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ON THE UNITARY GROUP OF THE JIANG–SU ALGEBRA

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ABSTRACT. We determine the topological group structure of the unitary group of the Jiang–Su algebra $\mathcal{Z}$. As a consequence, we also determine the topological group structure of $\mathrm{Aut}(\mathcal{Z})$, the automorphism group of $\mathcal{Z}$.

KEYWORDS: *Jiang–Su algebra, unitary group, simplicity, Elliott classification program, nuclear C^*-algebras.*

MSC (2000): 46L35.

1. INTRODUCTION

A topological group G is said to be *simple* if G has no proper nontrivial closed normal subgroups. Simple topological groups play a fundamental role in many places in mathematics — some examples being the connected simple Lie groups with trivial centre, for which there are a complete classification as well as much knowledge of the representation theory. In this paper, one of our first goals is to study the simplicity of certain topological groups associated to simple C^*-algebras. Perhaps the most basic examples of this are the full matrix algebras $\mathbb{M}_n(\mathbb{C})$ (the simple finite dimensional C^*-algebras). In this case, the unitary group $U(\mathbb{M}_n(\mathbb{C}))$ is *not* simple. However, when “moded out" by the scalar unitaries (or the centre), the quotient group $U(\mathbb{M}_n(\mathbb{C}))/\mathbb{T}$ is a simple topological group. We are interested in infinite-dimensional generalizations of this result, which will necessarily involve interesting nonlocally compact topological groups. Kadison generalized the finite-dimensional result to the case of simple von Neumann factors. Recall that the infinite-dimensional von Neumann factors which are simple are exactly the ones which are either type II_1 or type III. In [15], Kadison showed that if $\mathcal{M}$ is a type II_1 or a type III factor then $U(\mathcal{M})/\mathbb{T}$ is a simple topological group. (Here, the topology on $U(\mathcal{M})$ is the norm topology.)

In the C^*-algebra context, de la Harpe and Skandalis showed that if $\mathcal{A}$ is a unital simple AF-algebra then $U(\mathcal{A})/\mathbb{T}$ is a simple topological group [6]. (Again, $U(\mathcal{A})$ is given the norm topology.) Generalizing this, Elliott and Rørdam showed that if $\mathcal{A}$ is a unital simple C^*-algebra of real rank zero which is either purely infinite, or has stable rank one and weak unperforation, then $U(\mathcal{A})_0$ (given the same topology as before) is a simple topological group. (Here, $U(\mathcal{A})_0$ is the group of unitaries in $\mathcal{A}$ that are in the (path) connected component of the identity.) Elliott and Rørdam used their result to understand the structure of the topological group $\mathrm{Aut}(\mathcal{A})$, the automorphism group of $\mathcal{A}$ [8]. ($\mathrm{Aut}(\mathcal{A})$ is given the strong topology.) Among other things, they showed that the group $\overline{\mathrm{Inn}(\mathcal{A})}$ of approximately inner automorphisms of $\mathcal{A}$ is a simple topological group. ($\overline{\mathrm{Inn}(\mathcal{A})}$ is given the strong topology.) With this, they raised the following question:

QUESTION 1.1. *Let $\mathcal{A}$ be a simple unital separable C^*-algebra. Is it the case that $\overline{\mathrm{Inn}(\mathcal{A})}$, given the strong topology, is a simple topological group?*

Thomsen showed that the results of Elliott and Rørdam [8] cannot be directly generalized to the nonreal rank zero case. Specifically, Thomsen showed that there are examples of simple unital AH-algebras with bounded dimension growth and real rank one such that $U(\mathcal{A})_0/\mathbb{T}$ (with same topology as before) is *not* a simple topological group [23]. However, in the same paper, Thomsen showed that for certain classes of simple unital AH-algebras with bounded dimension growth (namely for those where the spectra of the building blocks were compact connected metric spaces with covering dimension less than or equal to two and with second cohomology group being zero), $CU(\mathcal{A})/\mathbb{T}$ is a simple topological group. ($CU(\mathcal{A})$ is the closure of the commutator subgroup of $U(\mathcal{A})$. $CU(\mathcal{A})$ is given the norm topology from $\mathcal{A}$. We note that it is a nontrivial result that $\mathbb{T}$ (the scalar multiples of the identity, where the scalars have absolute value one) is a subgroup of $CU(\mathcal{A})$. For the case of the Jiang–Su algebra, we prove this in Lemma 2.1.)

