J. OPERATOR THEORY
64:2(2010), 453–468

SUBFACTORS AND HADAMARD MATRICES

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Communicated by Şerban Strătilă

ABSTRACT. To any complex Hadamard matrix H one associates a spin model commuting square, and therefore a hyperfinite subfactor. The standard invariant of this subfactor captures certain "group-like" symmetries of H. To gain some insight, we compute the first few relative commutants of such subfactors for Hadamard matrices of small dimensions. Also, we show that subfactors arising from Dita–Haagerup type matrices have intermediate subfactors, and thus their standard invariants have some extra structure besides the Jones projections.

KEYWORDS: *Commuting squares, complex Hadamard matrices, subfactors, von Neumann algebras, Jones projection.*

MSC (2000): 46L10, 46L37, 15A57, 05B20.

1. INTRODUCTION

A complex Hadamard matrix is a matrix $H \in M_n(\mathbb{C})$ having all entries of absolute value 1 and all rows mutually orthogonal. Equivalently, $\frac{1}{\sqrt{n}}H$ is a unitary matrix with all entries of the same absolute value. For example, the *Fourier matrix* $F_n = (\omega^{ij})_{1\leqslant i,j\leqslant n}$, $\omega = \mathrm{e}^{2\pi \mathrm{i}/n}$, is a Hadamard matrix.

In the recent years, complex Hadamard matrices have found applications in various topics of mathematics and physics, such as quantum information theory, error correcting codes, cyclic n-roots, spectral sets and Fuglede's conjecture. A general classification of real or complex Hadamard matrices is not available. A catalogue of most known complex Hadamard matrices can be found in [18]. The complete classification is known for $n \leqslant 5$ ([7]) and for self-adjoint matrices of order 6 ([4]).

The connection between Hadamard matrices and von Neumann algebras arose from an observation of Popa ([15]): a unitary matrix U is of the form $\frac{1}{\sqrt{n}}H$, H Hadamard matrix, if and only if the algebra of $n \times n$ diagonal matrices $\mathcal{D}_n$ is orthogonal onto $U\mathcal{D}_n U^*$, with respect to the inner product given by the trace on $M_n(\mathbb{C})$. Equivalently, the square of inclusions:

$$\mathfrak{C}(H) = \begin{pmatrix} \mathcal{D}_n & \subset & M_n(\mathbb{C}) & \\ \cup & & \cup & ,\tau \\ \mathbb{C} & \subset & U\mathcal{D}_n U^* & \end{pmatrix}$$

is a *commuting square*, in the sense of [16], [15]. Here τ denotes the trace on $M_n(\mathbb{C})$, normalized such that $\tau(1) = 1$.

Such commuting squares are called *spin models*, the name coming from statistical mechanical considerations (see [11]). By iterating Jones' basic construction, one can construct a hyperfinite, index n subfactor from H (see for instance [11]). The subfactor associated to H can be used to capture some of the symmetries of H, and thus to classify H to a certain extent (see [1], [10], [3]).

Let $N \subset M$ be an inclusion of II_1 factors of finite index, and let $N \subset M \overset{e_1}{\subset} M_1 \overset{e_2}{\subset} M_2 \subset \cdots$ be the tower of factors constructed by iterating Jones' basic construction (see [9]), where $e_1, e_2, \ldots$ denote the Jones projections. The *standard invariant* $\mathcal{G}_{N,M}$ is then defined as the trace preserving isomorphism class of the following sequence of *commuting squares* of inclusions of finite dimensional $*$-algebras:

$$\begin{array}{ccccccccc} \mathbb{C} = N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & N' \cap M_2 & \subset & \cdots \\ & & \cup & & \cup & & \cup & & \\ & & M' \cap M & \subset & M' \cap M_1 & \subset & M' \cap M_2 & \subset & \cdots \end{array}.$$

The Jones projections $e_1, e_2, \ldots, e_n$ are always contained in $N' \cap M_n$. If the index of the subfactor $N \subset M$ is at least 4, they generate the Temperley–Lieb algebra of order n, denoted TL_n. In a lot of situations the relative commutant $N' \cap M_n$ has some interesting extra structure, besides TL_n. For instance, the five non-equivalent real Hadamard matrices of order 16 yield different dimensions for the second relative commutant $N' \cap M_1$, and thus are classified by these dimensions ([1]).

In this paper we investigate the relation between Hadamard matrices and their subfactors. We look at Hadamard matrices of small dimensions or of special types. The paper is organized as follows: in the second section we recall, in our present framework, several results of [10], [11] regarding computations of standard invariants for spin models.

In the third section we study the subfactors associated to Hadamard matrices of Dita–Haagerup type. These are matrices that arise from a construction of [5], which is a generalization of a construction of Haagerup ([7]). Most known parametric families of Hadamard matrices are of Dita type. We show that the associated subfactors have intermediate subfactors.

In the last section we present a list of computations of the second and third relative commutants $N' \cap M_1$, $N' \cap M_2$, for complex Hadamard matrices of small dimensions. We make several remarks and conjectures regarding the structure of the standard invariant. Most of the computations included were done using computers, with the help of the Mathematica and GAP software.

2. SUBFACTORS ASSOCIATED TO HADAMARD MATRICES

Let H be a complex $n \times n$ Hadamard matrix and let $U = \frac{1}{\sqrt{n}}H$. U is a unitary matrix, with all entries of the same absolute value. One associates to U the square of inclusions:

$$\mathfrak{C}(H) = \begin{pmatrix} \mathcal{D}_n & \subset & M_n(\mathbb{C}) & \\ \cup & & \cup & ,\tau \\ \mathbb{C} & \subset & U\mathcal{D}_nU^* & \end{pmatrix}$$

where $\mathcal{D}_n$ is the algebra of diagonal $n \times n$ matrices and τ is the trace on $M_n(\mathbb{C})$, normalized such that $\tau(1) = 1$.

Since H is a Hadamard matrix, $\mathfrak{C}(H)$ is a *commuting square* in the sense of [16], [15], i.e. $E_{\mathcal{D}_n}E_{U\mathcal{D}_nU^*} = E_{\mathbb{C}}$. The notation E_A refers to the τ-invariant conditional expectation from $M_n(\mathbb{C})$ onto the $*$-subalgebra A.

Recall that two complex Hadamard matrices are said to be *equivalent* if there exist unitary diagonal matrices D_1, D_2 and permutation matrices P_1, P_2 such that $H_2 = P_1D_1H_1D_2P_2$. It is easy to see that H_1, H_2 are equivalent if and only if $\mathfrak{C}(H_1), \mathfrak{C}(H_2)$ are isomorphic as commuting squares, i.e. conjugate by a unitary from $M_n(\mathbb{C})$.

