J. OPERATOR THEORY
65:1(2011), 47–70

NUCLEARITY-RELATED PROPERTIES FOR NONSELFADJOINT ALGEBRAS

DAVID P. BLECHER and BENTON L. DUNCAN

Communicated by Kenneth R. Davidson

ABSTRACT. In analogy with the C^*-algebra theory, we study variants appropriate to nonselfadjoint algebras of nuclearity, the local lifting property, exactness, and the weak expectation property. In addition, we study the relationships between these notions, and how they are connected with the classical C^*-algebra theory through the use of C^*-algebras generated by the algebra.

KEYWORDS: *Nonselfadjoint operator algebras, tensor products, nuclearity, exactness, weak expectation property WEP, LLP, unique extension property, C^*-envelope.*

MSC (2000): Primary 46L06, 46L07, 47L30, 47L40; Secondary 46B28, 46L09, 47L55, 46M05.

1. INTRODUCTION

The concept of *nuclearity* is fundamental in the study of C^*-algebras. It is often defined in terms of tensor products, as also are the slightly less well known, but also fundamental, properties known as (Lance's) *weak expectation property* (WEP), and (Kirchberg's) *exactness* and *local lifting property* (LLP). These are intimately related properties which a C^*-algebra may or may not possess, and the relations between them and ensuing theory are rich and profound. The goal of the present paper is to find appropriate generalizations of these notions to possibly nonselfadjoint operator algebras; and to illuminate some of the good, the bad, and the ugly that ensues. More precisely, some of the elegant basic implications and arguments in the C^*-algebraic theory are still valid in the nonselfadjoint case, while others only seem to be true for very special classes of algebras. That is, for extremely general classes of algebras some of these properties may be a little restrictive; and in this sense the present investigation is not as successful as some of our previous generalizations of C^*-algebraic notions to nonselfadjoint algebras. Nonetheless, the properties we introduce are natural, and almost all do not seem to have been considered before. Moreover, en route we present several new and very basic results of independent interest. There also seems to be some hope, as the reader may see at points in our paper, that they could lead in the future to a new approach to Kirchberg's famous conjectures (and also give variants of his conjectures for nonselfadjoint algebras which may be easier to solve). Another of our motivations was to find properties that imply that Ext of the C^*-envelope of a nonselfadjoint algebra is a group, and this angle is discussed in a paper [6] in preparation. We recall from [17] that the LLP is more than intimately connected with Ext of a separable C^*-algebra B being a group (the first implies the second, and the converse is an open question).

The new concepts introduced here are called *C^*-nuclearity*, the *algebra weak expectation property* (AWEP), the *homomorphism local lifting property* (HLLP), *$\mathbb{B}$-nuclearity*, and *subexactness*. Motivated primarily by Kirchberg's astonishing paper [17], and its operator space sequel due to Pisier, Effros, Ruan, and others, we try to build connections between our new variants that are similar to the classical C^*-algebraic theory. Throughout, we use generated C^*-algebras to relate these notions to their classical counterpart. For instance, if A is *Dirichlet* (that is, a unital algebra with $A + A^*$ norm dense in its C^*-envelope), then A has each of the five properties above if and only if the C^*-envelope of A has the matching C^*-algebra property (with one exception in one direction: we are not sure if A having HLLP implies that the C^*-envelope has LLP).

It is important for us to say that of course a significant part of modern operator space theory is devoted to linear analogues of some of the properties mentioned above; very strikingly some of the above-mentioned properties and their beautiful theory generalizes to operator spaces (see e.g. [15], [29]). However, with one exception the operator space versions of these properties turn out not to be appropriate for nonselfadjoint operator algebras, at least for the approach taken here.

Turning to notation, by an operator algebra, we mean a closed, not necessarily selfadjoint, algebra of operators on a Hilbert space. We will sometimes silently be using very basic principles from the theory of operator algebras, all of which are explained in [10]. An operator algebra is *unital* if it has an identity of norm 1, and is *approximately unital* if it has a contractive two-sided approximate identity (cai). For simplicity we will usually assume that our operator algebras are unital or approximately unital, but in many of the results this restriction is not necessary, by the usual trick of considering the unitization. We write A^1 for the unitization of a nonunital operator algebra (see Section 2.1 of [10]). A *unital-subalgebra* is a subalgebra containing the identity of the bigger algebra. All ideals are assumed to be two-sided and closed. Our morphisms will be linear completely contractive homomorphisms $\theta : A \to B$ between operator algebras. If $\theta(1) = 1$ we say that θ is *unital*. As usual, *UCP* means unital and completely positive. A *C^*-cover* of A is a C^*-algebra containing a copy of A completely isometrically as a subalgebra, which is generated by this copy. There are two "universal" C^*-covers of A, a "smallest" and a "largest": the *C^*-envelope* $C^*_e(A)$, and the *maximal C^*-dilation* $C^*_{\max}(A)$. We refer the reader to [10] for a discussion of these two notions; they have the extremal universal properties which the reader would expect. A new term: we will say that A is *C^*-split* if there exists a linear complete contraction $u : C^*_e(A) \to C^*_{\max}(A)$ extending the identity map on A. An example of this is any Dirichlet operator algebra (this may be seen by Arveson's Proposition 1.2.8 of [3]). By the well-known "rigidity" property of $C^*_e(A)$, it follows that u is a right inverse for the epimorphism $C^*_{\max}(A) \to C^*_e(A)$. We write $\mathbb{B}$ and $\mathbb{K}$ for respectively the bounded and compact linear operators on ℓ^2. We refer the reader to Chapter 6 of [10] or [27] for the tensor products of operator algebras used here. In places we also use notation and results from the paper [11] on extensions of nonselfadjoint algebras. By an *extension in the sense of* [11], we will mean a short exact sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

of nontrivial operator algebras, with A approximately unital; and α, β are completely contractive homomorphisms, with α completely isometric, and β a complete quotient map. Applications of our work to the theory of extensions are presented in a paper [6] in preparation, which is a sequel to [11]. For example, if A is subexact and C^*-nuclear then Ext is a group.

2. C^*- AND $\mathbb{B}$-NUCLEARITY

DEFINITION 2.1. An operator algebra A is *C^*-nuclear* (respectively *$\mathbb{B}$-nuclear*) if

$$A \otimes_{\max} B = A \otimes_{\min} B$$

for every C^*-algebra B (respectively for $B = \mathbb{B} = B(\ell^2)$).

It is important to note that if A is not selfadjoint then allowing B in the definition of C^*-nuclear to be nonselfadjoint yields a vacuous class, by Corollary 7.1.8 of [10].

In 6.2.5 of [10] it is remarked that any Dirichlet uniform algebra, such as the disk algebra $A(\mathbb{D})$, is C^*-nuclear in our terminology. Of course if A is a C^*-algebra then A is C^*-nuclear if and only if A is nuclear, and A is $\mathbb{B}$-nuclear if and only if A has the local lifting property (LLP), see [17], [29]. As in the C^*-algebra case, we will see that C^*-nuclearity implies the other new properties mentioned in the introduction, with the possible exception of subexactness (C^*-nuclearity does imply exactness).

Since we will use it many times we restate Lemma 2.8 of [11]:

LEMMA 2.2. *For an approximately unital operator algebra A, and a C^*-algebra B, we have $A \otimes_{\max} B \subset C^*_{\max}(A \otimes_{\max} B) = C^*_{\max}(A) \otimes_{\max} B$ completely isometrically.*

It is not important that the Hilbert space appearing in the definition of $\mathbb{B}$-nuclearity be separable:

LEMMA 2.3. *An approximately unital operator algebra A is $\mathbb{B}$-nuclear if and only if $A \otimes_{\max} B(H) = A \otimes_{\min} B(H)$ for every infinite dimensional Hilbert space H.*

Proof. Note that since $\mathbb{B}$ can be embedded as a complemented corner of $B(H)$ we see from Lemma 2.2 and e.g. 6.1.10 of [10] that we have canonical complete isometries

$$A \otimes_{\max} \mathbb{B} \subset C^*_{\max}(A) \otimes_{\max} \mathbb{B} \subset C^*_{\max}(A) \otimes_{\max} B(H) = C^*_{\max}(A \otimes_{\max} B(H)).$$

Thus $A \otimes_{\max} \mathbb{B} \subset A \otimes_{\max} B(H)$. Clearly $A \otimes_{\min} \mathbb{B} \subset A \otimes_{\min} B(H)$ and thus the reverse implication holds.

For the forward direction suppose A is $\mathbb{B}$-nuclear, and let $u = \sum_{k=1}^{n} a_k \otimes T_k$ for $a_k \in A, T_k \in B(H)$. Let D be the C^*-algebra generated by 1 and the T_k. This is separable and so there is a unital $*$-isomorphism π from D onto a C^*-algebra in $\mathbb{B}$, carrying T_k to S_k say. The inverse of this $*$-isomorphism extends to a UCP map $\theta : \mathbb{B} \to B(H)$. Then

$$\|u\|_{A \otimes_{\max} B(H)} = \Big\| \sum_{k=1}^{n} a_k \otimes \theta(S_k) \Big\|_{A \otimes_{\max} B(H)} \leqslant \Big\| \sum_{k=1}^{n} a_k \otimes S_k \Big\|_{A \otimes_{\max} \mathbb{B}}.$$

Since A is $\mathbb{B}$-nuclear, the last norm equals

$$\Big\| \sum_{k=1}^{n} a_k \otimes \pi(T_k) \Big\|_{A \otimes_{\min} \mathbb{B}} = \Big\| \sum_{k=1}^{n} a_k \otimes T_k \Big\|_{A \otimes_{\min} D} = \Big\| \sum_{k=1}^{n} a_k \otimes T_k \Big\|_{A \otimes_{\min} B(H)},$$

by injectivity of $\otimes_{\min}$. The result is now clear. ∎

One theme of our paper is that an operator space or C^*-algebraic property such as nuclearity or the LLP for a C^*-cover of an algebra A, often says something about C^*-nuclearity or $\mathbb{B}$-nuclearity for A; or vice versa. The following is a fairly superficial result of this kind. In the Examples section we shall show that there are commonly met operator algebras for which $C^*_{\max}(A)$ has the LLP.

