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RIESZ SUMMABILITY OF ORTHOGONAL SERIES IN NONCOMMUTATIVE L_2-SPACES

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ABSTRACT. A Riesz summability method is defined by means of a sequence $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_n \to \infty$ of real numbers. The following theorem is known in commutative L_2-spaces: If a sequence $\{\xi_n : n = 0, 1, \ldots\}$ of pairwise orthogonal functions in some $L_2 = L_2(X, \mathcal{F}, \mu)$ over a positive measure space is such that

$$\sum_{n:\lambda_n \geqslant 4} (\log\log \lambda_n)^2 \|\xi_n\|^2 < \infty,$$

then the series $\sum \xi_n$ is Riesz summable almost everywhere to its sum in the norm of L_2.

In this paper, we extend this theorem to noncommutative $L_2(\mathfrak{A}, \phi)$ spaces, where $\mathfrak{A}$ is a von Neumann algebra, ϕ is a faithful, normal state acting on $\mathfrak{A}$, and bundle convergence plays the role of almost everywhere convergence. An interesting corollary of our Theorem 2.1 reads as follows: For any sequence $\{A_n : n = 0, 1, \ldots\}$ of pairwise orthogonal operators in a von Neumann algebra $\mathfrak{A}$ with a faithful, normal state ϕ acting on $\mathfrak{A}$ for which $\sum \phi(|A_n|^2) < \infty$, there exists a Riesz method of summability such that the series $\sum \pi(A_n)\omega$ is summable in the sense of bundle convergence, where π is a one-to-one $*$-homomorphism of $\mathfrak{A}$ into the algebra of all bounded linear operators on L_2 and ω is a cyclic, separating vector in L_2 according to the Gelfand–Naimark–Segal representation theorem.

KEYWORDS: *von Neumann algebra $\mathfrak{A}$, faithful and normal state ϕ, completion $L_2 = L_2(\mathfrak{A}, \phi)$, Gelfand–Naimark–Segal representation theorem, bundle convergence, orthogonal series, Riesz summability.*

MSC (2000): Primary 46L53; Secondary 46L10, 42C15.

INTRODUCTION

Given a sequence of real numbers:

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty \quad \text{as } n \to \infty, \tag{0.1}$$

a series $\sum_{k=0}^{\infty} u_k$ of complex numbers is said to be Riesz summable, or summable $(R, \lambda_n, 1)$, to a sum s if

$$\lim_{n\to\infty} \sum_{k=0}^{n} \Big(1 - \frac{\lambda_k}{\lambda_{n+1}}\Big) u_k = s.$$

Clearly, s is uniquely determined if exists. Since

$$\frac{1}{\lambda_{n+1}} \sum_{k=0}^{n} (\lambda_{n+1} - \lambda_k) u_k = \frac{1}{\lambda_{n+1}} \sum_{k=0}^{n} (\lambda_{k+1} - \lambda_k) s_k,$$

where $s_k := u_0 + u_1 + \cdots + u_k$ is the kth partial sum of the series $\sum u_k$ in question, the infinite matrix of the Riesz method of summability is regular in the sense of Toeplitz (see, e.g., pp. 74–75 of [6]). Consequently, the ordinary convergence of the series $\sum u_k$ implies its Riesz summability to the same sum.

We note that the special case $\lambda_n := n$ gives rise to summability by the first arithmetic means, or shortly: summability $(C, 1)$.

Let $(X, \mathcal{F}, \mu)$ be an arbitrary positive measure space, $\{\xi_n : n = 0, 1, \dots\}$ a sequence of pairwise orthogonal functions in $L_2 := L_2(X, \mathcal{F}, \mu)$, and set

$$s_n := \sum_{k=0}^{n} \xi_k, \quad \sigma_n := \sum_{k=0}^{n} \Big(1 - \frac{\lambda_k}{\lambda_{n+1}}\Big) \xi_k, \quad n = 0, 1, \dots.$$

The following theorem is due to Zygmund [5] (see also p. 141 of [1]).

THEOREM 0.1. *If a sequence $\{\xi_n := 0, 1, \dots\}$ of pairwise orthogonal functions in $L_2 = L_2(X, \mathcal{F}, \mu)$ over a positive measure space is such that*

$$\sum_{n:\lambda_n \geqslant 4} (\log\log \lambda_n)^2 \|\xi_n\|^2 < \infty, \tag{0.2}$$

then

$$\lim_{n\to\infty} \sigma_n(x) = s(x) \quad a.e.,$$

where s is the sum of the series $\sum_{n=0}^{\infty} \xi_n$ in the norm of L_2.

In this paper, logarithms are to the base 2.

REMARK 0.2. We note that if the sequence $\{\lambda_n\}$ grows too fast in the sense that

$$\liminf_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} > 1,$$

then it follows from the celebrated Rademacher–Menshov theorem (see, e.g., p. 80 of [1]) that the orthogonal series $\sum \xi_n$ converges a.e., a fortiori it is Riesz summable a.e.

