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DOMINATION AND COMMUTATIVITY IN PONTRYAGIN SPACES

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ABSTRACT. We show a sufficient condition for selfadjointness and spectral commutation of a system of unbounded symmetric operators in a Pontryagin space. The main concepts are domination and the set of bounded vectors of an operator. We investigate also operator matrices with unbounded entries and prove a Pontryagin space version of the Nelson's criterion.

KEYWORDS: *Symmetric operator, Pontryagin space, operator matrix.*

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INTRODUCTION

An example by Nelson ([17]) shows that two essentially selfadjoint operators in a Hilbert space are not necessarily jointly selfadjoint, i.e. their closures do not have to commute spectrally. In his paper Nelson gave also several sufficient conditions for joint selfadjointness, which involved commutators and the notion of domination. Since that time the topic has evolved, one should mention here the works of Poulsen [18] and Stochel and Szafraniec [21]. We refer to the latter paper for an overview of the history of the research.

Our motivation and starting point is Theorem 10 of [21]. The goal of the present paper is to extend it onto the classes of symmetric and selfadjoint operators in a Pontryagin space. Apparently, this can be done without any additional assumptions (Theorem 4.2). As a consequence we get a Nelson criterion of polynomial type for joint selfadjointness in a Pontryagin space (Theorem 7.1) and a generalization of the classical Nelson's sufficient condition for joint selfadjointness of two operators (Theorem 7.2).

The result from [21] has found applications in approximation theory ([23]) and in multidimensional moment problems ([21], see also [19] for another approach to multidimensional moment problems). The version presented below might be a good tool for investigating multidimensional indefinite moment problems. We refer the reader to [3], [12] for the classical one dimensional indefinite moment problem.

Although the way to Theorem 4.2 repeats some parts of its Hilbert space analogue, we find it worth publishing. It is based on a technique of bounded vectors, which needs some adjustment in the Pontryagin space environment.

The proof of the indefinite Nelson criterion of polynomial type appears to be much more complicated than its definite counterpart. We propose a reasoning based on matrix decomposition of a symmetric operator in a Pontryagin space. Although we will not obtain exact analogues of the results of Stochel and Szafraniec ([21], Propositions 29, 30), we are able to generalize the famous Nelson criterion ([17]) onto Pontryagin space without any additional assumptions.

We begin our paper in a Krein space environment and prove some general statements on bounded vectors of definitizable operators (Proposition 2.1, Corollary 2.2 and Theorem 3.1). These are preliminary results and we formulate them in a very general context.

In the fourth section we prove the main theorem of the paper. It can be reasonably formulated only in a Pontryagin space and we resign there from the global setting. It should be mentioned that a straightforward reduction of Theorem 4.2 to the Hilbert space case is not possible, mainly because commutation of symmetric operators in a Pontryagin space: $ABf = BAf$ (for f belonging to some dense subspace $\mathcal{E}$) does not imply commutation of the operators JA and JB, where J is any fundamental symmetry. One should think rather about some results from the theory of noncommuting domination ([5], [24], [25]), where the commutator is not zero but is bounded in some sense. However, we will not follow that path but present an independent proof.

The next two sections are devoted to operator matrices with unbounded entries, which are considered as operators in an orthogonal sum of two Hilbert spaces. This is an essential tool in the proof of the Nelson criterion, which is formulated again in Pontryagin space terms.

1. PRELIMINARIES

Let $(\mathcal{K}, [\cdot,\cdot])$ be a Krein space and let A be an operator in $\mathcal{K}$. By $\mathcal{D}(A)$ and $\mathcal{R}(A)$ we understand the domain and range of A, respectively. The sum and product of unbounded operators is understood in a standard way, see e.g. [10]. We also set

$$\mathcal{D}^\infty(A) := \bigcap_{k=0}^{\infty} \mathcal{D}(A^k).$$

We say that an operator A in $\mathcal{K}$ is *bounded* if $\|Af\| \leqslant c\|f\|$ for some $c \geqslant 0$ and for all $f \in \mathcal{D}(A)$. The norm appearing above is one of the equivalent complete norms in the Krein space, such that the scalar product $[\cdot,\cdot]$ is continuous (see

[4], [13]). All the topological notions will refer to that norm. The main results do not depend on the choice of the norm but in the proofs very often a suitable fundamental symmetry ([4]) has to be chosen.

We write $\mathbf{B}(\mathcal{K})$ for the space of all bounded operators with the domain equal $\mathcal{K}$. The symbols $\sigma(A)$ and $\rho(A)$ stand, respectively, for the spectrum and the resolvent set of a closed operator A in $\mathcal{K}$. We say that $\mathcal{E} \subseteq \mathcal{D}(A)$ is a *core for A* if the graph of A is contained in the closure of the graph $A|_{\mathcal{E}}$.

A^+ denotes the adjoint of a densely defined operator A in a Krein space. We say that a symmetric operator A in $\mathcal{K}$ is *essentially selfadjoint on $\mathcal{E}$* if $\mathcal{E}$ is a dense linear subspace of $\mathcal{D}(A)$ and $(A|_{\mathcal{E}})^+ = \overline{A|_{\mathcal{E}}}$. By maximality of selfadjoint operators, a symmetric operator A is selfadjoint on $\mathcal{E}$ if and only if $\overline{A}$ is selfadjoint and $\mathcal{E}$ is a core for A (see e.g [5], [25]). If A is an operator in $\mathcal{K}$ and $\mathcal{E}$ is a subspace of $\mathcal{K}$ then by $A|_{\mathcal{E}}$ we mean the restriction of A to the subspace $\mathcal{E} \cap \mathcal{D}(A)$.

Let $(\mathcal{K}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and let A be a densely defined, closable operator in $\mathcal{K}$. We say that a closed subspace $\mathcal{L}$ of $\mathcal{K}$ *reduces* A, if $PA \subseteq AP$, where P denotes the orthogonal (in the sense of the inner product $\langle\cdot,\cdot\rangle$) projection from $\mathcal{H}$ to $\mathcal{L}$. If $\mathcal{L}$ is contained in $\mathcal{D}(A) \cap \mathcal{D}(A^*)$ then it reduces A if and only if it is invariant for A and A^* (see [20] for more interesting problems of that type). If $(\mathcal{K}, [\cdot,\cdot])$ is a Krein space and J is a fundamental symmetry in $\mathcal{K}$ then we say that a closed subspace $\mathcal{L}$ of $\mathcal{K}$ *J-reduces* an operator A if it reduces A in the sense of the Hilbert space inner product $[J\cdot,\cdot]$ on $\mathcal{K}$.

In the rest of this section A, B are two closable operators in $\mathcal{K}$ and $\mathcal{E}$ is a linear subspace of $\mathcal{K}$.

We say that *A dominates B on $\mathcal{E}$* if $\mathcal{E} \subseteq \mathcal{D}(A) \cap \mathcal{D}(B)$ and

$$\|Bf\| \leqslant c(\|f\| + \|Af\|), \quad f \in \mathcal{E}. \tag{1.1}$$

If $\mathcal{E} = \mathcal{D}(A)$ then we will just say that *A dominates B*. Observe that if A dominates B on $\mathcal{E}$, which is a core for A then $\mathcal{D}(\overline{A}) \subseteq \mathcal{D}(\overline{B})$ and $\overline{A}$ dominates $\overline{B}$. On the other hand, if $\mathcal{D}(\overline{A}) \subseteq \mathcal{D}(\overline{B})$ then, by the closed graph theorem, $\overline{A}$ dominates $\overline{B}$.