We denote by $\mathfrak{C}^{\mathrm{t}}(H)$ the commuting square obtained by flipping the upper left and lower right corners of $\mathfrak{C}(H)$:

$$\mathfrak{C}^{\mathrm{t}}(H) = \begin{pmatrix} U\mathcal{D}_nU^* & \subset & M_n(\mathbb{C}) & \\ \cup & & \cup & ,\tau \\ \mathbb{C} & \subset & \mathcal{D}_n & \end{pmatrix}.$$

We have: $\mathfrak{C}^{\mathrm{t}}(H) = \mathrm{Ad}(U)\mathfrak{C}(H^*)$. Thus, $\mathfrak{C}^{\mathrm{t}}(H)$ and $\mathfrak{C}(H)$ are isomorphic as commuting squares if and only if H, H^* are equivalent as Hadamard matrices.

We now recall the construction of a subfactor from a commuting square. By iterating Jones' basic construction ([9]), one obtains from $\mathfrak{C}^{\mathrm{t}}(H)$ a tower of commuting squares of finite dimensional $*$-algebras:

$$\begin{matrix} U\mathcal{D}_nU^* & \subset & M_n(\mathbb{C}) & \overset{g_3}{\subset} & \mathcal{X}_1 & \overset{g_4}{\subset} & \mathcal{X}_2 & \overset{g_5}{\subset} & \cdots \\ \cup & & \cup & & \cup & & \cup & & \\ \mathbb{C} & \subset & \mathcal{D}_n & \overset{g_3}{\subset} & \mathcal{Y}_1 & \overset{g_4}{\subset} & \mathcal{Y}_2 & \overset{g_5}{\subset} & \cdots \end{matrix} \tag{2.1}$$

together with the extension of the trace, which we will still denote by τ, and Jones projections $g_{i+2} \in \mathcal{Y}_i$, $i = 1, 2, \ldots$.

Let M_H be the weak closure of $\bigcup_i \mathcal{X}_i$, with respect to the trace τ, and let N_H be the weak closure of $\bigcup_i \mathcal{Y}_i$. N_H, M_H are hyperfinite II_1 factors, and the trace τ

extends continuously to the trace of M_H, which we will still denote by τ. It is well known that $N_H \subset M_H$ is a subfactor of index n, which we will call the subfactor associated to the Hadamard matrix H.

The standard invariant of $N_H \subset M_H$ can be expressed in terms of commutants of finite dimensional algebras, by using Ocneanu's compactness argument (5.7 in [11]). Consider the basic construction for the commuting square $\mathfrak{C}(H)$:

$$\begin{array}{ccccccccc} \mathcal{D}_n & \subset & M_n(\mathbb{C}) & \overset{e_3}{\subset} & \mathcal{P}_1 & \overset{e_4}{\subset} & \mathcal{P}_2 & \overset{e_5}{\subset} & \cdots \\ \cup & & \cup & & \cup & & \cup & & \\ \mathbb{C} & \subset & U\mathcal{D}_nU^* & \overset{e_3}{\subset} & \mathcal{Q}_1 & \overset{e_4}{\subset} & \mathcal{Q}_2 & \overset{e_5}{\subset} & \cdots \end{array} . \tag{2.2}$$

Ocneanu's compactness theorem asserts that the first row of the standard invariant of $N_H \subset M_H$ is the row of inclusions:

$$\mathcal{D}_n' \cap U\mathcal{D}_nU^* \subset \mathcal{D}_n' \cap \mathcal{Q}_1 \subset \mathcal{D}_n' \cap \mathcal{Q}_2 \subset \mathcal{D}_n' \cap \mathcal{Q}_3 \subset \cdots .$$

More precisely, if

$$N_H \subset M_H \overset{e_3}{\subset} M_{H,1} \overset{e_4}{\subset} M_{H,2} \overset{e_5}{\subset} \cdots$$

is the Jones tower obtained from iterating the basic construction for the inclusion $N_H \subset M_H$, then:

$$D_n' \cap \mathcal{Q}_i = N_H' \cap M_{H,i}, \quad \text{for all } i \geqslant 1.$$

Thus, the problem of computing the standard invariant of the subfactor associated to H is equivalent to the computation of $\mathcal{D}_n' \cap \mathcal{Q}_i$. However, such computations seem very hard, and even for small i and for matrices H of small dimensions they seem to require computer use. Jones ([10]) provided a diagrammatic description of the relative commutants $\mathcal{D}_n' \cap \mathcal{Q}_i$ (see also [11]), which we express below in the framework of this paper.

Let $\mathcal{P}_0 = M_n(\mathbb{C})$ and let $(e_{i,j})_{1\leqslant i,j\leqslant n}$ be its canonical matrix units. Let

$$e_2 = \frac{1}{n}\sum_{i,j=1}^{n} e_{i,j}\,.$$

It is easy to check that e_2 is a projection. Moreover: $\langle \mathcal{D}_n, e_2\rangle = M_n(\mathbb{C})$ and $e_2xe_2 = E_{\mathbb{C}}(x)e_2$ for all $x \in \mathcal{D}_n$. Thus, e_2 is realizing the basic construction

$$\mathbb{C} \subset \mathcal{D}_n \overset{e_2}{\subset} M_n(\mathbb{C}).$$

Let $e_{k,l} \otimes e_{i,j}$ denote the $n^2 \times n^2$ matrix having only one non-zero entry, equal to 1, at the intersection of row $(i-1)n+k$ and column $(j-1)n+l$. Thus, $e_{k,l} \otimes e_{i,j}$ are matrix units of $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$. In what follows, we will assume that the embedding of $M_n(\mathbb{C})$ into $M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ is realized as $e_{k,l} \to e_{k,l} \otimes I_n$, where $e_{k,l} \otimes I_n = \sum_{i=1}^{n} e_{k,l} \otimes e_{i,i}$.

LEMMA 2.1. *Let* $\mathcal{P}_1 = M_n(\mathbb{C}) \otimes \mathcal{D}_n$, $\mathcal{P}_2 = M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$, $e_3 = \sum_{i=1}^{n} e_{ii} \otimes e_{ii} \in \mathcal{P}_1$ *and* $e_4 = I_n \otimes e_2 \in \mathcal{P}_2$. *Then*

$$\mathcal{D}_n \subset M_n(\mathbb{C}) \overset{e_3}{\subset} \mathcal{P}_1$$

is a basic construction with Jones projection e_3 *and*

$$M_n(\mathbb{C}) \subset \mathcal{P}_1 \overset{e_4}{\subset} \mathcal{P}_2$$

is a basic construction with Jones projection e_4.