PROPOSITION 2.4. *If $C^*_{\max}(A)$ is nuclear (respectively has the* LLP*) then A is C^*-nuclear (respectively is $\mathbb{B}$-nuclear).*

Proof. For a C^*-algebra B (respectively $B = \mathbb{B}$) we have canonical complete isometries

$$A \otimes_{\min} B \to C^*_{\max}(A) \otimes_{\min} B = C^*_{\max}(A) \otimes_{\max} B = C^*_{\max}(A \otimes_{\max} B)$$

which compose to a map whose range is in $A \otimes_{\max} B$. ∎

We now look at stability of C^*-nuclearity and $\mathbb{B}$-nuclearity under the usual operator algebra constructions.

PROPOSITION 2.5. *If A is an approximately unital ideal in an approximately unital operator algebra B, and if D is any approximately unital operator algebra, then $A \otimes_{\max} D \subset B \otimes_{\max} D$ completely isometrically.*

Proof. Take two nondegenerate commuting completely contractive representations $\pi : A \to B(H)$ and $\theta : D \to B(H)$. By 2.6.13 of [10] we can extend π to a completely contractive representation $\widetilde{\pi} : B \to B(H)$ with

$$\theta(d)\widetilde{\pi}(b)\pi(a)\zeta = \theta(d)\pi(ba)\zeta = \pi(ba)\theta(d)\zeta = \widetilde{\pi}(b)\theta(d)\pi(a)\zeta,$$

for all $a \in A, b \in B, d \in D$, and $\zeta \in H$. It follows that $\widetilde{\pi}$ commutes with θ and hence the closure of $A \otimes D$ in $B \otimes_{\max} D$ will have the desired property for $A \otimes_{\max} D$ (see 6.1.1 and 6.1.11 in [10].) ∎

COROLLARY 2.6. *If A is an approximately unital ideal in a C^*-nuclear (respectively $\mathbb{B}$-nuclear) approximately unital algebra B, then A is C^*-nuclear (respectively $\mathbb{B}$-nuclear).*

REMARK 2.7. If A is a subalgebra of an operator algebra B, and if D is a nonselfadjoint operator algebra, then it need not be the case that $A \otimes_{\max} D$ is contained in $B \otimes_{\max} D$ completely isometrically. Note that it follows from Lemma 2.2 that $A \otimes_{\max} D \subset C^*_{\max}(A) \otimes_{\max} D$ for any C^*-algebra D, but this relation can be false if D is nonselfadjoint. To see this let A be an operator algebra for which $A \otimes_{\min} A(\mathbb{D})$ is not $A \otimes_{\max} A(\mathbb{D})$ (see Corollary 7.1.8 of [10]). Now $A(\mathbb{D})$ is C^*-nuclear as we remarked earlier, and hence we must have $C^*_{\max}(A) \otimes_{\min} A(\mathbb{D})$ equal to $C^*_{\max}(A) \otimes_{\max} A(\mathbb{D})$. This contains $A \otimes_{\min} A(\mathbb{D})$, and so it cannot contain $A \otimes_{\max} A(\mathbb{D})$.

To deal with quotients we will need the following. We do not know if the result is true for nonselfadjoint D.

LEMMA 2.8. *If D is a C^*-algebra and if A is an approximately unital ideal in an operator algebra B, then $(B \otimes_{\max} D)/(A \otimes_{\max} D) \cong (B/A) \otimes_{\max} D$ completely isometrically. The same relation holds if D is an approximately unital operator algebra and B is a C^*-algebra with ideal A.*

Proof. The assertions have similar proofs so we prove only the first. We know from Lemma 2.7 of [11] that $C^*_{\max}(A)$ is an approximately unital ideal in $C^*_{\max}(B)$, and that $C^*_{\max}(B)/C^*_{\max}(A) \cong C^*_{\max}(B/A)$. Setting $C = B/A$, and using the usual C^*-algebra result, we have the extension

$$0 \longrightarrow C^*_{\max}(A) \otimes_{\max} D \longrightarrow C^*_{\max}(B) \otimes_{\max} D \longrightarrow C^*_{\max}(C) \otimes_{\max} D \longrightarrow 0.$$

We will show that

$$0 \longrightarrow A \otimes_{\max} D \longrightarrow B \otimes_{\max} D \longrightarrow C \otimes_{\max} D \longrightarrow 0$$

is a *subextension* of the first extension, in the sense of Section 3.6 of [11]. To see this first note that, by Lemma 2.2, each term in the last sequence is a subalgebra of the matching term in the C^*-algebra extension. Moreover the intersection $(C^*_{\max}(A) \otimes_{\max} D) \cap (B \otimes_{\max} D)$ in $C^*_{\max}(B) \otimes_{\max} D$ is the closure of $A \otimes D$, as may be seen by the following argument. If u is in this intersection, and if (e_t) (respectively (f_s)) is a cai for A (respectively D), then $(e_t \otimes f_s)u$ is in the closure of $A \otimes D$. Taking the limit, and using the fact that (f_s) is a cai for $C^*_{\max}(D)$, we see that u is in the closure of $A \otimes D$ in $C^*_{\max}(B) \otimes_{\max} D$. But this closure is in the image of $A \otimes_{\max} D$ since by the previous result, $A \otimes_{\max} D \subset B \otimes_{\max} D \subset C^*_{\max}(B) \otimes_{\max} D$. By Proposition 3.6 of [11] we have a subextension, proving the result. ∎

LEMMA 2.9. *Every C^*-nuclear approximately unital operator algebra is exact, hence is locally reflexive.*

Proof. If D is a C^*-nuclear approximately unital operator algebra, and if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a C^*-extension, then by Lemma 2.8 we have an exact sequence

$$0 \longrightarrow C^*_{\max}(D) \otimes_{\max} A \longrightarrow C^*_{\max}(D) \otimes_{\max} B \longrightarrow C^*_{\max}(D) \otimes_{\max} C \longrightarrow 0.$$

As in the proof of Lemma 2.8 we have that

$$0 \longrightarrow D \otimes_{\max} A \longrightarrow D \otimes_{\max} B \longrightarrow D \otimes_{\max} C \longrightarrow 0$$

is a subextension in the sense of Section 3.6 of [11], hence it is a "1-exact" sequence in the sense of [15]. Thus by Theorem 14.4.1 of [15] we see that D is exact. ∎

We are now ready for some results concerning quotients. It is not clear whether in the following proposition a weaker condition on B/A than the completely contractive approximation property will suffice.

PROPOSITION 2.10. *Let A be an approximately unital ideal in a C^*-nuclear approximately unital operator algebra B. If B/A is separable and has the completely contractive approximation property, then B/A is C^*-nuclear.*

Proof. By Proposition 2.6 we know that A is C^*-nuclear too. By Lemma 2.8 we have $(B/A) \otimes_{\max} D \cong (B \otimes_{\max} D)/(A \otimes_{\max} D)$ for any C^*-algebra D. Since C^*-nuclearity implies local reflexivity by Lemma 2.9, the extension of B/A by A satisfies the condition of the lifting theorem in [14]. Hence by that result there is a completely contractive linear splitting map $B/A \to B$. From this it follows easily that $(B/A) \otimes_{\min} D \cong (B \otimes_{\min} D)/(A \otimes_{\min} D) \cong (B/A) \otimes_{\max} D$. ∎

THEOREM 2.11. *If C is a C^*-nuclear approximately unital operator algebra, and if*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an extension, then for every C^*-algebra D, the associated sequence

$$0 \longrightarrow A \otimes_{\min} D \longrightarrow B \otimes_{\min} D \longrightarrow C \otimes_{\min} D \longrightarrow 0$$

is an extension (these are extensions in the sense of [11]*). So $(B \otimes_{\min} D)/(A \otimes_{\min} D) \cong (B/A) \otimes_{\min} D$ completely isometrically.*

Proof. The canonical morphism from $B \otimes_{\max} D$ to $(B \otimes_{\min} D)/(A \otimes_{\min} D)$ annihilates the closure of $A \otimes D$, which by Proposition 2.5 is equal to $A \otimes_{\max} D$. Thus by Lemma 2.8 we have canonical completely contractive morphisms

$$C \otimes_{\max} D \cong (B \otimes_{\max} D)/(A \otimes_{\max} D) \to (B \otimes_{\min} D)/(A \otimes_{\min} D) \to C \otimes_{\min} D$$

which compose to the identity map since $C \otimes_{\min} D \cong C \otimes_{\max} D$. Since the range of the first "arrow" is dense, we see that $C \otimes_{\min} D \cong (B \otimes_{\min} D)/(A \otimes_{\min} D)$ via the canonical map. This completes the proof. ∎

REMARK 2.12. There are several conditions equivalent to an extension having the "tensorizing with every C^*-algebra" property in the last theorem. These are studied in [6].