In [18] and [17], the results of Elliott, Rørdam, and Thomsen were generalized to the case of an arbitrary simple unital AH-algebra $\mathcal{A}$ with bounded dimension growth. (There are real rank one such algebras.) Among other things, it was shown that $CU(\mathcal{A})/\mathbb{T}$ is simple and $\overline{\mathrm{Inn}(\mathcal{A})}$ is simple (thus answering Elliott and Rørdam's Question 1.1 for such algebras). The structures of $U(\mathcal{A})$ and $\mathrm{Aut}(\mathcal{A})$ (as topological groups) were determined.

We note that in all of the above results, the C^*-algebras involved had abundant projections. In this paper, we study the Jiang–Su algebra $\mathcal{Z}$. This C^*-algebra is completely different from the previous ones in that $\mathcal{Z}$ has *no* projections other than 0 and 1.

The Jiang–Su algebra $\mathcal{Z}$ is the unique simple unital inductive limit of dimension drop algebras such that $\mathcal{Z}$ has the same K-theory invariant as the complex numbers (i.e., $(K_0(\mathcal{Z}), K_0(\mathcal{Z})_+, K_1(\mathcal{Z}), [1_{\mathcal{Z}}]) = (\mathbb{Z}, \mathbb{Z}_+ \cup \{0\}, \{0\}, 1)$ and $\mathcal{Z}$ has unique tracial state). $\mathcal{Z}$ is currently the centre of much attention in the Elliott classification program for simple nuclear C^*-algebras. Among other things, it has been suggested that all simple unital separable nuclear $\mathcal{Z}$-stable C^*-algebras can be classified using K-theory invariants. (A C^*-algebra $\mathcal{A}$ is *$\mathcal{Z}$-stable* if $\mathcal{A} \otimes \mathcal{Z} \cong \mathcal{A}$.) All simple nuclear C^*-algebras which have so far been classified are $\mathcal{Z}$-stable, and many examples from applications are $\mathcal{Z}$-stable. Much exciting progress has been made in this direction in recent years ([3], [7], [11], [12], [20], [25], [26], [28], [29], [30], [21]). Hence, $\mathcal{Z}$ is a good C^*-algebra to which to generalize the theory of Elliott, Rørdam and Thomsen.

In this paper, we prove the following:

THEOREM 1.2. *Let $\mathcal{Z}$ be the Jiang–Su algebra. Then we have the following:*

(i) $CU(\mathcal{Z})/\mathbb{T}$ *is a simple topological group.*

(Here, $\mathbb{T}$ is the scalar multiples of the identity, where the scalars have absolute value one. It will be shown in Lemma 2.1 *that $\mathbb{T}$ is a subgroup of $CU(\mathcal{Z})$.)*

(ii) *By the work of Thomsen* [24], *there is an exact sequence*

$$\{1\} \to CU(\mathcal{Z}) \to U(\mathcal{Z}) \to \mathbb{T} \to \{1\}.$$

This, together with (i), *give the topological normal subgroup structure of $U(\mathcal{Z})$.*
(We note that the copy of the circle $\mathbb{T}$ in (ii) *is different from the copy of $\mathbb{T}$ in* (i).*)*

(iii) $\mathrm{Aut}(\mathcal{Z}) = \overline{\mathrm{Inn}(\mathcal{Z})}$ *is a simple topological group.*

Note that Theorem 1.2 part (iii) answers Elliott and Rørdam's Question 1.1 for $\mathcal{Z}$.

The argument for Theorem 1.2 is quite technical, but it is a first step in generalizing the theory of Elliott, Rørdam and Thomsen to a large class of interesting C^*-algebras. In particular, with our result for $\mathcal{Z}$, we raise the following question:

QUESTION 1.3. *Can we generalize Theorem* 1.2 *(with appropriate modifications) to arbitrary simple unital separable nuclear $\mathcal{Z}$-stable C^*-algebras?*

2. NOTATION AND PRELIMINARIES

For positive integers p, q, we let $\mathcal{Z}_{p,q}$ be the dimension drop algebra

$$\mathcal{Z}_{p,q} = \{f \in C[0,1] \otimes \mathbb{M}_{pq} : f(0) \in \mathbb{M}_p \otimes 1_{\mathbb{M}_q}\ f(1) \in 1_{\mathbb{M}_p} \otimes \mathbb{M}_q\}.$$

