J. OPERATOR THEORY
65:1(2011), 87–113

INJECTIVITY OF THE MODULE TENSOR PRODUCT OF SEMI-RUAN MODULES

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Communicated by Şerban Strătilă

ABSTRACT. We show that the projective module tensor product of a certain class of contractive left respectively right modules over properly infinite C^*-algebras is injective, i.e. the module tensor product of isometric morphisms is an isometric linear map.

Helemskii introduced *Ruan* $\mathcal{B}$-bimodules and left or right *semi-Ruan* $\mathcal{B}$-modules, where $\mathcal{B} = \mathcal{B}(L)$ and L a separable Hilbert space. Then he shows that certain $\mathcal{B}$-modules have a flatness property with respect to *(semi-) Ruan* $\mathcal{B}$-modules. We generalize this program to properly infinite C^*-algebras $\mathcal{A}$ and show that the projective module tensor product of arbitrary left and right *semi-Ruan* $\mathcal{A}$-modules is injective; i.e. they are flat in the sense of Helemskii. The proof starts with the special case of *cyclic* semi-Ruan modules and then uses an exhaustion argument. As an application we obtain a generalization of the extension theorem for completely bounded C^*-bimodule morphisms and a proof for the injectivity of the module Haagerup tensor product of operator C^*-modules. Semi-Ruan modules have a minimal isometric isomorphic representation as a submodule of $\mathcal{B}(K, H)$ for some Hilbert spaces H, K.

KEYWORDS: *Projective module tensor product, properly infinite C^*-algebra, operator module, module Haagerup tensor product, completely bounded operator.*

MSC (2000): 46L07, 46L05.

INTRODUCTION

We shall denote by $\mathcal{A}$, $\mathcal{C}$ properly infinite C^*-algebras, by X a right semi-Ruan $\mathcal{A}$-module, by Y a left semi-Ruan $\mathcal{A}$-module (or sometimes a $\mathcal{C}$-module) and by W a Ruan $\mathcal{A}$-$\mathcal{C}$-bimodule. These spaces are not necessarily complete. $X \underset{\mathcal{A}}{\otimes} Y$ denotes the uncompleted projective module tensor product.

We denote $\mathcal{B} = \mathcal{B}(L)$, $\mathcal{K} = \mathcal{K}(L)$ and $\mathcal{F} = \mathcal{F}(L)$ the operators of finite rank in $\mathcal{K}$, where L is a fixed separable infinite-dimensional Hilbert space (for instance ℓ_2 or so). We use $\mathcal{F} \dot{\otimes} W$ to denote the uncompleted minimal (spatial) tensor product of $\mathcal{F}$ and a matricial operator space W and $\breve{\otimes}$ denotes the completed minimal tensor product. $\underset{h}{\otimes}$ respectively $\underset{hA}{\otimes}$ denotes the uncompleted Haagerup tensor product respectively the uncompleted module Haagerup tensor product.

We take the definition and the basic results of/on one-sided semi-Ruan $\mathcal{A}$-modules and Ruan $\mathcal{A}$-$\mathcal{C}$-bimodules from Helemskii's paper [9]. Helemskii introduced semi-Ruan $\mathcal{B}$-modules and Ruan $\mathcal{B}$-bimodules. The preparing results ([9], Chapter 0-2) on these modules hold for arbitrary properly infinite C^*-algebras. The aim of this paper is to generalize Helemskii's results on flatness properties of certain semi-Ruan $\mathcal{B}$-modules to arbitrary semi-Ruan $\mathcal{A}$-modules.

The theory of Ruan $\mathcal{B}$-bimodules gives another axiomatic way to the quantization of Banach spaces and the theory of Ruan $\mathcal{B}$-bimodules is equivalent to operator space theory. The category of general Ruan bimodules is equivalent to the category of operator C^*-bimodules.

The main objects of this paper are the one-sided semi-Ruan $\mathcal{A}$-modules. We shall see that a left semi-Ruan $\mathcal{A}$-module has an isometric isomorphic representation as a submodule of some space $\mathcal{B}(K, H)$ where H, K are suitable Hilbert spaces and H a contractive left $\mathcal{A}$-module. Thus semi-Ruan modules have isometric quantizations but in general the quantization will not be unique (this gives an interpretation of the term "*semi-Ruan*"). Our construction gives the minimal quantization. For a maximal quantization see Anantharaman-Delaroche and Pop ([1], Proposition 1.14) and Lambert ([11], Section 4).

Lambert [11] developed a related theory of operator sequence spaces which is roughly speaking equivalent to the theory of semi-Ruan $\mathcal{B}$-modules. Both theories are based on a one-sided contractive module operation, Lambert uses the sequence of normed spaces E^n, $(n = 1, 2, \dots)$, and scalar matrices M_n. Helemskii considers modules over the algebra $\mathcal{B}$. Both theories postulate the quadratic inequality $\|x + y\|^2 \leqslant \|x\|^2 + \|y\|^2$ for orthogonal elements. The translation goes as follows: Starting with Lambert's theory, then the closure of $\bigcup_n E^n \subset E^\infty$ is a semi-Ruan $\mathcal{B}$-module. On the other hand, given a semi-Ruan $\mathcal{B}$-module Y, then there is a unique family of norms on Y^n, $(n = 1, 2, \dots)$, such that Y with these norms is an operator sequence space and the Y^n are contractive $M_n(\mathcal{B})$ modules.

I started this research with a generalization of Lambert's work to one-sided operator sequence C^*-modules. Then from Helemskii's paper [9] I became acquainted with semi-Ruan modules and found this *non-matricial* and *non-coordinate* approach more convenient.

We shall show in Section 6 that the projective module tensor product $X \underset{\mathcal{A}}{\otimes} Y$ of arbitrary right and left semi-Ruan $\mathcal{A}$-modules X respectively Y is injective, i.e. preserves isometric embeddings. For the proof we use the abstract Definition of semi-Ruan $\mathcal{A}$-modules and no representation theory. As an application we obtain a generalization of the extension theorem [24], [21], [15] for completely bounded C^*-bimodule morphisms and a proof for the injectivity of the module Haagerup tensor product of operator C^*-modules. The latter result was originally proved in Theorem 7.1 of [1] and Theorem 3.6.5 of [3].

Then we show that every left semi-Ruan $\mathcal{A}$-module Y has an isometric representation $\theta : Y \to \mathcal{B}(K, H)$ via a $*$-representation $\pi : \mathcal{A} \to \mathcal{B}(H)$. Since semi-Ruan modules are representable modules this is a special case of a more general result of Magajna [14] and Pop [18]. We give a proof for the special situation of one-sided modules and thereby gain some additional information. In the first step we construct for a given linear functional f in the dual unit sphere $S_{Y'}$ a uniquely determined majorizing state denoted by $|f|$. This generalizes the notion of the absolute value of a functional on a C^*-algebra. The minimal representation is an isometric functor into the category of left operator modules i.e. bounded morphisms of semi-Ruan modules become completely bounded morphisms with the same norm. This functor is naturally isomorphic to the functor which assigns to Y the Ruan $\mathcal{A}$-$\mathcal{B}$-bimodule $\mathcal{F}(L, Y)$ of bounded linear mappings of finite rank.

CONCLUDING REMARKS. Lambert uses his theory among other things to generalize the column quantization of Hilbert space to arbitrary Banach spaces E. Then one may mimic the formula $B(H) =_{\mathrm{cb}} \mathcal{C}B(H_{\mathrm{c}})$ to quantize $\mathcal{B}(E)$. Lambert–Neufang–Runde [12] showed that this column quantization of L_p-spaces and the corresponding quantization of $\mathcal{B}(L_p)$ are the right tools to generalize Ruan's results on the amenability of the quantized Fourier algebra $A(G)$ to $A_p(G)$. It might be promising to study the column quantization or the corresponding (semi-)Ruan modules for representable one-sided modules.

It might be possible to extend the injectivity of the module tensor product to a larger classes of "infinite" C^*-algebras; see Anantharaman-Delaroche and Pop [1] for related results.

1. PRELIMINARIES AND NOTATION

Let A be an arbitrary unital C^*-algebra and Z a left A-module. The term *module* always mean unital module. We recall that Z is called a *contractive* left A-module if Z is a normed space and if the module product satisfies $\|az\| \leqslant \|a\|\|z\|$ for all $a \in A$ and $z \in Z$. We denote the module product with az and in case of doubt with $a \cdot z$ as in the following remark. Given a contractive left A-module the module product defines a contractive morphism $\pi : A \to \mathcal{B}(Z)$ by $\pi(a) : z \mapsto a \cdot z$. Given a contractive unital morphism $\pi : A \to \mathcal{B}(Z)$ we define an A-module action on Z by $a \cdot z := \pi(a)z$ and say Z *is an A-module via* π. If Z happens to be a Hilbert space and simultaneously a contractive A-module then the corresponding contractive morphism $\pi\colon A \to \mathcal{B}(Z)$ is automatically a $*$-homomorphism (see Proposition A.5.8 of [3]), i.e. $\pi(a)^*z = a^* \cdot z$. Hence the meaning of * is unambiguous if we denote the module product for short with a^*z. We use the remarks above for right modules with the obvious modifications.

Given an C^*-algebra A and a contractive A-bimodule Z, then we cancel the bimodule operation and define $Z/A := Z/[A, Z]^-$ to be the quotient space by the closure of the subspace spanned by the commutators $za - az$. If Z is an operator space and a contractive A-bimodule then Z/A denotes the corresponding quotient operator space.

Clearly Z/A has the following universal property: given a balanced (i.e. $\Phi(az) = \Phi(za)$) bounded linear operator Φ from Z to a normed space E then there exist a unique bounded linear operator $\widetilde{\Phi}\colon Z/A \to E$ such that $\Phi = \widetilde{\Phi} \circ \kappa$, where $\kappa\colon Z \to Z/A$ is the quotient map. Clearly $\|\widetilde{\Phi}\| = \|\Phi\|$. Observe, Z/A may be the zero space.

Suppose that Z_1, Z_2 are a right respectively a left contractive A-modules. A bounded bilinear operator $\Phi\colon Z_1 \times Z_2 \to E$, where E is a normed space, is called balanced if $\Phi(z_1 a, z_2) = \Phi(z_1, a z_2)$ for all $a \in A$, $z_1 \in Z_1$, $z_2 \in Z_2$. The projective tensor product of the normed spaces $Z_1 \otimes_p Z_2$ is a contractive A-bimodule with the module multiplication defined by $a_1 \cdot (z_1 \otimes z_2) \cdot a_2 := (z_1 a_2) \otimes (a_1 z_2)$. The projective module tensor product $Z_1 \underset{A}{\otimes} Z_2$ is defined as the quotient of the projective tensor product $Z_1 \otimes_p Z_2$ of normed linear spaces by the closure of the commutators $[a, z_1 \otimes z_2] = z_1 \otimes a z_2 - z_1 a \otimes z_2$. We write $z_1 \underset{A}{\otimes} z_2$ for the equivalence class of $z_1 \otimes z_2$ in $Z_1 \underset{A}{\otimes} Z_2$. The space $Z_1 \underset{A}{\otimes} Z_2$ has the universal property that it *linearizes* bounded balanced bilinear mappings to linear mappings with the same norm.

In the two-sided case we consider a contractive C-A-bimodule Z and a contractive A-C-bimodule W. A bounded bilinear operator $\Phi\colon Z \times W \to E$ is called A-C-balanced if $\Phi(cza, w) = \Phi(z, azc)$ for all $a \in A$, $c \in C$, $z \in Z$, $w \in W$. The projective bimodule tensor product $Z \underset{A-C}{\otimes} W$ is defined as the quotient of the projective tensor product $Z \otimes_p W$ of normed linear spaces by the closure of the subspace spanned by terms of the form $za \otimes w - z \otimes aw$ and $cz \otimes w - z \otimes wc$. We write $z \underset{A-C}{\otimes} w$ for the equivalence class of $z \otimes w$ in $Z \underset{A-C}{\otimes} W$. The space $Z \underset{A-C}{\otimes} W$ has the universal property that it *linearizes* bounded A-C-balanced bilinear mappings to linear mappings with the same norm.

LEMMA 1.1. *Let A, C be unital C^*-algebras and Z a contractive $A \breve{\otimes} C$-bimodule, then Z is a contractive C-bimodule, Z/C is a contractive A-bimodule and $Z/(C \breve{\otimes} A) = (Z/C)/A$.*

Proof. We define the module operation $c_1 \cdot z \cdot c_2$ in the obvious way by $(c_1 \otimes \mathbf{1}_A) z (c_2 \otimes \mathbf{1}_A)$. Similarly Z is a contractive A-bimodule and $[C, Z]^-$ is an A-subbimodule. Thus Z/C is a contractive A-bimodule and the canonical map $\kappa_C\colon Z \to (Z/C)$ is an A-bimodule morphism.

The bounded operator $\Phi\colon Z \to Z/(C \breve{\otimes} A)$ taking z to $z + [C \breve{\otimes} A, Z]^-$ is C-balanced. Φ induces a bounded linear map by $\widetilde{\Phi}\colon Z/C \to Z/(C \breve{\otimes} A)$. Since $\widetilde{\Phi}$ is A-balanced there exists the induced map $\widetilde{\widetilde{\Phi}}\colon (Z/C)/A \to Z/(C \breve{\otimes} A)$.