In the sequel, we assume that the sequence $\{\lambda_n\}$ of real numbers is such that both conditions (0.1) and (0.3) are satisfied, where

$$\limsup_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} =: C < \infty. \tag{0.3}$$

1. BACKGROUND

Let H be a separable complex Hilbert space, $\mathfrak{A}$ a von Neumann algebra over H, and ϕ a faithful, normal state acting on $\mathfrak{A}$. Then

$$\langle A|B\rangle := \phi(B^*A), \quad A, B \in \mathfrak{A},$$

defines an inner product on $\mathfrak{A}$, and $(\mathfrak{A}, \langle\cdot|\cdot\rangle)$ is a complex prehilbert space. We denote by $L_2 = L_2(\mathfrak{A}, \phi)$ its completion, by $(\cdot|\cdot)$ and $\|\cdot\|$ the inner product and norm in L_2, respectively.

According to the Gelfand–Naimark–Segal representation theorem, there exist a one-to-one $*$-homomorphism π of $\mathfrak{A}$ into the algebra of all bounded linear operators on L_2 and a cyclic, separating vector ω in L_2 such that

$$\phi(A) = (\pi(A)\omega|\omega).$$

In particular, it follows that

$$\|\pi(A)\omega\|^2 = \phi(A^*A), \quad A \in \mathfrak{A}. \tag{1.1}$$

In 1996 Hensz, Jajte and Paszkiewicz [2] introduced the notion of bundle convergence for sequences of operators in $\mathfrak{A}$ as well as for sequences of vectors in L_2. To present their definition, we consider a sequence $\{D_k : k = 0, 1, 2, \dots\}$ of operators in $\mathfrak{A}_+$, the cone of all positive operators in $\mathfrak{A}$, for which

$$\sum_{k=0}^{\infty} \phi(D_k) < \infty, \tag{1.2}$$

and associate with it a so-called bundle $\mathcal{P} = \mathcal{P}(D_k)$ defined as follows:

$$\mathcal{P} := \Big\{ P \in \operatorname{Proj}\mathfrak{A} : \sup\Big\{ \Big\| P\Big(\sum_{k=0}^{n} D_k\Big)P \Big\|_\infty : n \geqslant 0 \Big\} < \infty \quad \text{and } \|PD_nP\|_\infty \to 0 \text{ as } n \to \infty \Big\}, \tag{1.3}$$

where $\operatorname{Proj}\mathfrak{A}$ is the class of all selfadjoint projections P in $\mathfrak{A}$ and $\|\cdot\|_\infty$ is the usual operator norm (of the bounded operators over H).

Now, a sequence $\{A_n : n = 0, 1, 2, \dots\}$ of operators in $\mathfrak{A}$ is said to be bundle convergent to some A in $\mathfrak{A}$, in symbol:

$$A_n \overset{\text{b},\mathfrak{A}}{\to} A \quad \text{as } n \to \infty, \tag{1.4}$$

if there exists a bundle $\mathcal{P}$ such that for each P in $\mathcal{P}$ we have

$$\|(A_n - A)P\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Furthermore, a sequence $\{\xi_n : n = 0, 1, 2, \dots\}$ of vectors in L_2 is said to be bundle convergent to some ξ in L_2, in symbol:

$$\xi_n \xrightarrow{\text{b}} \xi \quad \text{as } n \to \infty,$$

if there exists a sequence $\{A_n : n = 0, 1, 2, \dots\}$ of operators in $\mathfrak{A}$ for which

$$A_n \xrightarrow{\text{b},\mathfrak{A}} O \quad \text{as } n \to \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \|\xi_n - \xi - \pi(A_n)\omega\|^2 < \infty, \tag{1.5}$$

where O is the zero operator in $\mathfrak{A}$.

It is clear that if for some sequence $\{A_n\}$ in $\mathfrak{A}$ the limit relation (1.4) holds with $A = O$, then

$$\pi(A_n)\omega \xrightarrow{\text{b}} o,$$

where o is the zero vector in L_2.

Bundle convergence is an appropriate substitute for almost everywhere (in measure theory) or almost sure (in probability theory) convergence in a noncommutative setting (see p. 29 of [2]).

2. RIESZ SUMMABILITY IN VON NEUMANN ALGEBRAS

We will extend Theorem 0.1 for orthogonal sequences $\{A_n : n = 0, 1, \dots\}$ of operators in a von Neumann algebra $\mathfrak{A}$.

THEOREM 2.1. *Let $\mathfrak{A}$ be a von Neumann algebra, ϕ a faithful and normal state acting on $\mathfrak{A}$, and $\{\lambda_n : n = 0, 1, \dots\}$ a sequence of real numbers satisfying conditions* (0.1) *and* (0.3)*. If $\{A_n : n = 0, 1, \dots\}$ is a sequence of pairwise orthogonal operators in $\mathfrak{A}$ such that*

$$\sum_{n:\lambda_n \geqslant 4} (\log\log \lambda_n)^2 \phi(|A_n|^2) < \infty, \tag{2.1}$$

then

$$\sum_{k=0}^{n} \Big(1 - \frac{\lambda_k}{\lambda_{n+1}}\Big) \pi(A_k)\omega \xrightarrow{\text{b}} \sigma \quad \text{as } n \to \infty, \tag{2.2}$$

where σ is the sum of the series $\sum_{n=0}^{\infty} \pi(A_n)\omega$ in the norm of $L_2 = L_2(\mathfrak{A}, \phi)$.