We say that A and B *commute pointwise on $\mathcal{E}$* if

$$\mathcal{E} \subseteq \mathcal{D}(BA) \cap \mathcal{D}(AB), \quad BAf = ABf \quad \text{for } f \in \mathcal{E}.$$

If the resolvent sets of A and B are non-empty and the bounded operators $(A - z_1)^{-1}$, $(B - z_2)^{-1}$ commute pointwise on $\mathcal{K}$ for some (equivalently: for every) $z_1 \in \rho(A)$, $z_2 \in \rho(B)$ then we say that *A and B commute spectrally*.

If A and B are symmetric, $\mathcal{E} \subseteq \mathcal{D}(A) \cap \mathcal{D}(B)$ and $[Af, Bg] = [Bf, Ag]$ for $f, g \in \mathcal{E}$ then we say that *A and B commute weakly on $\mathcal{E}$*.

We finish this section with a lemma, the proof goes the same as the proof of Proposition 2 in [21] with the symbol " $^+$" instead of " *" .

LEMMA 1.1. *If A and B are closable operators in a Krein space $(\mathcal{K}, [\cdot,\cdot])$ which commute pointwise on a dense linear subspace $\mathcal{E}$ of $\mathcal{K}$ and $\mathcal{D}(A^+B^+) \cap \mathcal{D}(B^+A^+)$ is dense in $\mathcal{K}$, then $\overline{A}$ and $\overline{B}$ commute pointwise on $\mathcal{D}(\overline{A}) \cap \mathcal{D}(\overline{B})$.*

2. BOUNDED VECTORS

For an operator A in a Krein space $(\mathcal{K}, [\cdot,\cdot])$ we define:

$$\mathcal{B}_a(A) := \{f \in \mathcal{D}^\infty(A) : \exists c > 0 \ \forall n \in \mathbb{N} \ \|A^n f\| \leqslant c a^n\}, \quad a \geqslant 0.$$

Note that the definition above does not depend on the choice of an equivalent norm on $\mathcal{K}$. We say that $f \in \mathcal{D}^\infty(A)$ is *a bounded vector for* A if $f \in \mathcal{B}_a(A)$ for some $a \geqslant 0$. We denote the linear space of all bounded vectors by $\mathcal{B}(A)$. The spaces $\mathcal{B}_a(A)$ ($a \geqslant 0$) and $\mathcal{B}(A)$ are invariant for A. Moreover, $\mathcal{B}_a(A) \subseteq \mathcal{B}_b(A)$ if $0 \leqslant a \leqslant b$. For a normal operator N in a Hilbert space we have the following description of the set of its bounded vectors (cf. [6], [21]):

$$\mathcal{B}_a(N) = \mathcal{R}(E(\{z \in \mathbb{C} : |z| \leqslant a\})), \quad a \geqslant 0, \tag{2.1}$$

where E denotes the spectral measure of N. In particular $\mathcal{B}_a(N)$ ($a \geqslant 0$) is a closed subspace of $\mathcal{H}$ which reduces N and $\mathcal{B}(N)$ is a core for N.

Let A be a selfadjoint operator in a Krein space $(\mathcal{K}, [\cdot,\cdot])$. In what follows we will frequently assume that there exists a fundamental symmetry J and a pair of closed subspaces $\mathcal{H}_0$ and $\mathcal{H}_1$ of $\mathcal{K}$ such that:

(j1) the Hilbert space $(\mathcal{K}, [J\cdot,\cdot])$ decomposes as an $[J\cdot,\cdot]$-orthogonal sum (in another words: $(\mathcal{H}_0, [\cdot,\cdot])$ is a Krein space, $(\mathcal{H}_1, [\cdot,\cdot])$ is a Hilbert space, they are mutually $[\cdot,\cdot]$-orthogonal and $[J\cdot,\cdot]$-orthogonal and $\mathcal{K}$ is a direct sum of $\mathcal{H}_0$ and $\mathcal{H}_1$) $\mathcal{K} = \mathcal{H}_0 \oplus \mathcal{H}_1$, $J|_{\mathcal{H}_1} = I_{\mathcal{H}_1}$;

(j2) both spaces $\mathcal{H}_0$, $\mathcal{H}_1$ are invariant and J-reducing for the operator A;

(j3) $\mathcal{H}_0 \subseteq \mathcal{D}(A)$, $A|_{\mathcal{H}_0} \in \mathbf{B}(\mathcal{H}_0)$ and $A|_{\mathcal{H}_0}$ is selfadjoint in the Krein space $(\mathcal{H}_0, [\cdot,\cdot])$;

(j4) $A|_{\mathcal{H}_1}$ is selfadjoint in the Hilbert space $(\mathcal{H}_1, [\cdot,\cdot])$.

Obviously, in nontrivial cases there are infinitely many triplets $(J, \mathcal{H}_0, \mathcal{H}_1)$ satisfying the conditions (j1)–(j4). By theorem of Langer ([11]) such a triplet $(J, \mathcal{H}_0, \mathcal{H}_1)$ exists for a selfadjoint operator in a Pontryagin space and this information is sufficient for the purposes of the present paper. However, it is possible to make a similar construction in the case when A is a definitizable operator without a critical point at infinity ([8], [13]). Furthermore, one can even take under consideration some more general classes, like (bounded perturbations of) locally definitizable operators (see e.g. [2], [9]). We leave the research on the sets of bounded vectors in those cases for subsequent papers.

We provide the following description of the set of bounded vectors of a selfadjoint operator with a triplet satisfying (j1)–(j4). The symbol "$\oplus$" ("$\ominus$") below and in Corollary 2.2 means the orthogonal sum (difference) with respect to the Hilbert space inner product $[J\cdot,\cdot]$ on $\mathcal{K}$.

PROPOSITION 2.1. *Let A be a selfadjoint operator in a Krein space $(\mathcal{K}, [\cdot,\cdot])$ and let a triplet $(J, \mathcal{H}_0, \mathcal{H}_1)$ fulfill the conditions* (j1)–(j4)*. Then there exists a number $a_0 \geqslant 0$* (*depending on A and on the choice of the triplet*) *such that for $a \geqslant a_0$:*

(i) $\mathcal{B}_a(A) = \mathcal{H}_0 \oplus \mathcal{B}_a(A|_{\mathcal{H}_1})$;

(ii) $J(\mathcal{B}_a(A)) = \mathcal{B}_a(A)$ *and* $J|_{\mathcal{K}\ominus\mathcal{B}_a(A)} = I_{\mathcal{K}\ominus\mathcal{B}_a(A)}$;

(iii) *the space* $(\mathcal{B}_a(A), [\cdot,\cdot])$ *is a Krein space and its* $[J\cdot,\cdot]$*-orthogonal complement is a Hilbert space;*

(iv) $\mathcal{B}_a(A)$ *is* J*-reducing space for* A*;*

(v) *the operator* $A|_{\mathcal{B}_a(A)}$ *belongs to* $\mathbf{B}(\mathcal{B}_a(A))$ *and is selfadjoint in the Krein space* $(\mathcal{B}_a(A), [\cdot,\cdot])$.