COROLLARY 2.13. *Let*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an extension of operator algebras in the sense of [11]. *If both A and C are C^*-nuclear (respectively $\mathbb{B}$-nuclear) then B is C^*-nuclear (respectively $\mathbb{B}$-nuclear).*

Proof. Let D be a C^*-algebra. By Lemma 2.8 we have an extension

$$0 \longrightarrow A \otimes_{\max} D \longrightarrow B \otimes_{\max} D \longrightarrow C \otimes_{\max} D \longrightarrow 0 \,.$$

By Theorem 2.11 (or a variant of it in the $\mathbb{B}$-nuclear case) we have an extension

$$0 \longrightarrow A \otimes_{\min} D \longrightarrow B \otimes_{\min} D \longrightarrow C \otimes_{\min} D \longrightarrow 0 \,.$$

Now apply the "five lemma" from Lemma 3.2 of [11]. ∎

COROLLARY 2.14. *Let A be an approximately unital operator algebra. Then A is C^*-nuclear (respectively $\mathbb{B}$-nuclear) if and only if A^1 is C^*-nuclear (respectively $\mathbb{B}$-nuclear)*

Proof. We have the short exact sequence

$$0 \longrightarrow A \longrightarrow A^1 \longrightarrow \mathbb{C} \longrightarrow 0$$

with A an approximately unital ideal in A^1. The forward direction now follows from the previous result, and the reverse follows from Corollary 2.6. ∎

It is characteristic of the present paper that one gets much better results by restricting the class of operator algebras:

PROPOSITION 2.15. *If A is a Dirichlet operator algebra then A is C^*-nuclear (respectively $\mathbb{B}$-nuclear) if and only if $C^*_e(A)$ is nuclear (respectively has the* LLP*). The ($\Leftarrow$) implications are also true if A is merely C^*-split.*

Proof. If B is a C^*-algebra, then $A \otimes_{\max} B \subset C^*_{\max}(A) \otimes_{\max} B$ by Lemma 2.2. If A is C^*-split then there is a right inverse to the canonical map $C^*_{\max}(A) \otimes_{\max} B \to C^*_e(A) \otimes_{\max} B$. If the latter equals $C^*_e(A) \otimes_{\min} B$, then it follows that $A \otimes_{\max} B = A \otimes_{\min} B$. Conversely if the latter holds, then $A \otimes_{\max} B \subset C^*_e(A) \otimes_{\max} B$, since the inclusion of $A \otimes B$ induces a tensor norm which must then agree with $\otimes_{\max}$. If A is Dirichlet, then Arveson's Proposition 1.2.8 of [3] gives a complete isometry from the closure of $A \otimes_{\min} B + A^* \otimes_{\min} B$ which is $C_e^*(A) \otimes_{\min} B$, into the closure of $A \otimes_{\max} B + A^* \otimes_{\max} B$ in $C_e^*(A) \otimes_{\max} B$, which is $C_e^*(A) \otimes_{\max} B$. ∎

In Corollary 5.9 we will improve the last assertion of the last proposition.

3. THE HOMOMORPHISM LOCAL LIFTING PROPERTY AND $\mathbb{B}$-NUCLEARITY

In analogy with the C^*-algebraic theory of the LLP, one would expect a relationship between $\mathbb{B}$-nuclearity and lifting properties. At present we only see one direction of the relationship, which will be presented in the next theorem.

DEFINITION 3.1. An operator algebra C has *the homomorphism local lifting property (HLLP)* if for every operator algebra B, and any approximately unital ideal A in B, and any completely contractive homomorphism $u : C \to B/A$ and finite dimensional subspace $E \subset C$, there is a complete contraction from E to B which is a lift of $u|_E$.

Our original motivation in studying the HLLP, is that it has some connections with the topic of when Ext is a group, and this is studied in [6]. For example, we show in [6] that every extension in the sense of [11] of a separable operator algebra with the HLLP, by $\mathbb{K}$ (or by any C^*-algebra with a property described there) is "semisplit" in the sense of [6].

THEOREM 3.2. *Let C be an approximately unital operator algebra. If C is $\mathbb{B}$-nuclear then C has the* HLLP.

We defer the proof momentarily to prove a lemma which is of independent interest (solving an open question about tensor products of M-ideals in a special case: see the discussion before Proposition 1.7 in [1]).

LEMMA 3.3. *Suppose that A is an approximately unital ideal in an operator algebra B, and that E is an operator space. Then $A \otimes_{\min} E$ is a complete M-ideal in $B \otimes_{\min} E$, and in particular is proximinal.*

Proof. We know that $A^{\perp\perp} = eB^{**}$ for a central projection $e \in B^{**}$. Notice that $B \otimes_{\min} E$ is a left operator B-module (see e.g. the second paragraph of the Notes for Section 3.4 in [10]). Thus $(B \otimes_{\min} E)^{**}$ is a left dual operator B^{**}-module, by 3.8.9 in [10]. Hence $e \in B^{**}$ may be regarded as a left M-projection on $(B \otimes_{\min} E)^{**}$, and we claim that if we do so then $(A \otimes_{\min} E)^{\perp\perp} = e(B \otimes_{\min} E)^{**}$. It is routine to see that $(A \otimes_{\min} E)^{\perp\perp} \subset e(B \otimes_{\min} E)^{**}$. On the other hand, if $a_t \to e$ weak* in B^{**}, with $a_t \in A$, then since $a_t(b \otimes x) \in A \otimes E \subset (A \otimes_{\min} E)^{\perp\perp}$ for $b \in B$ and $x \in E$, it follows by separate weak* continuity and density that $e(B \otimes_{\min} E)^{**} \subset (A \otimes_{\min} E)^{\perp\perp}$. This proves the claim, and shows that $A \otimes_{\min} E$

is a complete right M-ideal in $B \otimes_{\min} E$. Similarly, it is a complete left M-ideal in $B \otimes_{\min} E$, and so it is a complete M-ideal [7]. ∎

Proof of Theorem 3.2. We will adapt a proof due to Pisier of Kirchberg's result that $\mathbb{B}$-nuclearity implies the LLP for C^*-algebras. The reader should follow along with the proof of (iii) implies (i) in Theorem 16.2 of [29]. We begin with $s \in (A/I) \otimes E^*$, as in that proof. Suppose $s = \sum_{k=1}^{n} [a_k] \otimes \psi_k$, for $a_k \in A, \psi_k \in E^* \subset B(H)$. The first change that needs to be made is that instead of appealing to (11.1) one uses the functoriality of the $\otimes_{\max}$ tensor product of operator algebras. (Note: this is the only place where u being a homomorphism is used.) The appeal to Exercise 11.2 is replaced by Lemma 2.8 above. One obtains $\left\| \sum_{k=1}^{n} [a_k] \otimes \psi_k \right\| \leqslant 1$ in $(A/I) \otimes_{\max} B(H)$. Then $\left\| \sum_{k=1}^{n} a_k \otimes \psi_k \right\| \leqslant 1$ in $(A \otimes_{\max} B(H))/(I \otimes_{\max} B(H))$, and it follows easily that $\left\| \sum_{k=1}^{n} a_k \otimes \psi_k + (I \otimes_{\min} B(H)) \right\| \leqslant 1$ in $(A \otimes_{\min} B(H))/(I \otimes_{\min} B(H))$. The proof of Lemma 2.4.8 in [29] may be easily adapted to our case, if one uses the known fact (see e.g. [12], [1] or Proposition 6.3 of [8]) that any ideal with cai has an approximate identity of the form $(1 - x_t)$ with $\|x_t\| \to 1$. This implies that $\left\| \sum_{k=1}^{n} a_k \otimes \psi_k + (I \otimes_{\min} E^*) \right\| \leqslant 1$ in $(A \otimes_{\min} E^*)/(I \otimes_{\min} E^*)$. The proof is completed as in [29] by an appeal to Lemma 3.3. ∎

We now turn to operator algebras A with the *unique extension property (UEP)* considered in [13]: that is, A is a unital-subalgebra A of a C^*-algebra B, such that for every Hilbert space H and every unital $*$-homomorphism $\pi : B \to B(H)$, there is a unique UCP map $\Psi : B \to B(H)$ extending $\pi_{|A}$. Clearly, we can replace $B(H)$ in this definition, by "every unital C^*-algebra". It follows from the proof of Theorem 2.7 of [9] that $B = C^*_{\mathrm{e}}(A)$. Examples of algebras with the UEP include logmodular and Dirichlet algebra (see p. 159–161 in [10]), and some nest algebras and crossed products [13]. As Elias Katsoulis has pointed out, it is an easy consequence of Choi's "multiplicative domain" trick (see e.g. Lemma 14.2 of [29]), that any unital operator algebra generated by unitaries will have UEP.

LEMMA 3.4. *If A and B are operator algebras with the* UEP, *then $A \otimes_{\min} B$ has the* UEP.

Proof. Since A and B are unital there are completely isometric inclusions $A \subset A \otimes_{\min} B$ and $B \subset A \otimes_{\min} B$. Now let $\pi : C^*_{\mathrm{e}}(A \otimes_{\min} B) \to B(H)$ be a unital $*$-homomorphism, and let $\sigma : C^*_{\mathrm{e}}(A \otimes_{\min} B) \to B(H)$ be a unital completely positive map satisfying $\pi|_{A \otimes_{\min} B} = \tau|_{A \otimes_{\min} B}$. We know by Theorem 2.10 of [11] that $C^*_{\mathrm{e}}(A \otimes_{\min} B) = C^*_{\mathrm{e}}(A) \otimes_{\min} C^*_{\mathrm{e}}(B)$. By the unique extension property $\sigma|_{C^*_{\mathrm{e}}(A)} = \pi|_{C^*_{\mathrm{e}}(A)}$. A similar result holds for $C^*_{\mathrm{e}}(B)$, and hence $C^*_{\mathrm{e}}(A)$ and $C^*_{\mathrm{e}}(B)$ are contained in the multiplicative domain (see e.g. [26] or Lemma 14.2 of [29]) for σ. Since $C^*_e(A)$ and $C^*_e(B)$ generate $C^*_e(A \otimes_{\min} B)$, we have that $C^*_e(A \otimes_{\min} B)$ is contained in the multiplicative domain for σ. Thus σ is a $*$-homomorphism, and so $\pi = \sigma$. ∎

LEMMA 3.5. *If A is an operator algebra with the* UEP, *and N is a nonunital nuclear C^*-algebra, then $(A \otimes_{\min} N)^1$ has the* UEP.

Proof. Recall from [11] that

$$C^*_e((N \otimes_{\min} A)^1) = C^*_e(N \otimes_{\min} A)^1 = (N \otimes_{\min} C^*_e(A))^1.$$

Let $\theta : (N \otimes_{\min} C^*_e(A))^1 \to B(H)$ be a unital $*$-homomorphism, and let $\Phi : (N \otimes_{\min} C^*_e(A))^1 \to B(H)$ be a UCP map extending $\theta_{|(N \otimes_{\min} C^*_e(A))^1}$. We need to show that $\Phi = \theta$.