We recall that the symbol $|\cdot|$ is defined by

$$|A| := (A^*A)^{1/2}, \quad A \in \mathfrak{A}. \tag{2.3}$$

The square root makes sense, since $A^*A \in \mathfrak{A}_+$. Unfortunately, the traditional triangle inequality does not hold in the noncommutative case. However, the following weaker substitute for the triangle inequality is available in any von Neumann algebra $\mathfrak{A}$ (see, e.g., p. 4 of [3]): *For any* $c_k \in \mathbb{C}$ *and* $A_k \in \mathfrak{A}$, $0 \leqslant k \leqslant n$, *we have*

$$\Big| \sum_{k=0}^{n} c_k A_k \Big|^2 \leqslant \sum_{k=0}^{n} |c_k|^2 \sum_{k=0}^{n} |A_k|^2. \tag{2.4}$$

This can be considered as a version of the Cauchy inequality in noncommutative setting.

REMARK 2.2. The following corollary of Theorem 2.1 is interesting in itself: *Let* $\mathfrak{A}$ *be a von Neumann algebra and* ϕ *a faithful, normal state acting on* $\mathfrak{A}$. *If a sequence* $\{A_n : n = 0, 1, \ldots\}$ *of pairwise orthogonal operators in* $\mathfrak{A}$ *is such that* $\sum \phi(|A_n|^2) < \infty$, *then there exists a Riesz method of summability by which the series* $\sum \pi(A_n)\omega$ *is summable in the sense of bundle convergence.*

Indeed, first we construct a strictly increasing sequence $\{\kappa_n : n = 1, 2, \ldots\}$ of positive real numbers tending to ∞ such that

$$\sum_{n=1}^{\infty} \kappa_n^2 \phi(|A_n|^2) < \infty \quad \text{and} \quad \limsup_{n\to\infty} (2^{\kappa_{n+1}} - 2^{\kappa_n}) < \infty,$$

then set

$$\lambda_0 := 0, \quad \lambda_n := 2^{2^{\kappa_n}}, \quad n = 1, 2, \ldots.$$

Clearly, this sequence $\{\lambda_n : n = 0, 1, \ldots\}$ satisfies conditions (0.1) and (0.3). Applying Theorem 2.1 furnishes the corollary stated above.

Proof of Theorem 2.1. *Part* (i). Given a sequence $\{\lambda_n\}$ of real numbers with (0.1) and (0.3), we define two sequences of integers:

$$0 < \nu_0 < \nu_1 < \cdots \quad \text{and} \quad 2 \leqslant p_0 < p_1 < \cdots$$

such that the following two conditions are satisfied:

$$2^{p_n} \leqslant \lambda_{\nu_n} < 2^{p_n+1}; \tag{2.5}$$

$$\lambda_{\nu_0 - 1} < 4 \quad \text{and} \quad \lambda_{\nu_{n+1}-1} < 2^{p_n+1}, \quad n = 0, 1, \ldots. \tag{2.6}$$

It is clear that $p_n \geqslant n$. Taking into account the left-hand inequality in (2.5), we have

$$2^{p_n} \leqslant \lambda_k \quad \text{for } k \geqslant \nu_n, n \geqslant 2.$$

Combining this inequality with (2.1) and the fact that $p_n \geqslant n$ gives

$$\sum_{n=2}^{\infty} (\log n)^2 \sum_{k=\nu_n+1}^{\nu_{n+1}} \phi(|A_k|^2) \leqslant \sum_{k=\nu_2+1}^{\infty} (\log\log \lambda_k)^2 \phi(|A_k|^2) < \infty. \tag{2.7}$$

If we take into account the right-hand inequality in (2.5), we find that

$$\lambda_k^2 \sum_{n:\nu_n \geqslant k} \frac{1}{\lambda_{\nu_n}^2} \leqslant \lambda_{\nu_{n_0}}^2 \sum_{n=n_0}^{\infty} \frac{1}{\lambda_{\nu_n}^2} \leqslant 2^{2(p_{n_0}+1)} \sum_{n=n_0}^{\infty} \frac{1}{2^{2p_n}} = \frac{16}{3}, \tag{2.8}$$

where $\nu_{n_0-1} < k \leqslant \nu_{n_0}$.