Proof. To show that $\mathcal{D}_n \subset M_n(\mathbb{C}) \overset{e_3}{\subset} \mathcal{P}_1$ is a basic construction it is enough to check that $\langle M_n(\mathbb{C}), e_3\rangle = \mathcal{P}_1$ and e_3 is implementing $E^{\mathcal{P}_1}_{M_n(\mathbb{C})}$. First part is clear, since $e_{k,i}e_3e_{i,l} = e_{k,l} \otimes e_{i,i}$ are a basis for $\mathcal{P}_1 = M_n(\mathbb{C}) \otimes \mathcal{D}_n$. To check that e_3 implements the conditional expectation, let $X = (x_{i,j}) \in M_n(\mathbb{C})$. We have:

$$\begin{aligned} e_3(X \otimes I_n)e_3 &= \sum_{i,j=1}^{n} (e_{ii} \otimes e_{ii})(X \otimes I_n)(e_{jj} \otimes e_{jj}) \\ &= \sum_{i=1}^{n} e_{ii}Xe_{ii} \otimes e_{ii} = \sum_{i=1}^{n} (e_{ii}Xe_{ii} \otimes I_n)e_3 = E_{\mathcal{D}_n \otimes I_n}(X)e_3. \end{aligned} \tag{2.3}$$

Since $\mathbb{C} \subset \mathcal{D}_n \overset{e_2}{\subset} M_n(\mathbb{C})$ is a basic construction, after tensoring to the left by $M_n(\mathbb{C})$ it follows that $M_n(\mathbb{C}) \subset \mathcal{P}_1 \overset{e_4}{\subset} \mathcal{P}_2$ is a basic construction, with $e_4 = I_n \otimes e_2$. ∎

PROPOSITION 2.2. *The algebras* $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \ldots$ *constructed in* (2.2) *are given by*

$$\mathcal{P}_{2k} = \bigotimes_{i=1}^{k+1} M_n(\mathbb{C}), \quad \mathcal{P}_{2k+1} = \mathcal{P}_{2k} \otimes \mathcal{D}_n,$$

with the Jones projections

$$e_{2k+2} = \bigotimes_{i=1}^{k} I_n \otimes e_2, \quad e_{2k+3} = \bigotimes_{i=1}^{k} I_n \otimes e_3.$$

Proof. Follows from the previous lemma, by tensoring successively by $M_n(\mathbb{C})$. ∎

PROPOSITION 2.3. *Let H be a complex $n \times n$ Hadamard matrix, let* $U = \frac{1}{\sqrt{n}} H$, *and*

$$D_U = \sqrt{n} \sum_{i,j=1}^{n} \overline{u}_{i,j} e_{j,j} \otimes e_{i,i}, \quad U_1 = UD_U.$$

Then the algebras $\mathcal{Q}_1, \mathcal{Q}_2, \mathcal{Q}_3, \ldots$ *constructed in* (2.2) *are given by*

$$\mathcal{Q}_k = U_k \mathcal{P}_{k-1} U_k^*, \quad k \geqslant 1,$$

where $U_k \in \mathcal{P}_k$ are the unitary elements:

$$U_{2k+1} = \prod_{i=0}^{k} (\otimes^i I_n \otimes U_1 \otimes^{k-i} I_n), \quad U_{2k} = U_{2k-1}(\otimes^k I_n \otimes U), \quad k \geqslant 1.$$

Proof. The unitary U_1 satisfies:

$$(\mathrm{Ad}U_1)(\mathcal{D}_n) = (\mathrm{Ad}U)(\mathcal{D}_n)$$

since $U^* U_1 = D_U \in \mathcal{D}_n$. Moreover, we have:

$$\begin{aligned}(\mathrm{Ad}U_1)(e_2) &= (\mathrm{Ad}U)\mathrm{Ad}\Big(\sum_{i,j=1}^{n} \overline{u}_{i,j} e_{j,j} \otimes e_{i,i}\Big)\Big(\frac{1}{n}\sum_{k,l=1}^{n} e_{k,l}\Big) \\ &= (\mathrm{Ad}U)\Big(\sum_{i,k,l=1}^{n} \overline{u}_{i,k} u_{i,l} e_{k,l} \otimes e_{i,i}\Big) = (\mathrm{Ad}U)(\mathrm{Ad}U^*(e_3)) = e_3.\end{aligned} \tag{2.4}$$

It follows that $\mathrm{Ad}U_1$ takes the basic construction $\mathbb{C} \subset \mathcal{D}_n \overset{e_2}{\subset} M_n(\mathbb{C})$ onto the inclusion $\mathbb{C} \subset U\mathcal{D}_n U^* \overset{e_3}{\subset} U_1 M_n(\mathbb{C}) U_1^*$. Thus this is also a basic construction, which shows that $\mathcal{Q}_1 = U_1 M_n(\mathbb{C}) U_1^*$. Moreover, it follows that each $\mathrm{Ad}U_i$ takes the basic construction $\mathcal{P}_{i-1} \subset \mathcal{P}_i \subset \mathcal{P}_{i+1}$ onto $\mathcal{Q}_i \subset \mathcal{Q}_{i+1} \subset \mathcal{Q}_{i+2}$, which ends the proof. ∎

The first relative commutant $\mathcal{D}_n' \cap U\mathcal{D}_n U^*$ is equal to $\mathbb{C}$, since the commuting square condition implies $\mathcal{D}_n \cap U\mathcal{D}_n U^* = \mathbb{C}$. Thus the subfactor $N_H \subset M_H$ is irreducible. In the following proposition we realize the higher relative commutants of the subfactor $N_H \subset M_H$ as the commutants of some matrices P_i, $i \geqslant 1$, in the algebras $\mathcal{D}_n' \cap \mathcal{P}_i$.

PROPOSITION 2.4. *With the previous notations, let P_i denote the projection $U_i e_{i+3} U_i^* \in \mathcal{P}_{i+1}$, $i \geqslant 1$. Then we have the following formula for the $(i+1)$-th relative commutant:*

$$\mathcal{D}_n' \cap \mathcal{Q}_i = P_i' \cap (\mathcal{D}_n' \cap \mathcal{P}_i).$$

Proof. We have:

$$\begin{aligned}\mathcal{D}_n' \cap \mathcal{Q}_i &= \mathcal{D}_n' \cap \mathrm{Ad}U_i(\mathcal{P}_{i-1}) = \mathcal{D}_n' \cap \mathrm{Ad}U_i(e_{i+3}' \cap \mathcal{P}_i) \\ &= \mathcal{D}_n' \cap P_i' \cap \mathrm{Ad}U_i(\mathcal{P}_i) = \mathcal{D}_n' \cap P_i' \cap \mathcal{P}_i.\end{aligned} \tag{2.5}$$

We used the fact that $\mathcal{P}_{i-1} \subset \mathcal{P}_i \overset{e_{i+3}}{\subset} \mathcal{P}_{i+1}$ is a basic construction, and thus $e_{i+3}' \cap \mathcal{P}_i = \mathcal{P}_{i-1}$. ∎

REMARK 2.5. The $n^2 \times n^2$ matrix $P_1 = U_1 e_4 U_1^*$ can be written as

$$P_1 = \sum_{a,b,c,d=1}^{n} p_{a,b}^{c,d} e_{a,b} \otimes e_{c,d}, \quad \text{where } p_{a,b}^{c,d} = \sum_{i=1}^{n} u_{a,i} \overline{u}_{b,i} \overline{u}_{c,i} u_{d,i}.$$

This matrix is used in the theory of Hadamard matrices and it is called the *profile* of H. It is a result of Jones ([10]) that the matrices P_{2i+1}, $i \geqslant 1$, depend only on P_1.

