathrm{d}\nu(z) \geqslant 2, \tag{2.24}$$

then one would be able to get the conclusion of Theorem 2.8 also for the values of p for which (2.24) holds.

Proof of Theorem 2.8. If $\|x\|_2 = \|y\|_2 = \infty$, there is nothing to prove.

If $\|x\|_2 < \infty$ and $\|y\|_2 < \infty$, then the fact that $\|x\|_2 = \|y\|_2$ follows from Lemma 2.7 with $n = 2$, $m = 1$ and $(\varepsilon_1, \varepsilon_2) = (*, 1)$ (and hence $\alpha = 1$). Indeed, by the hypothesis (2.22), the left-hand side in equation (2.19) does not change if one takes $x_1 = x_2 = x \in L_p(\mathcal{M})$ or $x_1 = x_2 = y \in L_p(\mathcal{N})$. Therefore the right-hand sides are also equal:

$$\tau(x^*x)\Big(2\binom{p/2}{2} + \binom{p/2}{1}\Big) = \tau(y^*y)\Big(2\binom{p/2}{2} + \binom{p/2}{1}\Big).$$

This implies that $\|x\|_2^2 = \tau(x^*x) = \tau(y^*y) = \|y\|_2^2$ since $2\binom{p/2}{2} + \binom{p/2}{1} = p^2/4 \neq 0$.

Hence we only have to prove that if $\|x\|_2 < \infty$, then $\|y\|_2 < \infty$.

Denote by $C(x)$ and $C(y)$ the following operators:

$$C(x) = \sum_{\omega \in \{1,i,-1,-i\}} (|1+\omega x|^p + |1+\omega x^*|^p - 2),$$
$$C(y) = \sum_{\omega \in \{1,i,-1,-i\}} (|1+\omega y|^p + |1+\omega y^*|^p - 2).$$

By Lemma 2.9, $C(x)$ and $C(y)$ are positive operators, and by Lemma 2.4, $C(rx)/r^2$ (respectively $C(ry)/r^2$) converges in measure to $p^2(x^*x+xx^*)$ (respectively $p^2(y^*y+yy^*)$) as $r \to 0$. Moreover, for any r, the hypothesis (2.22) implies that

$$\tau(C(rx)) = 2 \sum_{\omega \in \{1,i,-1,-i\}} \|1+\omega rx\|_p^p - 8 = \widetilde{\tau}(C(ry)).$$

By Fatou's Lemma ([4], Theorem 3.5), we thus have that

$$2p^2\|y\|_2^2 = \widetilde{\tau}(p^2(y^*y+yy^*)) \leqslant \liminf_{r\to 0} \tau\Big(\frac{C(rx)}{r^2}\Big).$$

We now use Fack and Kosaki's dominated convergence theorem ([4], Theorem 3.6) to prove that $\tau(C(rx))/r^2 \to 2p^2\|x\|_2^2$. By the domination Lemma 2.6 and the property (2.13) of singular numbers, there are constants $C, K > 0$ such that

$$\mu_t\Big(\frac{C(rx)}{r^2}\Big) \leqslant C(\mu_{t/K}(x)^2 + \mu_{t/K}(x)^p).$$

As in the proof of Lemma 2.7, this is enough to deduce that

$$\lim_{r\to 0} \tau\Big(\frac{C(rx)}{r^2}\Big) = \tau\Big(\lim_{r\to 0} \frac{C(rx)}{r^2}\Big) = 2p^2\|x\|_2^2.$$

This concludes the proof. ∎

2.4. BOUNDEDNESS ON $E \cap L_{2n}$ FOR ALL n OF 2-ISOMETRIES ON $E \subset L_p$. Here we prove that a unital 2-isometric map between unital subspaces of noncommutative L_p-spaces maps a bounded operator to a bounded operator. The general idea is to prove by induction on n that such a map is also an isometry for the q-norm when $q = 2n$ and to make n grow to ∞. The idea of the proof is similar to the proof of Theorem 2.8. The precise statement is:

THEOREM 2.11. *Let $0 < p < \infty$, p not an even integer. Let $x \in L_p(\mathcal{M}, \tau)$ and $y \in L_p(\mathcal{N}, \widetilde{\tau})$ such that, for any $a \in M_2(\mathbb{C})$*

$$\|1 + a \otimes x\|_p = \|1 + a \otimes y\|_p.$$

Then for any $n \in \mathbb{N}^* \cup \{\infty\}$, $x \in L_{2n}(\mathcal{M})$ if and only if $y \in L_{2n}(\mathcal{N})$, and when this holds $\|x\|_{2n} = \|y\|_{2n}$.

The theorem is proved with the use of a classical 2 by 2 matrix trick, Fatou's Lemma and expansions in power series of operators of the form $|1+a|^p$ for a satisfying $a^2 = 0$. More precisely, for such an a, we derive an expression of the following form (Corollary 2.14 and Lemma 2.15):

$$|1+a|^p + |1-a|^p + |1+a^*|^p + |1-a^*|^p \simeq \sum_{n=0}^{N} \lambda_n |a|^{2n} + \lambda_n |a^*|^{2n}$$

and are able to use a qualitative study of differential equations (Lemma 2.16) to prove the positivity (or negativity) of the difference of the two above terms.

LEMMA 2.12. *Let a be an element of a $*$-algebra such that $a^2 = 0$. Then if one denotes by a_1, a_2, a_3, a_4 the expressions*

$$a_1 = a^*a + a + a^*, \quad a_2 = a^*a - a - a^*, \quad a_3 = aa^* + a + a^*, \quad a_4 = aa^* - a - a^*,$$

then for any integer $m \geqslant 1$,

$$\sum_{j=1}^{4} a_j^m = 2P_m(a^*a) + 2P_m(aa^*),$$

where the polynomial P_m is defined by

$$P_m(X) = \Big(\frac{X + \sqrt{X^2+4X}}{2}\Big)^m + \Big(\frac{X - \sqrt{X^2+4X}}{2}\Big)^m \quad \textit{for any } X \in \mathbb{R}^+. \tag{2.25}$$

Proof. We can assume that a is a free element satisfying $a^2 = 0$, so that there are well defined polynomials A_m, B_m, C_m and D_m in $\mathbb{R}[X]$ such that

$$(a_1)^m = A_m(a^*a) + aB_m(a^*a) + C_m(a^*a)a^* + aD_m(a^*a)a^*.$$

Thus we can write, with $P_m = A_m + XD_m \in \mathbb{R}[X]$,

$$\sum_{j=1}^{4} a_j^m = 2P_m(a^*a) + 2P_m(aa^*).$$

It is easy to check that the sequences of polynomials $(A_m)_m$ and $(D_m)_m$ (and hence (P_m)) satisfy the following induction relations:

$$\begin{aligned} A_{m+2}(X) &= X(A_{m+1}(X) + A_m(X)) \quad \text{if } m \geqslant 0,\\ D_{m+2}(X) &= X(D_{m+1}(X) + D_m(X)) \quad \text{if } m \geqslant 1,\\ P_{m+2}(X) &= X(P_{m+1}(X) + P_m(X)) \quad \text{if } m \geqslant 1. \end{aligned}$$