On the other hand, consider the bounded map $\Psi = \kappa_A \circ \kappa_C \colon Z \to Z/C \to (Z/C)/A$. By definition we have

$$\begin{aligned}\Psi((c\otimes a)z) &= \kappa_A \circ \kappa_C((c\otimes \mathbf{1}_A)((\mathbf{1}_C \otimes a)z)) = \kappa_A(a\cdot \kappa_C(z(c\otimes \mathbf{1}_A)))\\ &= \kappa_A(\kappa_C(z(c\otimes \mathbf{1}_A))\cdot a) = \kappa_A(\kappa_C((z(c\otimes \mathbf{1}_A))(\mathbf{1}_C\otimes a))) = \Psi(z(c\otimes a)).\end{aligned}$$

Thus Ψ is balanced with respect to the algebraic tensor product $C \otimes A$. Since Ψ is bounded, it is $A \breve{\otimes} C$-balanced and thus there exists the induced bounded linear map $\widetilde{\Psi} \colon Z/(A \breve{\otimes} C) \colon \to (Z/C)/A$.

From the construction follows that $\widetilde{\Psi}$ and $\widetilde{\widetilde{\Phi}}$ are inverse to each other. ∎

COROLLARY 1.2. *Let A, C be unital C^*-algebras and X and Y right respectively left contractive $A \breve{\otimes} C$-modules, then $X \underset{C\breve{\otimes}A}{\otimes} Y = (X \underset{C}{\otimes} Y)/A$.*

2. MODULE TENSOR PRODUCT RELATIVE TO PROPERLY INFINITE C^*-ALGEBRAS

In this preparing section $\mathcal{A}$ denotes a properly infinite unital C^*-algebra. The module tensor product $X \underset{\mathcal{A}}{\otimes} Y$ of a a right contractive $\mathcal{A}$-module X and a left contractive $\mathcal{A}$-module Y has some simple but useful properties.

A unital C^*-algebra $\mathcal{A}$ is called *properly infinite* if it contains two isometries S_1, S_2 with orthogonal left support, i.e. the final projections $P_k := S_k S_k^*$, $(k = 1, 2)$, are orthogonal. Then

$$S_k = P_k S_k, \quad S_k^* = S_k^* P_k \quad \text{and} \quad S_k^* S_l = \begin{cases} 0 & \text{for } k \neq l, \\ \mathbf{1}_A & \text{for } k = l. \end{cases} \tag{2.1}$$

Clearly it suffices to have this property for $n = 2$. If S_1, S_2 are isometries with $S_1 S_1^* \perp S_2 S_2^*$ then $S_2^{n-1} S_1$, $(n = 1, 2, \dots)$, is a sequence of isometries with pairwise orthogonal left supports.

Given a contractive left $\mathcal{A}$-module Y and an isometry $S \in \mathcal{A}$, then without mentioning we will use the following simple estimates:

$$\|y\| = \|S^* S y\| \leqslant \|S^*\| \|Sy\| \leqslant \|S\| \|y\| = \|y\| \tag{2.2}$$

for all $y \in Y$; i.e. $\|Sy\| = \|y\|$. Similarly for a right contractive $\mathcal{A}$-module X holds $\|xS^*\| = \|x\|$.

Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{C}$ a C^*-subalgebra with $\mathbf{1}_{\mathcal{C}} = \mathbf{1}_{\mathcal{A}}$. If $\mathcal{C}$ is properly infinite then $\mathcal{A}$ is properly infinite. We will use this remark in the following situation: If A is an arbitrary unital C^*-algebra, then the spatial tensor product $\mathcal{A} := \mathcal{B} \breve{\otimes} A$ is properly infinite. If V is a left operator A-module in the sense of [4] then $\mathcal{K} \breve{\otimes} V$ is a contractive $(\mathcal{B} \breve{\otimes} A)$-$\mathcal{B}$-bimodule.

Given a left contractive $\mathcal{A}$-module Y, we consider the cyclic submodules $\mathcal{A}y \subset Y$, $(y \in Y)$. We define a partial order on the elements of Y by $y \preceq z$ if $\mathcal{A}y \subset \mathcal{A}z$. We denote the inclusion mappings by $i_y \colon \mathcal{A}y \hookrightarrow Y$ and by $i_{y,z} \colon \mathcal{A}y \hookrightarrow \mathcal{A}z$ for

$y \preceq z$. Then $i_y = i_z i_{y,z}$. We take inductive limits in the category of normed spaces and contractive linear maps.

LEMMA 2.1. *Given a properly infinite C^*-algebra $\mathcal{A}$ and a contractive left $\mathcal{A}$-module then the family $\{\mathcal{A}y : y \in Y\}$ of cyclic submodules is directed upwards and $(Y, (i_y)_{y \in Y})$ is the inductive limit, simply a union, of the system $(i_{y,z} : \mathcal{A}y \to \mathcal{A}z)_{y \preceq z}$.*

Proof. Given $y_1, y_2 \in Y$. Choose two isometries $S_1, S_2 \in \mathcal{A}$ with orthogonal final projections $S_1S_1^* \perp S_2S_2^*$ and define $y_3 = S_1y_1 + S_2y_2$. Then $\mathcal{A}y_i = \mathcal{A}S_i^*(S_1y_1 + S_2y_2) \subset \mathcal{A}y_3$ for $i = 1, 2$. Since $\mathcal{A}$ is a unital algebra $\bigcup_{y \in Y} \mathcal{A}y = Y$. ∎

We will use the system of linear contractions $\mathbf{1}_X \underset{\mathcal{A}}{\otimes} i_y \colon X \underset{\mathcal{A}}{\otimes} \mathcal{A}y \to X \underset{\mathcal{A}}{\otimes} Y$ and $\mathbf{1}_X \underset{\mathcal{A}}{\otimes} i_{y,z} \colon X \underset{\mathcal{A}}{\otimes} \mathcal{A}y \to Y \underset{\mathcal{A}}{\otimes} \mathcal{A}z$ for $y \preceq z$.

LEMMA 2.2. *Let X be a right contractive $\mathcal{A}$-module, $X_0 \subset X$ a submodule and Y a left contractive $\mathcal{A}$-module. Denote by $i_0 \colon X_0 \hookrightarrow X$ the isometric inclusion map. If for all cyclic submodules $\mathcal{A}y \subset Y$, $(y \in Y)$ the map $i_0 \underset{\mathcal{A}}{\otimes} \mathbf{1}_{\mathcal{A}y} \colon X_0 \underset{\mathcal{A}}{\otimes} \mathcal{A}y \to X \underset{\mathcal{A}}{\otimes} \mathcal{A}y$ is isometric, then $i_0 \underset{\mathcal{A}}{\otimes} \mathbf{1}_Y \colon X_0 \underset{\mathcal{A}}{\otimes} Y \to X \underset{\mathcal{A}}{\otimes} Y$ is an isometry.*

Proof. Clearly $i_0 \underset{\mathcal{A}}{\otimes} \mathbf{1}_Y$ is a contraction. In order to show that it is an isometry we construct to every bounded linear functional $\varphi \in (X_0 \underset{\mathcal{A}}{\otimes} Y)'$ an extension $\psi \in (X \underset{\mathcal{A}}{\otimes} Y)'$ such that $\psi \circ (i_0 \underset{\mathcal{A}}{\otimes} \mathbf{1}_Y) = \varphi$ and $\|\varphi\| = \|\psi\|$. It suffices to extend the corresponding bounded balanced bilinear form $f \colon X_0 \times Y \to \mathbb{C}$.

For $y \in Y$ denote by $\varphi_y = \varphi \circ (\mathbf{1}_{X_0} \underset{\mathcal{A}}{\otimes} i_y)$ and by $f_y \colon X_0 \times \mathcal{A}y \to \mathbb{C}$ the corresponding balanced bilinear form. Clearly $f_y = f|_{X_0 \times \mathcal{A}y}$ and $\|\varphi_y\| = \|f_y\| \leqslant \|f\| = \|\varphi\|$. By assumption φ_y has a linear extension $\psi_y \in (X \underset{\mathcal{A}}{\otimes} \mathcal{A}y)'$ such that $\psi_y \circ (i_0 \underset{\mathcal{A}}{\otimes} \mathbf{1}_{\mathcal{A}y}) = \varphi_y$ and $\|\psi_y\| = \|\varphi_y\|$. Let $e_y \colon X \times \mathcal{A}y \to \mathbb{C}$ be the corresponding bounded balanced bilinear form. Clearly $\|e_y\| = \|\psi_y\| \leqslant \|\varphi\|$ and $e_y|_{X_0 \times \mathcal{A}y} = f_y = f|_{X_0 \times \mathcal{A}y}$.

Recall the partial order on Y: we say $y_1 \preceq y_2$ if $\mathcal{A}y_1 \subset \mathcal{A}y_2$. From Lemma 2.1 this is an upwards directed system and $Y = \bigcup_{y \in Y} \mathcal{A}y$. Choose an ultra filter $\mathfrak{U}$ over the filter of tails of this directed system and define

$$e(x, z) := \lim_{z \preceq y \to \mathfrak{U}} e_y(x, z) \quad \text{for } x \in X, z \in Y.$$

Obviously $e \colon X \times Y \to \mathbb{C}$ is well defined bilinear balanced and an extension of f with $\|e\| = \|f\|$. The corresponding linear form $\psi \in (X \underset{\mathcal{A}}{\otimes} Y)'$ is an extension of φ with the same norm. ∎

The module tensor product relative to a properly infinite C^*-algebra $\mathcal{A}$ has some attractive properties. Helemskii ([9], Section 0-2) introduced these useful properties for $\mathcal{B}$-modules. It is easily seen that the results and the proofs of Section 0-2 in [9] hold for arbitrary properly infinite C^*-algebras. For the convenience of the reader we recall here and in the next section some of these results (see the Propositions 2.3, 3.8, 4.6).

PROPOSITION 2.3 ([9], Proposition 1). *Let X be a right contractive $\mathcal{A}$-module and Y a left contractive $\mathcal{A}$-module. Then every $u \in X \underset{\mathcal{A}}{\otimes} Y$ can be represented as a single elementary tensor $u = x \underset{\mathcal{A}}{\otimes} y$.*

3. SEMI-RUAN ONE-SIDED MODULES

DEFINITION 3.1. Given a properly infinite C^*-algebras $\mathcal{A}$, a contractive left $\mathcal{A}$-module Y is called a *left semi-Ruan $\mathcal{A}$-module* if it satisfies the following condition (the left semi-Ruan axiom): There exist two isometries $S_1, S_2 \in \mathcal{A}$ with orthogonal final projections $S_1S_1^* \perp S_2S_2^*$ such that

$$\|S_1y_1 + S_2y_2\| \leqslant (\|y_1\|^2 + \|y_2\|^2)^{1/2} \quad \text{for all } y_1, y_2 \in Y. \tag{lsR}$$

Similarly we define the notion of a *right semi-Ruan $\mathcal{A}$-module* X: There exist two isometries $S_1, S_2 \in \mathcal{A}$ with orthogonal final projections such that

$$\|x_1S_1^* + x_2S_2^*\| \leqslant (\|x_1\|^2 + \|x_2\|^2)^{1/2} \quad \text{for all } x_1, x_2 \in X. \tag{rsR}$$

Helemskii ([9], Definition 2) defines left semi-Ruan $\mathcal{B}$-modules Y, where $\mathcal{B} = \mathcal{B}(L)$, by the following formally stronger condition (lsR$'$) respectively a similar condition (rsR$'$) for right modules:

REMARK 3.2. Given a properly infinite C^*-algebra $\mathcal{A}$, a contractive left $\mathcal{A}$-module Y is a semi-Ruan $\mathcal{A}$-module if and only if

$$\|y_1 + y_2\| \leqslant (\|y_1\|^2 + \|y_2\|^2)^{1/2} \tag{lsR$'$}$$

for all y_1, y_2 with orthogonal left supports; i.e there exist orthogonal projections P_i such that $P_iy_i = y_i$, $(i = 1, 2)$. The equivalence of (lsR$'$) and (lsR) follows from the next proposition.

PROPOSITION 3.3. *Given a properly infinite C^*-algebra $\mathcal{A}$, a contractive left $\mathcal{A}$-module Y is a left semi-Ruan $\mathcal{A}$-module if and only if it satisfies the following condition (left representable module)*

$$\|a_1y_1 + a_2y_2\| \leqslant \|a_1a_1^* + a_2a_2^*\|^{1/2}(\|y_1\|^2 + \|y_2\|^2)^{1/2} \tag{lr}$$

for all $a_1, a_2 \in \mathcal{A}$, $y_1, y_2 \in Y$.

Proof. Choose two isometries S_1, S_2 with orthogonal final projections such that the estimate (lsR) holds and let $a = a_1S_1^* + a_2S_2^*$. The C^*-identity gives $\|a\|^2 = \|aa^*\| = \|a_1S_1^*S_1a_1^* + a_2S_2^*S_2a_2^*\| = \|a_1a_1^* + a_2a_2^*\|$. From the condition (lsR) follows $\|a_1y_1 + a_2y_2\| = \|a(S_1y_1 + S_2y_2)\| \leqslant \|a\|\|S_1y_1 + S_2y_2\| \leqslant \|a_1a_1^* + a_2a_2^*\|^{1/2}(\|y_1\|^2 + \|y_2\|^2)^{1/2}$.

On the other hand condition (lsR) is a special case of condition (lr). ∎

REMARK 3.4. Magajna ([14], Corollary 2.2) has shown that a left contractive module Y over an arbitrary C^*-algebra A is isometrically isomorphic to a left operator A-module in the sense of [4] if and only if it satisfies the condition (lr). Such a module is called *left representable*.