Indeed, one can check that

$$P_{2i+1} = \sum_{k_1,l_1,\ldots,k_i,l_i=1}^{n} p_{a,b}^{k_1,l_1} p_{k_1,l_1}^{k_2,l_2} \cdots p_{k_i,l_i}^{c,d} e_{a,b} \otimes e_{k_1,l_1} \otimes e_{k_2,l_2} \otimes \cdots \otimes e_{k_i,l_i} \otimes e_{c,d}.$$

Thus, all higher relative commutants of even orders are determined by P_1.

Let Γ_H denote the graph of vertices $\{1,2,\ldots,n\} \times \{1,2,\ldots,n\}$, in which the distinct vertices (a,c) and (b,d) are connected if and only if $p_{a,b}^{c,d} \neq 0$. The second relative commutant can be easily described in terms of Γ_H. We recall this in the following proposition, which is a reformulation of a result in [10] (see also [11]).

PROPOSITION 2.6. *The second relative commutant of the subfactor $N_H \subset M_H$ is Abelian, its minimal projections are in bijection with the connected components of Γ_H, and their traces are proportional to the sizes of the connected components.*

Proof. Let $\sum\limits_{i,j=1}^{n} \lambda_i^j e_{i,i} \otimes e_{j,j}$, $\lambda_i^j \in \{0,1\}$, be a projection in the second relative commutant $P_1' \cap (\mathcal{D}_n \otimes \mathcal{D}_n)$. We have:

$$\Big(\sum_{a,b,c,d=1}^{n} p_{a,b}^{c,d} e_{a,b} \otimes e_{c,d}\Big)\Big(\sum_{i,j=1}^{n} \lambda_i^j e_{i,i} \otimes e_{j,j}\Big) = \Big(\sum_{i,j=1}^{n} \lambda_i^j e_{i,i} \otimes e_{j,j}\Big)\Big(\sum_{a,b,c,d=1}^{n} p_{a,b}^{c,d} e_{a,b} \otimes e_{c,d}\Big).$$

Equivalently:

$$\sum_{a,c,i,j=1}^{n} \lambda_i^j p_{q,i}^{c,j} e_{a,i} \otimes e_{c,j} = \sum_{b,d,i,j=1}^{n} \lambda_i^j p_{i,b}^{j,d} e_{i,b} \otimes e_{j,d}.$$

By relabeling and identifying the set of indices, it follows:

$$(\lambda_a^c - \lambda_i^j) p_{a,i}^{c,j} = 0.$$

Thus, if the vertices (a,c) and (i,j) are connected then $\lambda_a^c = \lambda_i^j$. This ends the proof. ∎

3. SUBFACTORS ARISING FROM DITA–HAAGERUP MATRICES

In this section we investigate the standard invariant of subfactors associated to a particular class of Hadamard matrices, obtained by a construction of P. Dita ([5]), which is a generalization of an idea of U. Haagerup ([7]). These matrices have a lot of symmetries, and we show that for such matrices the second relative commutant has some extra structure besides the Jones projection.

Let n be non-prime, $n = km$ with $k, m \geqslant 2$. Let $A = (a_{i,j}) \in M_k(\mathbb{C})$ and $B_1, \ldots, B_k \in M_m(\mathbb{C})$ be complex Hadamard matrices. It is possible to construct an $n \times n$ Hadamard matrix from $A, B_1, \ldots, B_k$ by using an idea of [5] (see also [7], [14]). This construction is a generalization of the tensor product of two Hadamard matrices:

$$H = \begin{pmatrix} a_{1,1}B_1 & a_{1,2}B_2 & \cdots & a_{1,k}B_k \\ a_{2,1}B_1 & a_{2,2}B_2 & \cdots & a_{2,k}B_k \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{k,1}B_1 & a_{k,2}B_2 & \cdots & a_{k,k}B_k \end{pmatrix}. \tag{3.1}$$

Let $(f_{i,j})_{1\leqslant i,j\leqslant k}$ be the matrix units of $M_k(\mathbb{C})$. We identify $M_n(\mathbb{C})$ with the tensor product $M_m(\mathbb{C}) \otimes M_k(\mathbb{C})$, with the same conventions as before. Thus:

$$H = \sum_{i,j=1}^{k} a_{i,j} B_j \otimes f_{i,j}.$$

One can use construct multi-parametric families of non-equivalent Hadamard matrices, by replacing $B_1, \dots, B_k$ by $B_1 D_1, \dots, B_k D_k$, where $D_1, \dots, D_k$ are diagonal unitaries. Some of the families of Hadamard matrices of small orders considered in the next section arise from this construction.

Recall that the second relative commutant always contains the Jones projection $e_3 = \sum e_{ii} \otimes e_{ii}$. In the next proposition we show that the second relative commutant of a Dita type subfactor contains another projection $f \geqslant e_3$, so it has dimension at least 3.

PROPOSITION 3.1. *Let $H = (a_{i,j}B_j)_{1\leqslant i,j\leqslant k} \in M_n(\mathbb{C})$ be a Dita type matrix, where $A = (a_{i,j})_{1\leqslant i,j\leqslant k} \in M_k(\mathbb{C})$ and $B_1, \dots, B_k \in M_m(\mathbb{C})$ are complex Hadamard matrices, $n = mk$. Then the second relative commutant of the subfactor associated to H contains the projection:*

$$f = \sum_{1\leqslant i,j\leqslant n,\ i\equiv j (\mathrm{mod} m)} e_{i,i} \otimes e_{j,j} \in M_{n^2}(\mathbb{C}).$$

Proof. For $1 \leqslant i \leqslant n$, let $i_0 = (i-1)(\mathrm{mod\ m}) + 1$ and $i_1 = \frac{i-i_0}{m} + 1$. We will use similar notations for $1 \leqslant j \leqslant n$. Thus, the (i,j) entry of H is:

$$h_{i,j} = a_{i_1,j_1} b^{j_1}_{i_0,j_0},$$

where $b^t_{r,s}$ is the (r,s) entry of B_t, for all $1 \leqslant t \leqslant k$, $1 \leqslant r,s \leqslant m$.