But the right-hand side of (2.25) also satisfies the same relation, it is therefore enough (and trivial) to check that equality (2.25) holds for $m = 1$ and $m = 2$. ∎

LEMMA 2.13. *Let a be a closed densely defined operator affiliated with a von Neumann algebra $(\mathcal{M}, \tau)$ such that $a^2 = 0$. Let $a_i, i = 1,2,3,4$ be as in Lemma 2.12. Then* *for any continuous function* $f : [-1, \infty) \to \mathbb{R}$,

$$\sum_{j=1}^{4} f(a_i)^m = 2f\Big(\frac{a^*a + \sqrt{(a^*a)^2 + 4a^*a}}{2}\Big) + 2f\Big(\frac{a^*a - \sqrt{(a^*a)^2 + 4a^*a}}{2}\Big)$$

$$+2f\Big(\frac{aa^* + \sqrt{(aa^*)^2 + 4aa^*}}{2}\Big) + 2f\Big(\frac{aa^* - \sqrt{(aa^*)^2 + 4aa^*}}{2}\Big) - 4f(0). \tag{2.26}$$

Proof. Lemma 2.12 implies that (2.26) holds when f is a polynomial. By continuity of the continuous functional calculus (with respect to the measure topology when a is unbounded), (2.26) thus holds for any continuous f. ∎

COROLLARY 2.14. *Let* $0 < p < \infty$ *and* a, *as above, satisfying* $a^2 = 0$. *Then*

$$|1 + a|^p + |1 - a|^p + |1 + a^*|^p + |1 - a^*|^p = 2\psi(a^*a) + 2\psi(aa^*) - 4,$$

where ψ *is the function* $\psi : \mathbb{R}^+ \to \mathbb{R}$ *defined by*

$$\psi(t) = \Big(1 + \frac{t + \sqrt{t^2 + 4t}}{2}\Big)^{p/2} + \Big(1 + \frac{t - \sqrt{t^2 + 4t}}{2}\Big)^{p/2}.$$

Proof. This is immediate since with the notation above, $|1 + a|^p = (1 + a_1)^{p/2}$, $|1 - a|^p = (1 + a_2)^{p/2}$, $|1 + a^*|^p = (1 + a_3)^{p/2}$ and $|1 - a^*|^p = (1 + a_4)^{p/2}$. ∎

Let us study the function ψ.

PROPOSITION 2.15. *The following properties hold for* ψ:

(i) ψ *is a solution to the following differential equation on* $\mathbb{R}^+$:

$$(t^2 + 4t)y'' + (t + 2)y' - \frac{p^2}{4}y = 0. \tag{2.27}$$

(ii) ψ *has an expansion in power series around* 0, *more precisely for* $0 < |t| < 4$,

$$\psi(t) = \sum_{n \geqslant 0} \frac{2}{(2n)!} \prod_{k=0}^{n-1} \Big(\frac{p^2}{4} - k^2\Big) t^n = \sum_{n \geqslant 0} \lambda_n t^n. \tag{2.28}$$

(iii) *For any* $0 < p < \infty$, *for any* $t \in \mathbb{R}^+$ *and any* $N \in \mathbb{N}$,

$$\psi(t) - \sum_{n=0}^{N} \lambda_n t^n \begin{cases} \geqslant 0 & \text{if } p \geqslant 2N \text{ or } \lfloor N - p/2 \rfloor \text{ is odd,} \\ \leqslant 0 & \text{otherwise.} \end{cases} \tag{2.29}$$

In this proposition, for a real number t, the symbol $\lfloor t \rfloor$ denotes the largest integer smaller than or equal to t.

Proof. Checking (2.27) is just an easy computation, the details are left to the reader. It is also easy to see that ψ has an expansion in power series around 0, and (2.28) follows from the fact that both left-hand and right-hand sides of (2.28) satisfy (2.27) and have value 2 at $t = 0$.

Let us prove (2.29). Let us fix p and N. As a function of t, the left-hand side of (2.29) satisfies the following differential equation:

$$(t^2+4t)y''+(t+2)y'-\frac{p^2}{4}y=(2N+1)(2N+2)\lambda_{N+1}t^N.$$

Moreover, (2.28) shows that the left-hand side of (2.29) and its derivative has the same sign as λ_{N+1} when t is small (with $t>0$).

Note also that $\lambda_{N+1}\geqslant 0$ if $p\geqslant 2N$ or if $\lfloor N-p/2\rfloor$ is odd, that $\lambda_{N+1}\leqslant 0$ else.

The fact (2.29) thus follows from Lemma 2.16 applied to $\pm\Big(\psi(t)-\sum\limits_{n=0}^{N}\lambda_n t^n\Big)$ depending on the sign of λ_{N+1}. ∎

LEMMA 2.16. *Let a, b, c and d be continuous functions on $\mathbb{R}^+$ such that for any $t>0$,*

$$a(t)>0,\quad c(t)<0,\quad d(t)>0.$$

Let y be a C^2 function on $\mathbb{R}^+$ solution of $ay''+by'+cy=d$, and $t_0>0$ such that $y(t_0)>0$ and $y'(t_0)>0$. Then $y(t)>0$ for any $t\geqslant t_0$.

Proof. We prove that $y'(t)>0$ for any $t\geqslant t_0$. Assume that it is not true, and take $t_1=\min\{t>t_0, y'(t)=0\}$. Since $y'(t_1)=0$ and $y'(t)>0$ if $t_0<t<t_1$, we have that $y''(t_1)\leqslant 0$.

On the other hand, since $y'\geqslant 0$ on (t_0,t_1), $y(t_1)\geqslant y(t_0)>0$. Thus $y''(t_1)=(d(t_1)-c(t_1)y(t_1))/a(t_1)>0$, which is a contradiction. ∎

It is now possible to derive the main result of this part:

LEMMA 2.17. *Let $0<p<\infty$ and $(\lambda_n)_{n\geqslant 0}\in\mathbb{R}^{\mathbb{N}}$ defined by* (2.28). *Take $a\in L_p(\mathcal{M},\tau)$ such that $a^2=0$, and fix an integer $N>0$ with $\lambda_N\neq 0$.*

(i) *If $\|a\|_{2N-2}<\infty$*

$$\|a\|_{2N}^{2N}\leqslant\liminf_{t\to 0}\frac{1}{2\lambda_N t^{2N}}\Big(\|1+ta\|_p^p+\|1-ta\|_p^p-2-2\sum_{n=1}^{N-1}\lambda_n t^{2n}\|a\|_{2n}^{2n}\Big).\tag{2.30}$$

(ii) *Moreover, if $\|a\|_{2N}<\infty$, the previous inequality becomes an equality. More precisely,*

$$\|a\|_{2N}^{2N}=\lim_{t\to 0}\frac{1}{2\lambda_N t^{2N}}\Big(\|1+ta\|_p^p+\|1-ta\|_p^p-2-2\sum_{n=1}^{N-1}\lambda_n t^{2n}\|a\|_{2n}^{2n}\Big).\tag{2.31}$$

Proof. The first fact is a consequence of the properties of ψ and of Fatou's lemma.