Part (ii). We will use the following notations:

$$S_n:=\sum_{k=0}^{n} A_k,\ T_n:=\sum_{k=0}^{n}\Big(1-\frac{\lambda_k}{\lambda_{n+1}}\Big)A_k,\ \text{and } c_n:=(\phi(|A_n|)^2)^{1/2},\ n=0,1,\dots. \tag{2.9}$$

Since

$$S_{\nu_n}-T_{\nu_n}=\sum_{k=0}^{\nu_n}\frac{\lambda_k}{\lambda_{\nu_n+1}}A_k,$$

by orthogonality, we obtain

$$\phi(|S_{\nu_n}-T_{\nu_n}|^2)=\sum_{k=0}^{\nu_n}\frac{\lambda_k^2}{\lambda_{\nu_n+1}^2}c_k^2.$$

By (2.1) and (2.8), we conclude that

$$\sum_{n=0}^{\infty}\phi(|S_{\nu_n}-T_{\nu_n}|^2)=\sum_{n=0}^{\infty}\frac{1}{\lambda_{\nu_n+1}^2}\sum_{k=0}^{\nu_n}\lambda_k^2c_k^2=\sum_{k=0}^{\infty}\lambda_k^2c_k^2\sum_{n:\nu_n\geqslant k}\frac{1}{\lambda_{\nu_n+1}^2}\leqslant\frac{16}{3}\sum_{k=0}^{\infty}c_k^2<\infty.$$

It follows from Property 3.1, p. 30 of [2] that

$$S_{\nu_n}-T_{\nu_n}\overset{\mathrm{b},\mathfrak{A}}{\to}O\quad\text{as } n\to\infty. \tag{2.10}$$

By virtue of Theorem 4 in [4], it follows from (2.7) that

$$\pi(S_{\nu_n})\omega\overset{\mathrm{b}}{\to}\sigma\quad\text{as } n\to\infty. \tag{2.11}$$

Next, we will find an estimate for $|T_m-T_{\nu_n}|^2$, where the integers m and n are such that $\nu_n<m<\nu_{n+1}$. To this effect, we apply inequality (2.4) with appropriate scalars for the following representation of the difference:

$$T_m-T_{\nu_n}=\sum_{j=\nu_n}^{m-1}(T_{j+1}-T_j),\quad \nu_n<m<\nu_{n+1}.$$

As a result, we obtain

$$|T_m-T_{\nu_n}|^2\leqslant\sum_{j=\nu_n}^{\nu_{n+1}-2}\frac{\lambda_{j+2}-\lambda_{j+1}}{\lambda_{j+1}}\sum_{j=\nu_n}^{\nu_{n+1}-2}\frac{\lambda_{j+1}}{\lambda_{j+2}-\lambda_{j+1}}|T_{j+1}-T_j|^2.$$

By (0.3), (2.5) and (2.6), we find that

$$\sum_{j=\nu_n}^{\nu_{n+1}-2}\frac{\lambda_{j+2}-\lambda_{j+1}}{\lambda_{j+1}}\leqslant\frac{\lambda_{\nu_{n+1}}}{\lambda_{\nu_{n+1}-1}}+\frac{\lambda_{\nu_{n+1}-1}-\lambda_{\nu_n+1}}{\lambda_{\nu_n+1}}<C+\frac{\lambda_{\nu_{n+1}-1}}{\lambda_{\nu_n+1}}<C+2, \tag{2.12}$$

where the constant C occurs in (0.3). Thus, it follows that

$$|T_m-T_{\nu_n}|^2\leqslant(C+2)\sum_{j=\nu_n}^{\nu_{n+1}-2}\frac{\lambda_{j+1}}{\lambda_{j+2}-\lambda_{j+1}}|T_{j+1}-T_j|^2. \tag{2.13}$$

Now, we define the positive operators D_n as follows:

$$D_n := \sum_{j=\nu_n}^{\nu_{n+1}-2} \frac{\lambda_{j+1}}{\lambda_{j+2}-\lambda_{j+1}} |T_{j+1}-T_j|^2, \quad n=0,1,\ldots; \tag{2.14}$$

in case $\nu_{n+1} = \nu_n + 1$, the empty sum equals the zero operator O, by definition. Taking into account that

$$T_{j+1} - T_j = \frac{\lambda_{j+2}-\lambda_{j+1}}{\lambda_{j+1}\lambda_{j+2}} \sum_{k=1}^{j+1} \lambda_k A_k$$

and that the operators A_k are pairwise orthogonal, we obtain

$$\phi(|T_{j+1}-T_j|^2) = \Big(\frac{\lambda_{j+2}-\lambda_{j+1}}{\lambda_{j+1}\lambda_{j+2}}\Big)^2 \sum_{k=1}^{j+1} \lambda_k^2 c_k^2. \tag{2.15}$$

Putting together (2.14) and (2.15) gives

$$\begin{aligned} \sum_{n=0}^{\infty} \phi(D_n) &\leqslant \sum_{n=0}^{\infty} \frac{\lambda_{j+2}-\lambda_{j+1}}{\lambda_{j+1}\lambda_{j+2}^2} \sum_{j=\nu_n}^{\nu_{n+1}-2} \sum_{k=1}^{j+1} \lambda_k^2 c_k^2 \\ &\leqslant \sum_{j=\nu_0}^{\infty} \frac{\lambda_{j+2}-\lambda_{j+1}}{\lambda_{j+1}\lambda_{j+2}^2} \sum_{k=1}^{j+1} \lambda_k^2 c_k^2 \leqslant \sum_{k=1}^{\infty} \lambda_k^2 c_k^2 \sum_{j=k-1}^{\infty} \frac{\lambda_{j+2}-\lambda_{j+1}}{\lambda_{j+1}\lambda_{j+2}^2}. \end{aligned} \tag{2.16}$$