EXAMPLE AND REMARK 3.5. (i) Given a C^*-algebra $\mathcal{A}$ with a properly infinite unital subalgebra $\mathcal{A}_0$ and a contractive left $\mathcal{A}$-module Y, if Y is a semi-Ruan $\mathcal{A}_0$-module, then Y is a semi-Ruan $\mathcal{A}$-module.

(ii) Especially for an arbitrary Hilbert space H and a C^*-algebra $A \subset \mathcal{B}(H)$, the Hilbert space $L \otimes^2 H$ is a left semi-Ruan $(\mathcal{B} \breve{\otimes} A)$-module.

(iii) Given a $*$-representation $\pi\colon \mathcal{A} \to \mathcal{B}(H)$ of a properly infinite C^*-algebra $\mathcal{A}$, a normed space Y and an isometry $\theta\colon Y \to \mathcal{B}(K, H)$ such that $\pi(\mathcal{A})\theta(Y) \subset \theta(Y)$, define a left module action on Y by $ay := \theta^{-1}(\pi(a)\theta(y))$. Then Y is a left semi-Ruan module. We call θ a *representation* of the $\mathcal{A}$-module Y via π and Y a *representable semi-Ruan $\mathcal{A}$-module*. In Section 11 we shall show that every semi-Ruan module is representable.

(iv) A representation of a left semi-Ruan $\mathcal{A}$-module Y gives to Y a structure as a left operator $\mathcal{A}$-module in the sense of [4] (see Theorem 15.14 and Corollary 16.10 of [16] or Section 3.3.1 and Theorem 3.3.1 of [3]). In general the operator module structure of a semi-Ruan module is not unique, there is a minimal and a maximal one. This explains the term *semi-Ruan*.

(v) For example, $L \otimes^2 H$ is a left semi-Ruan $\mathcal{B}$-module. The minimal operator $\mathcal{B}$-module structure is given by the column operator Hilbert space $(L \otimes^2 H)_c$ and the maximal one by the maximal operator space over $L \otimes^2 H$.

(vi) Given a semi-Ruan $\mathcal{A}$-module Y and a closed $\mathcal{A}$-submodule Y_0, then the quotient space Y/Y_0 is a semi-Ruan $\mathcal{A}$-module.

(vii) The ℓ_p-direct-sum $(2 \leqslant p \leqslant \infty)$ of a family of semi-Ruan modules of the same type is also a semi-Ruan module.

(viii) Let E be a normed space and Y a left semi-Ruan $\mathcal{A}$-module, then $\mathcal{B}(E, Y)$ is a left semi-Ruan $\mathcal{A}$-module.

PROPOSITION 3.6. *The condition* (lsR) *is always an equality if and only if there is an inner product $\langle \cdot, \cdot \rangle$ on Y, such that $\|y\| = \langle y, y \rangle^{1/2}$. The completion H of Y is a Hilbert space and the module action gives a $*$-representation $\pi\colon A \to \mathcal{B}(H)$.*

Proof. Suppose S_1, S_2 are two isometries with orthogonal final projections P_1, P_2. Let $Q = \mathbf{1}_{\mathcal{A}} - P_1 - P_2$, then the element $U := 2^{-1/2}(\mathbf{1}_{\mathcal{A}} + \mathrm{i}Q + S_2S_1^* - S_1S_2^*)$ is unitary and $U(S_1y_1 + S_2y_2) = 2^{-1/2}(S_2(y_2 + y_1) + S_1(y_1 - y_2))$. Since by assumption the condition (lsR) is an equality we obtain

$$2\|y_1\|^2 + 2\|y_2\|^2 = 2\|S_1y_1 + S_2y_2\|^2 = 2\|U(S_1y_1 + S_2y_2)\| = \|y_1 + y_2\|^2 + \|y_1 - y_2\|^2.$$

The parallelogram equality holds and thus Y is an inner product space.

The module action gives a contractive morphism $\pi\colon \mathcal{A} \to \mathcal{B}(H)$. From Proposition A.5.8 of [3] follows that π is a $*$-homomorphism. ∎

REMARK 3.7. Given a left semi-Ruan $\mathcal{A}$-module Y then the dual space Y' with the multiplication $(f \cdot a)(y) := f(ay)$ for $f \in Y'$, $a \in \mathcal{A}$, $y \in Y$ is a right contractive $\mathcal{A}$-module with the following property:

$$\|f_1 + f_2\| \geqslant (\|f_1\|^2 + \|f_2\|^2)^{1/2}$$

for all $f_1, f_2 \in Y'$ with orthogonal right supports. Hence the dual of a semi-Ruan module Y is a semi-Ruan module if and only if Y is an inner product space.

PROPOSITION 3.8 ([9], Proposition 5). *Let X be a right semi-Ruan $\mathcal{A}$-module, Y a left semi-Ruan $\mathcal{A}$-module and $u \in X \underset{\mathcal{A}}{\otimes} Y$. Then*

$$\|u\| = \inf\{\|x\|\|y\|\},$$

where the infimum is taken over all possible representations of u in the form $u = x \underset{\mathcal{A}}{\otimes} y$, $x \in X$, $y \in Y$, (such representations exist by Proposition 2.3*).*

4. RUAN BIMODULES

DEFINITION 4.1. Given properly infinite C^*-algebras $\mathcal{A}$ and $\mathcal{C}$, then a contractive $\mathcal{A}$-$\mathcal{C}$-bimodule W is called a *Ruan $\mathcal{A}$-$\mathcal{C}$-bimodule* if it satisfies the following condition (the Ruan axiom):

There exist isometries $S_1, S_2 \in \mathcal{A}$ with orthogonal left supports and isometries $T_1, T_2 \in \mathcal{C}$ with orthogonal left supports such that

$$\text{(R)} \qquad \|S_1 w_1 T_1^* + S_2 w_2 T_2^*\| = \max\{\|w_1\|, \|w_2\|\} \quad \text{for all } w_1, w_2 \in W.$$

Given a contractive $\mathcal{A}$-module W and elements $w_1, w_2 \in W$ with orthogonal left supports P_1, P_2, then $\|w_i\| = \|P_i(w_1 + w_2)\| \leqslant \|P_i\|\|w_1 + w_2\| = \|w_1 + w_2\|$, $(i = 1, 2)$. Hence we have the following observation:

Instead of equation (R) the following condition is sufficient

$$\text{(R}'\text{)} \qquad \|S_1 w_1 T_1^* + S_2 w_2 T_2^*\| \leqslant \max\{\|w_1\|, \|w_2\|\} \quad \text{for all } w_1, w_2 \in W.$$

Helemskii defines Ruan $\mathcal{B}$-bimodules by the following stronger condition (R$''$). Given two elements $w_1, w_2 \in W$, then we say they have *orthogonal left supports*, if there exist projections $Q_1, Q_2 \in A$ such that $Q_1 \perp Q_2$ and $w_i = Q_i w_i$, $(i = 1, 2)$. Given a Ruan $\mathcal{A}$-$\mathcal{C}$-bimodule W then

$$\text{(R}''\text{)} \qquad \|w_1 + w_2\| = \max\{\|w_1\|, \|w_2\|\},$$

for all $w_1, w_2 \in W$ with orthogonal left and orthogonal right supports.

The equivalence of (R$''$) and (R) follows from the next proposition.

PROPOSITION 4.2. *Given properly infinite C^*-algebra $\mathcal{A},\mathcal{C}$, then a contractive $\mathcal{A}$-$\mathcal{C}$-module W is a Ruan $\mathcal{A}$-$\mathcal{C}$-module if and only if it satisfies the following condition (representable bimodule)*

(r) $\|a_1w_1c_1 + a_2w_2c_2\| \leqslant \|a_1a_1^* + a_2a_2^*\|^{1/2}(\max\{\|w_1\|, \|w_2\|\}\|c_1^*c_1 + c_2^*c_2\|^{1/2}$,

for all $a_1, a_2 \in A$, $c_1, c_2 \in \mathcal{C}$, $w_1, w_2 \in W$.

Proof. Choose isometries $S_1, S_2 \in \mathcal{A}$ with orthogonal final projections and isometries $T_1, T_2 \in \mathcal{C}$ with orthogonal final projections such that the equation (R) holds. Let $a = a_1S_1^* + a_2S_2^*$ and $c = T_1c_1 + T_2c_2$. The C^*-identity gives

$$\|a\|^2 = \|aa^*\| = \|a_1S_1^*S_1a_1^* + a_2S_2^*S_2a_2^*\| = \|a_1a_1^* + a_2a_2^*\|,$$

and similarly $\|c\|^2 = \|c_1^*c_1 + c_2^*c_2\|$. From the equation $a(S_1w_1T_1^* + S_2w_2T_2^*)c = a_1w_1c_1 + a_2w_2c_2$ and the condition (R) follows

$$\begin{aligned}\|a_1w_1c_1 + a_2w_2c_2\| &= \|a(S_1w_1T_1^* + S_2w_2T_2^*)c\| \leqslant \|a\|\max\{\|w_1\|, \|w_2\|\}\|c\| \\ &= \|a_1a_1^* + a_2a_2^*\|^{1/2}\max\{\|w_1\|, \|w_2\|\}\|c_1^*c_1 + c_2^*c_2\|^{1/2}.\end{aligned}$$

On the other hand condition (R′) is a special case of condition (r). ∎

REMARK 4.3. Magajna [14] and Pop [18] showed that a contractive bimodule W over arbitrary C^*-algebras A, C is isometrically isomorphic to an operator A-C-bimodule in the sense of [4] if and only if it satisfies the condition (r). Such a module is called *representable*. It has an isometric isomorphic representation as a submodule of some $\mathcal{B}(K, H)$, where H, K are Hilbert spaces and contractive left $\mathcal{A}$-modules respectively right $\mathcal{C}$-modules.

By a straight forward computation we obtain the following proposition:

PROPOSITION 4.4. *Given a Ruan $\mathcal{A}$-$\mathcal{C}$-bimodule W, then there exist a unique family of norms $\|\cdot\|_n$ on $M_n(W)$, $(n \in \mathbb{N})$ such that W becomes an operator $\mathcal{A}$-$\mathcal{C}$-bimodule (in the sense of* [4]*). These norms are given by the formula*

$$\|w\|_{M_n(W)} := \Big\| \sum_{i,j=1}^{n} S_iw_{i,j}T_j^* \Big\| \quad \textit{for } w = [w_{i,j}] \in M_n(W), \tag{4.1}$$

where $S_i \in \mathcal{A}$, $T_j \in \mathcal{C}$ are arbitrary n-tuples of isometries with pairwise orthogonal final projections. Every bounded morphism Φ between Ruan $\mathcal{A}$-$\mathcal{C}$-bimodules is completely bounded with $\|\Phi\|_{\mathrm{cb}} = \|\Phi\|$.

EXAMPLE AND REMARK 4.5. (i) Given C^*-algebras $\mathcal{A}$, $\mathcal{C}$ with properly infinite unital subalgebras $\mathcal{A}_0$ respectively $\mathcal{C}_0$ and a contractive $\mathcal{A}$-$\mathcal{C}$-bimodule W, if W is a Ruan $\mathcal{A}_0$-$\mathcal{C}_0$-bimodule, then W is a Ruan $\mathcal{A}$-$\mathcal{C}$-bimodule.

(ii) Given an arbitrary operator A-C-bimodule Z in the sense of [4] then the minimal (spatial) tensor product $W := \mathcal{K} \breve{\otimes} Z$ is a Ruan $\mathcal{A}$-$\mathcal{C}$-bimodule, where $\mathcal{A} = \mathcal{B} \breve{\otimes} A$ and $\mathcal{C} = \mathcal{B} \breve{\otimes} C$.

(iii) Let $\pi\colon \mathcal{A} \to \mathcal{B}(H)$ and $\rho\colon \mathcal{C} \to \mathcal{B}(K)$ be unital $*$-representations, W a normed space and $\theta\colon W \to \mathcal{B}(K,H)$ an isometry such that $\pi(\mathcal{A})\theta(W)\rho(\mathcal{C}) \subset \theta(W)$. Define the module action by $awc := \theta^{-1}(\pi(a)\theta(w)\rho(c))$. Then W is a Ruan $\mathcal{A}$-$\mathcal{C}$-bimodule. We call θ an isometric representation of W via (π,ρ) and W a *representable Ruan $\mathcal{A}$-$\mathcal{C}$-bimodule*. Every Ruan bimodule is representable (see Proposition 4.2 and the the proximate remark).

(iv) All isometric representation of a Ruan bimodule into some space $\mathcal{B}(K,H)$ give the same structure as an operator bimodule (see Proposition 4.4).

PROPOSITION 4.6 ([9], Proposition 6). *Let W be a Ruan $\mathcal{A}$-$\mathcal{C}$-bimodule, X a right semi-Ruan $\mathcal{A}$-module and Y a left semi-Ruan $\mathcal{C}$-module. Then $X \underset{\mathcal{A}}{\otimes} W$ is a right semi-Ruan $\mathcal{C}$-module respectively $W \underset{\mathcal{C}}{\otimes} Y$ a left semi-Ruan $\mathcal{A}$-module.*

The following proposition corresponds to the fact that the module Haagerup tensor product of operator modules is an operator module (see Chapter 8).