The restriction of θ to $N \otimes_{\min} C^*_e(A)$ is of the form $\pi \odot \rho$ for commuting $*$-representations $\pi : N \to B(H)$ and $\rho : C^*_e(A) \to B(H)$. If (e_t) is an increasing cai for N, then $\pi(e_t) \to q$ strongly for a projection q commuting with $\rho(1)$. Let $p = q\rho(1)$, and identify $pB(H)p = B(K)$, where $K = pH$. Since $\pi(f)\rho(a) = q\pi(f)\rho(a)\rho(1)$, we may replace π and ρ by $p\pi(\cdot)$ and $p\rho(\cdot)$. Then we may regard π and ρ as being $B(K)$-valued with commuting ranges, and now both of them are nondegenerate (for example $\pi(e_t) \to p = I_K$ strongly). Let $\pi^1 : N^1 \to B(K)$ be the "unitization" of π, then $\pi^1 \odot \rho$ is a unital $*$-homomorphism $N^1 \otimes_{\min} C^*_e(A) \to B(K)$. Let Ψ be the restriction of Φ to $N \otimes_{\min} C^*_e(A)$. Of course $\Phi(f \otimes 1_A) = \theta(f \otimes 1_A)$ for $f \in N$, so that it follows from a well known principle concerning completely positive maps, that $\Phi(fg \otimes a) = \theta(f \otimes 1_A)\Phi(g \otimes a) = \pi(f)p\Phi(g \otimes a)$ if $f, g \in N, a \in C^*_e(A)$. Thus Ψ may also be viewed as a $B(K)$-valued map.

It is well known that we can extend Ψ to a unital completely positive map from the subspace $N \otimes_{\min} C^*_e(A) + \mathbb{C}\, 1_{N^1} \otimes 1_A$ of $N^1 \otimes_{\min} C^*_e(A)$ to $B(K)$. We may then extend further to a unital completely positive map $\widetilde{\Psi} : N^1 \otimes_{\min} C^*_e(A) \to B(K)$. We claim that for $f \in N^1$ we have $\widetilde{\Psi}(f \otimes 1_A) = \pi^1(f)$. Indeed for $f \in N$ we have $\widetilde{\Psi}(f \otimes 1_A) = \Psi(f \otimes 1_A) = \pi(f) = \pi^1(f)$. The claim is also true for $f = 1_{N^1}$, and hence it is true in full generality. By the "well known principle" used a few lines earlier, we have $\widetilde{\Psi}(f \otimes a) = \pi^1(f)T(a)$ if $f \in N^1, a \in C^*_e(A)$, where $T(a) = \widetilde{\Psi}(1 \otimes a)$. Note that T is UCP. Also if $a \in A$ then

$$\pi(e_t)T(a) = \widetilde{\Psi}(e_t \otimes a) = \Phi(e_t \otimes a) = \pi(e_t)\rho(a),$$

and taking a limit shows that $T(a) = \rho(a)$. Since A has the UEP, $T = \rho$, and so

$$\Phi(f \otimes a) = \Psi(f \otimes a) = \widetilde{\Psi}(f \otimes a) = \pi^1(f)\rho(a) = \theta(f \otimes a),$$

for $f \in N, a \in C^*_e(A)$. It follows that $\Phi = \theta$ as desired. ∎

PROPOSITION 3.6. *If C is a separable unital $\mathbb{B}$-nuclear operator algebra with the* UEP, *then $C^*_e(C)$ has the* LLP.

Proof. It is shown in [6], using the HLLP and in particular the fact mentioned above Theorem 3.2, that these hypotheses imply that $\mathrm{Ext}_{\mathrm{u}}(C_{\mathrm{e}}^*(A))$ is a group. If $S(C)$ is the "unitized suspension" as in [17], then $S(C)$ is a separable, unital algebra. It is easy to see that it is $\mathbb{B}$-nuclear using Corollary 2.14. By the previous lemma we also have that $S(C)$ has the UEP, so that by the above we deduce that $\mathrm{Ext}_{\mathrm{u}}(C_{\mathrm{e}}^*(S(C)))$ is a group. Since $C_{\mathrm{e}}^*(S(C)) = S(C_{\mathrm{e}}^*(C))$ (see Corollary 2.11 of [11]), it now follows from a result of Kirchberg [17] that $C_{\mathrm{e}}^*(C)$ has the LLP. ∎

It follows that if A is C^*-split, or is separable and has UEP, then $C_{\mathrm{e}}^*(A)$ having LLP implies that A has HLLP (see Theorem 3.2 and Proposition 2.15).

4. WEAK EXPECTATION

We turn to themes connected with the weak expectation property. For simplicity, unless stated otherwise we assume that all algebras are unital, and that all subalgebras are "unital-subalgebras". We leave the nonunital case to the reader using the usual unitization results (e.g. as in Section 2.1, 6.1.6, and 6.1.11 of [10], and the remark after the next definition). This definition reduces to the WEP property of Lance if A is selfadjoint.

DEFINITION 4.1. We say that an operator algebra A has the *algebra weak expectation property* (AWEP) if $A \otimes_{\max} D \subset B \otimes_{\max} D$ completely isometrically for every (possibly nonunital) C^*-algebras B, D and completely isometric embedding of A as a subalgebra of B.

PROPOSITION 4.2. *If an operator algebra is C^*-nuclear then it has the* AWEP*.*

Proof. If A is C^*-nuclear then for all C^*-algebras D the embedding $A \otimes D \to B \otimes_{\max} D$ induces a tensor norm on $A \otimes D$ which must be the maximal tensor norm. ∎

REMARK 4.3. We may assume in the last definition that B is unital, and A is a unital-subalgebra of B. For suppose that $A \otimes_{\max} D \subset B \otimes_{\max} D$ for every unital C^*-algebra B containing A completely isometrically, and every C^*-algebra D. If C is a general C^*-algebra containing A as a subalgebra completely isometrically, and if $p \in C$ is the identity for A, then define $B = pCp$. The canonical projection of C onto B induces canonical complete contractions

$$A \otimes_{\max} D \longrightarrow B \otimes_{\max} D \longrightarrow C \otimes_{\max} D$$

which compose to a complete isometry, and hence A has AWEP.

Following the C^*-algebraic theory (see e.g. Theorem 3.3 of [24]), we restate the tensorial condition in terms of a "weak expectation":

PROPOSITION 4.4. *If A is a unital-subalgebra of a unital C^*-algebra B, then*

$$A \otimes_{\max} D \subset B \otimes_{\max} D$$

completely isometrically, for every C^**-algebra* D*, if and only if there is a* UCP *map* $\varphi : B \to C^*_{\max}(A)^{**}$ *such that* $\varphi|_A$ *is the "identity map" on* A*. This is also equivalent to: for every* C^**-algebra* G *containing* A *completely isometrically and as a unital-subalgebra, there is a* UCP *map* $\varphi : B \to G^{**}$ *such that* $\varphi|_A$ *is the "identity map" on* A*.*

Proof. We follow the usual C^*-algebraic proof. Suppose that the first condition holds, and that G^{**} is a von Neumann subalgebra $M \subset B(H)$. Let $D = M' \subset B(H)$. We have $A \otimes_{\max} D \subset B \otimes_{\max} D$, by the hypothesis. The canonical product map $A \otimes_{\max} D \to B(H) : a \otimes d = \widehat{a}d$ extends to a completely contractive unital, and hence completely positive, map $\Phi : B \otimes_{\max} D \to B(H)$. Define $T : B \to B(H)$ by $T(x) = \Phi(x \otimes 1)$. Since $\Phi(1 \otimes d) = d$ for all $d \in D$, a well known lemma implies that $T(x)d = \Phi(x \otimes d) = dT(x)$ for all $d \in D, x \in B$. That is, $T : B \to D' = M = G^{**}$. Finally, $T(a) = \Phi(a \otimes 1) = \widehat{a}$ for $a \in A$.

The other direction is much easier, essentially just as in [24]. Namely, we consider the canonical sequence

$$A \otimes_{\max} D \to B \otimes_{\max} D \to C^*_{\max}(A)^{**} \otimes_{\max} D,$$

and use the fact that $A \otimes_{\max} D \subset C^*_{\max}(A)^{**} \otimes_{\max} D$ (see Lemma 2.2 and Exercise 11.6 of [29]). ∎

REMARK 4.5. In the last line of the previous result one may replace B by B^{**}.

By the remark after Proposition 4.2, and by Proposition 4.4, A having the AWEP is equivalent to the conditions in Proposition 4.4 holding for every unital C^*-algebra B containing A completely isometrically as a unital-subalgebra. It is easy to see that this is equivalent to the conditions in Proposition 4.4 holding for $B = B(H)$, for all Hilbert spaces H and for all embeddings of A in $B(H)$ completely isometrically as a unital-subalgebra. One may replace the words "for all" in the last line with "for one fixed", to obtain condition (iv) in the next theorem. However this is shown there to be equivalent to the AWEP.

THEOREM 4.6. *For a unital operator algebra* A*, consider the following conditions:*

(i) $C^*_{\max}(A)$ *has the* WEP.

(ii) A *has the* AWEP.

(iii) *There exists an injective operator space* $R \subset C^*_{\max}(A)^{**}$ *containing the canonical copy of* A*.*

(iii′) *For every* C^**-cover* B *of* A*, there exists an injective operator space* $R \subset B^{**}$ *containing the canonical copy of* A*.*

(iv) *For every* C^**-algebra* B *containing* A *completely isometrically as a unital subalgebra, there exists a Hilbert space* H *and a completely isometric unital homomorphism* $\pi : A \to B(H)$*, and a* UCP *map* $T : B(H) \to B^{**}$*, such that* $T \circ \pi = I_A$*.*

(v) $C^*_e(A)$ *has the* WEP.

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iii′) $\Leftrightarrow$ (iv) $\Rightarrow$ (v).

Proof. (i) $\Rightarrow$ (iii) If (i) holds then there is an injective operator system R between $C^*_{\max}(A)$ and $C^*_{\max}(A)^{**}$.

(ii) $\Rightarrow$ (iv) This is a corollary of Proposition 4.4.