With these notations, the projection f can be written as

$$f = \sum_{i,j=1}^{n} \lambda^j_i e_{i,i} \otimes e_{j,j},$$

where $\lambda^j_i = 1$ if $i_0 = j_0$ and $\lambda^j_i = 0$ for all other i,j.

According to Proposition 2.6, showing that f is in the second relative commutant is equivalent to showing that $p^{j,d}_{i,c} = 0$ whenever $c_0 \neq d_0$. Using the formula for the entries of P_1 and the fact that $i_0 = j_0$ we obtain:

$$
\begin{aligned}
p_{i,c}^{j,d} &= \sum_{x=1}^{n} u_{i,x}\overline{u}_{c,x}\overline{u}_{j,x}u_{d,x} = \frac{1}{n^2}\sum_{x=1}^{n} h_{i,x}\overline{h}_{c,x}\overline{h}_{j,x}h_{d,x} \\
&= \frac{1}{n^2}\sum_{x=1}^{n} a_{i_1,x_1}b_{i_0,x_0}^{x_1}\overline{a}_{c_1,x_1}\overline{b}_{c_0,x_0}^{x_1}\overline{a}_{j_1,x_1}\overline{b}_{j_0,x_0}^{x_1}a_{d_1,x_1}b_{d_0,x_0}^{x_1} \\
&= \frac{1}{n^2}\sum_{x=1}^{n} a_{i_1,x_1}\overline{a}_{c_1,x_1}\overline{b}_{c_0,x_0}^{x_1}\overline{a}_{j_1,x_1}a_{d_1,x_1}b_{d_0,x_0}^{x_1} \\
&= \frac{1}{n^2}\sum_{x_1=1}^{k} \Big(a_{i_1,x_1}\overline{a}_{c_1,x_1}\overline{a}_{j_1,x_1}a_{d_1,x_1}\Big(\sum_{x_0=1}^{m}\overline{b}_{c_0,x_0}^{x_1}b_{d_0,x_0}^{x_1}\Big)\Big) \\
&= \frac{1}{n^2}\sum_{x_1=1}^{k} a_{i_1,x_1}\overline{a}_{c_1,x_1}\overline{a}_{j_1,x_1}a_{d_1,x_1}\delta_{c_0}^{d_0} = 0,
\end{aligned}
\tag{3.2}
$$

whenever $c_0 \neq d_0$. ∎

We show that in fact the subfactor $N_H \subset M_H$ associated to the Dita matrix H has an intermediate subfactor $N_H \subset R_H \subset M_H$, and the projection f is the Bisch projection (in the sense of [2]) corresponding to R_H.

PROPOSITION 3.2. *Let $H = \sum_{1\leqslant i,j\leqslant k} a_{i,j}B_j \otimes f_{i,j} \in M_n(\mathbb{C})$ be a Dita type matrix, where $A = (a_{i,j})_{1\leqslant i,j\leqslant k} \in M_k(\mathbb{C})$ and $B_1, \dots, B_k \in M_m(\mathbb{C})$ are complex Hadamard matrices, $n = mk$. Then:*

(i) *The commuting square $\mathfrak{C}(H)$ can be decomposed into two adjacent symmetric commuting squares:*

$$
\begin{array}{ccc}
\mathcal{D}_m \otimes D_k & \subset & M_m(\mathbb{C}) \otimes M_k(\mathbb{C}) \\
\cup & & \cup \\
\mathcal{D}_m \otimes I_k & \subset & U(M_m(\mathbb{C}) \otimes \mathcal{D}_k)U^*. \\
\cup & & \cup \\
\mathbb{C} & \subset & U(\mathcal{D}_m \otimes D_k)U^*
\end{array}
$$

(ii) *The commuting square $\mathfrak{C}^{\mathrm{t}}(H)$ can be decomposed into two adjacent symmetric commuting squares:*

$$
\begin{array}{ccc}
U(\mathcal{D}_m \otimes D_k)U^* & \subset & M_m(\mathbb{C}) \otimes M_k(\mathbb{C}) \\
\cup & & \cup \\
U(I_m \otimes \mathcal{D}_k)U^* & \subset & \mathcal{D}_m \otimes M_k(\mathbb{C}) \quad . \\
\cup & & \cup \\
\mathbb{C} & \subset & \mathcal{D}_m \otimes D_k
\end{array}
$$

Proof. (i) We first show that $\mathcal{D}_m \otimes I_k \subset U(M_m(\mathbb{C}) \otimes \mathcal{D}_k)U^*$. Equivalently, we check that $U^*(\mathcal{D}_m \otimes I_k)U \subset (M_m(\mathbb{C}) \otimes \mathcal{D}_k)$. Indeed, for $D \in \mathcal{D}_m$ we have:

$$\begin{aligned}
U^*(D\otimes I_k)U &= \frac{1}{n}\Big(\sum_{1\leqslant i',j'\leqslant k} \overline{a}_{i',j'} B_{j'}^* \otimes f_{j',i'}\Big)(D\otimes I_k)\Big(\sum_{1\leqslant i,j\leqslant k} a_{i,j} B_j \otimes f_{i,j}\Big)\\
&= \frac{1}{n}\sum_{1\leqslant i,j,j'\leqslant k} \overline{a}_{i,j'} a_{i,j} B_{j'}^* D B_j \otimes f_{j',j}\\
&= \frac{1}{n}\sum_{1\leqslant j,j'\leqslant k} \Big(\sum_{i=1}^{k} \overline{a}_{i,j'} a_{i,j}\Big) B_{j'}^* D B_j \otimes f_{j',j} \qquad (3.3)\\
&= \frac{1}{n}\sum_{1\leqslant j,j'\leqslant k} \delta_j^{j'} B_{j'}^* D B_j \otimes f_{j',j}\\
&= \frac{1}{n}\sum_{1\leqslant j\leqslant k} B_j^* D B_j \otimes f_{j,j} \in (M_m(\mathbb{C}) \otimes \mathcal{D}_k).
\end{aligned}$$

The lower square of inclusions is clearly a commuting square, since $\mathfrak{C}(H)$ is a commuting square. We check that

$$\begin{array}{ccc}
\mathcal{D}_m \otimes D_k & \subset & M_m(\mathbb{C}) \otimes M_k(\mathbb{C})\\
\cup & & \cup\\
\mathcal{D}_m \otimes I_k & \subset & U(M_m(\mathbb{C}) \otimes \mathcal{D}_k)U^*
\end{array}$$

is a commuting square. For $X \in M_m(\mathbb{C})$ and $D \in \mathcal{D}_k$ we have:

$$\begin{aligned}
U(X\otimes D)U^* &= \frac{1}{n}\Big(\sum_{1\leqslant i,j\leqslant k} a_{i,j} B_j \otimes f_{i,j}\Big)(X\otimes D)\Big(\sum_{1\leqslant i',j'\leqslant k} \overline{a}_{i',j'} B_{j'}^* \otimes f_{j',i'}\Big) \qquad (3.4)\\
&= \frac{1}{n}\sum_{1\leqslant i,i',j\leqslant k} \overline{a}_{i',j} a_{i,j} B_j X B_j^* \otimes D_{j,j} f_{i,i'}.
\end{aligned}$$