PROPOSITION 4.7. *Let V, W be Ruan $\mathcal{A}$-$\mathcal{C}$- respectively $\mathcal{C}$-$\mathcal{D}$-bimodules. Then $V \underset{\mathcal{C}}{\otimes} W$ is a Ruan $\mathcal{A}$-$\mathcal{D}$-bimodule.*

Proof. Clearly $V \underset{\mathcal{C}}{\otimes} W$ is a contractive $\mathcal{A}$-$\mathcal{D}$-bimodule. Let $u_1, u_2 \in V \underset{\mathcal{C}}{\otimes} W$ have orthogonal left supports $Q_1, Q_2 \in \mathcal{A}$ and orthogonal right supports $R_1, R_2 \in \mathcal{D}$. For $\varepsilon > 0$ there exist representations $u_i = v_i \underset{\mathcal{C}}{\otimes} w_i$ such that $\|v_i\|\|w_i\| \leqslant \|u_i\| + \varepsilon$, $(i = 1, 2)$, (see Proposition 3.8). Obviously we may suppose that $\|v_i\| = \|w_i\|$ and $v_i = Q_i v_i$, $w_i = w_i R_i$, $(i = 1, 2)$.

Choose two isometries $T_1, T_2 \in \mathcal{C}$ with orthogonal final supports P_1, P_2. Then we have $u_1 + u_2 = (v_1 T_1^* + v_2 T_2^*) \underset{\mathcal{C}}{\otimes} (T_1 w_1 + T_2 w_2)$. Now the elements $v_i T_i^*$ have orthogonal left supports Q_i and orthogonal right supports P_i and the elements $T_i w_i$ have orthogonal left supports P_i and orthogonal right supports R_i, $(i = 1, 2)$. From equation (R$''$) we have

$$\begin{aligned}\|u_1 + u_2\| &\leqslant \|v_1 T_1^* + v_2 T_2^*\| \|T_1 w_1 + T_2 w_2\| \\ &= \max\{\|v_1 T_1^*\|, \|v_2 T_2^*\|\} \max\{\|T_1 w_1\|, \|T_2 w_2\|\} \\ &= \max\{\|v_1\|, \|v_2\|\} \max\{\|w_1\|, \|w_2\|\} \\ &= \max\{\|v_1\|\|w_1\|, \|v_2\|\|w_2\|\} \leqslant \max\{\|u_1\|, \|u_2\|\} + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, $V \underset{\mathcal{C}}{\otimes} W$ has the Ruan property (R$''$). ∎

5. FROM SEMI-RUAN MODULES TO RUAN MODULES AND VICE VERSA

PROPOSITION 5.1 (see Proposition 4 of [9]). *Every Ruan $\mathcal{A}$-$\mathcal{C}$-bimodule, considered as a left $\mathcal{A}$-module or a right $\mathcal{C}$-module, is a respective one-sided semi-Ruan module.*

REMARK 5.2. (i) Given a Ruan $\mathcal{A}$-$\mathcal{C}$-bimodule and a $c \in \mathcal{C}$ then $Wc \subset W$ is a left semi-Ruan $\mathcal{A}$-submodule of W.

(ii) Let A be a C^*-algebra on a Hilbert space L and Y a left contractive A-module. The space $\mathcal{F}(L, Y)$ of bounded operators of finite rank is a contractive A-$\mathcal{B}$-bimodule. Given $\xi \in L$, $\|\xi\| = 1$ and $p = \xi' \otimes \xi \in \mathcal{B}(L)$ the corresponding minimal projection then the mapping $\mathcal{F}(L, Y) \cdot p \ni vp \to v\xi \in Y$ is an isometric A-isomorphism $\mathcal{F}(L, Y) \cdot p \cong Y$. Further on $\mathcal{F}(L, Y) = \mathrm{lin}(\mathcal{F}(L, Y) \cdot p \cdot \mathcal{B})$.

(iii) Observe $\mathcal{B}(\ell_2, Y)$ is isometric isomorphic to the *weak ℓ_2-direct sum* of countable many copies of Y, i.e. the norm of the mapping $v\colon \xi \mapsto \sum_{i=1}^{\infty} \xi_i y_i$ is $\|v\| = \sup_f \Big(\sum_{i=1}^{\infty} |f(y_i)|^2 \Big)^{1/2}$ where the supremum runs over all $f \in Y'$, $\|f\| \leqslant 1$.

(iv) Consider the special case where $\pi : A \to \mathcal{B}(H)$ is a $*$-representation on a Hilbert space H, $\mathcal{A} := \mathcal{B} \breve{\otimes} A$. Then $Y = L \otimes^2 H$ is a left semi-Ruan $\mathcal{A}$-module and $\mathcal{F}(L) \dot{\otimes} B(\mathbb{C}, H) = \mathcal{F}(L \otimes_2 \mathbb{C}, L \otimes_2 H) \subset \mathcal{F}(L, Y) \subset \mathcal{K}(L, Y) = \mathcal{K}(L \otimes^2 \mathbb{C}, L \otimes^2 H) = \mathcal{K}(L) \breve{\otimes} \mathcal{B}(\mathbb{C}, H)$. Both the left or right hand side give the usual column Hilbert space quantization of H (see Chapter 2 of [17]). The following proposition generalizes this column quantization (see Lambert, Chapter 5 of [11]).

PROPOSITION 5.3. *Given a left semi-Ruan $\mathcal{A}$-module Y and a separable Hilbert space L, then $\mathcal{F}(L, Y)$ is a Ruan $\mathcal{A}$-$\mathcal{B}$-bimodule, where $\mathcal{B} = \mathcal{B}(L)$. Let $p \in \mathcal{B}$ be a minimal projection, then Y is isometric $\mathcal{A}$-isomorphic to the semi-Ruan $\mathcal{A}$-module $\mathcal{F}(L, Y) \cdot p$.*

Proof. Clearly $\mathcal{F}(L, Y)$ is a contractive $\mathcal{A}$-$\mathcal{B}$-bimodule. Choose isometries $S_1, S_2 \in \mathcal{A}$ and isometries $T_1, T_2 \in \mathcal{B}$ with orthogonal final projections. Hence

$$\begin{aligned}
\|(S_1 w_1 T_1^* + S_2 w_2 T_2^*)\xi\| &\leqslant (\|S_1 w_1 T_1^* \xi\|^2 + \|S_2 w_2 T_2^* \xi\|^2)^{1/2} = (\|w_1 T_1^* \xi\|^2 + \|w_2 T_2^* \xi\|^2)^{1/2} \\
&\leqslant \max\{\|w_1\|, \|w_2\|\} (\|T_1^* \xi\|^2 + \|T_2^* \xi\|^2)^{1/2} \\
&= \max\{\|w_1\|, \|w_2\|\} \langle (T_1 T_1^* + T_2 T_2^*)\xi, \xi \rangle^{1/2} \\
&\leqslant \max\{\|w_1\|, \|w_2\|\} \|\xi\|
\end{aligned}$$

for all $w_1, w_2 \in \mathcal{B}(L, Y)$, $\xi \in L$. The rest follows from remark (ii) above. ∎

$\mathcal{F}(L, Y)$ is the *minimal* Ruan $\mathcal{A}$-$\mathcal{B}$-bimodule V such that (Vp) is isometric $\mathcal{A}$-isomorphic to Y. The following Lemma 5.4 states this more precisely.

LEMMA 5.4. *Given a Ruan $\mathcal{A}$-$\mathcal{B}$-bimodule W and a minimal projection $p = \xi' \otimes \xi \in \mathcal{B}$. If $W = \mathrm{lin}\{(Wp) \cdot \mathcal{B}\}$ then the linear mapping*

$$\Lambda : W \to \mathcal{F}(L, Wp), \quad \Lambda(wp \cdot b)\colon \zeta \mapsto \langle b\zeta, \xi \rangle wp,$$

for $w \in W$, $b \in \mathcal{B}$, $\zeta \in L$, is a contractive bimorphism of Ruan $\mathcal{A}$-$\mathcal{B}$-bimodules.

Proof. Given $\sum_\nu w_\nu p \cdot b_\nu \in \operatorname{lin}\{Wp \cdot \mathcal{B}\}$ and $\zeta \in L$, then

$$\Lambda\Big(\sum_\nu w_\nu p \cdot b_\nu\Big)(\zeta) = \sum_\nu \langle b_\nu \zeta, \xi\rangle w_\nu p = \sum_\nu w_\nu p \cdot b_\nu z$$

where $z\colon \eta \mapsto \langle \eta, \xi\rangle \zeta$ and thus

$$\Big\|\Lambda\Big(\sum_\nu w_\nu p \cdot b_\nu\Big)(\zeta)\Big\| = \Big\|\sum_\nu w_\nu p \cdot b_\nu z\Big\| \leqslant \Big\|\sum_\nu w_\nu p \cdot b_\nu\Big\| \|z\|.$$

Since $\|z\| = \|\zeta\|$ the mapping Λ is welldefined and contractive. Clearly, Λ is an $\mathcal{A}$-$\mathcal{B}$-bimorphism. ∎

6. INJECTIVITY OF THE PROJECTIVE MODULE TENSOR PRODUCT

Our aim is to show that the projective module tensor product of right and left semi-Ruan modules preserves embeddings.

THEOREM 6.1. *Given a properly infinite C^*-algebra $\mathcal{A}$, right semi-Ruan $\mathcal{A}$-modules X_0, X, left semi-Ruan $\mathcal{A}$-modules Y_0, Y and isometric morphisms $\alpha\colon X_0 \to X$ and $\beta\colon Y_0 \to Y$, of right respectively left $\mathcal{A}$-modules, then the linear operator $\alpha \underset{\mathcal{A}}{\otimes} \beta\colon X_0 \underset{\mathcal{A}}{\otimes} Y_0 \to X \underset{\mathcal{A}}{\otimes} Y$ is isometric.*

Proof. Since $\alpha \underset{\mathcal{A}}{\otimes} \beta = (\alpha \underset{\mathcal{A}}{\otimes} \mathbf{1}_Y)(\mathbf{1}_{X_0} \underset{\mathcal{A}}{\otimes} \beta)$ it suffices to show that both factors are isometric. We will prove this for the first factor (see Corollary 6.6 below). It is obvious that the proof for the second factor follows in a "*symmetric way*". Helemskii ([9], Proposition 7) gives a formal proof for the latter argument using complex conjugate opposite modules. ∎

We start the proof that $\alpha \underset{\mathcal{A}}{\otimes} \mathbf{1}_Y$ is isometric, with the special case where Y is a cyclic module. The general case then follows from Lemma 2.1 that the cyclic submodules are directed upwards and Lemma 2.2 about the limit of their tensor products.

In order to simplify the notation we consider the isometric morphism as an embedding of a submodule Z of X and write $Z \subset X$. In some places we need a name for the embedding map and denote it by $i_{Z,X}\colon Z \to X$. In the following $\mathcal{A}$ denotes a properly infinite C^*-algebra.

LEMMA 6.2. *Given a right semi-Ruan $\mathcal{A}$-module X, a left semi-Ruan $\mathcal{A}$-module Y and a fixed element $\widetilde{y} \in Y$, define*

$$X \ni x \mapsto q_{\widetilde{y}}(x) := \inf_{x = \widetilde{x}a} \|\widetilde{x}\| \|a\widetilde{y}\| \tag{6.1}$$

where $a \in \mathcal{A}$ and $\widetilde{x} \in X$ arbitrary. Then $q_{\widetilde{y}}$ is a seminorm on X. Furthermore we may suppose in the definition of $q_{\widetilde{y}}$ that a is invertible.

Proof. Clearly $q_{\widetilde{y}}(\lambda x) = |\lambda| q_{\widetilde{y}}(x)$ for $\lambda \in \mathbb{C}$. For $\varepsilon > 0$ and $x_1, x_2 \in X$ choose decompositions $x_i = \widetilde{x}_i a_i$, $(i = 1, 2)$, such that $\|\widetilde{x}_i\| \|a_i \widetilde{y}\| \leqslant q_{\widetilde{y}}(x_i) + \varepsilon$. We may assume that $\|\widetilde{x}_i\| = \|a_i \widetilde{y}\|$. Choose isometries $S_1, S_2 \in \mathcal{A}$ with orthogonal final projections. Then $S_i^* S_j = \delta_{i,j} \mathbf{1}_{\mathcal{A}}$ for $i = 1, 2$ and hence

$$x_1 + x_2 = (\widetilde{x}_1 S_1^* + \widetilde{x}_2 S_2^*)(S_1 a_1 + S_2 a_2).$$

From the conditions (rsR) and (lsR) and equation (2.2) we obtain the following estimation:

$$\begin{aligned} q_{\widetilde{y}}(x_1 + x_2) &\leqslant \|\widetilde{x}_1 S_1^* + \widetilde{x}_2 S_2^*\| \|S_1 a_1 \widetilde{y} + S_2 a_2 \widetilde{y}\| \\ &\leqslant (\|\widetilde{x}_1 S_1^*\|^2 + \|\widetilde{x}_2 S_2^*\|^2)^{1/2} (\|S_1 a_1 \widetilde{y}\|^2 + \|S_2 a_2 \widetilde{y}\|^2)^{1/2} \\ &= (\|\widetilde{x}_1\|^2 + \|\widetilde{x}_2\|^2)^{1/2} (\|a_1 \widetilde{y}\|^2 + \|a_2 \widetilde{y}\|^2)^{1/2} \\ &= \|\widetilde{x}_1\| \|a_1 \widetilde{y}\| + \|\widetilde{x}_2\| \|a_2 \widetilde{y}\| \leqslant q_{\widetilde{y}}(x_1) + q_{\widetilde{y}}(x_2) + 2\varepsilon. \end{aligned}$$

Since ε is arbitrary $q_{\widetilde{y}}$ is subadditive.