(iv) $\Rightarrow$ (iii$'$) Suppose that A is a unital-subalgebra of a C^*-algebra B, and that B^{**} is a unital-subalgebra of $B(K)$. By injectivity of $B(H)$ we can extend the map π in (iv) to a completely contractive unital, hence UCP, map $\widetilde{\pi} : B(K) \to B(H)$. If T is as in (iv), let $\widetilde{T} = T \circ \widetilde{\pi} : B(K) \to B^{**}$. We use notation from e.g. [16] or Section 4.2 of [10]. Let Θ be a minimal $\widehat{A}$-projection on $B(K)$. Then $\|\widetilde{T}(\Theta(\cdot))\|$ is a $\widehat{A}$-seminorm on $B(K)$ which is dominated by $\|\Theta(\cdot)\|$, and hence the two must coincide. By Lemma 4.2.2 of [10] we deduce that $\widetilde{T} \circ \Theta$ is idempotent. Since the range of $\widetilde{T} \circ \Theta$ is contained in B^{**} we deduce that there is an injective operator system Z with $A \subset Z \subset B^{**}$.

(iii$'$) $\Rightarrow$ (iii) Trivial.

(iii) $\Rightarrow$ (ii) If A is a unital-subalgebra of a C^*-algebra B, we can extend the inclusion $A \subset C^*_{\max}(A)$ to a UCP map $B \to R$. Hence A satisfies the hypotheses in Proposition 4.4, which yields (ii).

(iii$'$) $\Rightarrow$ (v) There is an injective Z with $A \subset Z \subset C^*_e(A)^{**} \subset B(H)$. Let $T : B(H) \to Z$ be a UCP idempotent. By the rigidity property of $C^*_e(A)$, we have that $R = T|_{C^*_e(A)}$ is a complete isometry onto $W = T(C^*_e(A))$. Then $R^{-1} : W \to C^*_e(A)$ extends to a complete contraction $\mu : C^*_e(A)^{**} \to Z \subset C^*_e(A)^{**}$ such that $\mu|_A = I_A$. Then $\mu \circ T : B(H) \to C^*_e(A)^{**}$ with $\mu(T(x)) = R^{-1}(T(x)) = R^{-1}(R(x)) = x$ for all $x \in C^*_e(A)$. Hence $C^*_e(A)$ has the WEP. ∎

REMARK 4.7. (i) Variants of the above proof shows that if some C^*-algebra generated by A has the WEP then so does $C^*_e(A)$; and that $C^*_e(A)$ has the WEP if and only if there exists an injective $R \subset C^*_e(A)^{**}$ containing the canonical copy of A.

(ii) As in Proposition 4.4, one may replace B in (iv) by $C^*_{\max}(A)$.

PROPOSITION 4.8. *Let A and B be approximately unital operator algebras, with A $\mathbb{B}$-nuclear and B having the* AWEP. *We have $A \otimes_{\min} B = A \otimes_{\max} B$ if either A or B is a C^*-algebra.*

Proof. If A is a C^*-algebra with the LLP and if B has the AWEP, then $A \otimes_{\max} B$ is contained in $A \otimes_{\max} C^*_e(B)$. By Theorem 4.6 we have $C^*_e(B)$ has the WEP, and so by the matching theorem of Kirchberg ([17], Proposition 1.1(i)) we have $A \otimes_{\min} C^*_e(B)$ equal to $A \otimes_{\max} C^*_e(B)$. From this the result is clear.

If $B \subset B(H)$ is a C^*-algebra with the WEP, and if A is $\mathbb{B}$-nuclear, then using Lemma 2.2 we have

$$A \otimes_{\max} B \subset C^*_{\max}(A) \otimes_{\max} B \subset C^*_{\max}(A) \otimes_{\max} B(H) = C^*_{\max}(A \otimes_{\max} B(H)).$$

Since $A \otimes_{\min} B(H) = A \otimes_{\max} B(H)$ and $A \otimes_{\min} B \subset A \otimes_{\min} B(H)$ we are done. ∎

PROPOSITION 4.9. *Suppose that A is a C^*-split unital operator algebra. Then A has* AWEP *if and only if $C^*_e(A)$ has* WEP. *Also, A has* AWEP *if $C^*(F) \otimes_{\min} A = C^*(F) \otimes_{\max} A$ completely isometrically for every discrete free group F. The converse of the last assertion holds too, if A is Dirichlet.*

Proof. If $C^*_e(A)$ has WEP then there is an injective between $C^*_e(A)$ and its second dual. If also A is C^*-split then there is an injective operator space between A and $C^*_{\max}(A)^{**}$. The rest of the first "if and only if" follows from Theorem 4.6. The rest follows from Kirchberg's matching result for C^*-algebras [17] and the proof of Proposition 2.15. ∎

REMARK 4.10. We do not know if either direction of the last assertion of the proposition is true for general operator algebras. It is however easy to see that like the AWEP, the condition involving $C^*(F)$ holds if $C^*_{\max}(A)$ has the WEP. See also Proposition 5.7 for another result concerning this property.

5. SOME CONNECTIONS WITH EXACTNESS

We recall that a C^*-algebra is nuclear if and only if it is both exact and has the WEP. The reader familiar with Kirchberg's work on exactness of C^*-algebras (see e.g. [17], [18]), will expect that we need to consider the following notion of exactness for nonselfadjoint operator algebras. Fortunately, this coincides with the usual operator space variant of exactness studied by Pisier [28], as we shall soon see.

DEFINITION 5.1. We say that an operator algebra D is *OA-exact* if for every extension

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in the sense of [11], the induced sequence

$$0 \longrightarrow A \otimes_{\min} D \longrightarrow B \otimes_{\min} D \longrightarrow C \otimes_{\min} D \longrightarrow 0$$

is an extension.

PROPOSITION 5.2. *For a subalgebra $D \subset B(H)$. The following are equivalent:*
(i) *D is* OA-*exact.*
(ii) *The induced sequence*

$$0 \longrightarrow \mathbb{K} \otimes_{\min} D \longrightarrow \mathbb{B} \otimes_{\min} D \longrightarrow (\mathbb{B} \,/\, \mathbb{K}) \otimes_{\min} D \longrightarrow 0$$

is an extension in the sense of [11].
(iii) *D is exact as an operator space.*
(iv) *If $u : D \to B(H)$ is the inclusion map, then for every approximately unital operator algebra A the map $I_A \otimes u$ extends to a contraction from $A \otimes_{\min} D$ to $A \otimes_{\max} B(H)$.*

Proof. By Theorem 14.4.1 of [29] any OA-exact operator algebra is exact as an operator space. Conversely, if D is exact as an operator space, and if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an extension in the sense of [11], then we have the induced sequence

$$0 \longrightarrow C^*_{\max}(A) \longrightarrow C^*_{\max}(B) \longrightarrow C^*_{\max}(C) \longrightarrow 0$$

is an extension of C^*-algebras by Lemma 2.7 of [11]. By Theorem 14.4.1(iv) of [15] we have the extension

$$0 \longrightarrow C^*_{\max}(A) \otimes_{\min} D \longrightarrow C^*_{\max}(B) \otimes_{\min} D \longrightarrow C^*_{\max}(C) \otimes_{\min} D \longrightarrow 0$$

and we can apply Proposition 3.6 of [11] and the idea in the proof of Lemma 2.8 to see that D is OA-exact.

That (ii) and (iii) are equivalent follows from e.g. Theorem 14.4.2 of [15]. Further, Theorem 17.1 of [29] gives (iv) $\Rightarrow$ (iii). Finally, assuming (iii), recall from Lemma 2.2 that $A \otimes_{\max} B(H) \subset C^*_{\max}(A) \otimes_{\max} B(H)$ completely isometrically. Then applying Theorem 17.1 of [29], the map $I_{C^*_{\max}(A)} \otimes u$ extends to a contraction

$$C^*_{\max}(A) \otimes_{\min} D \to C^*_{\max}(A) \otimes_{\max} B(H).$$

Restricting this map to $A \otimes_{\min} D$ we obtain (iv). ∎

Any exact operator space is a subspace of an exact unital operator algebra. To see this let E be an exact operator space and let $\mathcal{U}(E)$ be the universal algebra for E as in 2.2.10–2.2.11 of [10]. Using the well known characterization of exactness in terms of subspaces of M_n, we see that $\mathcal{U}(E)$ is exact if E is exact as an operator space by using the following variant of Proposition 2.2.11 of [10]. In our case we apply the next result to both T and T^{-1} to see that any finite dimensional subspace of $\mathcal{U}(E)$ can be embedded as a subspace of M_{2n}.

PROPOSITION 5.3. *If $T : E \to F$ is a linear map between operator spaces with $\|T\|_{\mathrm{cb}} \geqslant 1$, then the induced unital map $\theta_T : \mathcal{U}(E) \to \mathcal{U}(F)$ satisfies $\|\Theta_T\|_{\mathrm{cb}} = \|T\|_{\mathrm{cb}}$.*

Proof. Let $M = \|T\|_{\mathrm{cb}}$. Then $u = T/M$ is completely contractive, so that the map θ_{u} in Proposition 2.2.11 of [10] is completely contractive. But $\theta_T = A\theta_{\mathrm{u}}A^{-1}$ where A is the diagonal scalar matrix with entries M and 1. From this it is clear that $\|\theta_T\|_{\mathrm{cb}} \leqslant M$. ∎

We will say more about $\mathcal{U}(E)$ in the final subsection of our paper.

We now consider a stronger property than exactness. We say that an operator algebra A is *subexact* if it is a subalgebra of an exact C^*-algebra. We show in Section 6.4 that an exact operator algebra need not be subexact. The following is obvious:

PROPOSITION 5.4. *A is subexact if and only if $C^*_{\mathrm{e}}(A)$ is exact.*

The next result (and also Corollary 5.9) suggest that C^*-nuclearity is not as strong a condition as might at first appear if one views it from a "commutant lifting theorem" perspective, see Proposition 2.5 of [27].

THEOREM 5.5. *A unital operator algebra A is C^*-nuclear if and only if A is exact and has the* AWEP.