Proof. (i) In the calculations below we will use the norm $\|f\| := [Jf, f]$ ($f \in \mathcal{K}$). We put $a_0 := \max\{\|A|_{\mathcal{H}_0}\|, 1\}$. It results from (j2) and (j3) that $\mathcal{D}^\infty(A) = \mathcal{H}_0 \oplus \mathcal{D}^\infty(A|_{\mathcal{H}_1})$. Let $f = f_0 \oplus f_1 \in \mathcal{B}_a(A)$, $f_i \in \mathcal{H}_i$ ($i = 0, 1$). For some $d \geqslant 0$ we have

$$d^2 a^{2n} \geqslant \|A^n f\|^2 = \|(A|_{\mathcal{H}_0})^n f_0\|^2 + \|(A|_{\mathcal{H}_1})^n f_1\|^2.$$

Hence, $\|(A|_{\mathcal{H}_1})^n f_1\| \leqslant da^n$, which finishes the proof of $\mathcal{B}_a(A) \subseteq \mathcal{H}_0 \oplus \mathcal{B}_a(A|_{\mathcal{H}_1})$. To see the reverse inclusion let first $f_0 \in \mathcal{H}_0$. Then $\|A^n f_0\| = \|(A|_{\mathcal{H}_0})^n f_0\| \leqslant a_0^n \|f_0\| \leqslant a^n \|f_0\|$. If $f_1 \in \mathcal{B}_a(A|_{\mathcal{H}_1})$ then $\|A^n f_1\| = \|(A|_{\mathcal{H}_1})^n f_1\| \leqslant ca^n$ for some $c \geqslant 0$. These two inequalities lead easily to the desired inclusion.

Point (ii) follows from (j1) and (i). Claim (iii) results directly from (i) and (ii). To prove (iv) it is enough to note that the space $\mathcal{B}_a(A|_{\mathcal{H}_1})$ J-reduces the selfadjoint operator $A|_{\mathcal{H}_1}$ in $\mathcal{H}_1$ (cf. (2.1)) and use (j2) and (i).

Point (v) is a consequence of the closed graph theorem and invariance of $\mathcal{B}_a(A)$ for A. ∎

COROLLARY 2.2. *Let A be a selfadjoint operator in a Krein space $(\mathcal{K}, [\cdot,\cdot])$ such that there exists a triplet $(J, \mathcal{H}_0, \mathcal{H}_1)$ satisfying* (j1)–(j4). *Then there exists $a_0 \geqslant 0$ such that for all $a \geqslant a_0$ the space $(\mathcal{B}_a(A), [\cdot,\cdot])$ is a Krein space, in particular it is closed in $\mathcal{K}$. Moreover, $\mathcal{B}(A)$ is a core for A.*

Proof. Only the second sentence of the statement requires proof. Proposition 2.1(i) gives us

$$\mathcal{B}(A) = \mathcal{H}_0 \oplus \mathcal{B}(A|_{\mathcal{H}_1}). \tag{2.2}$$

Let $f = f_0 \oplus f_1 \in \mathcal{D}(A)$, ($f_i \in \mathcal{H}_i$, $i = 0, 1$). Since $f_1 \in \mathcal{D}(A|_{\mathcal{H}_1})$, there exists a sequence $(g_n)_{n=0}^\infty \subseteq \mathcal{B}(A|_{\mathcal{H}_1})$ tending to f_1 in graph norm of $A|_{\mathcal{H}_1}$. Hence, $(f_0 \oplus g_n)_{n=0}^\infty \subseteq \mathcal{B}(A)$ tends to f in the graph norm of A, which completes the proof. ∎

3. DOMINATION AND POINTWISE COMMUTATIVITY IN KREIN SPACES

The following result is an indefinite inner product version of Theorem 7 of [21]. Although the proof of (i) repeats many arguments from the proof of that theorem, we present it for the convenience of the reader. In the proof of (ii) one can observe some differences between the Hilbert and the Krein space case.

THEOREM 3.1. *Let A, B be symmetric operators in a Krein space $(\mathcal{K}, [\cdot,\cdot])$ and let $\mathcal{E} \subseteq \mathcal{D}(A) \cap \mathcal{D}(B)$ be such a dense subspace of $\mathcal{K}$ that*

- *A is essentially selfadjoint on $\mathcal{E}$,*
- *A dominates B on $\mathcal{E}$,*
- *A and B commute weakly on $\mathcal{E}$.*

Then:

(i) *$\overline{A}\mathcal{D}^\infty(\overline{A}) \subseteq \mathcal{D}^\infty(\overline{A})$, and $\overline{B}\mathcal{D}^\infty(\overline{A}) \subseteq \mathcal{D}^\infty(\overline{A})$; moreover $\overline{AB}f = \overline{BA}f$ for $f \in \mathcal{D}^\infty(\overline{A})$;*

(ii) *if $a \geqslant 0$ and the space $\mathcal{B}_a(\overline{A})$ is closed, then it is contained in $\mathcal{B}(\overline{B})$ and invariant for $\overline{B}$;*

(iii) *if for all $a \geqslant 0$ there exists $a' \geqslant a$ such that space $\mathcal{B}_{a'}(\overline{A})$ is closed, then $\mathcal{B}(\overline{A}) \subseteq \mathcal{B}(\overline{B})$, and $\mathcal{B}(\overline{A})$ is invariant for $\overline{B}$.*

Note that if (j1)–(j4) are satisfied in $\mathcal{K}$ (with some $J, \mathcal{H}_0, \mathcal{H}_1$) then the assumption in (ii) is satisfied for $a \geqslant a_0$ and the assumption in (iii) is satisfied automatically (cf. Corollary 2.2).

Proof. We fix a Hilbert space norm $\|\cdot\|$ on $\mathcal{K}$.

(i) Since $\mathcal{E}$ is a core for A and A dominates B on $\mathcal{E}$ we have

$$\mathcal{D}(\overline{A}) \subseteq \mathcal{D}(\overline{B}), \quad \|\overline{B}f\| \leqslant c(\|f\| + \|\overline{A}f\|), \quad f \in \mathcal{D}(\overline{A}), \tag{3.1}$$

with some $c \geqslant 0$ (see Section 1).