Let $\mathbf{p}, \mathbf{q}$ be supernatural numbers, we let $\mathcal{Z}_{\mathbf{p},\mathbf{q}}$ be the generalized dimension drop algebra

$$\mathcal{Z}_{\mathbf{p},\mathbf{q}} =_{\mathrm{df}} \{f \in C[0,1] \otimes \mathbb{M}_{\mathbf{p}} \otimes \mathbb{M}_{\mathbf{q}} : f(0) \in \mathbb{M}_{\mathbf{p}} \otimes 1_{\mathbb{M}_{\mathbf{q}}}\ f(1) \in 1_{\mathbb{M}_{\mathbf{p}}} \otimes \mathbb{M}_{\mathbf{q}}\}.$$

(In the above, $\mathbb{M}_{\mathbf{p}}$ is the UHF-algebra with supernatural number $\mathbf{p}$. Similar for $\mathbb{M}_{\mathbf{q}}$. See [30].)

We say that a supernatural number $\mathbf{p}$ is of *infinite type* if any prime number that occurs in $\mathbf{p}$ occurs infinitely many times.

Let $\mathbf{p}$ be a supernatural number of infinite type. Let $\{P_k\}_{k=1}^{\infty}$ be a sequence of natural numbers. We say $\{P_k\}_{k=1}^{\infty}$ *generates* $\mathbf{p}$ *exponentially* if the following conditions hold:

(i) $\mathbf{p} = P_1 P_2 P_3 \cdots$;
(ii) $(P_k)^2$ divides P_{k+1} for all $k \geqslant 1$.

(See [30].)

We let $\mathbb{T}$ denote the unit circle of the complex plane (i.e., $\mathbb{T} =_{\mathrm{df}} \{z \in \mathbb{C} : |z| = 1\}$). For a unital C^*-algebra $\mathcal{A}$, $U(\mathcal{A})$ denotes the unitary group of $\mathcal{A}$ and $CU(\mathcal{A})$ denotes the closure of the commutator subgroup of $U(\mathcal{A})$. $U_0(\mathcal{A})$ (or $U(\mathcal{A})_0$) denotes the elements of $U(\mathcal{A})$ that are in the (path) connected component of the identity. $\mathrm{Aut}(\mathcal{A})$ is the automorphism group of $\mathcal{A}$, given the strong topology, and $\overline{\mathrm{Inn}(\mathcal{A})}$ is the subgroup of $\mathrm{Aut}(\mathcal{A})$ consisting of the approximately inner automorphisms.

With notation as in the previous paragraph, throughout this paper, $U(\mathcal{A})$ and its subgroups are given the norm topology. $\mathrm{Aut}(\mathcal{A})$ and its subgroups are given the strong topology.

If $\mathcal{A} = \mathbb{M}_n(C[0,1])$ or if $\mathcal{A}$ is a dimension drop algebra or a generalized dimension drop algebra, then $U(\mathcal{A})_{\mathrm{e}}$ consists of those elements $f \in U(\mathcal{A})$ such that $f(0) = 1$ and $f(1) = 1$. $CU(\mathcal{A})_{\mathrm{e}}$ is the closure of the commutator subgroup of $U(\mathcal{A})_{\mathrm{e}}$.

For a C^*-algebra $\mathcal{C}$ and for finitely many elements $a_1, a_2, \ldots, a_n \in \mathcal{C}$, we let $\prod_{i=1}^{n} a_i$ denote the finite product $\prod_{i=1}^{n} a_i =_{\mathrm{df}} a_1 a_2 \cdots a_n$. Also, if $a, b \in \mathcal{C}$ then we let (a, b) denote the commutator $(a, b) = aba^* b^*$.

Also, we note that in $\mathbb{M}_2(\mathbb{C})$, every unitary of the form $\mathrm{diag}(\alpha, \overline{\alpha})$, where $\alpha = \mathrm{e}^{\mathrm{i}\theta}$ has absolute value one (so $\theta \in \mathbb{R}$), is in $CU(\mathbb{M}_2(\mathbb{C}))$, since

$$\begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta/2} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta/2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\theta/2} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\theta/2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Also,

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

And also, for $\theta \in \mathbb{R}$,

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\theta} \end{pmatrix} = (1/\sqrt{2}) \begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix} \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} (1/\sqrt{2}) \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \text{ is an element of } CU(\mathbb{M}_2(\mathbb{C})). \text{ Hence, for } n \geqslant 1,$$

$$SU(\mathbb{M}_n(\mathbb{C})) = CU(\mathbb{M}_n(\mathbb{C})) \text{ and } U(\mathbb{M}_n(\mathbb{C}))/\mathbb{T} = CU(\mathbb{M}_n(\mathbb{C}))/\mathbb{T}. \tag{2.1}$$

(Note also that the latter two are simple as (algebraic) groups.)