Since

$$\frac{\lambda_{j+2}-\lambda_{j+1}}{\lambda_{j+1}\lambda_{j+2}^2} = \frac{\lambda_{j+2}^2-\lambda_{j+1}^2}{\lambda_{j+1}\lambda_{j+2}^2(\lambda_{j+2}+\lambda_{j+1})} < \frac{\lambda_{j+2}^2-\lambda_{j+1}^2}{2\lambda_{j+1}^2\lambda_{j+2}^2} = \frac{1}{2}\Big(\frac{1}{\lambda_{j+1}^2} - \frac{1}{\lambda_{j+2}^2}\Big), \tag{2.17}$$

we conclude from (2.16) and (2.1) that

$$\sum_{n=0}^{\infty} \phi(D_n) \leqslant \frac{1}{2} \sum_{k=1}^{\infty} \lambda_k^2 c_k^2 \sum_{j=k-1}^{\infty} \Big(\frac{1}{\lambda_{j+1}^2} - \frac{1}{\lambda_{j+2}^2}\Big) = \frac{1}{2} \sum_{k=1}^{\infty} c_k^2 := \frac{1}{2} \sum_{k=1}^{\infty} \phi(|A_k|^2) < \infty.$$

This last inequality means that the sequence $\{D_n : n = 0,1,\ldots\}$ determines a bundle, which we denote by $\mathcal{P}$.

For each integer $m \geqslant \nu_0$, let $n(m)$ be the unique integer for which

$$\nu_{n(m)} \leqslant m < \nu_{n(m)+1}. \tag{2.18}$$

By (2.13) and (2.14), we have

$$|T_m - T_{\nu_{n(m)}}|^2 \leqslant (C+2) D_{\nu(m)}, \quad m = 0,1,\ldots, \tag{2.19}$$

whence it follows that for any projection P in the bundle $\mathcal{P}$,

$$\|P|T_m - T_{\nu_{n(m)}}|^2 P\|_\infty \leqslant 2\|PD_{n(m)}P\|_\infty \to 0 \quad \text{as } m \to \infty.$$

By definition, this means that

$$T_m - T_{\nu_{n(m)}} \stackrel{\mathrm{b},\mathfrak{A}}{\to} O \quad \text{as } m \to \infty. \tag{2.20}$$

Finally, we consider the representation

$$\pi(T_m)\omega = \pi(T_m - T_{\nu_{n(m)}})\omega + \pi(T_{\nu_{n(m)}} - S_{\nu_{n(m)}})\omega + \pi(S_{\nu_{n(m)}})\omega. \tag{2.21}$$

By using (2.10), (2.11), (2.20) as well as Property 3.6, p. 31 of [2] and the additivity property of bundle convergence, we conclude from (2.21) that

$$\pi(T_m)\omega \xrightarrow{\text{b}} \sigma \quad \text{as } m \to \infty,$$

which is (2.2) to be proved (see the notation for T_n in (2.9)). ∎

REMARK 2.3. We point out that Theorem 2.1 remains valid if (0.3) is omitted from the conditions. A brief sketch of this claim is the following. It is easy to see that both (2.10) and (2.11) also hold true with $\nu_n - 1$ in place of ν_n. These facts make it possible that in the definition (2.14) of D_n we may put $\sum\limits_{j=\nu_n}^{\nu_{n+1}-3}$ in place of $\sum\limits_{j=\nu_n}^{\nu_{n+1}-2}$ (with the agreement that the empty sum equals O). In doing so, the upper limit $\nu_{n+1} - 2$ of the summation in (2.12) may be replaced by $\nu_{n+1} - 3$. As a result, conditions (2.5) and (2.6) are enough to ensure the boundedness of the sums in (2.12). The reader may trace out with ease these changes in the formulas involved.

3. RIESZ SUMMABILITY IN L_2-SPACES

Things become more complicated in the case of L_2-spaces. This is the reason that we have to impose an additional condition on the sequence $\{\lambda_n\}$ to be increasing in such a way that

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} < \infty. \tag{3.1}$$

THEOREM 3.1. *Let $\mathfrak{A}$ be a von Neumann algebra, ϕ a faithful and normal state acting on $\mathfrak{A}$, and $\{\lambda_n\}$ a sequence of real numbers with properties* (0.1), (0.3) *and* (3.1). *If $\{\xi_n : n = 0, 1, \dots\}$ is a sequence of pairwise orthogonal vectors in $L_2 := L_2(\mathfrak{A}, \phi)$ such that*

$$\sum_{n:\lambda_n \geqslant 4} (\log\log \lambda_n)^2 \|\xi_n\|^2 < \infty, \tag{3.2}$$

then

$$\tau_n := \sum_{k=0}^{n} \Big(1 - \frac{\lambda_k}{\lambda_{n+1}}\Big)\xi_k \xrightarrow{\text{b}} \sigma \quad \text{as } n \to \infty, \tag{3.3}$$

where σ is the sum of the series $\sum\limits_{n=0}^{\infty} \xi_n$ in the norm of L_2.