The inclusion $\mathcal{D}(\overline{A}) \subseteq \mathcal{D}(\overline{B})$ implies that

$$\mathcal{D}(\overline{A}^{k+1}) \subseteq \mathcal{D}(\overline{BA}^{k}), \quad k \in \mathbb{N}. \tag{3.2}$$

Now let $g \in \mathcal{D}(\overline{A})$. Since $\mathcal{E}$ is a core A, there exists a sequence $(g_n)_{n=1}^\infty \subseteq \mathcal{E}$ such that $g_n \to g$ and $\overline{A}g_n \to \overline{A}g$ with $n \to \infty$. It follows from (3.1) that $Bg_n \to \overline{B}g$ with $n \to \infty$. Consequently, by weak commutativity of A and B on $\mathcal{E}$,

$$[Af, \overline{B}g] = [Bf, \overline{A}g], \quad f \in \mathcal{E},\, g \in \mathcal{D}(\overline{A})(\subseteq \mathcal{D}(\overline{B})). \tag{3.3}$$

As a next step of the proof we show that for each $n \in \mathbb{N}$ we have

$$\mathcal{D}(\overline{A}^{n+1}) \subseteq \mathcal{D}(\overline{A}^{n}\overline{B}) \cap \mathcal{D}(\overline{BA}^{n}), \quad \overline{B}(\mathcal{D}(\overline{A}^{n+1})) \subseteq \mathcal{D}(\overline{A}^{n}), \tag{3.4}$$

$$\overline{BA}^{n}f = \overline{A}^{n}\overline{B}f, \quad f \in \mathcal{D}(\overline{A}^{n+1}). \tag{3.5}$$

The proof goes by induction with respect to $n \in \mathbb{N}$. The case $n = 0$ is obvious. Suppose that (3.4) and (3.5) hold for some $n \in \mathbb{N}$. Let $f \in \mathcal{D}(\overline{A}^{n+2})$. Then $f \in \mathcal{D}(\overline{A}^{n+1})$ and, by induction, we get:

$$[\overline{A}^{n}\overline{B}f, \overline{A}h] \overset{\text{ind.}}{=} [\overline{BA}^{n}f, Ah] \overset{(3.3)}{=} [\overline{A}^{(n+1)}f, Bh] \overset{(3.2)}{=} [\overline{BA}^{(n+1)}f, h], \quad h \in \mathcal{E}.$$

Since $\mathcal{E}$ is a core for A, left hand side and right hand side of the above are equal for $h \in \mathcal{D}(\overline{A})$. Therefore, we obtain that $\overline{A}^{n}\overline{B}f \in \mathcal{D}(A^+) = \mathcal{D}(\overline{A})$ (equivalently: $f \in \mathcal{D}(\overline{A}^{(n+1)}\overline{B})$, so $\overline{B}f \in \mathcal{D}(\overline{A}^{n+1})$), and $\overline{BA}^{(n+1)}f = \overline{A}^{(n+1)}\overline{B}f$, which finishes the induction argument.

The inclusion $\overline{A}\mathcal{D}^\infty(\overline{A}) \subseteq \mathcal{D}^\infty(\overline{A})$ follows from the definition of $\mathcal{D}^\infty(\overline{A})$. Formula (3.4) implies that $\overline{B}\mathcal{D}^\infty(\overline{A}) \subseteq \mathcal{D}^\infty(\overline{A})$. Moreover, by (3.5), we have $\overline{A}\overline{B}f = \overline{B}\overline{A}f$ for $f \in \mathcal{D}^\infty(\overline{A})$.

(ii) It follows from (i) that $\mathcal{B}_a(\overline{A}) \subseteq \mathcal{D}^\infty(\overline{A}) \subseteq \mathcal{D}(\overline{B})$. By the closed graph theorem, the operator $\overline{B}|_{\mathcal{B}_a(\overline{A})}\colon \mathcal{B}_a(\overline{A}) \to \mathcal{K}$ is bounded (($\mathcal{B}_a(\overline{A}), \|\cdot\|$) is a Banach space as a closed subspace of $\mathcal{K}$. However, we do not know if $(\mathcal{B}_a(\overline{A}), [\cdot,\cdot])$ is a Krein space, if we do not assume anything like (j1)–(j4)). Let $\phi(a)$ denote its norm. If $f \in \mathcal{B}_a(\overline{A})$ then, by (i),

$$\|\overline{A}^n\overline{B}f\| = \|\overline{B}\overline{A}^n f\| \leqslant \phi(a)\|\overline{A}^n f\|, \quad n \in \mathbb{N}.$$

Hence $\overline{B}f \in \mathcal{B}_a(\overline{A})$ and so $\mathcal{B}_a(\overline{A})$ is invariant for $\overline{B}$. This and the closed graph theorem imply that $\mathcal{B}_a(\overline{A}) \subseteq \mathcal{B}(\overline{B})$.

(iii) follows directly from (ii). ∎

COROLLARY 3.2. *Assume that $(\mathcal{K}, [\cdot,\cdot])$ is a Krein space and the operators A and B are as in Theorem* 3.1. *Let $(J, \mathcal{H}_0, \mathcal{H}_1)$ satisfy* (j1)–(j4) *(with $\overline{A}$ instead of A) and let a_0 be chosen as in Proposition* 2.1. *Then*

(i) $\overline{B}|_{\mathcal{B}_a(\overline{A})}$ *is selfadjoint and bounded in the Krein space $(\mathcal{B}_a(\overline{A}), [\cdot,\cdot])$ and commutes pointwise (on $\mathcal{K}$) with $\overline{A}|_{\mathcal{B}_a(\overline{A})}$;*

(i) *the space $\mathcal{B}_a(\overline{A})$ J-reduces $\overline{B}$.*

Proof. The closed graph theorem implies that $\overline{B}|_{\mathcal{B}_a(\overline{A})} \in \mathbf{B}(\mathcal{B}_a(\overline{A}))$. Since the operator $\overline{B}$ is symmetric, $\overline{B}|_{\mathcal{B}_a(\overline{A})}$ is selfadjoint in $(\mathcal{B}_a(\overline{A}), [\cdot,\cdot])$ (which is a Krein space by Corollary 2.2). Theorem 3.1(i) gives us pointwise commutativity of the operators $\overline{A}|_{\mathcal{B}_a(\overline{A})}$ and $\overline{B}|_{\mathcal{B}_a(\overline{A})}$ belonging to $\mathbf{B}(\mathcal{B}_a(\overline{A}))$.

We show now that $\mathcal{B}_a(\overline{A})$ J-reduces $\overline{B}$. Let $\overline{B}^*$ denote the adjoint of $\overline{B}$ in the Hilbert space $(\mathcal{K}, [J\cdot,\cdot])$, which means that $\overline{B}^* = J\overline{B}^+J$. Note, that $J(\mathcal{B}_a(\overline{A})) = \mathcal{B}_a(\overline{A})$ (Proposition 2.1(ii)). Therefore, $\mathcal{B}_a(\overline{A}) \subseteq \mathcal{D}(\overline{B}^*)$ and

$$\overline{B}^*(\mathcal{B}_a(\overline{A})) = J\overline{B}^+J(\mathcal{B}_a(\overline{A})) = J\overline{B}(\mathcal{B}_a(\overline{A})) \overset{\text{Thm.3.1(ii)}}{\subseteq} J(\mathcal{B}_a(\overline{A})) = \mathcal{B}_a(\overline{A}).$$

Hence, the space $\mathcal{B}_a(\overline{A})$ is contained in the domain and is invariant for the operators $\overline{B}$ and $\overline{B}^*$. This means that it J-reduces $\overline{B}$. ∎

4. DOMINATION AND SPECTRAL COMMUTATIVITY IN PONTRYAGIN SPACES

The following simple lemma is left to the reader.