We also see, from the computations in (2.1),

that if $h : [0,1] \to \mathbb{R}$ is a continuous function then

$$\begin{pmatrix} \mathrm{e}^{\mathrm{i}h} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}h} \end{pmatrix} \text{ and } \begin{pmatrix} \cos(h) & \sin(h) \\ -\sin(h) & \cos(h) \end{pmatrix} \text{ are elements of } CU(\mathbb{M}_2(C[0,1])). \tag{2.2}$$

We end this section with a preliminary lemma which is important in understanding the topological normal subgroup structure of the Jiang–Su algebra:

LEMMA 2.1. *Let $\mathcal{Z}$ be the Jiang–Su algebra. Then the following hold:*
(i) $CU(\mathcal{Z}) \subseteq U_0(\mathcal{Z})$;
(ii) $\mathbb{T} \subseteq CU(\mathcal{Z})$ *(i.e., the scalar unitaries are contained in $CU(\mathcal{Z})$).*

Proof. (i) By [14], $K_1(\mathcal{Z}) = 0$. By [20], $\mathcal{Z}$ has stable rank one. Hence, by Theorem 10.12 of [19] and Corollary 7.14 of [1], we see that $U(\mathcal{Z}) = U_0(\mathcal{Z})$. In particular, $CU(\mathcal{Z}) \subseteq U_0(\mathcal{Z})$ as required.

(ii) Let $\alpha \in \mathbb{T}$ be a complex number with absolute value one. We will show that $\alpha 1_{\mathcal{Z}} \in CU(\mathcal{Z})$.

Let $\varepsilon > 0$ be given. Let $p, q \geqslant 2$ be relatively prime positive integers, and let $r, s \geqslant 1$ be positive integers with $0 < r < p$ and $0 < s < q$ such that the following hold:

$$\begin{aligned} &\text{(i) } r/p < s/q; \\ &\text{(ii) } |r/p - s/q| < \varepsilon/100; \\ &\text{(iii) for } r/p \leqslant t \leqslant s/q, |\mathrm{e}^{\mathrm{i}2\pi t} - \alpha| < \varepsilon/100. \end{aligned} \tag{2.3}$$

Let $\mathbf{p}, \mathbf{q}$ be relatively prime supernatural numbers such that p divides $\mathbf{p}$ and q divides $\mathbf{q}$. Then by Proposition 3.3 of [21], $\mathcal{Z}_{\mathbf{p},\mathbf{q}}$ embeds unitally into $\mathcal{Z}$. Since $\mathcal{Z}_{p,q}$ embeds unitally into $\mathcal{Z}_{\mathbf{p},\mathbf{q}}$, we have a sequence of unital embeddings:

$$\mathcal{Z}_{p,q} \to \mathcal{Z}_{\mathbf{p},\mathbf{q}} \to \mathcal{Z}. \tag{2.4}$$

Now let $u \in U(\mathcal{Z}_{p,q})$ be the unitary that is given by:
(i) $u(t) = \mathrm{e}^{\mathrm{i}2\pi(r/p)} 1_{\mathbb{M}_p \otimes \mathbb{M}_q}$ for $t \in [0, 1/3]$;
(ii) $u(t) = \mathrm{e}^{\mathrm{i}2\pi(r/p)(-3t+2)} 1_{\mathbb{M}_p \otimes \mathbb{M}_q}$ for $t \in [1/3, 2/3]$;
(iii) $u(t) = 1_{\mathbb{M}_p \otimes \mathbb{M}_q}$ for $t \in [2/3, 1]$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_p : [0,1] \to \mathbb{T}$ be continuous functions given by:

$$\begin{aligned} &\text{(i) } \lambda_1(t) = \mathrm{e}^{\mathrm{i}2\pi(r/p)} \text{ for } t \in [0, 1/3]; \\ &\text{(ii) } \lambda_1(t) = \mathrm{e}^{\mathrm{i}2\pi(r/p)(-3t+2)} \text{ for } t \in [1/3, 2/3]; \\ &\text{(iii) } \lambda_1(t) = 1 \text{ for } t \in [2/3, 1]; \\ &\text{(iv) for } 2 \leqslant n \leqslant p-1, \lambda_n \text{ is the unique function such that } \lambda_1 = \overline{\lambda_{n-1}}\lambda_n. \end{aligned} \tag{2.5}$$