REMARK 3.2. Our "noncommutative" Theorem 3.1 is not so general as the "commutative" Theorem 0.1, owing to the fact that the additional condition (3.1) is imposed on the sequence $\{\lambda_n\}$ which determines the Riesz summability method in question. However, such a condition is not too restrictive. For example, ([3], Theorem 5.2.8. on p. 71) relating to $\lambda_n := n$, $n = 0, 1, \ldots$, is a particular case of our Theorem 3.1.

Proof of Theorem 3.1. *Part* (i). Given a sequence $\{\lambda_n\}$ of real numbers with (0.1) and (0.3), we consider the sequences $\{\nu_n\}$ and $\{p_n\}$ of integers satisfying conditions (2.5)–(2.8) (see Part (i) in the proof of Theorem 2.1). In addition to the notation in (3.3), we also use the following ones:

$$\sigma_n := \sum_{k=0}^{n} \xi_k \quad \text{and} \quad c_n := \|\xi_n\|, \quad n = 0, 1, \ldots.$$

In the same way as in the proof of Theorem 2.1, we conclude

$$\sigma_{\nu_n} - \tau_{\nu_n} \xrightarrow{\text{b}} o \quad \text{and} \quad \sigma_{\nu_n} \xrightarrow{\text{b}} \sigma \quad \text{as } n \to \infty.$$

By the additive property of bundle convergence, it follows that

$$\tau_{\nu_n} \xrightarrow{\text{b}} \sigma \quad \text{as } n \to \infty. \tag{3.4}$$

Part (ii). Now, we introduce an appropriate sequence $\{A_n : n = 0, 1, \ldots\}$ of operators in $\mathfrak{A}$ for the approximation to the vectors of the given sequence $\{\xi_n\}$, while taking care of the fulfillment of the second condition in (1.5). We set (cf. (2.9))

$$S_n := \sum_{k=0}^{n} A_k, \quad T_n := \sum_{k=0}^{n} \Big(1 - \frac{\lambda_k}{\lambda_{n+1}}\Big) A_k, \quad \text{and} \quad \eta_n := \xi_n - \pi(A_n)\omega, \quad n = 0, 1, \ldots.$$

For each integer $m \geqslant \nu_0$, we denote by $n(m)$ the unique integer $n(m)$ for which condition (2.18) is satisfied.

Now, we consider the representation

$$\tau_m = [\tau_m - \tau_{\nu_{n(m)}} - \pi(T_m)\omega + \pi(T_{\nu_{n(m)}})\omega] + [\pi(T_m)\omega - \pi(T_{\nu_{n(m)}})\omega] + \tau_{\nu_{n(m)}}. \tag{3.5}$$

By (3.4), the last term on the right-hand side of (3.5) is bundle convergent to σ as $m \to \infty$ (and, consequently, as $n(m) \to \infty$). So, it remains to show that the expressions in the first two brackets on the right-hand side of (3.5) are bundle convergent to o as $m \to \infty$.

Part (iii). We start with the following representation of the expression in the first brackets:

$$\begin{aligned}\tau_m - \tau_{\nu_{n(m)}} - \pi(T_m)\omega + \pi(T_{\nu_{n(m)}})\omega &= \sum_{k=0}^{m}\Big(1-\frac{\lambda_k}{\lambda_{m+1}}\Big)\eta_k - \sum_{k=0}^{\nu_{n(m)}}\Big(1-\frac{\lambda_k}{\lambda_{\nu_{n(m)}+1}}\Big)\eta_k \\ &= \sum_{k=\nu_{n(m)}+1}^{m}\eta_k - \frac{1}{\lambda_{m+1}}\sum_{k=0}^{m}\lambda_k\eta_k + \frac{1}{\lambda_{\nu_{n(m)}+1}}\sum_{k=0}^{\nu_{n(m)}}\lambda_k\eta_k \qquad (3.6)\\ &=: \alpha_m + \beta_m + \gamma_m, \quad m=0,1,\ldots.\end{aligned}$$

For the sake of brevity in writing, we introduce the notation

$$J_n := \{\nu_n, \nu_n+1, \ldots, \nu_{n+1}-1\}, \quad n=0,1,\ldots.$$

It is easy to see that

$$\sum_{m=\nu_0}^{\infty}\|\alpha_m\|^2 = \sum_{n=0}^{\infty}\sum_{m\in J_n}\|\alpha_m\|^2 \leqslant \sum_{n=0}^{\infty}\sum_{m\in J_n}(m-\nu_n)\sum_{k=\nu_n+1}^{m}\|\eta_k\|^2.$$

We will assume that

$$\|\eta_k\|^2 \leqslant \frac{1}{(n+1)^2(\nu_{n+1}-\nu_n)^3} \quad \text{for } k\in J_n; \quad n=0,1,\ldots, \tag{3.7}$$

then

$$\sum_{k=\nu_n+1}^{m}\|\eta_k\|^2 \leqslant \frac{1}{(n+1)^2(\nu_{n+1}-\nu_n)^2} \quad \text{for } m\in J_n,$$

and, consequently, we have

$$\sum_{m=\nu_0}^{\infty}\|\alpha_m\|^2 \leqslant \sum_{n=0}^{\infty}\frac{1}{(n+1)^2} < \infty.$$