We denote temporarily $\widetilde{q}_{\widetilde{y}}(x) := \inf\limits_{x = \widetilde{x} a} \|\widetilde{x}\| \|a \widetilde{y}\|$ where $\widetilde{x} \in X$, $a \in \mathcal{A}$ and a is invertible. We will show that $\widetilde{q}_{\widetilde{y}} = q_{\widetilde{y}}$.

Obviously $q_{\widetilde{y}} \leqslant \widetilde{q}_{\widetilde{y}}$. For $\varepsilon > 0$ there are $\widetilde{x} \in X$, $a \in \mathcal{A}$ such that $x = \widetilde{x} a$ and $(\|\widetilde{x}\| \|a \widetilde{y}\|) \leqslant q_{\widetilde{y}}(x) + \varepsilon$. We may assume $\|\widetilde{x}\| = 1$. With isometries S_1, S_2 as above we have $x = \widetilde{x} S_1^* (S_1 a + \varepsilon S_2)$ where $(S_1 a + \varepsilon S_2)^* (S_1 a + \varepsilon S_2) = a^* a + \varepsilon^2 \mathbf{1}_{\mathcal{A}}$ is invertible. Let $v_\varepsilon |a_\varepsilon|$ be the polar decomposition of $a_\varepsilon = S_1 a + \varepsilon S_2$. Then $|a_\varepsilon|$ is invertible, $v_\varepsilon = a_\varepsilon |a_\varepsilon|^{-1} \in \mathcal{A}$ and $x = (\widetilde{x} S_1 v_\varepsilon) |a_\varepsilon|$. Since $|a_\varepsilon| = v_\varepsilon^* a_\varepsilon$ we obtain

$$\begin{aligned} \widetilde{q}_{\widetilde{y}}(x) &\leqslant \|\widetilde{x} S_1^* v_\varepsilon\| \| |a_\varepsilon| \widetilde{y}\| = \|\widetilde{x}\| \|a_\varepsilon \widetilde{y}\| \leqslant \|\widetilde{x}\| (\|S_1 a \widetilde{y}\| + \varepsilon \|S_2 \widetilde{y}\|) \\ &= \|\widetilde{x}\| \|a \widetilde{y}\| + \varepsilon \|\widetilde{x}\| \|\widetilde{y}\| \leqslant q_{\widetilde{y}}(x) + \varepsilon(1 + \|\widetilde{y}\|). \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary $\widetilde{q}_{\widetilde{y}} \leqslant q_{\widetilde{y}}$. ∎

LEMMA 6.3. *Define the seminorm $q_{\widetilde{y}} \colon X \to [0, \infty)$ as in Lemma* 6.2. *Given a submodule $Z \subset X$ define analogously a seminorm $p_{\widetilde{y}}$ on Z (see equation* (6.1)*):*

$$Z \ni z \mapsto p_{\widetilde{y}}(z) := \inf_{z = \widetilde{z} a} \|\widetilde{z}\| \|a \widetilde{y}\|$$

where $\widetilde{z} \in Z$, $a \in \mathcal{A}$ arbitrary. Then $q_{\widetilde{y}}|_Z = p_{\widetilde{y}}$.

Proof. If $z \in Z$, then $q_{\widetilde{y}}(z) \leqslant p_{\widetilde{y}}(z)$ since there are more decompositions for z in X. On the other hand, for $z \in Z$ and $\varepsilon > 0$ there exists a decomposition $z = \widetilde{x} a$ as in Lemma 6.2 with $\widetilde{x} \in X$ and an invertible a such that $\|\widetilde{x}\| \|a \widetilde{y}\| \leqslant q_{\widetilde{y}}(z) + \varepsilon$. Then $\widetilde{x} = z a^{-1} \in Z$ and thus $p_{\widetilde{y}}(z) \leqslant \|\widetilde{x}\| \|a \widetilde{y}\| \leqslant q_{\widetilde{y}}(z) + \varepsilon$. ∎

PROPOSITION 6.4. *Let X be a right semi-Ruan, Y a left semi-Ruan $\mathcal{A}$-module and $\widetilde{y}$ a fixed element in Y. Define the semi norm $q_{\widetilde{y}}$ on X as in Lemma* 6.2. *Denote the kernel of $q_{\widetilde{y}}$ by $X_{\widetilde{y}} = \{x \in X : q_{\widetilde{y}}(x) = 0\}$ and by $\| \cdot \|_{q_{\widetilde{y}}}$ the induced norm on $X / X_{\widetilde{y}}$. The kernel of the linear map (tensor multiplication with $\widetilde{y}$)*

$$\tau_{\widetilde{y}} \colon X \ni x \to x \underset{\mathcal{A}}{\otimes} \widetilde{y} \in X \underset{\mathcal{A}}{\otimes} \mathcal{A} \widetilde{y}$$

is $X_{\widetilde{y}}$ *and the induced map* $\widetilde{\tau}_{\widetilde{y}}\colon (X/X_{\widetilde{y}}, \|\cdot\|_{\widetilde{y}}) \to X \underset{\mathcal{A}}{\otimes} \mathcal{A}\widetilde{y}$ *is a surjective isometry.*

Proof. Clearly $\tau_{\widetilde{y}}$ is a linear map. Choose a decomposition $x = \widetilde{x}a$, $\widetilde{x} \in X$, $a \in \mathcal{A}$ of x, then

$$\tau_{\widetilde{y}}(x) = \widetilde{x}a \underset{\mathcal{A}}{\otimes} \widetilde{y} = \widetilde{x} \underset{\mathcal{A}}{\otimes} a\widetilde{y} \quad \text{and} \quad \|\tau_{\widetilde{y}}(x)\| \leqslant \|\widetilde{x}\|\|a\widetilde{y}\|.$$

Taking the infimum over all decompositions we obtain $\|\tau_{\widetilde{y}}(x)\| \leqslant q_{\widetilde{y}}(x)$. Hence $\tau_{\widetilde{y}}$ induces a contraction $\widetilde{\tau}_{\widetilde{y}}\colon (X/X_{\widetilde{y}}, \|\cdot\|_{q_{\widetilde{y}}}) \to X \underset{\mathcal{A}}{\otimes} \mathcal{A}\widetilde{y}$. By Proposition 2.3 $\widetilde{\tau}_{\widetilde{y}}$ is surjective.

For $\widetilde{x} \in X$ there is a linear functional $\varphi : X \to \mathbb{C}$ such that $\varphi(\widetilde{x}) = q_{\widetilde{y}}(\widetilde{x})$ and $|\varphi(x)| \leqslant q_{\widetilde{y}}(x)$ for all $x \in X$. We define a function $f\colon X \times \mathcal{A}\widetilde{y} \to \mathbb{C}$ by

$$f(x, a\widetilde{y}) := \varphi(xa).$$

We show that f is well defined. If $a_1\widetilde{y} = a_2\widetilde{y}$ then

$$|\varphi(x(a_1 - a_2))| \leqslant q_{\widetilde{y}}(x(a_1 - a_2)) \leqslant \|x\|\|(a_1 - a_2)\widetilde{y}\| = 0.$$

Similarly follows that f is linear in the second variable: If $a\widetilde{y} + b\widetilde{y} = c\widetilde{y}$ then

$$\begin{aligned} |f(x, a\widetilde{y}) + f(x, b\widetilde{y}) - f(x, c\widetilde{y})| = |\varphi(x(a + b - c))| &\leqslant q_{\widetilde{y}}(x(a + b - c)) \\ &\leqslant \|x\|\|(a + b - c)\widetilde{y}\| = 0. \end{aligned}$$

Thus f is bilinear and balanced:

$$f(xa, b\widetilde{y}) = \varphi((xa)b) = \varphi(x(ab)) = f(x, a(b\widetilde{y}))$$

for all $x \in X$ and $a, b \in \mathcal{A}$. By definition $|f(x, a\widetilde{y})| \leqslant q_{\widetilde{y}}(xa) \leqslant \|x\|\|a\widetilde{y}\|$, i.e. $f\colon X \times \mathcal{A}\widetilde{y} \to \mathbb{C}$ is contractive. Hence there is a bounded linear functional $\widetilde{\varphi} \in (X \underset{\mathcal{A}}{\otimes} \mathcal{A}\widetilde{y})'$ such that $\widetilde{\varphi}(x \underset{\mathcal{A}}{\otimes} a\widetilde{y}) = f(x, a\widetilde{y})$ and $\|\widetilde{\varphi}\| = \|f\| \leqslant 1$. Now

$$q_{\widetilde{y}}(\widetilde{x}) = \varphi(\widetilde{x}) = f(\widetilde{x}, \widetilde{y}) = \widetilde{\varphi}(\widetilde{x} \underset{\mathcal{A}}{\otimes} \widetilde{y}) = \widetilde{\varphi}(\tau_{\widetilde{y}}(\widetilde{x})) \leqslant \|\widetilde{\varphi}\|\|\tau_{\widetilde{y}}(\widetilde{x})\|$$

and thus $\|\widetilde{x} + X_{\widetilde{y}}\|_{q_{\widetilde{y}}} = q_{\widetilde{y}}(\widetilde{x}) \leqslant \|\widetilde{\tau}_{\widetilde{y}}(\widetilde{x} + X_{\widetilde{y}})\|_{Y \underset{\mathcal{A}}{\otimes} \mathcal{A}\widetilde{y}}$. Since $\widetilde{x}$ is arbitrary $\widetilde{\tau}_{\widetilde{y}}$ is an isometry. ∎

COROLLARY 6.5. *Let* $i_{Z,X}\colon Z \hookrightarrow X$ *be a submodule and* $\widetilde{y}$ *a fixed element in* Y. *The map* $i_{Z,X} \underset{\mathcal{A}}{\otimes} \mathbf{1}_{\mathcal{A}\widetilde{y}}\colon Z \underset{\mathcal{A}}{\otimes} \mathcal{A}\widetilde{y} \to X \underset{\mathcal{A}}{\otimes} \mathcal{A}\widetilde{y}$ *is an isometry.*

Proof. Consider the seminorms $q_{\widetilde{y}}$ and $p_{\widetilde{y}} = q_{\widetilde{y}}|_Z$ as defined in Lemma 6.3 and their kernels as defined in Proposition 6.4:

$$Z_{\widetilde{y}} = \{y \in Z : p_{\widetilde{y}}(y) = 0\} = X_{\widetilde{y}} \cap Z.$$

Hence the canonical linear map $Z/Z_{\widetilde{y}} = Z/(Z \cap X_{\widetilde{y}}) \overset{\kappa}{\hookrightarrow} X/X_{\widetilde{y}}$ is injective and an isometry for the induced norms $\|\cdot\|_{p_{\widetilde{y}}}$ and $\|\cdot\|_{q_{\widetilde{y}}}$. Now we have the commutative diagram

$$\begin{array}{ccc} Z/Z_{\widetilde{y}} & \xrightarrow{\kappa} & X/X_{\widetilde{y}} \\ {\scriptstyle \widetilde{\tau}_{\widetilde{y}}}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{\tau}_{\widetilde{y}}} \\ Z \underset{\mathcal{A}}{\otimes} \mathcal{A}\widetilde{y} & \xrightarrow[i_{Z,X} \underset{\mathcal{A}}{\otimes} \mathbf{1}_{\mathcal{A}\widetilde{y}}]{} & X \underset{\mathcal{A}}{\otimes} \mathcal{A}\widetilde{y} \end{array}$$

where the vertical maps are isometric isomorphisms (see Proposition 6.4) and κ is an isometry. Hence $i_{Z,X} \underset{\mathcal{A}}{\otimes} \mathbf{1}_{\mathcal{A}\widetilde{y}}$ is an isometry too. ∎

The following corollary is the last step in the proof of Theorem 6.1:

COROLLARY 6.6. *Given a submodule $i_{Z,X} \colon Z \hookrightarrow X$, then the map $i_{Z,X} \underset{\mathcal{A}}{\otimes} \mathbf{1}_Y$: $Z \underset{\mathcal{A}}{\otimes} Y \to X \underset{\mathcal{A}}{\otimes} Y$ is an isometry.*

Proof. For $y_1 \preceq y_2 \preceq y_3 \preceq \cdots$ we have the upwards directed system of subspaces (see Lemma 2.1)

$$\mathcal{A}y_1 \subset \mathcal{A}y_2 \subset \mathcal{A}y_3 \;\cdots \qquad \text{with union } Y.$$

Let $i_{1,2} \colon \mathcal{A}y_1 \hookrightarrow \mathcal{A}y_2$ and $i_y \colon \mathcal{A}y \hookrightarrow Y$ denote the inclusion mappings. Building the module tensor product with the inclusion mapping $i_{Z,X} \colon Z \hookrightarrow X$ we obtain a commutative diagram, where all vertical maps on the left side are isometric

$$\begin{array}{ccccccc} Z \underset{\mathcal{A}}{\otimes} \mathcal{A}y_1 & \xrightarrow{\mathbf{1}_Z \underset{\mathcal{A}}{\otimes} i_{1,2}} & Z \underset{\mathcal{A}}{\otimes} \mathcal{A}y_2 & \xrightarrow{\mathbf{1}_Z \underset{\mathcal{A}}{\otimes} i_{2,3}} & Z \underset{\mathcal{A}}{\otimes} \mathcal{A}y_3 & \longrightarrow & \cdots\; Z \underset{\mathcal{A}}{\otimes} Y \\ {\scriptstyle i_{Z,X} \underset{\mathcal{A}}{\otimes}}\big\downarrow{\scriptstyle \mathbf{1}_{\mathcal{A}y_1}} & & {\scriptstyle i_{Z,X} \underset{\mathcal{A}}{\otimes} \mathbf{1}_{\mathcal{A}y_2}}\big\downarrow & & {\scriptstyle i_{Z,X} \underset{\mathcal{A}}{\otimes} \mathbf{1}_{\mathcal{A}y_3}}\big\downarrow & & {\scriptstyle i_{Z,X} \underset{\mathcal{A}}{\otimes} \mathbf{1}_Y}\big\downarrow \\ X \underset{\mathcal{A}}{\otimes} \mathcal{A}y_1 & \xrightarrow[\mathbf{1}_X \underset{\mathcal{A}}{\otimes} i_{1,2}]{} & X \underset{\mathcal{A}}{\otimes} \mathcal{A}y_2 & \xrightarrow[\mathbf{1}_X \underset{\mathcal{A}}{\otimes} i_{2,3}]{} & X \underset{\mathcal{A}}{\otimes} \mathcal{A}y_3 & \longrightarrow & \cdots\; X \underset{\mathcal{A}}{\otimes} Y \end{array} \tag{6.2}$$