Also, A is both subexact and has the AWEP, *if and only if both A is C^*-nuclear and $C^*_e(A)$ is nuclear.*

Proof. We know from Lemma 2.9 that C^*-nuclearity implies exactness, and from Proposition 4.2 we know that C^*-nuclearity implies the AWEP. Conversely, suppose A is exact and has the AWEP. For any C^*-algebra D we have by (6.3) of [10] that $A \otimes_{\max} D \subset C^*_{\max}(A) \otimes_{\max} D$. Indeed by an argument similar to that of (6.3) in [10], using the universal property of $\otimes_{\max}$ and Corollary 2.5.6 of [10], we have

$$A \otimes_{\max} D \subset C^*_{\max}(A) \otimes_{\max} D \subset C^*_{\max}(A)^{**} \otimes_{\max} D$$

completely isometrically. On the other hand, the composition of the maps in the last string agrees with the composition of the following canonical maps (induced by the maps in Theorem 4.6(iv) with $B = C^*_{\max}(A)$):

$$A \otimes_{\max} D \to B(H) \otimes_{\max} D \to C^*_{\max}(A)^{**} \otimes_{\max} D.$$

This forces $A \otimes_{\max} D \subset B(H) \otimes_{\max} D$ completely isometrically. By part (iv) of Proposition 5.2, the exactness of A gives $A \otimes_{\min} D \subset B(H) \otimes_{\max} D$. It is now easy to see that $A \otimes_{\max} D = A \otimes_{\min} D$. That is, A is C^*-nuclear.

For the second equivalence notice that if A is a subalgebra of a nuclear C^*-algebra N, then the C^*-algebra generated by A in N is exact and hence so is its quotient $C^*_e(A)$. If in addition A has the AWEP, then by Theorem 4.6 we have that $C^*_e(A)$ has the WEP. Hence $C^*_e(A)$ is nuclear by Exercise 17.1 of [29]. Since $C^*_e(A)$ is nuclear we know that $C^*_e(A) \otimes_{\min} D = C^*_e(A) \otimes_{\max} D$ for all C^*-algebras D. Hence $A \otimes_{\min} D \subset C^*_e(A) \otimes_{\max} D$ completely isometrically for all C^*-algebras D. This, by part (iv) of Proposition 5.2, forces A to be exact. Thus by the first chain of equivalences we have that A is C^*-nuclear. Finally, if $C^*_e(A)$ is nuclear, then A is clearly subexact; and if A is C^*-nuclear then A has the AWEP. ∎

REMARK 5.6. By Theorem 12.6 of [29], A is C^*-nuclear if and only if there is a net of finite rank contractions $v_t : A \to M_{n_t}$ and maps $w_t : M_{n_t} \to C^*_{\max}(A)$ with $\|w_t\|_{\rm dec} \leqslant 1$ for all t, such that $w_t v_t$ converges pointwise to the natural inclusion map of A into $C^*_{\max}(A)$. This is because A is C^*-nuclear if and only if the canonical map $A \otimes D \to C^*_{\max}(A) \otimes_{\max} D$ is a complete isometry with respect to $\otimes_{\min}$ for every C^*-algebra D.

The following result is a variant of the last theorem.

PROPOSITION 5.7. *If A is an exact approximately unital operator algebra then A is C^*-nuclear if and only if $C^*(F) \otimes_{\min} A = C^*(F) \otimes_{\max} A$ completely isometrically for every discrete free group F.*

Proof. If A is exact then the fact that any C^*-algebra B is a quotient of $C^*(F)$ for some F, forces exactness of the sequence

$$0 \longrightarrow A \otimes_{\min} J \longrightarrow A \otimes_{\min} C^*(F) \longrightarrow A \otimes_{\min} B \longrightarrow 0.$$

Applying Lemma 2.8, we have the exact sequence

$$0 \longrightarrow A \otimes_{\max} J \longrightarrow A \otimes_{\max} C^*(F) \longrightarrow A \otimes_{\max} B \longrightarrow 0.$$

If $C^*(F) \otimes_{\min} A = C^*(F) \otimes_{\max} A$ it follows that $A \otimes_{\min} B = A \otimes_{\max} B$. ∎

COROLLARY 5.8. *If A is C^*-nuclear and approximately unital, and if either A is subexact or A is generated by unitaries, then $C^*_{\mathrm{e}}(A)$ is nuclear.*

Proof. If A is C^*-nuclear then so is A^1 by Corollary 2.14. Similarly if A is subexact then so is A^1, since the unitization of an exact C^*-algebra is exact. By Theorem 5.5 it follows that $C^*_{\mathrm{e}}(A^1) = C^*_{\mathrm{e}}(A)^1$ is nuclear and hence so is $C^*_{\mathrm{e}}(A)$. On the other hand if A is generated by unitaries, then so is $C^*_{\mathrm{e}}(A)$, and so $C^*_{\mathrm{e}}(A)$ is nuclear by Theorem 13.4 of [29]. ∎

COROLLARY 5.9. *If a unital operator algebra A is C^*-split, then A is C^*-nuclear if and only if it is both exact and $C^*_{\mathrm{e}}(A)$ has* WEP.

Proof. This is a combination of Theorem 5.5 and Proposition 4.9. ∎

6. EXAMPLES

This main purpose of this section is to illuminate connections (or lack thereof) of the properties studied above, in the case of some extremely commonly encountered examples, to the matching C^*-algebraic properties for their C^*-covers.

6.1. THE DISK ALGEBRA. It is well known (see e.g. 6.2.5 of [10]) that $A(\mathbb{D})$ is C^*-nuclear. Hence it has the AWEP and the HLLP, etc. We shall show that $C^*_{\max}(A(\mathbb{D}))$ has the LLP but is not nuclear (nor exact). This shows amongst other things that the converse of the first assertion in Proposition 2.4 fails.

To see that $C^*_{\max}(A(\mathbb{D}))$ is not exact, we will use the fact that $C^*_{\max}(A(\mathbb{D}))$ is the universal C^*-algebra generated by a contraction. Let B be any separable C^*-algebra which is not exact. Since $\mathbb{K} \otimes_{\min} B$ contains a complemented copy of B and exactness, viewed as an operator space property, would pass to this copy, it follows that $\mathbb{K} \otimes_{\min} B$ is not exact. By [22], $\mathbb{K} \otimes_{\min} B$ is singly generated as a C^*-algebra by a contractive element, call it x. Since $C^*_{\max}(A(\mathbb{D}))$ is the universal C^*-algebra generated by a contraction, there exists a $*$-representation

$\pi : C^*_{\max}(A(\mathbb{D})) \to \mathbb{K} \otimes_{\min} B$ which is onto. Since the exactness is preserved by C^*-quotients it must be the case that $C^*_{\max}(A(\mathbb{D}))$ is not exact, and hence is not nuclear.

One may identify $C^*_{\max}(A(\mathbb{D}))$ with $C^*_u\langle\mathbb{C}\rangle$, the universal C^*-algebra for the operator space $\mathbb{C}$, by comparing their universal properties (see [23]). With this identification we can use Theorem 16.5 of [29] to see that $C^*_{\max}(A(\mathbb{D}))$ does in fact have the LLP.

In Proposition 16.13 of [29] one finds the striking fact that $C^*_{\max}(A(\mathbb{D}))$ having the WEP is equivalent to Kirchberg's important conjecture from [17] that LLP implies WEP. It is easy to argue that this is also equivalent to whether

$$C^*_{\max}(A(\mathbb{D})) \otimes_{\min} C^*_{\max}(A(\mathbb{D})) = C^*_{\max}(A(\mathbb{D})) \otimes_{\max} C^*_{\max}(A(\mathbb{D})).$$

Indeed this follows from Kirchberg's remarkable result from [17] that a C^*-algebra B has WEP if and only if $B \otimes_{\min} B^{\mathrm{op}} = B \otimes_{\max} B^{\mathrm{op}}$; together with the fact that $C^*_{\max}(A(\mathbb{D})) = C^*_{\max}(A(\mathbb{D}))^{\mathrm{op}}$. The latter is a special case of the more general fact that for an operator algebra A, $C^*_{\max}(A^{\mathrm{op}}) = C^*_{\max}(A)^{\mathrm{op}}$; whose proof is left as a simple exercise.

We remark that it is easy to see that

$$C^*_{\max}(A(\mathbb{D}) \otimes_{\max} A(\mathbb{D})) \neq C^*_{\max}(A(\mathbb{D})) \otimes_{\max} C^*_{\max}(A(\mathbb{D})).$$

Indeed by (6.9) in [10], we have $A(\mathbb{D}) \otimes_{\max} A(\mathbb{D}) = A(\mathbb{D}) \otimes_{\min} A(\mathbb{D})$ is the bidisk algebra (see 6.2). A pair of commuting contractive representations of $A(\mathbb{D})$ gives rise to a representation of $A(\mathbb{D}) \otimes_{\max} A(\mathbb{D})$, and hence to a $*$-representation π of $C^*_{\max}(A(\mathbb{D}) \otimes_{\max} A(\mathbb{D}))$. It is easy to construct an example of such representations (even two dimensional, taking z to E_{21}, where z is the usual generator of $A(\mathbb{D})$) such that $\pi(z \otimes 1)$ does not commute with $\pi(1 \otimes \bar{z})$. On the other hand, these would have to commute for any representation π of the C^*-algebra $C^*_{\max}(A(\mathbb{D})) \otimes_{\max} C^*_{\max}(A(\mathbb{D}))$, by Corollary 6.1.7 of [10].

6.2. THE BIDISK ALGEBRA. Using a result of Parrott [25] one may see that the bidisk algebra $A(\mathbb{D}^2)$ is C^*-nuclear (see e.g. p. 266 in [10]). Thus it has the AWEP and the HLLP, etc. On the other hand, it is easy to argue from the universal property of $C^*_{\max}$ applied to obvious maps between $A(\mathbb{D})$ and $A(\mathbb{D}^2)$, that $C^*_{\max}(A(\mathbb{D}))$ is contained in $C^*_{\max}(A(\mathbb{D}^2))$. Hence $C^*_{\max}(A(\mathbb{D}^2))$ is not exact. We do not know if $C^*_{\max}(A(\mathbb{D}^2))$ has the LLP, or if $C^*_{\max}(A(\mathbb{D})) \otimes_{\max} C^*_{\max}(A(\mathbb{D}))$ has the LLP.