Denote by $b(t,a)$ the following (unbounded) operator affiliated with $\mathcal{M}$:

$$b(t,a)=\frac{1}{\lambda_N t^{2N}}\Big(\psi(t^2a^*a)-\sum_{n=0}^{N-1}\lambda_n t^{2n}(a^*a)^n\Big).$$

In this equation, $(a^*a)^0$ is equal to $1_{\mathcal{M}}$. Note that the operators $b(t,a)$ are affiliated with the commutative von Neumann algebra generated by a^*a, which is isomorphic to the space of (classes of) bounded measurable functions on some probability space (Ω, μ).

Then (2.28) implies that $b(t,a) \to (a^*a)^N$ in the measure topology as $t \to \infty$ (in fact the convergence holds almost surely if the operators are viewed as functions on Ω). But (2.29) also implies that $b(t,a) \geqslant 0$. Thus one can apply Fatou's lemma to conclude that

$$\|a\|_{2N}^{2N} = \tau((a^*a)^N) \leqslant \liminf_{t \to 0} \tau(b(t,a)). \tag{2.32}$$

Replace a by a^* in the preceding inequality, and add the two equations to get (using $\|a^*\|_q = \|a\|_q$ for any real q)

$$2\|a\|_{2N}^{2N} \leqslant \liminf_{t\to 0} \frac{1}{\lambda_N t^{2N}} \tau\Big(\psi(t^2 a^*a) + \psi(t^2 aa^*) - \sum_{n=0}^{N-1} \lambda_n t^{2n}((a^*a)^n + (aa^*)^n)\Big).$$

Applying Corollary 2.14 and the linearity of the trace yields to the desired conclusion (since $(aa^*)^n$ and $(a^*a)^n$ belong to $L_1(\mathcal{M})$ for $n \leqslant N-1$).

To prove the second fact, we prove that if $\|a\|_{2N} < \infty$, then equality holds in (2.32). But this follows from the (classical) dominated convergence theorem since $\Big|\psi(t) - \sum_{n=0}^{N} \lambda_n t^n\Big| \leqslant C(t^N + t^{p/2})$ for some constant C not depending on $t \in \mathbb{R}$. ∎

The proof of Theorem 2.11 follows:

Proof of Theorem 2.11. First note that the statement for $n = \infty$ follows from the one for $n \in \mathbb{N}$, since $\|x\|_\infty = \lim_{n\to\infty} \|x\|_{2n}$. So we focus on the case when n is a positive integer.

The idea is to construct operators related to x and y of zero square by putting then in a corner of a 2 by 2 matrix, and then to use Lemma 2.17. So let us denote $a(x)$ and $a(y)$ the operators

$$a(x) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \in M_2(L_p(\mathcal{M})) \simeq L_p(M_2(\mathcal{M})),$$

$$a(y) = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \in M_2(L_p(\mathcal{M})) \simeq L_p(M_2(\mathcal{M})).$$

Note that $a(x)^2 = 0$, that for any $q \in \mathbb{R} \cup \{\infty\}$, $\|a(x)\|_q = 2^{-1/q}\|x\|_q$, and that the same holds for y. Moreover $\|1 + ta(x)\|_p = \|1 + ta(y)\|_p$ for any $t \in \mathbb{R}$. It is thus enough to prove that if $\|a(x)\|_{2n} < \infty$, then $\|a(y)\|_{2n} < \infty$ and $\|a(y)\|_{2n} = \|a(x)\|_{2n}$. We prove this by induction on n.

So take $N > 0$, assume that the aforementioned statement holds for any $n < N$ (note that we assume nothing if $N = 1$). Suppose that $\|a(x)\|_{2N} < \infty$. Then by induction hypothesis for any $n < N$, $\|y\|_{2n} = \|x\|_{2n}$. Thus the right-hand side of (2.30) is the same when a is replaced by $a(y)$ or by $a(x)$. But for $a = a(x)$, it is equal, by (2.31), to $\|a(x)\|_{2N}$. Hence (2.30) proves that $\|a(y)\|_{2N} \leqslant \|a(x)\|_{2N} < \infty$.

Applying (2.31) again with $a(y)$ yields to $\|a(y)\|_{2N} = \|a(x)\|_{2N}$. ∎

3. PROOF OF THEOREM 0.3

In this section we develop some consequences of Lemma 2.2. We are given $(\mathcal{M}, \tau)$ and $(\mathcal{N}, \widetilde{\tau})$ two von Neumann algebras with normal faithful tracial states.

Let $x_1, \dots, x_n \in \mathcal{M}$ and $y_1, \dots, y_n \in \mathcal{N}$. The noncommutative analogue (in the bounded case) of Theorem 0.1 is:

THEOREM 3.1. *Let $0 < p < \infty$ such that $p \neq 2, 4, 6, \dots$ is not an even integer. Suppose that for all $m \in \mathbb{N}$ and all $a_1, \dots, a_n \in M_m$,*

$$\Big\|1 + \sum a_i \otimes x_i\Big\|_p = \Big\|1 + \sum a_i \otimes y_i\Big\|_p.$$

Then the n-tuples $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ have the same $*$-distributions. More precisely, for all $P \in \mathbb{C}\langle X_1, \dots, X_{2n}\rangle$ polynomial in $2n$ noncommuting variables,

$$\tau(P(x_1, \dots, x_n, x_1^*, \dots, x_n^*)) = \widetilde{\tau}(P(y_1, \dots, y_n, y_1^*, \dots, y_n^*)). \tag{3.1}$$

This theorem relies on Lemma 2.2 and on the following lemma:

LEMMA 3.2. *Let $N, \alpha \in \mathbb{N}$ be integers such that $N \geqslant 1$ and $\alpha \leqslant N/2$. Then if p is a positive number such that $p \notin 2\mathbb{N}$ or $p \geqslant 2(N - \alpha)$, then*

$$\sum_{k=0}^{\alpha} (N-k) \binom{p/2}{N-k} \binom{\alpha}{k} \neq 0.$$

Proof. Take α, N and p as in the lemma. Since $(N-k)\binom{p/2}{N-k} = p/2\binom{p/2-1}{N-k-1}$, showing the lemma is the same as showing that

$$\sum_{k=0}^{\alpha} \binom{p/2-1}{N-k-1} \binom{\alpha}{k} \neq 0. \tag{3.2}$$

For every real number β, let us consider the left-hand side of (3.2) where $p/2 - 1$ is replaced by β. Since $\binom{\beta}{n}$ is a polynomial function in β of degree n, the expression $P(\beta) \overset{\text{def}}{=} \sum\limits_{k=0}^{\alpha} \binom{\beta}{N-k-1}\binom{\alpha}{k}$ is a polynomial in β of degree $N-1$. To prove that it takes nonzero values for $\beta = p/2 - 1$, we show that it has $N-1$ roots different from $p/2 - 1$. More precisely, we show that if β is an integer such that $-\alpha \leqslant \beta \leqslant N - \alpha - 2$, then $P(\beta) = 0$.