From Lemma 2.2 follows, that the "limit map" $i_{Z,X} \underset{\mathcal{A}}{\otimes} \mathbf{1}_Y$ is isometric too. ∎

After finishing the proof of Theorem 6.1 we may apply the result to the top and bottom row of the diagram (6.2):

REMARK 6.7. The mappings on the left-hand side of the top and bottom row of diagram (6.2) are all isometric. The family of isometries $\mathbf{1}_X \underset{\mathcal{A}}{\otimes} i_y \colon X \underset{\mathcal{A}}{\otimes} \mathcal{A}y \hookrightarrow X \underset{\mathcal{A}}{\otimes} Y$, $(y \in Y)$, is the inductive limit of the bottom row and analogously for the top row. Now $i_{Z,X} \underset{\mathcal{A}}{\otimes} \mathbf{1}_Y$ is the inductive limit of the vertical maps. We may consider this as an exhaustion by an upwards directed system of subspaces

$$X \underset{\mathcal{A}}{\otimes} Y = \bigcup_{y \in Y} X \underset{\mathcal{A}}{\otimes} \mathcal{A}y.$$

7. AN EXTENSION THEOREM FOR BOUNDED BIMODULE MORPHISMS

Given a right contractive $\mathcal{A}$-module X and a left contractive $\mathcal{C}$-module Y, then the the projective tensor product $Y \otimes_p X$ of the normed spaces is a contractive $\mathcal{C}$-$\mathcal{A}$-bimodule and the dual $(Y \otimes_p X)'$ is a contractive $\mathcal{A}$-$\mathcal{C}$-bimodule. The module operation is given by

$$c \cdot (y \otimes x) \cdot a := (cy) \otimes (xa) \quad \text{and} \quad (a \cdot f \cdot c)(y \otimes x) = f(cy \otimes xa)$$

for all $a \in \mathcal{A}$, $c \in \mathcal{C}$, $y \in Y$, $x \in X$, $f \in (Y \otimes_p X)'$.

A short inspection of the proof shows that Helemskii's result ([9], Proposition 8) holds for arbitrary (semi-)Ruan (bi)modules with respect to properly infinite C^*-algebras. So using this proof we get the following result:

PROPOSITION 7.1. *Let X be a right semi-Ruan $\mathcal{A}$-module, Y a left semi-Ruan $\mathcal{C}$-module and W a Ruan $\mathcal{A}$-$\mathcal{C}$-bimodule. Given submodules $i_{X_0,X} \colon X_0 \hookrightarrow X$, $i_{Y_0,Y} \colon Y_0 \hookrightarrow Y$ and $i_{W_0,W} \colon W_0 \hookrightarrow W$ then the following is an isometry of normed spaces:*

$$(i_{Y_0,Y} \underset{p}{\otimes} i_{X_0,X}) \underset{\mathcal{A}-\mathcal{C}}{\otimes} i_{W_0,W} : (Y_0 \underset{p}{\otimes} X_0) \underset{\mathcal{A}-\mathcal{C}}{\otimes} W_0 \to (Y \underset{p}{\otimes} X) \underset{\mathcal{A}-\mathcal{C}}{\otimes} W. \tag{7.1}$$

Proof. Consider the following commutative diagram

$$\begin{array}{ccc}
(Y_0 \underset{p}{\otimes} X_0) \underset{\mathcal{A}-\mathcal{C}}{\otimes} W_0 & \xrightarrow{(i_{Y_0,Y} \underset{p}{\otimes} i_{X_0,X}) \underset{\mathcal{A}-\mathcal{C}}{\otimes} i_{W_0,W}} & (Y \underset{p}{\otimes} X) \underset{\mathcal{A}-\mathcal{C}}{\otimes} W \\
\big\downarrow & & \big\downarrow \\
(X_0 \underset{\mathcal{A}}{\otimes} W_0) \underset{\mathcal{C}}{\otimes} Y_0 & \xrightarrow{(i_{X_0,X} \underset{\mathcal{A}}{\otimes} i_{W_0,W}) \underset{\mathcal{C}}{\otimes} i_{Y_0,Y}} & (X \underset{\mathcal{A}}{\otimes} W) \underset{\mathcal{C}}{\otimes} Y
\end{array}$$

where the vertical arrows are isometric isomorphisms of normed spaces. In the case that we are considering, these isomorphisms are the shuffle maps defined on elementary tensors as $(y \otimes x) \underset{\mathcal{A}-\mathcal{C}}{\otimes} w \to (x \underset{\mathcal{A}}{\otimes} w) \underset{\mathcal{C}}{\otimes} y$ where $x \in X_0$, $y \in Y_0$ and $w \in W_0$ respectively $x \in X$, $y \in Y$ and $w \in W$ (see [10]). We see that it is sufficient to show that the operator $(i_{X_0,X} \underset{\mathcal{A}}{\otimes} i_{W_0,W}) \underset{\mathcal{C}}{\otimes} i_{Y_0,Y}$ is isometric. By Proposition 5.1 $i_{W_0,W}$ is an isometric morphism of left semi-Ruan $\mathcal{A}$-modules and therefore the operator $i_{X_0,X} \underset{\mathcal{A}}{\otimes} i_{W_0,W}$ is isometric (see Theorem 6.1). Moreover, by Proposition 4.6 it is an isometric morphism of right semi-Ruan $\mathcal{C}$-modules. Applying once more Theorem 6.1 we see that $(i_{X_0,X} \underset{\mathcal{A}}{\otimes} i_{W_0,W}) \underset{\mathcal{C}}{\otimes} i_{Y_0,Y}$ is isometric. ∎

Recall the isometric isomorphisms

$$((Y \underset{p}{\otimes} X) \underset{\mathcal{A}-\mathcal{C}}{\otimes} W)' \cong {}_{\mathcal{A}}\mathbf{h}_{\mathcal{C}}(W, (Y \underset{p}{\otimes} X)') \cong {}_{\mathcal{C}}\mathbf{h}_{\mathcal{A}}((Y \underset{p}{\otimes} X), W'), \tag{7.2}$$

where ${}_{\mathcal{A}}\mathbf{h}_{\mathcal{C}}$ denotes the normed space of bounded $\mathcal{A}$-$\mathcal{C}$-bimorphisms. By specializing we obtain from equations (7.1) and (7.2) *extension* and *lifting theorems* for bounded bimodule morphisms, for example:

THEOREM 7.2. *Given families of right semi-Ruan $\mathcal{A}$-module X_i, of left semi-Ruan $\mathcal{C}$-module Y_i, $(i \in I)$, a Ruan $\mathcal{A}$-$\mathcal{C}$-bimodule W and a subbimodule $W_0 \subset W$, then for every bounded $\mathcal{A}$-$\mathcal{C}$-morphism $\Phi\colon W_0 \to \Big(\overset{\ell_1}{\bigoplus\limits_{i\in I}} Y_i \otimes_p X_i\Big)'$ there exists an extension to an $\mathcal{A}$-$\mathcal{C}$-morphism $\widetilde{\Phi}\colon W \to \Big(\overset{\ell_1}{\bigoplus\limits_{i\in I}} Y_i \otimes_p X_i\Big)'$ with the same norm.*

In the terminology of Helemskii [9]: the space $\overset{\ell_1}{\bigoplus\limits_{i\in I}} Y_i \otimes_p X_i$ is ER-flat.

Proof. Clearly $(Y_i \otimes_p X_i)'$ has the extension property. Thus the ℓ_∞ direct sum of these spaces has the extension property and the ℓ_∞ direct sum is the dual of the ℓ_1 direct sum. ∎

Theorem 7.2 generalizes the well known extension Theorem 7.3 for operator valued completely bounded A-C-morphisms. For this purpose consider the following special case:

Let H, K be Hilbert spaces, $\pi : A \to \mathcal{B}(H)$, $\rho\colon C \to \mathcal{B}(K)$ representations of arbitrary unital C^*-algebras. K^{op} denotes the complex conjugate space with the *opposite* module structure, i.e. K^{op} is a right A-module. The Hilbert space $X := L^{\mathrm{op}} \otimes^2 K^{\mathrm{op}}$ is a right semi-Ruan $(\mathcal{B} \check{\otimes} A)$-module and $Y := L \otimes^2 H$ a left semi-Ruan $(\mathcal{B} \check{\otimes} C)$-module via the representation $\mathbf{1}_B \otimes \rho$ respectively $\mathbf{1}_B \otimes \pi$. The projective tensor product $(Y \otimes_p X)$ coincides with the space $\mathcal{T}(L \otimes^2 H, L \otimes^2 K)$ of trace class operators and the dual $(Y \otimes_p X)'$ with the Ruan $(\mathcal{B} \check{\otimes} A)$-$(\mathcal{B} \check{\otimes} C)$-bimodule $B(L \otimes^2 K, L \otimes^2 H)$. Now using the same line of arguments as Helemskii ([9], Section 4) we see that the extension Theorem 7.3 below follows from Theorem 7.2. I must confess that this is not the shortest proof of Theorem 7.3 but this was not the intention.

THEOREM 7.3. *Let A, C be unital C^*-algebras, H, K Hilbert spaces and $\pi\colon A \mapsto \mathcal{B}(H)$, $\rho : C \mapsto \mathcal{B}(K)$ unital $*$-representations. Given an operator A-C-bimodule W and a subbimodule W_0, every completely bounded A-C-morphism $\Phi\colon W_0 \mapsto \mathcal{B}(K, H)$ (i.e. $\Phi(avc) = \pi(a)\Phi(v)\rho(c)$ for $a \in A$, $v \in V$, $c \in C$) has an extension to a completely bounded A-C-morphism $\widetilde{\Phi} : W \mapsto \mathcal{B}(K, H)$ with the same norm $\|\widetilde{\Phi}\|_{\mathrm{cb}} = \|\Phi\|_{\mathrm{cb}}$.*

The extension theorem 7.3 above was first proved by the author [24] (see also Muhly and Na [15]) using a generalization of the Hahn–Banach theorem for operator valued sublinear functionals [23]. Suen [21] gave a proof for a concrete $\mathcal{A}$-$\mathcal{C}$-bimodule using the bimodule Paulsen system (see Section 3.6.1 of [3]). For this proof and related results see Blecher and Le Merdy ([3], Chapter 3.6 and the notes in Chapter 3.9). The extension theorem holds for arbitrary C^*-algebras by replacing A, C with the unitizations A^1, C^1 if necessary. The extension theorem is not valid in general with A or C replaced by a nonselfadjoint operator algebra. See for example Proposition 7.2.11 of [3] and Example 3.5 of [20].

8. INJECTIVITY OF THE MODULE HAAGERUP TENSOR PRODUCT

Let V respectively W be a right respectively left operator A-module. The uncompleted Haagerup tensor product $V \underset{h}{\otimes} W$ is an uncompleted operator space and a contractive A-bimodule with the multiplication $a_1 \cdot (x \otimes y) \cdot a_2 := xa_2 \otimes a_1 y$. The uncompleted module Haagerup tensor product is defined by $V \underset{hA}{\otimes} W := (V \underset{h}{\otimes} W)/A$ (see Section 1). The injective operator space tensor product $\dot{\otimes}$ with $\mathcal{K} = \mathcal{K}(L)$ is projective (see Definition 2.4.3 and Remark(iii) of [17]). Hence we have a completely isometric isomorphism of operator spaces and an isomorphism of Ruan $\mathcal{B}$-bimodules

$$\mathcal{K} \dot{\otimes} (V \underset{hA}{\otimes} W) = \mathcal{K} \dot{\otimes} ((V \underset{h}{\otimes} W)/A) = (\mathcal{K} \dot{\otimes} (V \underset{h}{\otimes} W))/A. \tag{8.1}$$

For further properties of the module Haagerup tensor product see Sections 3.4 and 3.6 of [3]. The module Haagerup tensor product is injective for operator A-modules (see Anantharaman-Delaroche and Pop Chapter 7 of [1] and Theorem 3.6.5 of [3]). By the last remark in Section 3.1.11 of [3] one may assume that A is unital (by replacing A with the unitization A^1 if necessary). Now the injectivity follows from Theorem 6.1 and the following result of Helemskii:

THEOREM 8.1 ([8], Theorem 3). *Let V, W be operator spaces. The balanced bounded bilinear operator $(\mathcal{K} \dot{\otimes} V) \times (\mathcal{K} \dot{\otimes} W) \to K \dot{\otimes} (V \underset{h}{\otimes} W)$ uniquely defined by*

$$(b_1 \otimes v, b_2 \otimes w) \mapsto b_1 b_2 \otimes (v \otimes w) \tag{8.2}$$

extends to an isometric isomorphism of Ruan $\mathcal{B}$-bimodules $(\mathcal{K} \dot{\otimes} V) \underset{\mathcal{B}}{\otimes} (\mathcal{K} \dot{\otimes} W) \cong K \dot{\otimes} (V \underset{h}{\otimes} W)$.