Hence:

$$\begin{aligned}
E_{\mathcal{D}_n}(U(X\otimes D)U^*) &= E_{\mathcal{D}_n}\Big(\frac{1}{n}\sum_{1\leqslant i,i',j\leqslant k} \overline{a}_{i',j} a_{i,j} B_j X B_j^* \otimes D_{j,j} f_{i,i'}\Big)\\
&= \frac{1}{n}\sum_{1\leqslant i,i',j\leqslant k} E_{\mathcal{D}_m}(\overline{a}_{i',j} a_{i,j} B_j X B_j^*) \otimes D_{j,j}\delta_i^{i'} f_{i,i} \qquad (3.5)\\
&= \frac{1}{n}\sum_{1\leqslant i,j\leqslant k} D_{j,j} E_{\mathcal{D}_m}(B_j X B_j^*) \otimes f_{i,i}\\
&= \frac{1}{n}\sum_{1\leqslant j\leqslant k} D_{j,j} E_{\mathcal{D}_m}(B_j X B_j^*) \otimes I_k \in \mathcal{D}_m \otimes \mathcal{I}_k\,.
\end{aligned}$$

The lower commuting square is symmetric, since the product of the dimensions of its upper left and lower right corners equals the dimension of its upper right corner. This also implies that the upper commuting square is symmetric, since $\mathfrak{C}(H)$ is symmetric.

(ii) The proof is similar to the proof of part (i). ∎

COROLLARY 3.3. *The subfactors associated to Dita matrices have intermediate subfactors.*

Proof. By iterating the basic construction for the decomposition of $\mathfrak{C}^{\mathrm{t}}(H)$ in commuting squares, we obtain the towers of algebras:

$$\begin{array}{ccccccccc}
U(\mathcal{D}_m \otimes D_k)U^* & \subset & M_m(\mathbb{C}) \otimes M_k(\mathbb{C}) & \overset{e_3}{\subset} & \mathcal{X}_1 & \overset{e_4}{\subset} & \mathcal{X}_2 & \overset{e_5}{\subset} & \cdots \\
\cup & & \cup & & \cup & & \cup & & \\
U(I_m \otimes \mathcal{D}_k)U^* & \subset & \mathcal{D}_m \otimes M_k(\mathbb{C}) & \overset{e_3}{\subset} & \mathcal{R}_1 & \overset{e_4}{\subset} & \mathcal{R}_2 & \overset{e_5}{\subset} & \cdots, \\
\cup & & \cup & & \cup & & \cup & & \\
\mathbb{C} & \subset & \mathcal{D}_m \otimes D_k & \overset{e_3}{\subset} & \mathcal{Y}_1 & \overset{e_4}{\subset} & \mathcal{Y}_2 & \overset{e_5}{\subset} & \cdots
\end{array}$$

where $\mathcal{R}_i = \langle \mathcal{R}_{i-1}, e_{i+2} \rangle \subset \mathcal{X}_i$. Let R_H be the weak closure of $\bigcup_i \mathcal{R}_i$. We have $N_H \subset R_H \subset M_H$ and R_H is a II_1 factor since the subfactor $N_H \subset M_H$ is irreducible. ∎

REMARK 3.4. It is immediate to check that the projection $f \in M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ from Proposition 3.1 implements the conditional expectation from $M_n(\mathbb{C}) \otimes I_n = M_n(\mathbb{C})$ onto $D_m \otimes M_k(\mathbb{C})$. It follows that f is the Bisch projection for the intermediate subfactor $N_H \subset R_H \subset M_H$.

4. MATRICES OF SMALL ORDER

In this section we compute the second relative commutants of the subfactors associated to Hadamard matrices of small dimensions. For some of the matrices considered we also specify the dimension of the third relative commutant. Most computations included were done with the help of computers, using GAP and Mathematica.

Let H be an $n \times n$ complex Hadamard matrix and $N_H \subset M_H$ its associated hyperfinite subfactor. It is well known in subfactor theory that the dimension of the second relative commutant is at most n, with equality if and only if H is equivalent to a tensor product of Fourier matrices. In this case the subfactor $N_H \subset M_H$ is well understood, being a cross-product subfactor. For this reason, we exclude from our analysis tensor products of Fourier matrices.

Some of the matrices we present are parameterized and they yield continuous families of complex Hadamard matrices. In such cases, the strategy for computing the second relative commutant will be to determine which entries of the profile matrix P_1 depend on the parameters, and for what values of the parameters are these entries 0. According to Proposition 2.6, the second relative commutant will not change as long as the 0 entries of P_1 do not change. Thus, to compute the second relative commutant for any other value of the parameters, it is enough to compute it for some random value.

We will describe the second relative commutant by specifying its minimal projections. Each such projection p corresponds to a subset $S \subset \{1,2,\dots,n^2\}$: p is the $n^2 \times n^2$ diagonal matrix having 1 on position (i,i) if and only if $i \in S$, and 0 on all other positions. Since the Jones projection e_3 is always in the second relative commutant, one of the subsets of our partitions will always be $\{1, n+2, 2n+3, \dots, kn+k+1, \dots, n^2\}$.

4.1. COMPLEX HADAMARD MATRICES OF DIMENSION 4. There exists, up to equivalence, only one family of complex Hadamard matrices of dimension 4:

$$F_4(a) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & a & -1 & -a \\ 1 & -1 & 1 & -1 \\ 1 & -a & -1 & a \end{pmatrix}, \quad |a| = 1.$$

The entries of P_1 that depend on the parameter a are $\frac{1}{8} + \frac{a^2}{8}$, $\frac{1}{8} - \frac{a^2}{8}$, $\frac{1}{8} + \frac{1}{8a^2}$, $\frac{1}{8} - \frac{1}{8a^2}$. Thus, the second relative commutant is the same for all values of a that are not roots of these equations.

The roots $a = 1, a = -1$ yield matrices that are tensor products of 2×2 Fourier matrices. Thus the dimension of the second relative commutant is 4, and its minimal projections are given by the partition $\{1,6,11,16\}$, $\{2,5,12,15\}$, $\{3,8,9,14\}$, $\{4,7,10,13\}$.