First if β is an integer between 0 and $N - \alpha - 2$ included, then for any $0 \leqslant k \leqslant \alpha$, it is immediate to check from the definition (2.1) that $\binom{\beta}{N-k-1} = 0$, which implies $P(\beta) = 0$.

The second fact to check is that if l is an integer such that $1 \leqslant l \leqslant \alpha$, then $P(-l) = 0$. Let us fix such an l. Then writing $\binom{-l}{N-k-1} = (-1)^{N-k-1}\binom{N-k+l-2}{l-1}$ we get

$$P(-l) = \sum_{k=0}^{\alpha} \binom{-l}{N-k-1}\binom{\alpha}{k} = \sum_{k=0}^{\alpha} \binom{\alpha}{k}(-1)^{N-k-1}\binom{N-k+l-2}{l-1}.$$

It only remains to note that l and N being fixed, $\binom{N-k+l-2}{l-1}$ is (as a function of k) a polynomial of degree $l-1 < \alpha$. The equality $P(-l) = 0$ arises from the fact that if $1 \leqslant i < \alpha$,

$$\sum_{k=0}^{\alpha} \binom{\alpha}{k}(-1)^k k^i = 0. \quad \blacksquare$$

Theorem 3.1 follows:

Proof of Theorem 3.1. By linearity it is enough to prove (3.1) when P is a monomial. The fact to be proved is that for every finite sequence $i_1, \dots, i_N$ of indices between 1 and n, and for every sequence $\varepsilon_1, \dots, \varepsilon_N \in \{1, *\}$,

$$\tau\Big(\prod_k x_{i_k}^{\varepsilon_k}\Big) = \widetilde{\tau}\Big(\prod_k y_{i_k}^{\varepsilon_k}\Big).$$

But from Lemma 2.2, if α is the number of indices k such that $\varepsilon_k = *$ and $\varepsilon_{k+1 \mod N} = 1$, we have

$$\tau\Big(\prod_k x_{i_k}^{\varepsilon_k}\Big)\sum_{k=0}^{\alpha}(N-k)\binom{p/2}{N-k}\binom{\alpha}{k} = \widetilde{\tau}\Big(\prod_k y_{i_k}^{\varepsilon_k}\Big)\sum_{k=0}^{\alpha}(N-k)\binom{p/2}{N-k}\binom{\alpha}{k}.$$

This implies that $\tau\Big(\prod_k x_{i_k}^{\varepsilon_k}\Big) = \widetilde{\tau}\Big(\prod_k y_{i_k}^{\varepsilon_k}\Big)$ since from Lemma 3.2 $\sum_{k=0}^{\alpha}(N-k)\binom{p/2}{N-k}\binom{\alpha}{k} \neq 0$ if $p \notin 2\mathbb{N}$. $\blacksquare$

Theorem 0.3 is an immediate consequence of Theorem 3.1, Theorem 2.11 and of the following well-known lemma:

LEMMA 3.3. *Let $(\mathcal{M}, \tau)$ and $(\mathcal{N}, \widetilde{\tau})$ be two von Neumann algebras equipped with faithful normal tracial states, and let $(x_i)_{i\in I} \in \mathcal{M}$ and $(y_i)_{i\in I} \in \mathcal{N}$ be noncommutative random variables that have the same $*$-distribution. Then the von Neumann algebras generated respectively by the x_i's and the y_i's are isomorphic, with a normal isomorphism sending x_i on y_i and preserving the trace.*

Proof of Theorem 0.3. Let $(x_i)_{i\in I}$ be a family spanning E. If $y_i = u(x_i)$ for any $i \in I$, then Theorem 2.11 shows that $\|y_i\|_\infty < \infty$, which is equivalent to the fact that $y_i \in \mathcal{N}$. By Theorem 3.1, the families (x_i, x_i^*) and (y_i, y_i^*) have the same distribution and so by Lemma 3.3, u extends to a trace preserving isomorphism between the von Neumann algebras generated by the x_i's and y_i's respectively. $\blacksquare$

It is also possible to get some approximation results using ultraproducts:

3.1. APPROXIMATION RESULTS.

COROLLARY 3.4. *Let $(\mathcal{M}_\alpha, \tau_\alpha)_{\alpha\in A}$ be a net of von Neumann algebras equipped with normal faithful normalized traces. Let I be a set, and for all α, let $(x_i^\alpha)_{i\in I} \in {\mathcal{M}_\alpha}^I$ such that for all $i \in I$, the net $(x_i^\alpha)_\alpha$ is uniformly bounded, i.e.* $\sup_\alpha \|x_i^\alpha\| < \infty$. *Assume that there is a family $(y_i)_{i\in I}$ in a von Neumann algebra $(\mathcal{N}, \widetilde{\tau})$ and a $p \notin 2\mathbb{N}$ such that for all integer n and all finitely supported family $(a_i)_{i\in I} \in M_n$, the following holds:*

$$\lim_\alpha \Big\|1 + \sum_i a_i \otimes x_i^\alpha\Big\|_p = \Big\|1 + \sum_i a_i \otimes y_i\Big\|_p. \tag{3.3}$$

Then the net $((x_i^\alpha)_i)_\alpha$ converges in $*$-distribution to $(y_i)_i$. Moreover (3.3) *holds with p replaced by any* $0 < q < \infty$.

Proof. Let $\mathcal{U}$ be any ultraproduct on A finer that the net (α), and for $i \in I$ consider x_i the image of $(x_i^\alpha)_{\alpha\in A}$ in the von Neumann ultraproduct $\mathcal{M} = \prod_{\mathcal{U}} \mathcal{M}_\alpha$. If $\mathcal{M}$ is equipped with the tracial state $\tau = \lim_{\mathcal{U}} \tau_\alpha$, then the assumption (3.3) implies that for all m and all $a_i \in M_m$,

$$\Big\|1 + \sum_i a_i \otimes x_i\Big\|_p = \Big\|1 + \sum_i a_i \otimes y_i\Big\|_p.$$

Lemma 2.2 implies that $(x_i)_i$ and $(y_i)_i$ have the same $*$-distribution. This exactly means that $(x_i^\alpha)_i$ converges in $*$-distribution to $(y_i)_i$ as $\alpha \in \mathcal{U}$.

Since this holds for any ultrafilter $\mathcal{U}$ finer than the net (α), this proves the convergence in $*$-distribution of the net $((x_i^\alpha)_i)_\alpha$ to $(y_i)_i$. The fact that (3.3) then holds with p replaced by any $0 < q < \infty$ is immediate. ∎

Theorem 0.3 can also be reformulated in the operator space setting:

3.2. REFORMULATION IN THE OPERATOR SPACE SETTING. Let $\mathcal{M} \subset B(H)$ be a von Neumann algebra equipped with a normal faithful trace τ satisfying $\tau(1) = 1$. Let E be a linear subspace of $\mathcal{M}$. There are several "natural" operator space structures on E:

For all $1 \leqslant p \leqslant \infty$, the noncommutative L_p-spaces $L_p(\mathcal{M}, \tau)$ are equipped with a natural operator space structure (see Chapter 7 of [13]). (When $p = \infty$, $L_p(\mathcal{M}, \tau)$ is the von Neumann algebra $\mathcal{M}$ with its obvious operator space structure.)