Let $\{e_{i,j}\}_{1 \leqslant i,j \leqslant p}$ be a system of matrix units for $\mathbb{M}_p$. Then by (2.5), we see that

$$u = \lambda_1 e_{1,1} \otimes 1_{\mathbb{M}_q} + \sum_{j=2}^{p-1} \overline{\lambda_{j-1}} \lambda_j e_{j,j} \otimes 1_{\mathbb{M}_q} + \overline{\lambda_{p-1}} e_{p,p} \otimes 1_{\mathbb{M}_q}. \tag{2.6}$$

For $1 \leqslant i \leqslant p-1$, let $u_i \in \mathcal{Z}_{p,q}$ be given by

$$u_i =_{\mathrm{df}} \lambda_i e_{i,i} \otimes 1_{\mathbb{M}_q} + \overline{\lambda_i} e_{i+1,i+1} \otimes 1_{\mathbb{M}_q} + (1_{\mathbb{M}_p \otimes \mathbb{M}_q} - e_{i,i} \otimes 1_{\mathbb{M}_q} - e_{i+1,i+1} \otimes 1_{\mathbb{M}_q}).$$

Hence, we have that

$$u = u_1 u_2 \cdots u_n. \tag{2.7}$$

For $1 \leqslant i \leqslant p-1$, let $\gamma_i : [0,1] \to \mathbb{T}$ be a continuous function such that $(\gamma_i)^2 = \lambda_i$ and $\gamma_i(t) = 1$ for $t \in [2/3, 1]$. Let v_i, w_i be unitaries in $U(\mathcal{Z}_{p,q})$ such that

$$v_i =_{\mathrm{df}} \gamma_i e_{i,i} \otimes 1_{\mathbb{M}_q} + \overline{\gamma_i} e_{i+1,i+1} \otimes 1_{\mathbb{M}_q} + (1_{\mathbb{M}_p \otimes \mathbb{M}_q} - (e_{i,i} + e_{i+1,i+1}) \otimes 1_{\mathbb{M}_q})$$

and

$$w_i(t) = \begin{cases} e_{i,i+1} \otimes 1_{\mathbb{M}_q} + e_{i+1,i} \otimes 1_{\mathbb{M}_q} + (1_{\mathbb{M}_p \otimes \mathbb{M}_q} - (e_{i,i} + e_{i+1,i+1}) \otimes 1_{\mathbb{M}_q}) & t \in [0, 2/3], \\ 1_{\mathbb{M}_p \otimes \mathbb{M}_q} & t = 1. \end{cases}$$

Hence, for $1 \leqslant i \leqslant p-1$, we have that

$$u_i = v_i w_i (v_i)^* (w_i)^* = (v_i, w_i) \in CU(\mathcal{Z}_{p,q}). \tag{2.8}$$

Hence, $u = u_1 u_2 \cdots u_n \in CU(\mathcal{Z}_{p,q})$ as required.

Next, let $u' \in U(\mathcal{Z}_{p,q})$ be the unitary in $U(\mathcal{Z}_{p,q})$ that is given by:

(i) $u'(t) = 1_{\mathbb{M}_p \otimes \mathbb{M}_q}$ for $t \in [0, 1/3]$;

(ii) $u'(t) = \mathrm{e}^{\mathrm{i}2\pi(s/q)(3t-1)} 1_{\mathbb{M}_p \otimes \mathbb{M}_q}$ for $t \in [1/3, 2/3]$;

(iii) $u'(t) = \mathrm{e}^{\mathrm{i}2\pi(s/q)}$ for $t \in [2/3, 1]$.

By an argument similar to that for u, we have that $u' \in CU(\mathcal{Z}_{p,q})$. From this and (2.8), we have that $uu' \in CU(\mathcal{Z}_{p,q})$.