LEMMA 4.1. *Let $(\mathcal{H}_k)_{k=0}^\infty$ be a sequence of Hilbert spaces and let $S_k \in \mathbf{B}(\mathcal{H}_k)$ $(k \in \mathbb{N})$ be such that S_k is a selfadjoint operator in $\mathcal{H}_k$ for all but a finite number of*

$k \in \mathbb{N}$. Then the set $\Big(\bigcap\limits_{k=0}^{\infty} \rho(S_k)\Big) \setminus \mathbb{R}$ is nonempty, is contained in $\rho\Big(\bigoplus\limits_{k=0}^{\infty} S_k\Big)$, and

$$\Big(\bigoplus_{k=0}^{\infty} S_k - z\Big)^{-1} = \bigoplus_{k=0}^{\infty} \Big(S_k - z\Big)^{-1}, \quad z \in \Big(\bigcap_{k=0}^{\infty} \rho(S_k)\Big) \setminus \mathbb{R}. \tag{4.1}$$

Moreover, if $T_k \in \mathbf{B}(\mathcal{H}_k)$ ($k \in \mathbb{N}$) is a second sequence of operators with only finite number of entries which are not selfadjoint and if $T_k S_k = S_k T_k$ for $k \in \mathbb{N}$, then the operators $S := \bigoplus\limits_{k=0}^{\infty} S_k$ and $T := \bigoplus\limits_{k=0}^{\infty} T_k$ commute spectrally.

The theorem below is a Pontryagin space version of a criterion for selfadjointness (normality) from [21], cf. also [17], [18]. We need the following definition. If $A_0, \dots, A_n$ are operators in $\mathcal{K}$ then we say that $\mathcal{E} \subseteq \mathcal{D}(A_0) \cap \dots \cap \mathcal{D}(A_n)$ is a *joint core for the system* $(A_0, \dots, A_n)$ if

$$\{(f, A_0 f, \dots, A_n f) : f \in \mathcal{D}(A_0) \cap \dots \cap \mathcal{D}(A_n)\} \subseteq \overline{\{(f, A_0 f, \dots, A_n f) : f \in \mathcal{E}\}}\,.$$

For the basic properties of joint cores and joint graphs see e.g. [21].

THEOREM 4.2. *Suppose that $A_0, \dots, A_n$ ($n \geqslant 1$) are symmetric operators in a Pontryagin space $(\mathcal{K}, [\cdot, \cdot])$ and that $\mathcal{E}_{i,j}$, $0 \leqslant i < j \leqslant n$, are dense linear subspaces of $\mathcal{K}$ such that:*

(i) *A_j commutes weakly with A_0 on $\mathcal{E}_{0,j}$ for $j = 1, \dots, n$;*
(ii) *A_i commutes pointwise with A_j on $\mathcal{E}_{i,j}$ for $1 \leqslant i < j \leqslant n$;*
(iii) *A_0 is essentially selfadjoint on $\mathcal{E}_{0,j}$ for $j = 1, \dots, n$;*
(iv) *A_0 dominates A_j on $\mathcal{E}_{0,j}$ for $j = 1, \dots, n$.*

Then $\overline{A}_0, \dots, \overline{A}_n$ are spectrally commuting selfadjoint operators. Furthermore, if $\mathcal{E}_{0,j} = \mathcal{E}$ for $j = 1, \dots, n$, then $\mathcal{E}$ is a joint core for every subsystem of $(\overline{A}_0, \dots, \overline{A}_n)$.

Proof. Fix the fundamental symmetry J and the spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ satisfying the conditions (j1)–(j4) and define the number a_0 as in Proposition 2.1 (both for $A := \overline{A}_0$). The symbols "$\ominus$" and "$\oplus$" below mean respectively the orthogonal difference and sum in the Hilbert space $(\mathcal{K}, [J\cdot, \cdot])$. We put

$$\mathcal{L}_0 := \mathcal{B}_{a_0}(\overline{A}_0), \quad \mathcal{L}_k := \mathcal{B}_{a_0+k}(\overline{A}_0) \ominus \mathcal{B}_{a_0+k-1}(\overline{A}_0), \quad k \in \mathbb{N} \setminus \{0\}.$$

By Proposition 2.1 $(\mathcal{L}_0, [\cdot, \cdot])$ is Pontryagin space and $(\mathcal{L}_k, [\cdot, \cdot])$ ($k \in \mathbb{N} \setminus \{0\}$) is a Hilbert space. Corollary 2.2 implies that

$$\mathcal{K} = \bigoplus_{k=0}^{\infty} \mathcal{L}_k.$$

Since each of the spaces $\mathcal{B}_{a_0+k}(\overline{A}_0)$ ($k \in \mathbb{N}$) J-reduces each of $\overline{A}_0, \dots, \overline{A}_n$ to a bounded operator (Corollary 3.2), the spaces $\mathcal{L}_k$ ($k \in \mathbb{N}$) also have this property. Set

$$S_{i,k} := \overline{A}_i|_{\mathcal{L}_k}, \quad k \in \mathbb{N},\ i = 0, \dots, n.$$

For each $i = 0, 1, \dots, n$ the operator $S_{i,0}$ is selfadjoint in the Pontryagin space $(\mathcal{L}_0, [\cdot,\cdot])$ and $S_{i,k}$ is selfadjoint in the Hilbert space $(\mathcal{L}_k, [\cdot,\cdot])$ ($k \in \mathbb{N} \setminus \{0\}$). Furthermore, $S_{0,k}S_{i,k} = S_{i,k}S_{0,k}$ for $k \in \mathbb{N}$, $i = 0, \dots, n$ (Theorem 3.2(i)). Let $1 \leqslant i < j \leqslant n$. Since $A_iA_jf = A_jA_if$ for $f \in \mathcal{E}_{i,j}$, Lemma 1.1 and Theorem 3.1(ii) give us $S_{i,k}S_{j,k} = S_{j,k}S_{i,k}$ for $k \in \mathbb{N}$. All these facts allow us to apply Lemma 4.1 to each pair of operators $\bigoplus_{k=0}^{\infty} S_{i,k}$, $\bigoplus_{k=0}^{\infty} S_{j,k}$, $0 \leqslant i < j \leqslant n$. In consequence, the operators $\bigoplus_{k=0}^{\infty} S_{i,k}$, $i = 0, \dots, n$, commute spectrally in $\mathcal{K}$. Moreover, by Proposition 2.1,

$$\Big(\bigoplus_{k=0}^{\infty} S_{i,k}\Big)^{+} = J\Big(\bigoplus_{k=0}^{\infty} S_{i,k}\Big)^{*} J = J\Big(S_{i,0}^{*} \oplus \bigoplus_{k=1}^{\infty} S_{i,k}\Big) J = S_{i,0}^{+} \oplus \bigoplus_{k=1}^{\infty} S_{i,k} = \bigoplus_{k=0}^{\infty} S_{i,k}.$$

And so the operators $\bigoplus_{k=0}^{\infty} S_{i,k}$, ($i = 0, \dots, n$) are selfadjoint in $(\mathcal{K}, [\cdot,\cdot])$. Since $\bigoplus_{k=0}^{\infty} S_{i,k} \subseteq \overline{A}_i$ and selfadjoint operators are maximal among symmetric ones, we have

$$\bigoplus_{k=0}^{\infty} S_{i,k} = \overline{A}_i, \quad i = 0, \dots, n. \tag{4.2}$$

Therefore, $\overline{A}_0, \dots, \overline{A}_n$ are spectrally commuting selfadjoint operators.