Then the linear embedding $E \subset L_p(\mathcal{M}, \tau)$ allows to define, for all $1 \leqslant p \leqslant \infty$, an operator space structure on E, which we denote by $O_p(E)$.

In this setting, Theorem 0.3 states that if E is a linear subspace of $\mathcal{M}$ containing the unit and if $1 \leqslant p < \infty$ and $p \notin 2\mathbb{N}$, then the operator space structure $O_p(E)$ together with the unit entirely determines the von Neumann algebra generated by E and the trace on it. In particular it determines all of the other operator space structures $O_q(E)$ for all $1 \leqslant q \leqslant \infty$.

More precisely:

COROLLARY 3.5. *Let $1_{\mathcal{M}} \in E \subset \mathcal{M}$ be as above, $(\mathcal{N}, \widetilde{\tau})$ be another von Neumann algebra equipped with a normal faithful tracial state, $u : E \to \mathcal{N}$ be a unit preserving linear map and $1 \leqslant p < \infty$ with $p \notin 2\mathbb{N}$.*

If $u : O_p(E) \to L_p(\mathcal{N}, \widetilde{\tau})$ is a complete isometry, then u uniquely extends to an isomorphism between the von Neumann subalgebras generated by E and its image; moreover u is then trace preserving. In particular, for all $1 \leqslant q \leqslant \infty$, $u : O_q(E) \to L_q(\mathcal{N}, \widetilde{\tau})$ is a complete isometry.

Proof. The proof is a reformulation of Theorem 0.3 once we know the two following results from the theory of noncommutative vector valued L_p-spaces developed in [12]:

A map $u : X- > Y$ between two operator spaces is completely isometric if and only if for all n, the map $u \otimes \mathrm{id} : S_p^n(X) \to S_p^n(Y)$ is an isometry (Lemma 1.7 in [12]). More precisely, for any $n \in \mathbb{N}$ and any $1 \leqslant p \leqslant \infty$,

$$\|u \otimes \mathrm{id} : S_p^n(X) \to S_p^n(Y)\| = \|u \otimes \mathrm{id} : M_n(X) \to M_n(Y)\|.$$

The second result is Fubini's theorem, which states that isometrically (and even completely isometrically, but this is of no use here) $S_p^n(L_p(\mathcal{M}, \tau)) \simeq L_p(M_n \otimes A, \mathrm{tr}_n \otimes \tau)$. See Theorem 1.9 in [12].

These two results together prove that the hypotheses in Corollary 3.5 imply those in Theorem 0.3, and thus the result is proved. ∎

3.3. ON THE NECESSITY OF TAKING MATRICES OF ARBITRARY SIZE. Here we discuss the necessity of taking matrices of arbitrary size in Theorem 0.3. In view of Theorem 0.3 a natural question is thus:

Let $p \in \mathbb{R}$. Consider the class $\mathcal{E}_{p,1}$ of all linear maps u between subspaces of noncommutative L_p-spaces constructed on von Neumann algebras equipped with a n.f.f. normalized trace. Is there an integer n such that for any such $u : E \to F$, if (0.2) holds for all $x \in M_n(E)$, then it holds for any m and any $x \in M_m(E)$? The smallest such integer will be denoted by $n_{p,1}$.

A similar question is:

Let $p \in \mathbb{R}$. Consider the class $\mathcal{E}_p$ of all linear maps u between subspaces of noncommutative L_p-spaces constructed on von Neumann algebras equipped with a normal semifinite faithful normalized trace. Is there an integer n such that for any such $u \in \mathcal{E}_p$, if u is n-isometric, then u is completely isometric? The smallest such integer will be denoted by n_p.

As was noted in the introduction, the transposition map from M_n to M_n ($n \geqslant 2$) shows that, except for $p = 2$, we necessarily have $n_{p,1} > 1$ and $n_p > 1$.

When $p \notin 2\mathbb{N}$, it is not clear whether $n_{p,1} < \infty$ (or $n_p < \infty$).

In the opposite direction, as announced in the introduction, when $p = 2m \in 2\mathbb{N}$, then it is not hard to prove that $n_{p,1} \leqslant m$ and $n_p \leqslant m$.

THEOREM 3.6. *Let $p = 2m \in 2\mathbb{N}$. Let $(\mathcal{M}, \tau)$, $(\mathcal{N}, \widetilde{\tau})$ be as in Theorem 0.3.*
Let $E \subset L_p(\mathcal{M}, \tau)$ be a subspace and $u : E \to L_p(\mathcal{N}, \widetilde{\tau})$ be a linear map.

Assume that for all $x \in M_m(E)$, the following equality between the p-norms holds:

$$\forall x \in M_m(E), \quad \|1_m \otimes 1_{\mathcal{M}} + x\|_{2m} = \|1_m \otimes 1_{\mathcal{N}} + (\mathrm{id} \otimes u)(x)\|_{2m}. \tag{3.4}$$

Then in fact this equality holds for $x \in M_n(E)$ for every $n \in \mathbb{N}$:

$$\|1_n \otimes 1_{\mathcal{M}} + x\|_{2m} = \|1_n \otimes 1_{\mathcal{N}} + (\mathrm{id} \otimes u)(x)\|_{2m}.$$

THEOREM 3.7. *Let $p = 2m \in 2\mathbb{N}$. Let $(\mathcal{M}, \tau)$, $(\mathcal{N}, \widetilde{\tau})$ be (exceptionally) von Neumann algebras with normal faithful semifinite traces.*

Let $E \subset L_p(\mathcal{M}, \tau)$ be a subspace and $u : E \to L_p(\mathcal{N}, \widetilde{\tau})$ be a linear map.

If u is m-isometric (i.e. $\|x\|_{L_p(\tau^{(m)})} = \|(\mathrm{id} \otimes u)(x)\|_{L_p(\widetilde{\tau}^{(m)})}$ for any $x \in M_m(E)$), then u is completely isometric.

REMARK 3.8. Note that Theorem 3.6 is not a formal consequence of Theorem 3.7. Indeed, when $1 \notin E$, assuming (3.4) for any $x \in M_m(E)$ is stronger that assuming that $u : E \to L_p(\widetilde{\tau})$ is m-isometric, and is weaker than assuming that the map $\widetilde{u} : \mathrm{span}(1, E) \to L_p(\widetilde{\tau})$ that extends u by $\widetilde{u}(1) = 1$ is m-isometric. We therefore give a proof of the two results.