6.3. TRIANGULAR MATRICES. Let T_n denote the $n \times n$ upper triangular matrices, which are known to be C^*-nuclear [27], and hence it also has AWEP and the HLLP, etc. We will show that $C^*_{\max}(T_n)$ is not exact if $n \geqslant 3$, but it does have the LLP. Note that $C^*_{\max}(T_2)$ is nuclear, since it can be identified with the subalgebra of $C([0,1], M_2)$ consisting of matrices which are diagonal matrices at $t = 0$ (see 2.4.5 of [10]).

For $n \geqslant 3$ we will first show that T_n is essentially a free product of copies of T_2. Define for $1 \leqslant i \leqslant n-1$ the algebra

$$A_i := \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{i-1 \text{ copies}} \oplus T_2 \oplus \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{n-i-1 \text{ copies}} \subseteq M_n.$$

We will denote by ι_i the inclusion map of A_i into M_n and we will let D be the subalgebra of A_i given by the diagonal matrices.

LEMMA 6.1. *The algebra T_n is completely isometrically isomorphic to $*_D A_i$.*

Proof. For each i we have $A_i \subseteq T_n$ completely isometrically isomorphically. It follows that there is a completely contractive representation $*\iota_i$ of $*_D A_i$ into T_n. The range of $*\iota_i$ contains a generating set for T_n and hence the representation $*\iota_i$ maps onto T_n. Next let $\pi : T_n \to *_D A_i$ be given by defining

$$\pi(E_{i,i}) = \underbrace{0 \oplus \cdots \oplus 0}_{i-1 \text{ copies}} \oplus 1 \oplus \underbrace{0 \oplus \cdots \oplus 0}_{n-i \text{ copies}} \in D$$

and

$$\pi(E_{i,i+1}) = \underbrace{0 \oplus \cdots \oplus 0}_{i-1 \text{ copies}} \oplus E_{1,2} \oplus \underbrace{0 \oplus \cdots \oplus 0}_{n-i-1 \text{ copies}} \in A_i,$$

where $E_{i,j}$ is the usual elementary matrix. It is easy to see that π is well defined. Now extending using linearity and algebra operations we have a representation. Notice that $\pi \circ *\iota_i$ is trivial on generators, as is $*\iota_i \circ \pi$, and hence these two maps are inverses. The result will follow if we can show that π is completely contractive. This follows easily from the now standard result from [21] stating that it suffices to show that π is contractive on matrix units. But by construction, $\pi(E_{i,i+k}) = \pi(E_{i,i+1})\pi(E_{i+1,i+2}) \cdots \pi(E_{i+k-1,i+k})$ which is a product of contractions and hence is a contraction. This is true for $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant n-i$, and we are done. ∎

We now combine the last result with the fact that free products "commute" with $C^*_{\max}$ (which follows immediately from the universal properties, as in Proposition 2.2 of [4]), to obtain

$$C^*_{\max}(T_n) = \mathop{*}_i C^*_{\max}(A_i) = \mathop{*}_i \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{i-1 \text{ copies}} \oplus C^*_{\max}(T_2) \oplus \underbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}_{n-i-1 \text{ copies}}.$$

These free products are amalgamated over D.

Since $C^*_{\max}(T_n)$ is the n-fold free product of nuclear C^*-algebras, amalgamated over D, by Theorem 13.2 of [29] and the remark after Proposition 3.21 in [24], we have that $C^*_{\max}(T_n)$ has the LLP for all n. The fact that $C^*_{\max}(T_n)$ is not exact for $n \geqslant 3$ follows from the next lemma, and the fact that $C([0,1]) * C([0,1])$ is not exact. The latter is probably well known, but for the readers convenience we give a proof, using facts in Section 5 of [20] about the universal C^*-algebra $C^*_u(X)$ of an operator system X. Namely, $C^*_u(\ell^\infty_2) = C([0,1])$ and $C^*_u(\ell^\infty_3)$ is not exact. Define $\pi_k : \ell_2^\infty \to \ell_3^\infty$ by $\pi_1((\lambda,\mu)) = (\lambda,\lambda,\mu)$ and $\pi_2((\lambda,\mu)) = (\lambda,\mu,\mu)$. Clearly π_k is a unital complete isometry for $k = 1,2$. Notice also that the ranges of the π_k are jointly spanning. By the universal property for $C^*_{\mathrm{u}}(\ell_2^\infty)$, there are unital $*$-homomorphisms $\widetilde{\pi}_k : C([0,1]) \to C^*_{\mathrm{u}}(\ell_3^\infty)$ whose ranges together generate $C^*_{\mathrm{u}}(\ell_3^\infty)$. By the universal property for free products, there is a $*$-representation of $C([0,1]) * C([0,1])$ onto $C^*_{\mathrm{u}}(\ell_3^\infty)$. Since exactness passes to C^*-quotients, it follows that $C([0,1]) * C([0,1])$ is not exact.

LEMMA 6.2. *There is a completely isometric embedding of the amalgamated free product* $C([0,1]) * C([0,1])$ *into* $C^*_{\max}(T_n)$ *for* $n \geqslant 3$.

Proof. Without loss of generality we will stick to the case of $n = 3$. The proof $n > 3$ will follow in the same manner, or by noting $C^*_{\max}(T_3) \subset C^*_{\max}(T_n)$.

We have $C^*_{\max}(T_3) = (C^*_{\max}(T_2) \oplus \mathbb{C}) *_D (\mathbb{C} \oplus C^*_{\max}(T_2))$. Notice that the map $E_1 : \begin{bmatrix} f_{1,1} & f_{1,2} \\ f_{2,1} & f_{2,2} \end{bmatrix} \oplus \lambda \mapsto f_{2,2}$ is a conditional expectation of $C^*_{\max}(T_2) \oplus \mathbb{C}$ onto a copy of $C([0,1])$. Similarly $E_2 : \lambda \oplus \begin{bmatrix} g_{1,1} & g_{1,2}\, g_{2,1} & g_{2,2} \end{bmatrix} \mapsto g_{2,2}$ defines a conditional expectation onto a copy of $C([0,1])$. Lastly the map $d : D \to \mathbb{C}$ taking a 3×3 diagonal matrix to its 2-2 entry, is a conditional expectation of D onto a copy of $\mathbb{C}$. The result now follows from Proposition 2.4 of [2]. ∎

6.4. THE ALGEBRA $\mathcal{U}(X)$ FOR AN OPERATOR SPACE X. The operator algebra $\mathcal{U}(X)$ consists of 2×2 upper triangular matrices with elements from X in the 1-2 corner and scalars in the two diagonal entries; see 2.2.10–2.2.11 of [10]. We refer the reader to Section 6.4 of [10] for a discussion of Pisier's δ norm on tensor products.

LEMMA 6.3. *If X is an operator space, and if* $\mathcal{U}(X) \otimes_{\min} D = \mathcal{U}(X) \otimes_{\max} D$ *isometrically for a unital C^*-algebra D, then* $\delta = \min$ *on* $X \otimes D$. *The converse of this is true if* $D \cong M_n(D)$ *for all* $n \in \mathbb{N}$. *In particular, X has the* 1-OLLP *of Ozawa* [23], [29] *if and only if* $\mathcal{U}(X)$ *is* $\mathbb{B}$*-nuclear.*

Proof. If $\Phi : X \to B(H)$ and $\pi : D \to B(H)$ are commuting complete contractions, with π a representation, then by Proposition 2.2.11 of [10] we obtain a representation $\theta_\Phi : \mathcal{U}(X) \to B(H^{(2)})$ which commutes with $\pi^{(2)} : D \to B(H^{(2)})$. Thus if $\mathcal{U}(X) \otimes_{\min} D = \mathcal{U}(X) \otimes_{\max} D$ then for $x_k \in X, d_k \in D$, and with a_k the matrix with x_k in its 1-2 corner and 0 elsewhere, we have

$$\Big\| \sum_k \theta_\Phi(a_k)\pi^{(2)}(d_k) \Big\| = \Big\| \Phi(x_k)\pi(d_k) \Big\| \leqslant \Big\| \sum_k a_k \otimes d_k \Big\|_{\min} = \Big\| \sum_k x_k \otimes d_k \Big\|_{\min}.$$

Hence $\delta = \min$ on $X \otimes D$.

Conversely, suppose $\delta = \min$ on $X \otimes D$, and let θ and ρ be two commuting completely contractive representations of $\mathcal{U}(X)$ and D respectively. The diagonal projections in $\mathcal{U}(X)$ induce a decomposition of the Hilbert space as a sum $H \oplus K$ so that $\theta = \theta_\Phi$ for a complete contraction $\Phi : X \to B(K,H)$, and $\rho(d) = \pi_1(d) \oplus \pi_2(d)$, for $d \in D$ and two $*$-representations π_1, π_2 of D on H and K respectively, such that $\Phi(x)\pi_2(d) = \pi_1(d)\Phi(x)$ for $x \in X, d \in D$. Note that $\theta_\Phi \circ c$ commutes with ρ, where $c : X \to \mathcal{U}(X)$ is the canonical embedding. It follows that with notation as above, for $x_k \in X, d_k \in D$,

$$\Big\| \sum_k \theta_\Phi(c(x_k))\rho(d_k)\Big\| = \Big\| \sum_k \Phi(x_k)\pi_2(d_k)\Big\| \leqslant \Big\| \sum_k x_k \otimes d_k\Big\|_\delta = \Big\| \sum_k x_k \otimes d_k\Big\|_{\min}.$$

Notice that $W = \Phi(X)\pi_2(D)$ is an operator D-bimodule, a D-subbimodule of $B(K, H)$. Also, $X \otimes_{\min} D$ is an operator D-bimodule with the canonical actions. The computation above shows that the map $u : x \otimes d \mapsto \Phi(x)\pi_2(d)$ is a contractive D-bimodule map from $X \otimes_{\min} D$ to W. If $D \cong M_n(D)$, then we also have $\delta = \min$ on $X \otimes M_n(D)$, and it is easy to see from this that u is completely contractive. The map induced from the bimodule map u by (3.12) of [10] is also completely contractive. One may argue from this that for $b_1, \dots, b_n \in \mathcal{U}(X), d_1, \dots, d_n \in D$:

$$\Big\| \sum_k \theta(b_k)\rho(d_k)\Big\| \leqslant \Big\| \Big[\begin{array}{ccc} \sum_k \lambda_k 1 \otimes d_k & \sum_k x_k \otimes d_k \; 0 & \sum_k \mu_k 1 \otimes d_k \end{array} \Big] \Big\| = \Big\| \sum_k b_k \otimes d_k \Big\|_{\min}.$$

Here λ_k, x_k, μ_k are the three nonzero "corners" of b_k, and the norm of the middle matrix is taken in $M_2(B(H \otimes K)$, where $X \subset B(H), D \subset B(K)$. That is, $\mathcal{U}(X) \otimes_{\min} D = \mathcal{U}(X) \otimes_{\max} D$ isometrically.