COROLLARY 8.2. *Let V, W be a right respectively left operator A-modules. The balanced completely bounded bilinear operator given by equation* (8.2) *extends to an isometric isomorphism with dense image of Ruan $\mathcal{B}$-bimodules*

$$K \dot{\otimes} (V \underset{hA}{\otimes} W) \hookrightarrow (\mathcal{K} \breve{\otimes} V) \underset{\mathcal{B} \breve{\otimes} A}{\otimes} (\mathcal{K} \breve{\otimes} W). \tag{8.3}$$

Proof. From equation (8.1), Theorem 8.1 and Corollary 1.2 follows $K \dot{\otimes} (V \underset{hA}{\otimes} W) = (K \dot{\otimes} (V \underset{h}{\otimes} W))/A = ((\mathcal{K} \dot{\otimes} V) \underset{\mathcal{B}}{\otimes} (\mathcal{K} \dot{\otimes} W))/A \hookrightarrow ((\mathcal{K} \breve{\otimes} V) \underset{\mathcal{B}}{\otimes} (\mathcal{K} \breve{\otimes} W))/A = (\mathcal{K} \breve{\otimes} V) \underset{\mathcal{B} \breve{\otimes} A}{\otimes} (\mathcal{K} \breve{\otimes} W)$ isometrically. ∎

REMARK 8.3. Since the tensor product on the right-hand side of equation (8.3) is injective, the module Haagerup tensor product is injective and thus completely injective (see Proposition 4.4).

9. ABSOLUTE VALUE OF A LINEAR FUNCTIONAL

We consider now an arbitrary unital C^*-algebra A and a *representable* left A-module Y (see Remark 3.4). We shall show that every representable A-module has an isometric representation. In the first step we construct for a given linear functional f in the dual unit sphere $S_{Y'}$ a uniquely determined majorizing state denoted by $|f|$. This generalizes the notion of the absolute value of a normal functional on a W^*-algebra ([22], Proposition 4.6).

Let us start with a simple example. Given $f \in Y'$ with $\|f\| = 1$. Assume that there is a $\widetilde{y} \in Y$ with $\|\widetilde{y}\| = 1$ such that $\|f\| = f(\widetilde{y})$. Define a linear functional $|f|$ an A by $|f|\colon a \mapsto f(a\widetilde{y})$. Since $||f|(a)| = |f(a\widetilde{y})| \leqslant \|f\|\|a\|\|\widetilde{y}\| = \|a\|$ the functional has norm $\||f|\| \leqslant 1$. In addition $|f|(\mathbf{1}_{\mathcal{A}}) = f(\widetilde{y}) = \|f\| = 1$ and thus $|f|$ is a state. Proposition 9.3 says that $|f|$ is a *majorizing* state for f, i.e.

$$|f(ay)| \leqslant (|f|(aa^*)^{1/2}\|y\| \quad \text{for all } a \in A, y \in Y$$

and it is the unique state with this property.

Although the absolute value $|f|$ of a functional $f \in S_{Y'}$ will be unique and given by a simple formula (see equation (9.5)), I need the Hahn–Banach theorem for the proof. It would be nice to have a direct proof but I doubt whether this will be possible. There is a consequence of the uniqueness: the sublinear functional p defined by the curious equation (9.3) below is $\mathbb{R}$-linear and $p = \operatorname{Re}|f|$.

LEMMA 9.1. *Given a C^*-algebra A, a left representable A-module Y and $f \in Y'$ with $\|f\| = 1$, then there exists a state ψ on A such that*

$$|f(ay)| \leqslant \psi(aa^*)^{1/2}\|y\| \quad \text{for all } a \in A, y \in Y. \tag{9.1}$$

We call such a state ψ a *majorizing state* for f. In a next step we show that the majorizing state is unique and denote it by $|f|$.

Proof. Clearly, the multiplicative inequality (9.1) is equivalent to the following additive form: $2\operatorname{Re} f(ay) \leqslant \psi(aa^*) + \|y\|^2$ and the latter is equivalent to

$$\operatorname{Re}\psi(b - aa^*) \leqslant \|b\| + \|y\|^2 - 2\operatorname{Re} f(ay) \quad \text{for all } a, b \in A, y \in Y. \tag{9.2}$$

Given a decomposition $c = b - aa^*$ then $\operatorname{Re}\psi(c)$ is dominated by the right hand side of the inequality above.

To find a suitable state ψ we take the infimum of the right hand side of (9.2) and verify that this defines a sublinear functional on A:

$$c \mapsto p(c) := \inf\{\|b\| + \|y\|^2 - 2\operatorname{Re} f(ay)\} \tag{9.3}$$

where $c, b, a \in A$ such that $c = b - aa^*$ and $y \in Y$ arbitrary.

Step 1. We claim that we may choose a invertible in the definition of p. Let $\widetilde{p}(c) = \inf\{\|b\| + \|y\|^2 - 2\operatorname{Re} f(ay)\}$ where $c, b, a \in A$, $y \in Y$ as in (9.3) and in addition a invertible.

Obviously $p(c) \leqslant \widetilde{p}(c)$. For $t > p(c)$ there exist a decomposition $c = b - aa^*$ and an y such that $\|b\| + \|y\|^2 - 2\operatorname{Re} f(ay) < t$. For $\varepsilon > 0$ let $b_\varepsilon := b + \varepsilon\mathbf{1}_A$,

$a_\varepsilon := (aa^* + \varepsilon\mathbf{1}_A)^{1/2}$. Then a_ε is invertible and $c = b_\varepsilon - a_\varepsilon a_\varepsilon^*$. The element $v_\varepsilon := a_\varepsilon^{-1} a \in A$ and since

$$v_\varepsilon v_\varepsilon^* = a_\varepsilon^{-1} aa^* a_\varepsilon^{-1} \leqslant a_\varepsilon^{-1}(aa^* + \varepsilon\mathbf{1}_A)a_\varepsilon^{-1} = \mathbf{1}_A$$

we have $\|v_\varepsilon\| \leqslant 1$. Let $y_\varepsilon := v_\varepsilon y$ then $ay = a_\varepsilon y_\varepsilon$ and thus

$$\begin{aligned}\widetilde{p}(c) &\leqslant \|b_\varepsilon\| + \|y_\varepsilon\|^2 - 2\operatorname{Re} f(a_\varepsilon y_\varepsilon) = \|b + \varepsilon\mathbf{1}_A\| + \|v_\varepsilon y\|^2 - 2\operatorname{Re} f(ay)\\ &\leqslant \|b\| + \|y\|^2 - 2\operatorname{Re} f(ay) + \varepsilon < t + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ and $t > p(c)$ are arbitrary we have $\widetilde{p}(c) \leqslant p(c)$.

Step 2. If $c = 0$ then $b = aa^*$ and thus $2\operatorname{Re} f(ay) \leqslant 2\|a\|\|y\| \leqslant \|a\|^2 + \|y\|^2 = \|b\| + \|y\|^2$. Hence $p(0) = 0$.

Step 3. Given bounds $t_i > p(c_i)$ choose decompositions $c_i = b_i - a_i a_i^*$, a_i invertible, and y_i such that $\|b_i\| + \|y_i\|^2 - 2\operatorname{Re} f(a_i y_i) < t_i$, $(i = 1, 2)$. Then $a = (a_1 a_1^* + a_2 a_2^*)^{1/2}$ is invertible. Let $v_i = a^{-1} a_i$, $(i = 1, 2)$, then

$$\begin{aligned}&v_1 v_1^* + v_2 v_2^* = a^{-1}(a_1 a_1^* + a_2 a_2^*)a^{-1} = \mathbf{1}_A,\\ &c_1 + c_2 = (b_1 + b_2) - (a_1 a_1^* + a_2 a_2^*) = (b_1 + b_2) - aa^*.\end{aligned}$$

From the condition (lr) we obtain the following estimation:

$$\begin{aligned}p(c_1+c_2) &\leqslant \|b_1 + b_2\| + \|v_1 y_1 + v_2 y_2\|^2 - 2\operatorname{Re} f(a(v_1 y_1 + v_2 y_2))\\ &\leqslant \|b_1\|+\|b_2\|+\|v_1 v_1^*+v_2 v_2^*\|(\|y_1\|^2+\|y_2\|^2) - 2\operatorname{Re} f(a_1 y_1 + a_2 y_2) < t_1 + t_2.\end{aligned}$$

Since the bounds t_1, t_2 are arbitrary $p(c_1 + c_2) \leqslant p(c_1) + p(c_2)$.

Step 4. From $0 = p(0) = p(c - c) \leqslant p(c) + p(-c)$ follows $-\infty < p(c)$.

Step 5. Let $\lambda > 0$. For a decomposition $c = b - aa^*$ and y as in equation (9.3) holds $\lambda c = \lambda b - (\sqrt{\lambda}a)(\sqrt{\lambda}a)^*$ and

$$p(\lambda c) \leqslant \|\lambda b\| + \|\sqrt{\lambda}y\|^2 - 2\operatorname{Re} f(\sqrt{\lambda}a\sqrt{\lambda}y).$$

Taking the infimum over all decompositions of c gives $p(\lambda c) \leqslant \lambda p(c)$. Now $p(c) = p((1/\lambda)\lambda c) \leqslant (1/\lambda)p(\lambda c)$ and hence $p(\lambda c) = \lambda p(c)$.

Step 6. We compute the value of $p(-\mathbf{1}_A)$: On the right side of equation (9.3) consider the special values $b = 0$, $a = \mathbf{1}_A$ and an arbitrary $y \in Y$ with $\|y\| = 1$. Since $\inf_{\|y\|=1}\{\|y\|^2 - 2\operatorname{Re} f(\mathbf{1}_A y)\} = 1 - 2\|f\| = -1$ we have $p(-\mathbf{1}_A) \leqslant -1$. Clearly $p(c) \leqslant \|c\|$. From $-1 \leqslant -p(\mathbf{1}_A) \leqslant p(-\mathbf{1}_A)$ follows $p(-\mathbf{1}_A) = -1$.

Step 7. By the Hahn–Banach theorem there exists a linear functional ψ on A such that $\operatorname{Re}\psi(c) \leqslant p(c)$. From the definition of p follows $\operatorname{Re}\psi(c) \leqslant \|c\|$ and thus $\|\psi\| \leqslant 1$. On the other hand $-\operatorname{Re}\psi(\mathbf{1}_A) \leqslant p(-\mathbf{1}_A) = -1$ and thus $\operatorname{Re}\psi(\mathbf{1}_A) = 1 = \|\psi\|$, i.e. ψ is a state. By definition ψ satisfies the estimates (9.2) and (9.1). ∎

LEMMA 9.2. *Let A be a unital C^*-algebra and φ_n a sequence in A' with a common majorizing state ψ, i.e. $|\varphi_n(a)| \leqslant (\psi(aa^*))^{1/2}$ for $a \in A$ and $n \in \mathbb{N}$. If $\lim_{n\to\infty} \varphi_n(\mathbf{1}_A) = 1$, then $\lim_{n\to\infty} \varphi_n(a) = \psi(a)$ uniformly on the set $\{a \in A : \psi(aa^*) \leqslant 1\}$.*

Proof. From the GNS representation theorem we have a cyclic representation $\pi\colon A \to \mathcal{B}(H)$ with a cyclic unit vector ξ_0 such that $\psi(a) = \langle \pi(a)\xi_0, \xi_0\rangle$ for all $a \in A$. Since $|\varphi_n(a^*)| \leqslant \psi(a^*a)^{1/2} = \|\pi(a)\xi_0\|$ the mapping $H \ni \pi(a)\xi_0 \mapsto \varphi_n(a^*)$ is a well defined and contractive conjugate linear functional on the dense subspace $\pi(A)\xi_0$ of H. Thus, there exists a unique element $\eta_n \in H$ such that $\varphi_n(a^*) = \langle \eta_n, \pi(a)\xi_0\rangle$ for all $a \in A$. Clearly $\|\eta_n\| \leqslant 1$.

From the parallelogram identity follows

$$\begin{aligned} \|\eta_m - \eta_n\|^2 &= 2\|\eta_m\|^2 + 2\|\eta_m\|^2 - \|\eta_m + \eta_n\|^2 \\ &\leqslant 4 - |\langle \eta_m + \eta_n, \xi_0\rangle|^2 = 4 - |\varphi_m(\mathbf{1}_A) + \varphi_n(\mathbf{1}_A)|^2 \xrightarrow[m,n\to\infty]{} 0. \end{aligned} \tag{9.4}$$

Let $\eta := \lim_{n\to\infty} \eta_n$, then $1 = \langle \eta, \xi_0\rangle \leqslant \|\eta\|\|\xi_0\| \leqslant 1$. The equality in the Schwarz inequality holds if and only if $\eta = \xi_0$. Hence

$$\lim_{n\to\infty} \varphi_n(a) = \Big\langle \lim_{n\to\infty} \eta_n, \pi(a^*)\xi_0 \Big\rangle = \langle \pi(a)\xi_0, \xi_0\rangle = \psi(a)$$

for all $a \in \mathcal{A}$. The sequence $\langle \eta_n, \pi(a^*)\xi_0\rangle$ converges uniformly on the unit-ball of H, i.e. uniformly on the set $\{a : \psi(aa^*) \leqslant 1\}$. ∎

PROPOSITION AND DEFINITION 9.3. *Given a left representable A-module Y and $f \in Y'$ with $\|f\| = 1$, then there exists a unique majorizing state on A. We denote this state by $|f|$ and call it the absolute value of f. Given a sequence $(y_n)_n$ in Y with $\|y_n\| \leqslant 1$ such that $\|f\| = \lim_{n\to\infty} f(y_n)$, then*

$$|f|(a) = \lim_{n\to\infty} f(ay_n). \tag{9.5}$$

The convergence is uniform on the set $\{a \in A : |f|(a^*a) \leqslant 1\}$.