We first provide the proof of Theorem 3.7 which is simpler:

Proof of Theorem 3.7. Assume that u is m-isometric. By Lemma 3.3 it clearly suffices to prove that if $x_1, \dots, x_{2m} \in E$ and $y_i = u(x_i)$, then

$$\tau(x_1^* x_2 x_3^* x_4 \cdots x_{2m-1}^* x_{2m}) = \widetilde{\tau}(y_1^* y_2 y_3^* y_4 \cdots y_{2m-1}^* y_{2m}).$$

But this is easy to get if one takes $a_1 \cdots a_{2m} \in M_m$ satisfying (1.3) and one applies $\|x\|_{2m} = \|(\mathrm{id} \otimes u)(x)\|_{2m}$ to $x = x(z_1, \dots, z_{2m}) \in M_m(E)$ defined by the following, for any $(z_1, \dots, z_{2m}) \in \mathbb{C}^{2m}$:

$$x = \sum_{j=1}^{m} \overline{z_{2j-1}} a_{2j-1}^* \otimes x_{2j-1} + z_{2j} a_{2j} \otimes x_{2j-1}.$$

Indeed, $\|x\|_{2m}^{2m}$ is a polynomial in the complex numbers z_j and $\overline{z_j}$, the coefficient in front of $z_1 z_2 \cdots z_{2m}$ is $\tau(x_1^* x_2 x_3^* x_4 \cdots x_{2m-1}^* x_{2m})$. ∎

Proof of Theorem 3.6. Roughly, the idea of the proof is the same as the previous one: the $2m$ norm of $1 + \sum a_j \otimes x_j$, depends, as a function of the x_j's, only on a finite number of moments of the x_j's. And Lemma 2.2 shows that these moments can be computed from the $2m$-norm of $1 + y$ when y describes the set of $m \times m$ matrices with values in the linear space generated by the x_j's.

But the description of these particular moments is not as simple as in Theorem 3.7, and the computations are more complicate.

Take $x \in M_n(E)$, say $x = 1 + \sum_{j=1}^{N} a_j \otimes x_j$ where $a_j \in M_n$ and $x_j \in E$. Denote by $y_j = u(x_j)$. First compute

$$\|1 + x\|_{2m}^{2m} = \tau^{(n)}((1 + x + x^* + x^* x)^m).$$

The same kind of enumeration as in the proof of Lemma 2.1 shows that for any integer j,

$$\tau^{(n)}((x+x^*+x^*x)^j) = \sum_{k=j}^{2j} \sum_{(\varepsilon_1,\dots,\varepsilon_k)\in\{1,*\}^k} \frac{j}{k}\binom{\alpha(\varepsilon)}{k-j}\tau^{(m)}(x^{\varepsilon_1}x^{\varepsilon_2}\cdots x^{\varepsilon_k}).$$

Multiplying the above equation by $\binom{m}{j}$ and summing on j yields to

$$\begin{aligned}(3.5)\quad & \tau^{(n)}((1+x+x^*+x^*x)^m) \\ & = \sum_{k=0}^{2m} \sum_{\substack{\varepsilon_1,\dots,\varepsilon_k\in\{1,*\} \\ i_1,\dots,i_k\in\{1,\dots,N\}}} \mathrm{tr}_n(a_{i_1}^{\varepsilon_1}\cdots a_{i_k}^{\varepsilon_k})\tau(x_{i_1}^{\varepsilon_1}\cdots x_{i_k}^{\varepsilon_k}) \sum_{0\leqslant j\leqslant k}\frac{j}{k}\binom{m}{j}\binom{\alpha(\varepsilon)}{k-j}.\end{aligned}$$

But the assumption (3.4) together with Lemma 2.2 (and Remark 2.3) imply that for any $k \leqslant 2m$, any $\varepsilon \in \{1,*\}^k$ and any $(i_1,\dots,i_k) \in \{1,\dots,N\}^k$,

$$\tau(x_{i_1}^{\varepsilon_1}\cdots x_{i_k}^{\varepsilon_k}) \sum_{0\leqslant j\leqslant k}\frac{j}{k}\binom{m}{j}\binom{\alpha(\varepsilon)}{k-j} = \widetilde{\tau}(y_{i_1}^{\varepsilon_1}\cdots y_{i_k}^{\varepsilon_k}) \sum_{0\leqslant j\leqslant k}\frac{j}{k}\binom{m}{j}\binom{\alpha(\varepsilon)}{k-j}.$$

Remembering (3.5), we get that

$$\|1+x\|_{L_p(\tau^{(n)})} = \|1+u^{(n)}(x)\|_{L_p(\widetilde{\tau}^{(n)})}.$$

Since this holds for any n and any $x \in M_n(E)$, we have the desired conclusion. ∎

Now we discuss the case of $p \notin 2\mathbb{N}$. We are unable to determine whether $n_p < \infty$ (or $n_{p,1} < \infty$), but we are able to show that the assertion $n_{p,1} < \infty$ is related to an assertion concerning the $*$-distributions of single matricial operators, which we detail below.

If $(x_i)_{i\in I} \in \mathcal{M}^I$ and $(y_i)_{i\in I} \in \mathcal{N}^I$ are two families of operators in von Neumann algebras with n.f.f. traces, the same arguments as in Lemma 1.1 show that these families have the same $*$-distribution if for any integer n, and any (finitely supported) family $(a_i)_{i\in I} \in M_n^I$,

$$(3.6)\qquad *\text{-dist}\Big(\sum_{i\in I} a_i\otimes x_i\Big) = *\text{-dist}\Big(\sum_{i\in I} a_i\otimes y_i\Big).$$

It is also natural to ask: is there an integer n such that (3.6) for all $a_i \in M_n$ imply that (x_i) and (y_i) have the same $*$-distribution? In the same way as above, the smallest such integer will be denoted by N. (If such integer does not exist, take $N = \infty$.)

Since (3.6) implies that $\|1+\sum a_i\otimes x_i\|_p = \|\sum 1 + a_i \otimes y_i\|_p$, Theorem 3.1 shows that when p is not an even integer, $N \leqslant n_{p,1}$. To show that $n_{p,1} = \infty$, it would thus be enough to show $N = \infty$.

4. OTHER APPLICATIONS

In this section we prove some other consequences of Lemma 2.2 and Lemma 2.7. In particular we prove a noncommutative (weaker) version of Rudin's Theorem 0.2: Theorem 4.1. A result of the same kind (dealing with bounded operators only) and using the same ideas has already been developed in [11]. The main difference is that in [11], the author stays at the Banach space level (as opposed to the operator space level, i.e. he does not allow matrix coefficients in (4.1)).

THEOREM 4.1. *Let $(\mathcal{M},\tau)$ and $(\mathcal{N},\widetilde{\tau})$ be as in Theorem 0.3. Let $0<p<\infty$ and $p\neq 2,4$. Let $M\subset L_p(\mathcal{M},\tau)$ be a subalgebra (not necessarily self-adjoint) of $L_p(\mathcal{M},\tau)$ containing $1_{\mathcal{M}}$, and let $u: M\to L_p(\mathcal{N},\widetilde{\tau})$ be a linear map such that $u(1)=1$.*

Assume that $u^{(2)}=\mathrm{id}\otimes u : M_2(M)\to M_2(L_p(\mathcal{N},\widetilde{\tau}))$ is an isometry for the p-"norms":

$$\forall a\in M_2(M)\quad \|a\|_p=\|u^{(2)}(a)\|_p. \tag{4.1}$$

Assume moreover that $M\subset L_4(\mathcal{M},\tau)$.