The roots $a = \mathrm{i}, a = -\mathrm{i}$ yield the 4×4 Fourier matrix, thus the minimal projections are $\{1,6,11,16\}$, $\{2,7,12,13\}$, $\{3,8,9,14\}$, $\{4,5,10,15\}$.

Any other values of a, $|a| = 1$, yield relative commutants of dimension 3: $\{1,6,11,16\}$, $\{2,4,5,7,10,12,13,15\}$, $\{3,8,9,14\}$. This is not surprising, since this matrix is of Dita type (see Proposition 3.1).

The dimension of the third relative commutant is 10, and the dimension of the fourth relative commutant is 35 unless a is a primitive root of order 8 of unity, in which case the dimension is 36. Based on this evidence, we conjecture that the principal graph of the subfactor associated to $F_4(a)$ is $D_{2k}^{(1)}$ if a is a primitive root of order 2^k of unity, and $D_\infty^{(1)}$ otherwise.

4.2. COMPLEX HADAMARD MATRICES OF DIMENSION 6. The Fourier matrix F_6 is part of an affine 2-parameter family of Dita matrices:

$$F_6(a,b) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a\,\mathrm{e}^{(\mathrm{i}/3)\pi} & b\,\mathrm{e}^{(2\mathrm{i}/3)\pi} & -1 & \frac{a}{\mathrm{e}^{(2\mathrm{i}/3)\pi}} & \frac{b}{\mathrm{e}^{(\mathrm{i}/3)\pi}} \\ 1 & \mathrm{e}^{(2\mathrm{i}/3)\pi} & \mathrm{e}^{(-2\mathrm{i}/3)\pi} & 1 & \mathrm{e}^{(2\mathrm{i}/3)\pi} & \mathrm{e}^{(-2\mathrm{i}/3)\pi} \\ 1 & -a & b & -1 & a & -b \\ 1 & \mathrm{e}^{(-2\mathrm{i}/3)\pi} & \mathrm{e}^{(2\mathrm{i}/3)\pi} & 1 & \mathrm{e}^{(-2\mathrm{i}/3)\pi} & \mathrm{e}^{(2\mathrm{i}/3)\pi} \\ 1 & \frac{a}{\mathrm{e}^{(\mathrm{i}/3)\pi}} & \frac{b}{\mathrm{e}^{(2\mathrm{i}/3)\pi}} & -1 & a\,\mathrm{e}^{(2\mathrm{i}/3)\pi} & b\,\mathrm{e}^{(\mathrm{i}/3)\pi} \end{pmatrix}.$$

The entries of P_1 that depend on a, b are: $2\,(1+a^{-2}+b^{-2})$, $2+\frac{2\,(-1)^{2/3}}{a^2}-\frac{2\,(-1)^{1/3}}{b^2}$, $2-\frac{2\,(-1)^{1/3}}{a^2}+\frac{2\,(-1)^{2/3}}{b^2}$, $2\,(1+a^2+b^2)$, $2+2\,(-1)^{2/3}\,a^2-2\,(-1)^{1/3}\,b^2$, $2-2\,(-1)^{1/3}\,a^2+2\,(-1)^{2/3}\,b^2$.

Making one of these entries 0 yields the following possibilities: $a=-\frac{1}{2}-\frac{i}{2}\sqrt{3}, b=-\frac{1}{2}+\frac{i}{2}\sqrt{3}$ or $a=-\frac{1}{2}+\frac{i}{2}\sqrt{3}, b=-\frac{1}{2}-\frac{i}{2}\sqrt{3}$ or $a=\frac{1}{2}-\frac{i}{2}\sqrt{3}, b=\frac{1}{2}+\frac{i}{2}\sqrt{3}$ or $a=\frac{1}{2}+\frac{i}{2}\sqrt{3}, b=\frac{1}{2}-\frac{i}{2}\sqrt{3}$ or $a=-\frac{1}{2}-\frac{i}{2}\sqrt{3}, b=\frac{1}{2}-\frac{i}{2}\sqrt{3}$ or $a=-\frac{1}{2}+\frac{i}{2}\sqrt{3}, b=\frac{1}{2}+\frac{i}{2}\sqrt{3}$ or $a=\frac{1}{2}-\frac{i}{2}\sqrt{3}, b=-\frac{1}{2}-\frac{i}{2}\sqrt{3}$ or $a=\frac{1}{2}+\frac{i}{2}\sqrt{3}, b=-\frac{1}{2}+\frac{i}{2}\sqrt{3}$ or $a=-1, b=-1$ or $a=1, b=1$ or $a=-1, b=1$ or $a=1, b=-1$.

In each of these cases the matrix $F_6(a, b)$ is a tensor product of Fourier matrices.

For all other pairs (a, b) satisfying $|a|=|b|=1$, the second relative commutant has dimension 4: $\{1, 8, 15, 22, 29, 36\}$, $\{2, 4, 6, 7, 9, 11, 14, 16, 18, 19, 21, 23, 26, 28, 30, 31, 33, 35\}$, $\{3, 10, 17, 24, 25, 32\}$, $\{5, 12, 13, 20, 27, 34\}$.

The following family of self-adjoint, non-affine, complex Hadamard matrices was obtained in [4], one of the motivations being the search for Hadamard matrices of small dimensions that might yield subfactors with no extra structure in their relative commutants, besides the Jones projections:

$$BN_6(\theta)=\begin{pmatrix}1 & 1 & 1 & 1 & 1 & 1\\ 1 & -1 & \overline{x} & -y & -\overline{x} & y\\ 1 & x & -1 & t & -t & -x\\ 1 & -\overline{y} & \overline{t} & -1 & \overline{y} & -\overline{t}\\ 1 & -x & -\overline{t} & y & 1 & \overline{z}\\ 1 & \overline{y} & -\overline{x} & -t & z & 1\end{pmatrix},$$

where $\theta \in [-\pi, -\arccos(\frac{-1+\sqrt{3}}{2})] \cup [\arccos(\frac{-1+\sqrt{3}}{2}), \pi]$ and the variables x, y, z, t are given by:

$$y=\exp(i\theta), \quad z=\frac{1+2y-y^2}{y(-1+2y+y^2)}, \quad x=\frac{1+2y+y^2-\sqrt{2}\sqrt{1+2y+2y^3+y^4}}{1+2y-y^2},$$
$$t=\frac{1+2y+y^2-\sqrt{2}\sqrt{1+2y+2y^3+y^4}}{-1+2y+y^2}.$$

The entries of BN_6 do not depend linearly on the parameters, thus this is not a Dita-type family. The corresponding subfactors have the second relative commutant generated by the Jones projection. We conjecture that $BN_6(\theta)$ give supertransitive subfactors, i.e. all the relative commutants of higher orders are generated by the Jones projections.

There are other interesting complex Hadamard matrices of order 6, such as the one found by Tao in connection to Fuglede's conjecture ([19]), or the Haagerup matrix ([7], TZ). We computed the second and third relative commutants for these matrices, and they only contain the Jones projection.