The proof of the "Furthermore" part of the theorem goes essentially the same as the proof of the analogue part of Theorem 10 of [21]. The difference is that in our situation we need to define $\mathcal{X}$ as

$$\mathcal{X} := \bigcup_{k=0}^{\infty} \mathcal{B}_{a_0+k}(\overline{A}_0).$$

and instead of the spectral projections $E(\Delta_k)$, appearing in [21], we need to use the orthogonal (in the sense of the $[J\cdot,\cdot]$ inner product) projection onto $\mathcal{B}_{a_0+k}(\overline{A}_0)$. ∎

5. OPERATOR MATRICES WITH UNBOUNDED ENTRIES

The following approach to operator matrices comes from the papers by Szafraniec [22] and Szafraniec and Möller [14]. As a pioneering work in this topic we quote Nagel [15], [16]. Later on we apply the results from the previous section to a symmetric operator in a Pontryagin space. We also refer the reader to [1] for another view on powers of symmetric operators in Krein spaces.

In this section we will consider operators in the orthogonal sum of two Hilbert spaces $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1$ having in mind a matrix representation of a symmetric operator in a Pontryagin space. We say that $\boldsymbol{A} = (A_{ij})_{i,j=0,1}$ is an *operator matrix in* $\mathcal{H}_0 \oplus \mathcal{H}_1$ if A_{ij} is an operator (not necessarily bounded) acting from

$\mathcal{H}_j$ to $\mathcal{H}_i$ $(i,j = 0,1)$. Let P_i denote the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_i$ $(i = 0,1)$. With the matrix $\boldsymbol{A}$ we link an operator $\mathrm{M}_{\boldsymbol{A}}$ in $\mathcal{H}$, which we call an operator *generated by* $\boldsymbol{A}$, and which is defined in the following way:

$$\begin{aligned}\mathcal{D}(\mathrm{M}_{\boldsymbol{A}}) &:= (\mathcal{D}(A_{00}) \cap \mathcal{D}(A_{10})) \oplus (\mathcal{D}(A_{01}) \cap \mathcal{D}(A_{11})),\\ \mathrm{M}_{\boldsymbol{A}} f &= (A_{00}f_0 + A_{01}f_1) \oplus (A_{10}f_0 + A_{11}f_1), \quad f \in \mathcal{D}(\mathrm{M}_{\boldsymbol{A}}),\end{aligned}$$

where $f_i := P_i f$ $(i = 0,1)$. If $\boldsymbol{B} = (B_{ij})_{ij=0,1}$ is a second operator matrix in $\mathcal{H}_0 \oplus \mathcal{H}_1$ then

$$\mathrm{M}_{\boldsymbol{A}} + \mathrm{M}_{\boldsymbol{B}} = \mathrm{M}_{\boldsymbol{A}+\boldsymbol{B}}, \quad \text{where } \boldsymbol{A} + \boldsymbol{B} := (A_{ij} + B_{ij})_{ij=0,1}, \tag{5.1}$$

and the sum $\mathrm{M}_{\boldsymbol{A}} + \mathrm{M}_{\boldsymbol{B}}$ is understood as the sum of unbounded operators. One can also easily see that $\mathrm{M}_{\boldsymbol{A}}\,\mathrm{M}_{\boldsymbol{B}} \supseteq \mathrm{M}_{\boldsymbol{AB}}$, where $\boldsymbol{AB} := (A_{i0}B_{0j} + A_{i1}B_{1j})_{ij=0,1}$. The reverse inclusion does not hold in general, therefore later on we will not use the product $\boldsymbol{AB}$.

Suppose now that $A_{00} \in \mathbf{B}(\mathcal{H}_0)$, $A_{10} \in \mathbf{B}(\mathcal{H}_0, \mathcal{H}_1)$, $A_{01} \in \mathbf{B}(\mathcal{H}_1, \mathcal{H}_0)$ and that A_{11} is densely defined. Then it is easy to show (see e.g. Lemma 3.1 of [24]) that the operator $\mathrm{M}_{\boldsymbol{A}}$ is closable if and only if A_{11} is closable. Moreover, the closure of $\mathrm{M}_{\boldsymbol{A}}$ is generated by the matrix $\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & \overline{A_{11}} \end{pmatrix}$. The following lemma, concerning powers of the operator $\mathrm{M}_{\boldsymbol{A}}$, was proved in [24].

LEMMA 5.1. *Let $\boldsymbol{A} := (A_{ij})_{i,j=0,1}$ be an operator matrix such that $A_{00} \in \mathbf{B}(\mathcal{H}_0)$, $A_{10} \in \mathbf{B}(\mathcal{H}_0, \mathcal{H}_1)$, and $A_{01} \in \mathbf{B}(\mathcal{H}_1, \mathcal{H}_0)$. For each $m \in \mathbb{N} \setminus \{0\}$ the following conditions are equivalent:*

(a$_\mathrm{m}$) $\mathcal{H}_0 \subseteq \mathcal{D}((\mathrm{M}_{\boldsymbol{A}})^m)$;

(b$_\mathrm{m}$) $\mathcal{R}(A_{10}) \subseteq \mathcal{D}(A_{11}^{m-1})$;

(c$_\mathrm{m}$) $\mathcal{D}(A^m) = \mathcal{H}_0 \oplus \mathcal{D}(A_{11}^m)$.

It also apparent that if A is an operator in $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ such that $\mathcal{D}(A) = \mathcal{D}_0 \oplus \mathcal{D}_1$ with $\mathcal{D}_j \subseteq \mathcal{H}_j$ $(j = 0,1)$ then the operator A can be identified with an operator matrix, namely

$$A = \mathrm{M}_{\boldsymbol{A}}, \quad \text{where } \boldsymbol{A} = (A_{ij})_{i,j=0,1},\ A_{ij} = P_i A|_{\mathcal{H}_j}\ (i,j = 0,1). \tag{5.2}$$

Note that if $\mathcal{H}_0 \subseteq \mathcal{D}(A)$ (in what follows this will be a usual assumption), then the domain of A decomposes as $\mathcal{H}_0 \oplus \mathcal{D}_1$ with $\mathcal{D}_1 \subseteq \mathcal{H}_1$. Suppose now that $m \in \mathbb{N}$, $\boldsymbol{A}$ satisfies the assumptions of the above lemma, (a$_\mathrm{m}$) holds and the operator $(\mathrm{M}_{\boldsymbol{A}})^m$ is densely defined (apparently, we could proceed without this assumption, but we keep it here for simplicity) and all the operators $(\mathrm{M}_{\boldsymbol{A}})^k$ $(k = 0, \dots, m)$ are closable. Since

$$\mathcal{H}_0 \subseteq \mathcal{D}((\mathrm{M}_{\boldsymbol{A}})^m) \subseteq \mathcal{D}((\mathrm{M}_{\boldsymbol{A}})^k), \quad k = 0, \dots, m,$$

the first m powers of $\mathrm{M}_{\boldsymbol{A}}$ are generated by operator matrices, i.e. for $k = 0, \dots, m$

$$(\mathrm{M}_{\boldsymbol{A}})^k = \mathrm{M}_{\boldsymbol{S}(k)}, \text{ where } \boldsymbol{S}(k) = (S_{ij}(k))_{i,j=0,1} := (P_i (\mathrm{M}_{\boldsymbol{A}})^k|_{\mathcal{H}_j})_{i,j=0,1}.$$