Then for all $a,b\in M$

$$u(ab)=u(a)u(b).$$

Proof. The proof is based on Lemma 2.7. Theorem 2.11 implies that $u(M)\subset L_4(\mathcal{N},\widetilde{\tau})$.

If $a,b\in M$, note that

$$\begin{aligned}\|u(ab)-u(a)u(b)\|_2^2=&\widetilde{\tau}(u(b)^*u(a)^*u(a)u(b))+\widetilde{\tau}(u(ab)^*u(ab))\\ &-\widetilde{\tau}(u(b)^*u(a)^*u(ab))-\widetilde{\tau}(u(ab)^*u(a)u(b)).\end{aligned} \tag{4.2}$$

Apply Lemma 2.7 with $n=4$, $(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)=(*,*,1,1)$ (so that with the notation of Lemma 2.2, $\alpha=1$), $m=n/2=2$ and with $(x_1,x_2,x_3,x_4)=(b,a,a,b)$ on the one hand and $(x_1,x_2,x_3,x_4)=(u(b),u(a),u(a),u(b))$ on the other hand. One gets:

$$\tau(b^*a^*ab)\Big(4\binom{p/2}{4}+3\binom{p/2}{3}\Big)=\widetilde{\tau}(u(b)^*u(a)^*u(a)u(b))\Big(4\binom{p/2}{4}+3\binom{p/2}{3}\Big).$$

But $4\binom{p/2}{4}+3\binom{p/2}{3}=p^2(p/2-1)(p/2-2)/24\neq 0$ if $p\neq 0,2,4$. Thus,

$$\widetilde{\tau}(u(b)^*u(a)^*u(a)u(b))=\tau(b^*a^*ab).$$

The same argument yields to

$$\widetilde{\tau}(u(ab)^*u(a)u(b))=\tau((ab)^*ab)=\tau(b^*a^*ab),$$
$$\widetilde{\tau}(u(b)^*u(a)^*u(ab))=\tau(b^*a^*ab),\quad \widetilde{\tau}(u(ab)^*u(ab))=\tau(b^*a^*ab).$$

Thus, remembering (4.2), one gets

$$\|u(ab)-u(a)u(b)\|_2^2=0.$$

Since $\widetilde{\tau}$ is supposed to be faithful, this implies $u(ab)=u(a)u(b)$. ∎

For unital isometries defined on self-adjoint subspaces, the situation is also nice:

LEMMA 4.2. *Let* $1 \leqslant p < \infty$ *and* $p \neq 2$. *Let* $(\mathcal{M}, \tau)$ *and* $(\mathcal{N}, \widetilde{\tau})$ *be as in Theorem* 0.3. *Let* $E \subset L_p(\tau)$ *be a unital and self-adjoint subspace (i.e.* $x \in E \Rightarrow x^* \in E$*) and* $u : E \to L_p(\widetilde{\tau})$ *a unital isometric map.*

Then for any $x \in E$ *such that* $\|x\|_2 < \infty$, $u(x^*) = u(x)^*$.

Proof. The proof is of the same kind of the one above: take $x \in E \cap L_2(\mathcal{M})$, and first apply Theorem 2.8 to show that $\|u(x)\|_2, \|u(x^*)\|_2 < \infty$. Then the proof consists in applying Lemma 2.7 in order to prove that $\|u(x^*) - u(x)^*\|_2^2 = 0$. The details are not provided. ∎

When the unital completely isometric map u in Theorem 0.3 is defined on the whole L_p-space, we recover some very special cases of known results by Yeadon ([18], Theorem 2) for isometries and Junge, Ruan and Sherman ([9], Theorem 2) for 2-isometries:

THEOREM 4.3. *Let* $p \in \mathbb{R}^+$, $p \neq 2$. *Let* $u : L_p(\mathcal{M}, \tau) \to L_p(\mathcal{N}, \widetilde{\tau})$ *be a linear map such that* $u(1_{\mathcal{M}}) = 1_{\mathcal{N}}$.

(i) *If* $p \geqslant 1$ *or* u *maps self-adjoint operators to self-adjoint operators, and if* u *is isometric, then* u *maps* $\mathcal{M}$ *into* $\mathcal{N}$ *and preserves the trace, the adjoint and the Jordan product: for any* $a, b \in \mathcal{M}$

$$\widetilde{\tau}(u(a)) = \tau(a), u(a^*) = u(a)^* \quad \text{and} \quad u(ab + ba) = u(a)u(b) + u(b)u(a).$$

(ii) *If* u *is* 2*-isometric (i.e.* $u^{(2)}$ *is isometric) and* $p \neq 2$, *then the image of* $\mathcal{M}$ *is a von Neumann algebra and the restriction of* u *to* $\mathcal{M}$ *is a von Neumann algebra trace preserving isomorphism.*

Proof. We start by the first point. Take u as above. Note that by Lemma 4.2, if $p \geqslant 1$ then u preserves the adjoint.

For any $a \in \mathcal{M}$ such that $a^* = a$, apply the commutative Theorem 0.2 to the unital isometric map u from the commutative unital subalgebra of $L_p(\mathcal{M})$ generated by a into the commutative unital subalgebra of $L_p(\mathcal{N})$ generated by $u(a)$. One gets that $\|u(a)\|_\infty < \infty$ and that $u(a^2) = u(a)^2$ for any self-adjoint $a \in M$. By polarization, this implies that for any self-adjoint $a, b \in M$,

$$u(ab + ba) = u(a)u(b) + u(b)u(a).$$

By linearity this equality extends to any $a, b \in M$, and $\|a\|_\infty < \infty$. The fact that u preserves the trace is an application of Lemma 2.7 with $n = 1$.

Assume now that u is 2-isometric (with $0 < p \neq 2 < \infty$). By Theorem 2.11 and Lemma 4.2, u (and hence $u^{(2)}$) preserves the adjoint map. We can apply the isometric case above for $u^{(2)}$. Thus $u^{(2)}$ is a trace preserving Jordan map. If

$a, b \in \mathcal{M}$, the equation $u^{(2)}(\widetilde{a}\widetilde{b} + \widetilde{b}\widetilde{a}) = u^{(2)}(\widetilde{a})u^{(2)}(\widetilde{b}) + u^{(2)}(\widetilde{b})u^{(2)}(\widetilde{a})$ for

$$\widetilde{a} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{b} = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}$$

yields to $u(ab) = u(a)u(b)$ (and $u(ba) = u(b)u(a)$). Thus u is a $*$-isomorphism from the von Neumann algebra $\mathcal{M}$ onto its image, and it preserves the trace. This implies, by Lemma 3.3, that $u(\mathcal{M})$ is a von Neumann subalgebra of $\mathcal{N}$ and that u is a von Neumann algebra isomorphism. ∎

4.1. APPLICATIONS TO NONCOMMUTATIVE H^p-SPACES. The main result also applies in the setting of noncommutative H^p-spaces (see [2]).