We leave the remaining assertion to the reader. ∎

COROLLARY 6.4. *There exist unital operator algebras A with $C^*_e(A)$ nuclear (hence having the* LLP *and* WEP*), but A is neither $\mathbb{B}$-nuclear nor C^*-nuclear nor has the* AWEP.

Proof. If X is a minimal operator space without the 1-OLLP, and if $A = \mathcal{U}(X)$, then A is not $\mathbb{B}$-nuclear by Lemma 6.3, and hence not C^*-nuclear. But $C^*_e(A)$ is nuclear: it is a subalgebra of $M_2(B)$ for a commutative C^*-algebra B by Theorem 4.21 of [5], hence Type I, and so nuclear. Since A is exact but not C^*-nuclear it cannot have AWEP by Theorem 5.5. ∎

From Lemma 6.3 and facts in [23], it is easy to build $\mathbb{B}$-nuclear operator algebras with bad properties. Indeed for any finite dimensional operator space X with X^* 1-exact, we have that X has the 1-OLLP, so that $\mathcal{U}(X)$ is $\mathbb{B}$-nuclear.

REMARK 6.5. (i) We do not have an example of a C^*-nuclear algebra A with $C^*_e(A)$ not nuclear, but presumably they exist in abundance. Equivalently, we do not know if C^*-nuclearity implies subexactness.

(ii) For a given operator space it seems rather restrictive, and therefore probably uninteresting, for $\mathcal{U}(X)$ to be C^*-nuclear. Indeed, this is equivalent to saying that the δ tensor norm agrees with the spatial (minimal) norm on $X \otimes D$, for all C^*-algebras D. Of course this occurs if $X = \mathbb{C}$, and if X is a Hilbert row or column space (since $C_n \otimes_{\mathrm{h}} D = C_n \otimes_{\min} D$ for any C^*-algebra D, and so these also agree with $C_n \otimes_\delta D$, since $\delta \leqslant \mathrm{h}$), but probably in few other cases. We are indebted to Gilles Pisier and N. Ozawa for conversations on this matter, which is related to the discussion on p. 341 of [29] of exact spaces whose dual is exact too. If X is finite dimensional with $\mathcal{U}(X)$ C^*-nuclear, then as we saw in Lemma 2.9, $\mathcal{U}(X)$ is 1-exact and hence so is X. On the other hand, by [24], since X has 1-OLLP, X^* is 1-exact. There is only a small list of 1-exact finite dimensional spaces whose dual is known to be 1-exact too (see p. 341 of [29]).

We do not have an operator algebra version of Kirchberg's profound characterization of separable exact C^*-algebras as subalgebras of a fixed "universal" separable exact C^*-algebra (see e.g. [17], [18], [19]). It seems feasible that there does exist some such result, although the following rules out one approach:

PROPOSITION 6.6. *There exists a separable exact operator space that is not linearly completely isometric to a subspace of an exact C^*-algebra. There exists a separable unital exact operator algebra which is not subexact.*

Proof. It is shown in Theorem 18 of [20] that there exists a separable exact operator system $\mathcal{S}$ that is not a unital-subsystem of any unital separable exact C^*-algebra. Suppose that $\mathcal{S}$ was a subspace of an exact C^*-algebra A. We will use notation from [16] (see also p. 285–286 of [5]). Clearly $M_2(A)$ is exact, and hence so too is its C^*-subalgebra generated by the copy of the "Paulsen system" of $\mathcal{S}$. Since exactness also passes to C^*-quotients, the C^*-envelope of the latter system is exact, and hence so too is its upper right corner, the "ternary envelope" of $\mathcal{S}$ (see [16] or p. 286 of [5]). This envelope is completely isometric to $C^*_e(\mathcal{S})$, by the uniqueness of the ternary envelope. Thus $C^*_e(\mathcal{S})$ is exact, with $\mathcal{S}$ as a unital subsystem, contradicting the result cited from [20] above.

For the last part, consider $\mathcal{U}(\mathcal{S})$, an exact separable operator algebra by the remarks above Proposition 5.3, which is not a subspace of an exact C^*-algebra. ∎

REFERENCES

[1] A. ARIAS, H.P. ROSENTHAL, *M*-complete approximate identities in operator spaces, *Studia Math.* **141**(2000), 143–200.

[2] S. ARMSTRONG, K. DYKEMA, R. EXEL, H. LI, On embeddings of full amalgamated free product C^*-algebras, *Proc. Amer. Math. Soc.* **132**(2004), 2019–2030.

[3] W.B. ARVESON, Subalgebras of C^*-algebras, *Acta Math.* **123**(1969), 141–224.

[4] D.P. BLECHER, Modules over operator algebras, and the maximal C^*-dilation, *J. Funct. Anal.* **169**(1999), 251–288.

[5] D.P. BLECHER, The Shilov boundary of an operator space and the characterization theorems, *J. Funct. Anal.* **182**(2001), 280–343.

[6] D.P. BLECHER, Extensions of operator algebras. II, in preparation.

[7] D.P. BLECHER, E.G. EFFROS, V. ZARIKIAN, One-sided M-ideals and multipliers in operator spaces. I, *Pacific J. Math.* **206**(2002), 287–319.

[8] D.P. BLECHER, D.M. HAY, M. NEAL, Hereditary subalgebras of operator algebras, *J. Operator Theory* **59**(2008), 333–357.

[9] D.P. BLECHER, L.E. LABUSCHAGNE, Noncommutative function theory and unique extensions, *Studia Math.* **178**(2007), 177–195.

[10] D.P. BLECHER, C. LE MERDY, *Operator Algebras and their Modules — An Operator Space Approach*, Oxford Univ. Press, Oxford 2004.

[11] D.P. BLECHER, M. ROYCE, Extensions of operator algebras. I, *J. Math. Anal. Appl.* **339**(2008), 1451–1467.

[12] K.R. DAVIDSON, S.C. POWER, Best approximation in C^*-algebras, *J. Reine Angew. Math.* **368**(1986), 43–62.

[13] B.L. DUNCAN, C^*-envelopes of universal free products and semicrossed products for multivariable dynamics, *Indiana Univ. Math. J.* **57**(2008), 1781–1788.

[14] E.G. EFFROS, Z.-J. RUAN, Mapping spaces and liftings for operator spaces, *Proc. London Math. Soc.* **69**(1994), 171–197.

[15] E.G. EFFROS, Z.-J. RUAN, *Operator Spaces*, London Math. Soc. Monographs (N.S.), vol. 23, The Clarendon Press, Oxford Univ. Press, New York 2000.

[16] M. HAMANA, Triple envelopes and Silov boundaries of operator spaces, *Math. J. Toyama Univ.* **22**(1999), 77–93.

[17] E. KIRCHBERG, On nonsemisplit extensions, tensor products and exactness of group C^*-algebras, *Invent. Math.* **112**(1993), 449–489.

[18] E. KIRCHBERG, On subalgebras of the CAR-algebra, *J. Funct. Anal.* **129**(1995), 35–63.

[19] E. KIRCHBERG, N. PHILLIPS, Embedding of exact C^*-algebras in the Cuntz algebra O_2, *J. Reine Angew. Math.* **525**(2000), 17–53.

[20] E. KIRCHBERG, S. WASSERMANN, C^*-algebras generated by operator systems, *J. Funct. Anal.* **155**(1998), 324–351.

[21] M.J. MCASEY, P.S. MUHLY, Representations of nonselfadjoint crossed products, *Proc. London Math. Soc.* **47**(1983), 128–144.

[22] C. OLSEN, W. ZAME, Some C^*-algebras with a single generator, *Trans. Amer. Math. Soc.* **215**(1976), 205–217.

[23] N. OZAWA, On the lifting property for universal C^*-algebras of operator spaces, *J. Operator Theory* **46**(2001), 579–591.

[24] N. OZAWA, About the QWEP conjecture, *Internat. J. Math.* **15**(2004), 501–530.

[25] S. PARROTT, On a quotient norm and the Sz.-Nagy–Foias lifting theorem, *J. Funct. Anal.* **30**(1978), 311–328.

[26] V.I. PAULSEN, *Completely Bounded Maps and Operator Algebras*, Cambridge Stud. Adv. Math., vol. 78, Cambridge Univ. Press, Cambridge 2002.

[27] V.I. PAULSEN, S.C. POWER, Tensor products of non-self-adjoint operator algebras, *Rocky Mountain J. Math.* **20**(1990), 331–350.

[28] G. PISIER, Exact operator spaces. Recent advances in operator algebras (Orleans, 1992), *Asterisque* **232**(1995), 159–186.

[29] G. PISIER, *Introduction to Operator Space Theory*, London Math. Soc. Lecture Note Ser., vol. 294, Cambridge Univ. Press, Cambridge 2003.

DAVID P. BLECHER, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TX 77204-3008, U.S.A.
E-mail address: dblecher@math.uh.edu

BENTON L. DUNCAN, DEPARTMENT OF MATHEMATICS, NORTH DAKOTA STATE UNIVERSITY, NDSU DEPT # 2750 PO BOX 6050, FARGO, ND 58108-6050, U.S.A.
E-mail address: benton.duncan@ndsu.edu

Received April 1, 2008.