To sum up, under condition (3.7), it follows that

$$\alpha_m \xrightarrow{\text{b}} o \quad \text{as } m\to\infty. \tag{3.8}$$

By the triangle inequality, we estimate as follows:

$$\|\beta_m\|^2 = \frac{1}{\lambda_{m+1}^2}\Big\|\sum_{k=0}^{m}\lambda_k\eta_k\Big\|^2 \leqslant \frac{1}{\lambda_{m+1}^2}\Big(\sum_{k=0}^{m}\lambda_k\|\eta_k\|\Big)^2.$$

We will also assume that

$$\|\eta_k\| \leqslant \frac{1}{(k+1)^2\lambda_k}, \quad k=0,1,\ldots, \tag{3.9}$$

then it follows from (3.1) that

$$\sum_{m=\nu_0}^{\infty}\|\beta_m\|^2 \leqslant \sum_{m=\nu_0}^{\infty}\frac{1}{\lambda_{m+1}^2}\Big(\sum_{k=0}^{\infty}\frac{1}{(k+1)^2}\Big)^2 < \infty,$$

and, consequently, we have

$$\beta_m \xrightarrow{\text{b}} o \quad \text{as } m\to\infty. \tag{3.10}$$

It remains to observe that $\gamma_m = \beta_{\nu_{n(m)}}$, that is $\{\gamma_m\}$ is a subsequence of $\{\beta_m\}$. Thus, by (3.6), (3.8) and (3.10), we conclude

$$\tau_m - \tau_{\nu_{n(m)}} - \pi(T_m)\omega + \pi(T_{\nu_{n(m)}})\omega \xrightarrow{\text{b}} o \quad \text{as } m \to \infty, \tag{3.11}$$

provided that conditions (3.7) and (3.9) are satisfied.

Part (iv). Next, we deal with the expression in the second brackets on the right-hand side of (3.5). We start with the representation

$$\begin{aligned}\pi(T_{j+1})\omega - \pi(T_j)\omega = [\pi(T_{j+1})\omega - \tau_{j+1} - \pi(T_j)\omega + \tau_j] \\ + [\tau_{j+1} - \tau_j] =: \chi_j + (\tau_{j+1} - \tau_j), \quad j = 0, 1, \ldots .\end{aligned} \tag{3.12}$$

By the triangle inequality, we clearly have

$$\|\pi(T_{j+1})\omega - \pi(T_j)\omega\|^2 \leqslant 2(\|\chi_j\|^2 + \|\tau_{j+1} - \tau_j\|^2). \tag{3.13}$$

Since

$$\chi_j = -\sum_{k=0}^{j+1}\Big(1-\frac{\lambda_k}{\lambda_{j+2}}\Big)\eta_k + \sum_{k=0}^{j}\Big(1-\frac{\lambda_k}{\lambda_{j+1}}\Big)\eta_k = -\Big(1-\frac{\lambda_{j+1}}{\lambda_{j+2}}\Big)\eta_{j+1} + \Big(\frac{1}{\lambda_{j+2}} - \frac{1}{\lambda_{j+1}}\Big)\sum_{k=0}^{j}\lambda_k\eta_k,$$

we may estimate again as above:

$$\|\chi_j\|^2 \leqslant 2\Big[\|\eta_{j+1}\|^2 + \Big(\frac{1}{\lambda_{j+2}} - \frac{1}{\lambda_{j+1}}\Big)^2\Big\|\sum_{k=0}^{j}\lambda_k\eta_k\Big\|^2\Big]. \tag{3.14}$$

If (3.9) is satisfied, then we have

$$\Big\|\sum_{k=0}^{j}\lambda_k\eta_k\Big\|^2 \leqslant \Big(\sum_{k=0}^{j}|\lambda_k|\|\eta_k\|\Big)^2 \leqslant \Big(\frac{\pi^2}{6}\Big)^2 =: K, \quad j = 0, 1, \ldots . \tag{3.15}$$

Combining (3.12)–(3.15), while taking into account (1.1), yields

$$\begin{aligned}\phi(|T_{j+1} - T_j|^2) &= \|\pi(T_{j+1})\omega - \pi(T_j)\omega\|^2 \\ &\leqslant 4\Big[\|\eta_{j+1}\|^2 + \Big(\frac{1}{\lambda_{j+2}} - \frac{1}{\lambda_{j+1}}\Big)^2 K\Big] + 2\|\tau_{j+1} - \tau_j\|^2,\end{aligned} \tag{3.16}$$

provided that (3.9) is satisfied.