By the closed graph theorem $S_{00}(k) \in \mathbf{B}(\mathcal{H}_0)$ and $S_{10}(k) \in \mathbf{B}(\mathcal{H}_0, \mathcal{H}_1)$ for $k = 0, \ldots, m$. The other two entries of the matrices $\boldsymbol{S}(k)$ might be unbounded operators, but they are densely defined. In fact, by (c_k) we have $\mathcal{D}(S_{01}(k)) = \mathcal{D}(S_{11}(k)) = \mathcal{D}(A_{11}^k)$ ($k = 0, \ldots, m$). Assume additionally that $S_{01}(k)$ is bounded for $k = 0, \ldots, m$. Then $(\mathrm{M}_{\boldsymbol{A}})^k$ can be decomposed as a sum of a diagonal operator and a bounded one, namely if we set

$$\boldsymbol{S}'(k) := \begin{pmatrix} S_{00}(k) & \overline{S_{01}(k)} \\ S_{10}(k) & 0 \end{pmatrix}, \quad k = 0, \ldots, m,$$

then for $k = 0, \ldots, m$

$$(\mathrm{M}_{\boldsymbol{A}})^k = (0|_{\mathcal{H}_0} \oplus S_{11}(k)) + \mathrm{M}_{\boldsymbol{S}'(k)}, \quad \mathrm{M}_{\boldsymbol{S}'(k)} \in \mathbf{B}(\mathcal{H}). \tag{5.3}$$

We continue our reasoning with a lemma.

LEMMA 5.2. *Let $\boldsymbol{A}$ be like in Lemma 5.1 and let $m \in \mathbb{N} \setminus \{0\}$. If ($\mathrm{a_m}$) holds, $(\mathrm{M}_{\boldsymbol{A}})^m$ is densely defined, $(\mathrm{M}_{\boldsymbol{A}})^k$ is closable and $P_0(\mathrm{M}_{\boldsymbol{A}})^k|_{\mathcal{H}_1}$ is bounded for all $k \leqslant m$, then*

$$(\mathrm{M}_{\boldsymbol{A}})^k = (0|_{\mathcal{H}_0} \oplus A_{11}^k) + K_k$$

with some $K_k \in \mathbf{B}(\mathcal{H})$ for all $k = 0, \ldots, m$.

Proof. We will show by induction with respect to k that for $k = 1, 2, \ldots, m$

$$S_{11}(k) = A_{11}^k + B_k, \quad \text{where } B_k \in \mathbf{B}(\mathcal{H}_1),\ \mathcal{R}(B_k) \subseteq \mathcal{D}(A_{11}^{m-k}). \tag{5.4}$$

The above formula together with (5.3) will give us the claim. For $k = 1$ (5.4) is obvious. Suppose that it holds for some $k \in \{1, \ldots, m-1\}$. Note that, by the induction hypothesis, $\mathcal{D}(A_{11}B_k) = \mathcal{H}_1$. Hence, the closed graph theorem gives us

$$A_{11}B_k \in \mathbf{B}(\mathcal{H}). \tag{5.5}$$

Since $1 \leqslant k < m$ and we assumed ($\mathrm{a_m}$), (a_{k+1}) and in consequence (c_{k+1}) hold as well. We take $f \in \mathcal{D}(A_{11}^{k+1})$. Observe that f is in $\mathcal{D}(A_{11}^k)$, which by (c_k) equals $\mathcal{D}(A_{10}S_{01}(k))$. Therefore, f is in the domains of all three operators $A_{10}S_{01}(k)$, A_{11}^{k+1} and $A_{11}B_k$ and we can compute, using the induction assumption,

$$\begin{aligned} A_{10}S_{01}(k)f + A_{11}^{k+1}f + A_{11}B_k f &= A_{10}S_{01}(k)f + A_{11}(A_{11}^k + B_k)f \stackrel{\text{ind.}}{} \\ &= A_{10}S_{01}(k)f + A_{11}S_{11}(k)f = P_1(\mathrm{M}_{\boldsymbol{A}})(S_{01}(k)f + S_{11}(k)f) \\ &= P_1(\mathrm{M}_{\boldsymbol{A}})(\mathrm{M}_{\boldsymbol{A}})^k f = S_{11}(k+1)f. \end{aligned} \tag{5.6}$$

Put $C = \overline{S_{01}(k)}$, it belongs to $\mathbf{B}(\mathcal{H}_1, \mathcal{H}_0)$ by assumption. Take $B_{k+1} := A_{10}C + A_{11}B_k$, which belongs to $\mathbf{B}(\mathcal{H}_1)$, by (5.5). Equation (5.6) together with the fact that $\mathcal{D}(A_{11}^{k+1}) = \mathcal{D}(S_{11}(k+1))$ give us $S_{11}(k+1) = A_{11}^{k+1} + B_{k+1}$. To complete the proof we need to show that $\mathcal{R}(B_{k+1}) \subseteq \mathcal{D}(A_{11}^{m-k-1})$. From ($\mathrm{b_m}$) we obtain that $\mathcal{R}(A_{10}C) \subseteq \mathcal{R}(A_{10}) \subseteq \mathcal{D}(A_{11}^{m-1}) \subseteq \mathcal{D}(A_{11}^{m-k-1})$. Since $\mathcal{R}(B_k) \subseteq \mathcal{D}(A_{11}^{m-k})$, we have $\mathcal{R}(A_{11}B_k) \subseteq \mathcal{D}(A_{11}^{m-k-1})$, which finishes the proof. ∎

6. POWERS OF SYMMETRIC OPERATORS

Let A be a symmetric operator in a Pontriagin space $\mathcal{K}$. Note that a power of A does not have to be densely defined even if $\mathcal{K}$ is a Hilbert space. However, if $\mathcal{D}(A^m)$ is a dense in $\mathcal{K}$ then all the operators A^k ($k = 1, \dots, m$) are symmetric and hence closable. Moreover, by a well known fact ([4], Theorem IX.1.4, [7], Lemma 2.1), we can find a maximal negative space $\mathcal{H}_0 \subseteq \mathcal{D}(A^m)$. We put $\mathcal{H}_1 = \mathcal{H}_0^{[\perp]}$ and fix a fundamental symmetry $J(f_0 \,[\dot{+}]\, f_1) = -f_0 + f_1$, $f_j \in \mathcal{H}_j$, $j = 0, 1$. The Hilbert space $(\mathcal{K}, [J\cdot, \cdot])$ is an orthogonal sum of two its subspaces $\mathcal{K} = \mathcal{H}_0 \oplus \mathcal{H}_1$. With respect to this decomposition we can identify A with the operator matrix, see formula (5.2).