Proof. From Lemma 9.1 we have a majorizing state ψ of f. Then we apply Lemma 9.2 to the functionals $\varphi_n\colon a \mapsto f(ay_n)$ and get $\psi(a) = \lim_{n\to\infty} f(ay_n)$. Thus the majorizing state is uniquely determined. ∎

REMARK 9.4. Similar results hold for right representable modules and representable bimodules. Let V be a representable A-C-bimodule and $f \in S_{V'}$. There exist uniquely a left majorizing state $|f|_{\mathrm{l}}$ on A and a right majorizing state $|f|_{\mathrm{r}}$ on C such that $|f(avc)| \leqslant |f|_{\mathrm{l}}(aa^*)^{1/2}\|v\|\,|f|_{\mathrm{r}}(c^*c)^{1/2}$ for $a \in A$, $v \in v$ and $c \in C$. The corresponding sublinear functional (see equation (9.3)) on $A \oplus C$ is $p(a \oplus c) = \inf\{\|\widetilde{a}\| + \|\widetilde{c}\| - 2\operatorname{Re} f(bvd)\}$, where the infimum runs over all decompositions $a = \widetilde{a} - bb^*$, $c = \widetilde{c} - d^*d$ and $v \in V$, $\|v\| = 1$ (see also [6] and Lemma 2.3 of [18]). Thus $\operatorname{Re}(|f|_{\mathrm{l}}(a) + |f|_{\mathrm{r}}(c)) = p(a \oplus c)$ for $a \in A, c \in C$.

10. CYCLIC REPRESENTATION OF LEFT REPRESENTABLE MODULES

PROPOSITION 10.1. *Given a left representable A-module Y and $f \in Y'$ with $\|f\| = 1$, there exist a unital $*$-representation π of A on a Hilbert space H with a cyclic* *unit vector $\xi_0 \in H$ and a contractive A-morphism $\theta\colon Y \to H$ such that $\theta(ay) = \pi(a)\theta(y)$ and $f(y) = \langle \theta(y), \xi_0 \rangle$ for all $a \in A$, $y \in Y$. $\theta(Y)$ is dense in H. All cyclic contractive representations corresponding to f are unitary equivalent.*

Proof. We have the majorizing state $|f|$ on A (see Proposition 9.3). From the GNS representation theorem there is a corresponding representation $\pi = \pi_{|f|}$ of A on a Hilbert space $H = H_{|f|}$ with cyclic unit vector $\xi_0 = \xi_{|f|}$ satisfying $|f|(a) = \langle \pi(a)\xi_0, \xi_0 \rangle$ for all $a \in A$. Given an element $y \in Y$, then

$$|f(a^*y)| \leqslant |f|(a^*a)^{1/2}\|y\| = \|\pi(a)\xi_0\|\|y\|$$

and thus we have a well defined and bounded conjugate linear form

$$H \ni \pi(a)\xi_0 \mapsto f(a^*y)$$

on the dense subspace $\pi(A)\xi_0$ of H. Hence there exists a unique element $\theta(y) \in H$ for which $f(a^*y) = \langle \theta(y), \pi(a)\xi_0 \rangle$ for all $a \in A$. Clearly the map $\theta = \theta_f : Y \to H$ is linear and contractive. Further on

$$\langle \theta(by), \pi(a)\xi_0 \rangle = f(a^*by) = \langle \theta(y), \pi(b^*a)\xi_0 \rangle = \langle \pi(b)\theta(y), \pi(a)\xi_0 \rangle$$

for all $a, b \in A$. Hence $\theta(by) = \pi(b)\theta(y)$. The other assertions follow from the lemma below. ∎

LEMMA 10.2. *Given $f \in S_{Y'}$ and a contractive cyclic representation (θ, π) on a Hilbert space H with a cyclic unit vector ξ_0 as in Proposition* 10.1 *then the image $\theta(Y)$ is dense in H and $|f|(a) = \langle \pi(a)\xi_0, \xi_0 \rangle$ for all $a \in A$. All cyclic contractive representations corresponding to f are unitary equivalent.*

Proof. The space $H_0 := \mathrm{cl}\{\theta(Y)\}$ is invariant under $\pi(A)$ and the projection P_0 onto H_0 commutes with $\pi(A)$. Now

$$\|\xi_0\| = 1 = \|f\| = \sup_{\|y\|\leqslant 1} |f(y)| = \sup_{\|y\|\leqslant 1} |\langle \theta(y), \xi_0 \rangle| = \sup_{\|y\|\leqslant 1} |\langle \theta(y), P\xi_0 \rangle| \leqslant \|P_0\xi_0\| \leqslant \|\xi_0\|.$$

Hence $P_0\xi_0 = \xi_0$ and thus $H_0 = P_0H = \mathrm{cl}\{\pi(A)P_0\xi_0\} = \mathrm{cl}\{\pi(A)\xi_0\} = H$.

Clearly $a \mapsto \langle \pi(a)\xi_0, \xi_0 \rangle$ is a majorizing state of f and hence it is $|f|$. ∎

PROPOSITION 10.3. *If Y is a left representable A-module, then there are Hilbert spaces H, K, a $*$-representation π of A on H and an isometric A-morphism θ of Y into $\mathcal{B}(K, H)$. If $Y$is separable then we can let the Hilbert spaces H, K separable.*

Proof. We choose a norming set S_0 of the dual unit sphere $S_{Y'}$. Recall, a subset S_0 of $S_{Y'}$ is *norming* if $\sup_{f\in S_0}|f(y)| = \|y\|$ for all $y \in Y$. Then $|y\| = \sup_{f\in S_0}|f(y)| = \sup_{f\in S_0}|\langle \theta_f(y), \xi_{|f|} \rangle| \leqslant \sup_{f\in S_0} \|\theta_f(y)\| \leqslant \|y\|$.

Now we may set $H := \bigoplus_f H_{|f|}$, $K := \ell_2^{S_0}$. We define $\theta \in B(\ell_2^{S_0}, H)$ and a $*$-representation π of A on H by

$$\begin{array}{lll} (10.1) & \theta(y)\colon (\eta_f) \mapsto (\eta_f \theta_f(y)) & \text{for all } y \in Y,\ (\eta_f) \in \ell_2^{S_0}, \\ & \pi(a) = \bigoplus_f \pi_{|f|}(a) & \text{for all } a \in A, \end{array}$$

where the direct sum runs over the norming set S_0. Then $\theta\colon Y \to \mathcal{B}(K, H)$ is an isometric representation via π.

If Y is separable then $\theta_f(Y)$ is a separable dense subset of $H_{|f|}$. We may choose S_0 countable and then the Hilbert spaces H, K are separable. ∎

11. FUNCTORIAL PROPERTIES OF THE REPRESENTATION OF SEMI-RUAN MODULES

A semi-Ruan $\mathcal{A}$-module Y is representable and has an isometric representation θ into some $\mathcal{B}(K, H)$ (see Proposition 3.3 and Proposition 10.3). From this representation Y receives a structure as Ruan $\mathcal{A}$-$\mathcal{B}$-bimodule $L' \dot{\otimes} \theta(Y)$. The uncompleted spatial tensor product is defined by $L' \dot{\otimes} \theta(Y) \subset B(L, \mathbb{C}) \dot{\otimes} \mathcal{B}(K, H) \subset B(L \otimes^2 K, \mathbb{C} \otimes^2 H)$.

The following lemma and it corollary say that the representation θ given by equation (10.1) always generates the minimal Ruan bimodule structure. The letter is given by the minimal Ruan bimodule $L' \otimes Y \cong \mathcal{F}(L, Y)$ (see Proposition 5.3 and Lemma 5.4). We will show that the mapping $\mathbf{1}_{L'} \otimes \theta\colon F(L, Y) \to L' \dot{\otimes} \theta(Y)$ is an isometric isomorphism of Ruan $\mathcal{A}$-$\mathcal{B}$-bimodules.

LEMMA 11.1. *Given a left semi-Ruan $\mathcal{A}$-module Y and an isometric isomorphic representation (θ, π) into $\mathcal{B}(K, H)$ given by equation* (10.1)*, then the linear mapping $L' \otimes Y \to L' \dot{\otimes} \theta(Y)$ is an isometric isomorphism of Ruan $\mathcal{A}$-$\mathcal{B}$-modules $\mathcal{F}(L, Y) \to L' \dot{\otimes} \theta(Y)$ and a complete isometry.*

Proof. Recall $\theta(y) = \bigoplus_f \theta_f(y) \in \bigoplus_f B(\mathbb{C}, H_{|f|}) \subset \mathcal{B}\Big(\bigoplus_f \mathbb{C}, \bigoplus_f H_{|f|}\Big) = \mathcal{B}(K, H)$ and $\|\theta(y)\| = \sup_f \|\theta_f(y)\|$ for $y \in Y$ where f runs over a norming subset S_0 of the dual unit sphere. An element $w \in \mathcal{F}(L, \theta(Y))$ has the components $w_f \in \mathcal{F}(L, H_{|f|})$ and $\|w\| = \sup_f \|w_f\|$. Thus we have an isometric morphism $w \mapsto \bigoplus_f w_f$

$$\mathcal{F}(L, \theta(Y)) \hookrightarrow \mathcal{B}\Big(\bigoplus_f L, \bigoplus_f H_{|f|}\Big) = \mathcal{B}(L \otimes^2 K, \mathbb{C} \otimes^2 H).$$

This isomorphism takes an operator $\xi' \otimes \theta(y) \in \mathcal{F}(L, \theta(Y)$ to the corresponding elementary tensor in $\mathcal{B}(L, \mathbb{C}) \otimes \theta(Y)$ and thus

$$\mathcal{F}(L, \theta(Y)) \xrightarrow{\sim} \mathcal{B}(L, \mathbb{C}) \dot{\otimes} \theta(Y) \subset \mathcal{B}(L \otimes^2 K, \mathbb{C} \otimes^2 H)$$

is an isometry. Hence we have an isometric isomorphism of Ruan $\mathcal{A}$-$\mathcal{B}$-modules

$$\mathcal{F}(L,Y) \cong_1 \mathcal{F}(L,\theta(Y)) \xrightarrow{\sim} L' \dot{\otimes} \theta(Y).$$

Since Ruan bimodules have a unique structure of an operator bimodule (see Proposition 4.4) this isomorphism is a complete isometry. ∎

From Proposition 10.3 every left Ruan $\mathcal{A}$-module Y has an isometric isomorphic representation $Y \cong_1 \theta_Y(Y) \subset \mathcal{B}(K_Y, H_Y)$ and thus Y is isometrically isomorphic to an operator $\mathcal{A}$-module. The following corollary says, that the representations given by equation (10.1) induce an isometric functor from the category of semi-Ruan $\mathcal{A}$-modules into the category of operator $\mathcal{A}$-modules.

COROLLARY 11.2. *Given left semi-Ruan modules Y, Z, then choose representations θ_Y, θ_Z as in Proposition* 10.3 *and equation* (10.1). *Let $\Phi\colon Y \to Z$ be a bounded morphism of semi-Ruan $\mathcal{A}$-modules then the induced morphism $\widetilde{\Phi}\colon \theta_Y(Y) \to \theta_Z(Z)$ is completely bounded with the same norm $\|\widetilde{\Phi}\|_{\mathrm{cb}} = \|\Phi\|$.*

Proof. Choose an arbitrary unit vector $\xi' \in L'$ and define an isometry $\theta_Y(Y)$ to $L' \dot{\otimes} \theta_Y(Y)$ by $\theta_Y(y) \to \xi' \dot{\otimes} \theta_Y(y)$ and the corresponding isometry $\theta_Z(Z) \to L' \dot{\otimes} \theta_Z(Z)$. Then we have the commutative diagram

$$\begin{tikzcd}
\theta_Y(Y) \arrow[r, "\text{cb isometric}"] \arrow[d, "\widetilde{\Phi}"] & L' \dot{\otimes} \theta_Y(Y) \arrow[r, "\text{cb isometric}", "\cong"'] \arrow[d, "\mathbf{1}_{L'} \dot{\otimes} \widetilde{\Phi}"] & \mathcal{F}(L,Y) \arrow[d, "\widetilde{\widetilde{\Phi}}"] \\
\theta_Z(Z) \arrow[r, "\text{cb isometric}"] & L' \dot{\otimes} \theta_Z(Z) \arrow[r, "\text{cb isometric}", "\cong"'] & \mathcal{F}(L,Z)
\end{tikzcd}$$

where $\widetilde{\widetilde{\Phi}}\colon w \to \Phi \circ w$ is completely bounded with $\|\widetilde{\widetilde{\Phi}}\|_{\mathrm{cb}} = \|\Phi\|$ (see Proposition 4.4). Thus we have $\|\Phi\| = \|\widetilde{\Phi}\| \leqslant \|\widetilde{\Phi}\|_{\mathrm{cb}} \leqslant \|\widetilde{\widetilde{\Phi}}\|_{\mathrm{cb}} = \|\Phi\|$. ∎

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Received April 24, 2008.