Now, we consider the sequence $\{D_n : n = 0, 1, \ldots\}$ of positive operators in $\mathfrak{A}$ defined in (2.14) in the proof of Theorem 2.1. Combining (2.14) and (3.16) gives

$$\begin{aligned}\sum_{n=0}^{\infty}\phi(D_n) &\leqslant 4\sum_{n=0}^{\infty}\sum_{j=\nu_n}^{\nu_{n+1}-2}\frac{\lambda_{j+1}}{\lambda_{j+2} - \lambda_{j+1}}\Big[\|\eta_{j+1}\|^2 + \Big(\frac{1}{\lambda_{j+1}} - \frac{1}{\lambda_{j+2}}\Big)^2 K\Big] \\ &\quad + 2\sum_{n=0}^{\infty}\sum_{j=\nu_n}^{\nu_{n+1}-2}\frac{\lambda_{j+1}}{\lambda_{j+2} - \lambda_{j+1}}\|\tau_{j+1} - \tau_j\|^2 =: E + F, \quad \text{say.}\end{aligned} \tag{3.17}$$

In an analogous way as in the proof of Theorem 2.1 (cf. (2.15)), we obtain

$$\|\tau_{j+1} - \tau_j\|^2 = \Big(\frac{\lambda_{j+2} - \lambda_{j+1}}{\lambda_{j+1}\lambda_{j+2}}\Big)^2 \sum_{k=1}^{j+1} \lambda_k^2 c_k^2,$$

whence it follows (cf. (2.17)) that

$$F = 2\sum_{n=0}^{\infty}\sum_{j=\nu_n}^{\nu_{n+1}-2} \frac{\lambda_{j+2} - \lambda_{j+1}}{\lambda_{j+1}\lambda_{j+2}^2} \sum_{k=1}^{j+1} \lambda_k^2 c_k^2 \leqslant \sum_{j=\nu_0}^{\infty} \Big(\frac{1}{\lambda_{j+1}^2} - \frac{1}{\lambda_{j+2}^2}\Big) \sum_{k=1}^{j+1} \lambda_k^2 c_k^2 \tag{3.18}$$

$$\leqslant \sum_{k=1}^{\infty} \lambda_k^2 c_k^2 \sum_{j=k-1}^{\infty} \Big(\frac{1}{\lambda_{j+1}^2} - \frac{1}{\lambda_{j+2}^2}\Big) = \sum_{k=1}^{\infty} c_k^2 := \sum_{k=1}^{\infty} \|\xi_k\|^2 < \infty.$$

Next, we estimate E. Clearly, we have

$$E \leqslant 4\sum_{j=\nu_0}^{\infty} \frac{\lambda_{j+1}}{\lambda_{j+2} - \lambda_{j+1}} \Big[\|\eta_{j+1}\|^2 + \Big(\frac{1}{\lambda_{j+1}} - \frac{1}{\lambda_{j+2}}\Big)^2 K\Big]. \tag{3.19}$$

Since

$$\frac{\lambda_{j+1}}{\lambda_{j+2} - \lambda_{j+1}} \Big(\frac{1}{\lambda_{j+1}} - \frac{1}{\lambda_{j+2}}\Big)^2 = \frac{\lambda_{j+1}}{\lambda_{\lambda_{j+2}} - \lambda_{j+1}} \frac{(\lambda_{j+2} - \lambda_{j+1})^2}{\lambda_{j+1}^2\lambda_{j+2}^2}$$

$$= \frac{\lambda_{j+2} - \lambda_{j+1}}{\lambda_{j+1}\lambda_{j+2}^2} < \frac{1}{\lambda_{j+1}\lambda_{j+2}} < \frac{1}{\lambda_{j+1}^2},$$

it follows from (3.1) and (3.19) that

$$E \leqslant \sum_{j=\nu_0}^{\infty} \Big[\frac{\lambda_{j+1}}{\lambda_{j+2} - \lambda_{j+1}} \|\eta_{j+1}\|^2 + \frac{K}{\lambda_{j+1}^2}\Big] < \infty \tag{3.20}$$

if we impose upon $\{\eta_k\}$ the following third condition:

$$\sum_{j=0}^{\infty} \frac{\lambda_j}{\lambda_{j+1} - \lambda_j} \|\eta_j\|^2 < \infty. \tag{3.21}$$

From (3.17), (3.18) and (3.20) it follows that

$$\sum_{n=0}^{\infty} \phi(D_n) < \infty, \tag{3.22}$$

provided that conditions (3.9) and (3.21) are satisfied, where the sequence $\{D_n\}$ is defined in (2.14). Now, we can derive again the fulfillment of (2.20) exactly in the same way as in the proof of Theorem 2.1.

We recall that according to the Gelfand–Naimark–Segal representation theory, the closure of the set

$$\pi(\mathfrak{A})\omega := \{\pi(A)\omega : A \in \mathfrak{A}\}$$

in the norm $\|\cdot\|$ coincides with $L_2 = L_2(\mathfrak{A}, \phi)$ itself. Thus, we can define an approximating sequence $\{A_n\}$ of operators in $\mathfrak{A}$ to the vectors of the sequence

$\{\xi_n\}$ in such a way that conditions (3.7), (3.9) and (3.21) are satisfied for the vectors of the sequence $\{\eta_n := \xi_n - \pi(A_n)\omega\}$. By making use of (3.4), (3.5), (3.11), (2.20) as well as Property 3.6, p. 31 of [2] and the additivity property of bundle convergence, we conclude (3.3) with m in place of n. ∎

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