DEFINITION 4.4. Let $(\mathcal{M}, \tau)$ be, as above, a von Neumann algebra with a faithful normal normalized trace. A tracial subalgebra A of $(\mathcal{M}, \tau)$ is a weak-$*$ closed unital subalgebra such that the conditional expectation $\phi_A : \mathcal{M} \to A \cap A^*$ satisfies $\phi_A(ab) = \phi_A(a)\phi_A(b)$ for any $a, b \in A$.

A noncommutative H^p-space is the closure, denoted $[A]_p$, of a tracial subalgebra A in $L_p(\mathcal{M}, \tau)$.

Since by definition, a noncommutative H^p-space is a unital subspace of $L_p(\mathcal{M}, \tau)$ in which the subset of bounded operators is dense, Theorem 0.3 automatically implies the following, which gives a beginning of answer to a question raised in [2] and [11]:

THEOREM 4.5. *Let $p \notin 2\mathbb{N}$.*

A unital complete isometry between noncommutative H^p-spaces extends to an isomorphism between the von Neumann algebras they generate.

Moreover, if a non commutative H^p-space is unitally completely isometric to a unital subspace E of a noncommutative L_p-space, then E is a noncommutative H^p-space.

For a 2-isometric map between noncommutative H^p-spaces, we also get something:

THEOREM 4.6. *Let $p \in \mathbb{R}$, $p \neq 2, 4$.*

Let u be a 2-isometric and unital map from a noncommutative H^p-space $[A]_p$ into a noncommutative L_p-space $L_p(\mathcal{N}, \widetilde{\tau})$.

Then the image of A is a subalgebra B of $L_p(\mathcal{N})$ such that for any $a, b \in A$,

$$u(ab) = u(a)u(b).$$

Moreover, $B \cap B^* = u(A \cap A^*)$ is a von Neumann algebra such that the restriction to B of the conditional expectation $\Phi_B : L_p(\mathcal{N}) \to L_p(B \cap B^*)$ satisfies, for any $a \in M$

$$\Phi_B(u(a)) = u(\Phi_A(a)) \in N \cap B^*.$$

In the case when p is not an even integer, B is contained in $\mathcal{N}$.

Proof. The fact that $B \stackrel{\text{def}}{=} u(A)$ is contained in $\mathcal{N}$ when $p \notin 2\mathbb{N}$ follows from Theorem 2.11.

Theorem 4.1 implies that B is an algebra and that $u(ab) = u(a)u(b)$.

The fact that $B \cap B^* = u(A \cap A^*)$ is immediate from Lemma 4.2, and Theorem 4.3 shows that $u(A \cap A^*)$ is a von Neumann algebra.

Recall that the conditional expectation $\Phi_B : \mathcal{N} \to B \cap B^*$ coincides with the orthogonal projection from $L_2(\mathcal{N})$ to $L_2(B \cap B^*)$. To check the last equation

$$\Phi_B(u(a)) = u(\Phi_A(a))$$

we thus have to show that for any $x \in B \cap B^*$,

$$\widetilde{\tau}(xu(a)) = \widetilde{\tau}(xu(\Phi_A(a))).$$

Write $x = u(b)$ for $b \in \mathcal{M}$. The above equation arises from the definition of $\Phi_A(a)$, from the multiplicativity of u and from the fact that, by Lemma 2.7, $\widetilde{\tau} \circ u = \tau$:

$$\widetilde{\tau}(xu(a)) = \widetilde{\tau}(u(ba)) = \tau(ba) = \tau(b\Phi_A(a)) = \widetilde{\tau}(u(b\Phi_A(a))) = \widetilde{\tau}(xu(\Phi_A(a))).$$

This concludes the proof. ∎

We end this paper with some additional remarks and questions related Yeadon's result. The main theorem of [18] in particular contains the following:

LEMMA 4.7. *Let u be an isometry from an L_p-space $L_p(\mathcal{M})$ constructed on a von Neumann algebra $(\mathcal{M}, \tau)$ with a n.f.f. trace to an L_p-space $L_p(\mathcal{N})$ constructed on a von Neumann algebra $(\mathcal{N}, \sigma)$ with a normal semifinite faithful trace.*

Then if $u(1)$=b is positive, then b commutes with $u(L_p(\mathcal{M}))$ and has full support.

REMARK 4.8. This also holds if $\mathcal{N}$ does not carry a semifinite trace and $L_p(\mathcal{N})$ is Haagerup's generalized L_p-space (see Theorem 3.1 of [9]).

This fact allows to reduce the general case to the unital case. Here are the details: if one denotes by $b = u(1) \geqslant 0 \in L_p(\mathcal{M})$, by $s \in \mathcal{N}$ the support projection of b, and by $\widetilde{\mathcal{N}}$ the von Neumann subalgebra of $s\mathcal{N}s$ generated by $b^{-1}u(\mathcal{M})$, then $\widetilde{\mathcal{N}}$ carries a n.f.f. trace given by

$$\widetilde{\tau}(x) = \begin{cases} \sigma(b^p x) & \text{if } \sigma \text{ was a semifinite trace on } \mathcal{N}, \\ \mathrm{tr}(b^p x) & \text{in Haagerup's construction,} \end{cases}$$

(in Haagerup's construction, tr is the trace functional on $L_1(\mathcal{N})$).

The assumption that u is an isometry exactly means that (the unital linear map) $b^{-1} \cdot u$ is an isometry from $L_p(\mathcal{M}, \tau)$ to $L_p(\widetilde{\mathcal{N}}, \widetilde{\tau})$.

Thus Yeadon's Lemma 4.7 (respectively with the preceding remark) and Theorem 4.3 of this paper are enough to recover Yeadon's result (respectively Junge, Ruan and Sherman's result with the restriction that the first L_p-space be semifinite). Of course, all this is not surprising at all since Lemma 4.7 contains most of the results from [18], and we therefore do not provide more details. But this leads naturally to the question: to what extend Lemma 4.7 can be generalized when u is only defined on a unital subspace of $L_p(\mathcal{M}, \tau)$?

As justified above, it is natural to wonder whether the same result holds for isometries between subspaces of noncommutative L_p-spaces. More precisely, let $1 \in E \in L_p(\mathcal{M}, \tau)$ be a unital subspace of a noncommutative L_p-space with τ a n.f.f. trace. Let $u : E \to L_p(\mathcal{N})$ be an isometry between E and a subspace of an arbitrary noncommutative L_p-space such that $u(1) \geqslant 0$. Then is it true that $u(1)$ commutes with $u(E)$ and has full support in $u(E)$? (that is: if s is the support projection of $u(1)$, then $su(x) = u(x)s = u(x)$ for any $x \in E$). As noted above, this would allow to use all the results of this paper for u and would have several interesting consequences.

It should be noted that Yeadon's proof (as well as the generalization in [9]) consists in applying the equality condition in Clarkson's inequality for projections with disjoint supports. This of course is not possible for a general subspace of $L_p(\mathcal{M})$ since it may not contain any nontrivial projection.

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