4.3. COMPLEX HADAMARD MATRICES OF DIMENSION 7. The following one-parameter family was found in [14], providing a counterexample to a conjecture of Popa regarding the finiteness of the number of complex Hadamard matrices of prime dimension.

$$P_7(a) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a\,\mathrm{e}^{(\mathrm{i}/3)\pi} & \frac{a}{\mathrm{e}^{(2\mathrm{i}/3)\pi}} & \mathrm{e}^{(-\mathrm{i}/3)\pi} & -1 & -1 & \mathrm{e}^{(\mathrm{i}/3)\pi} \\ 1 & \frac{a}{\mathrm{e}^{(2\mathrm{i}/3)\pi}} & a\,\mathrm{e}^{(\mathrm{i}/3)\pi} & -1 & \mathrm{e}^{(-\mathrm{i}/3)\pi} & -1 & \mathrm{e}^{(\mathrm{i}/3)\pi} \\ 1 & \mathrm{e}^{(-\mathrm{i}/3)\pi} & -1 & \frac{\mathrm{e}^{(\mathrm{i}/3)\pi}}{a} & \frac{1}{a\,\mathrm{e}^{(2\mathrm{i}/3)\pi}} & \mathrm{e}^{(\mathrm{i}/3)\pi} & -1 \\ 1 & -1 & \mathrm{e}^{(-\mathrm{i}/3)\pi} & \frac{1}{a\,\mathrm{e}^{(2\mathrm{i}/3)\pi}} & \frac{\mathrm{e}^{(\mathrm{i}/3)\pi}}{a} & \mathrm{e}^{(\mathrm{i}/3)\pi} & -1 \\ 1 & -1 & -1 & \mathrm{e}^{(\mathrm{i}/3)\pi} & \mathrm{e}^{(\mathrm{i}/3)\pi} & \mathrm{e}^{(-2\mathrm{i}/3)\pi} & \mathrm{e}^{(-\mathrm{i}/3)\pi} \\ 1 & \mathrm{e}^{(\mathrm{i}/3)\pi} & \mathrm{e}^{(\mathrm{i}/3)\pi} & -1 & -1 & \mathrm{e}^{(-\mathrm{i}/3)\pi} & \mathrm{e}^{(-2\mathrm{i}/3)\pi} \end{pmatrix}.$$

The second relative commutant of the associated subfactors is generated by the Jones projection, for all $|a| = 1$. For $a = 1$ we also computed the third relative commutant, and it is just the Temperley–Lieb algebra TL_2. We conjecture that $P_7(a)$ yield subfactors with no extra structure in their higher order relative commutants, besides the Jones projections.

4.4. COMPLEX HADAMARD MATRICES OF DIMENSION 8. The following 5-parameter family of Hadamard matrices contains the Fourier matrix and is of Dita type:

$$F_8(a,b,c,d,z) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & a\mathrm{e}^{(\mathrm{i}/4)\pi} & \mathrm{i}b & c\mathrm{e}^{(3\mathrm{i}/4)\pi} & -1 & \frac{a}{\mathrm{e}^{(3\mathrm{i}/4)\pi}} & -\mathrm{i}b & \frac{c}{\mathrm{e}^{(\mathrm{i}/4)\pi}} \\ 1 & \mathrm{i}d & -1 & -\mathrm{i}d & 1 & \mathrm{i}d & -1 & -\mathrm{i}d \\ 1 & \mathrm{e}^{(3\mathrm{i}/4)\pi}z & -\mathrm{i}b & \frac{c\mathrm{e}^{(\mathrm{i}/4)\pi}z}{a} & -1 & \frac{z}{\mathrm{e}^{(\mathrm{i}/4)\pi}} & \mathrm{i}b & \frac{cz}{a\mathrm{e}^{(3\mathrm{i}/4)\pi}} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & \frac{a}{\mathrm{e}^{(3\mathrm{i}/4)\pi}} & \mathrm{i}b & \frac{c}{\mathrm{e}^{(\mathrm{i}/4)\pi}} & -1 & a\mathrm{e}^{(\mathrm{i}/4)\pi} & -\mathrm{i}b & c\mathrm{e}^{(3\mathrm{i}/4)\pi} \\ 1 & -\mathrm{i}d & -1 & \mathrm{i}d & 1 & -\mathrm{i}d & -1 & \mathrm{i}d \\ 1 & \frac{z}{\mathrm{e}^{(\mathrm{i}/4)\pi}} & -\mathrm{i}b & \frac{cz}{a\mathrm{e}^{(3\mathrm{i}/4)\pi}} & -1 & \mathrm{e}^{(3\mathrm{i}/4)\pi}z & \mathrm{i}b & \frac{c\mathrm{e}^{(\mathrm{i}/4)\pi}z}{a} \end{pmatrix}.$$

The list of possible values of a, b, c, d, z that yield 0 entries for P_1 is very long and we do not include it here. Outside these values, the second relative commutant has dimension 4 and it is given by $\{1, 10, 19, 28, 37, 46, 55, 64\}$, $\{2, 4, 6, 8, 9, 11, 13, 15, 18, 20, 22, 24, 25, 27, 29, 31, 34, 36, 38, 40, 41, 43, 45, 47, 50, 52, 54, 56, 57, 59, 61, 63\}$, $\{3, 7, 12, 16, 17, 21, 26, 30, 35, 39, 44, 48, 49, 53, 58, 62\}$, $\{5, 14, 23, 32, 33, 42, 51, 60\}$.

We analysed several other complex Hadamard matrices besides those included in this paper, such as those found by [12], [17]. We covered most known examples of complex Hadamard matrices of size $\leqslant 11$. We draw some conclusions:

(i) As shown in the previous section, matrices of Dita–Haagerup type yield subfactors with intermediate subfactors, and thus the second relative commutant has some extra structure besides the Jones projection. We note that parametric families of Dita–Haagerup matrices exist for every n non-prime, and they contain the Fourier matrix F_n.

(ii) All non-Dita, non-Fourier matrices we tested have the second relative commutant generated by the Jones projection. The third relative commutant is also generated by the first two Jones projections for all cases we could compute. It remains an open problem whether there exist such complex Hadamard matrices with non-trivial standard invariant. Such examples would be even more interesting if the second relative commutant contains just the Jones projections.

Acknowledgements. We would like to thank Teodor Banica, Kyle Beauchamp and Dietmar Bisch for fruitful discussions and correspondence. We would particularly like to thank Wes Camp for his help on computations using the Mathematica and GAP software. The author was supported in part by NSF under Grant No. DMS 0500933 and by the Romanian Government under Grant 2-CEx06-11-34/2006.

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Received February 13, 2008.