LEMMA 6.1. *Let A be a symmetric operator in a Pontryagin space $(\mathcal{K}, [\cdot, \cdot])$ and let q be a real polynomial of degree m. If A^m is densely defined, then $\mathcal{D}(\overline{q(A)}) = \mathcal{D}(\overline{A^m})$, i.e. $q(A)$ dominates A^m and conversely. If, additionally, p is a real polynomial of degree not greater than m, then $q(A)$ dominates $p(A)$.*

Proof. To prove both claims of the lemma it is enough to show that A^m dominates A^k for $k = 0, \dots, m$. We fix the spaces $\mathcal{H}_0$ and $\mathcal{H}_1$ and a fundamental symmetry as it was described before the lemma. Let $k \in \{0, \dots, m\}$. We apply Lemma 5.2 to the matrix $\boldsymbol{A}$ defined by (5.2) and get

$$A^m = (\mathrm{M}_{\boldsymbol{A}})^m = (0 \oplus A_{11}^m) + K_m, \quad K_m \in \mathbf{B}(\mathcal{K}), \tag{6.1}$$

$$A^k = (\mathrm{M}_{\boldsymbol{A}})^k = (0 \oplus A_{11}^k) + K_k, \quad K_k \in \mathbf{B}(\mathcal{K}). \tag{6.2}$$

The operators $(0 \oplus A_{11}^m)$ and $(0 \oplus A_{11}^k)$ are both symmetric in the Hilbert space $(\mathcal{K}, [J\cdot, \cdot])$ and they are both powers (respectively m-th and k-th) of the same symmetric operator $(0 \oplus A_{11})$. Using the spectral measure of the Naimark's extension of the latter operator one can easily show that $(0 \oplus A_{11}^m)$ dominates $(0 \oplus A_{11}^k)$. By (6.1) and (6.2), A^m dominates A^k. ∎

LEMMA 6.2. *Let $B^{(1)}, \dots, B^{(n)}$ ($n \geqslant 1$) be symmetric operators in a Pontryagin space $(\mathcal{K}, [\cdot, \cdot])$. If the operator*

$$B^{(0)} := (B^{(1)})^2 + \cdots + (B^{(n)})^2 \tag{6.3}$$

is densely defined, then it dominates each of the operators $B^{(j)}$ ($j = 1, \dots, n$).

Proof. Let $\mathcal{H}_0$ be a maximal negative space contained in $\mathcal{D}(B^{(0)})$. As in the proof of the previous lemma fix a Hilbert space inner product $[J\cdot, \cdot]$ on $\mathcal{K}$ and we write $\mathcal{K} = \mathcal{H}_0 \oplus \mathcal{H}_1$. Each of the operators $B^{(j)}$ ($j = 0, \dots, n$) is generated by some operator matrix $((B^{(j)})_{ij})_{i,j=0,1}$ (see again formula (5.2)). Put

$$C_j = (0|_{\mathcal{H}_0}) \oplus (B^{(j)})_{11}, \quad j = 0, \dots, n.$$

Obviously, $B^{(j)}$ is a bounded perturbation of C_j for $j = 0, \dots, n$. Lemma 5.2 implies that $B^{(0)}$ is a bounded perturbation of $C := C_1^2 + \cdots + C_n^2$. Hence, it is enough to show that C dominates each C_j ($j = 1, \dots, n$). Recall that these two operators are symmetric in the Hilbert space $(\mathcal{K}, [J\cdot, \cdot])$. Hence, (cf. proof of Proposition 30 in [21]):

$$\|C_j f\|^2 = [JC_j^2 f, f] \leqslant [JCf, f] \leqslant 2(\|Cf\|^2 + \|f\|^2), \quad f \in \mathcal{D}(C_j),\ j = 1, \dots, n. \quad \blacksquare$$

7. NELSON CRITERION

Compare the theorem below with Propositions 29 and 30 of [21].

THEOREM 7.1. *Let $\mathcal{D}$ be a dense subspace of a Pontryagin space $(\mathcal{K}, [\cdot, \cdot])$ and let $A_1, \dots, A_n : \mathcal{D} \to \mathcal{D}$ be symmetric operators, which commute pointwise on $\mathcal{D}$. Suppose also that $q_1, \dots, q_n$ are real polynomials of one variable. If the operator*

$$A_0 := q_1(A_1)^2 + \cdots + q_n(A_n)^2$$

is essentially selfadjoint on $\mathcal{D}$, then for all real polynomials of one variable $p_1, \dots, p_n$ of degrees satisfying $\deg p_j \leqslant \deg q_j$ ($j = 1, \dots, n$) *the operators $\overline{p_1(A_1)}, \dots, \overline{p_n(A_n)}$ are spectrally commuting, selfadjoint operators in $\mathcal{K}$ and $\mathcal{D}$ is a joint core for the system $(\overline{p_1(A_1)}, \dots, \overline{p_n(A_n)})$.*

Proof. Lemma 6.2 implies that A_0 dominates $q_j(A_j)$, Lemma 6.1 says that $q_j(A_j)$ dominates $p_j(A_j)$ ($j = 1, \dots, n$). In particular A_0 dominates $p_j(A_j)$ on $\mathcal{D}$ for $j = 1, \dots, n$. It is clear that A_0 commutes pointwise (thus weakly) on $\mathcal{D}$ with each of the operators $p_1(A_1), \dots, p_n(A_n)$. Therefore, we can apply Theorem 4.2 to the operators $A_0, p_1(A_1), \dots, p_n(A_n)$ and $\mathcal{E}_{i,j} = \mathcal{D}$ for $0 \leqslant i < j \leqslant n$ which finishes the proof. $\blacksquare$

The following theorem is well known in the Hilbert space context as the Nelson's criteria (cf. Corollary 9.2 of [17], see also Corollary 27 of [21]). Note that it is not a direct consequence of Theorem 7.1 above, since we dot assume that $\mathcal{D}$ is invariant for A and B.

THEOREM 7.2. *Let A and B be symmetric operators in a Pontryagin space $\mathcal{K}$ and let $\mathcal{D}$ be linear subspace of $\mathcal{K}$. If $A^2 + B^2$ is essentially selfadjoint on $\mathcal{D}$ and A and B commute pointwise on $\mathcal{D}$ then $\overline{A}$ and $\overline{B}$ are spectrally commuting selfadjoint operators.*

Proof. As in the last proof we obtain that the operator $C := A^2 + B^2$ dominates A and B on $\mathcal{D}$. To apply Theorem 4.2 to the operators C, A, B and the space $\mathcal{D}$ we need to show that C commutes weakly with A and B on $\mathcal{D}$. This can be seen as follows: note that the assumptions guarantee that $\mathcal{D} \subseteq \mathcal{D}(A^2) \cap \mathcal{D}(B^2) \cap \mathcal{D}(AB) \cap \mathcal{D}(BA)$. Hence, for $f, g \in \mathcal{D}$ we have

$$[B^2 f, Ag] = [Bf, BAg] = [Bf, ABg] = [ABf, Bg] = [BAf, Bg] = [Af, B^2 g],$$

which shows that C commutes weakly with A on $\mathcal{D}$. The proof of weak commutation of C and B on $\mathcal{D}$ goes in the same way. $\blacksquare$

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