Moreover, by (2.3), we have that $\|\alpha 1_{\mathbb{M}_p \otimes \mathbb{M}_q} - u(t)u'(t)\| < \varepsilon$ for $t \in [0,1]$. Hence, $\alpha 1_{\mathbb{M}_p \otimes \mathbb{M}_q}$ is in norm within ε of an element of $CU(\mathcal{Z}_{p,q})$. From this and the diagram in (2.8), we see that $\alpha 1_{\mathbb{M}_p \otimes \mathbb{M}_q}$ is in norm within ε of an element of $CU(\mathcal{Z})$. Since $\varepsilon > 0$ is arbitrary and since $CU(\mathcal{Z})$ is a closed subgroup of $U(\mathcal{Z})$, we see that $\alpha 1_{\mathbb{M}_p \otimes \mathbb{M}_q} \in CU(\mathcal{Z})$ as required. ∎

3. UNITARIES IN THE BUILDING BLOCKS

The first result follows from Theorem 9.1 of [6].

LEMMA 3.1. *Let $\mathcal{A}$ be a* UHF*-algebra. Let $H \subseteq U(\mathcal{A})$ be a noncentral (not necessarily closed) subgroup of $U(\mathcal{A})$ that is normalized by $CU(\mathcal{A})$.*

Then H contains $CU(\mathcal{A})$.

LEMMA 3.2. *Let $\mathbf{p}, \mathbf{q}$ be supernatural numbers. Let H be a closed normal subgroup of $CU(\mathcal{Z}_{\mathbf{p},\mathbf{q}})$ such that H contains a unitary u with $u(0)$ being a nonscalar unitary. For all nonzero orthogonal projections $p_0, q_0 \in \mathbb{M}_{\mathbf{p}} \otimes 1_{\mathbb{M}_{\mathbf{q}}}$ such that p_0 and q_0 are Murray–von Neumann equivalent in $\mathbb{M}_{\mathbf{p}} \otimes 1_{\mathbb{M}_{\mathbf{q}}}$, for every $\theta \in \mathbb{R}$, and for every $\varepsilon > 0$, there exist $v \in H$ and $0 < \delta < \varepsilon$ such that:*

(i) *$v(0) = \mathrm{e}^{\mathrm{i}\theta} p_0 + \mathrm{e}^{-\mathrm{i}\theta} q_0 + (1 - (p_0 + q_0))$;*

(ii) *$v(t) = 1$ for $t \in [\delta, 1]$;*

(iii) *$\|v(t) - (\mathrm{e}^{\mathrm{i}\theta(-(1/\delta)t+1)} p_0 + \mathrm{e}^{-\mathrm{i}\theta(-(1/\delta)t+1)} q_0 + (1 - (p_0 + q_0)))\| < \varepsilon$ for $t \in [0, \delta]$. (We can actually choose $\delta > 0$ to be arbitrarily small.)*

Proof. Since p_0, q_0 are Murray–von Neumann equivalent in $\mathbb{M}_{\mathbf{p}} \otimes 1_{\mathbb{M}_{\mathbf{q}}}$, let $w_0 \in \mathbb{M}_{\mathbf{p}} \otimes 1_{\mathbb{M}_{\mathbf{q}}}$ be a partial isometry with initial projection p_0 and range projection q_0. By Lemma 3.1, let $u \in H$ be such that

$$u(0) = w_0 + (w_0)^* + (1 - (p_0 + q_0)).$$

Choose $\delta > 0$ such that $\delta < \varepsilon$ and for $t \in [0, \delta]$,

$$\|u(t) - u(0)\| < \varepsilon/100. \tag{3.1}$$

Now let $x \in \mathbb{M}_{\mathbf{p},\mathbf{q}}$ be a unitary such that, for $t \in [0, \delta]$,

$$x(t) = \mathrm{e}^{\mathrm{i}\theta(-(1/\delta)t+1)/2} p_0 + \mathrm{e}^{-\mathrm{i}\theta(-(1/\delta)t+1)/2} q_0 + (1 - (p_0 + q_0))$$

and

$$x(t) = 1$$

for $t \in [\delta, 1]$. By arguments similar to those of (2.1) and (2.2), we see that $x \in CU(\mathbb{M}_{\mathbf{p},\mathbf{q}})$.

Then the following is an element of H which satisfies the required conditions:

$$v =_{\mathrm{df}} xux^*u^*. \quad ∎$$

The next lemma is the same as Lemma 3.2, with endpoints interchanged.

LEMMA 3.3. *Let $\mathbf{p}, \mathbf{q}$ be supernatural numbers. Let H be a closed normal subgroup of $CU(\mathcal{Z}_{\mathbf{p},\mathbf{q}})$ such that H contains a unitary u with $u(1)$ being a nonscalar unitary. For all nonzero orthogonal projections $p_0, q_0 \in 1_{\mathbb{M}_{\mathbf{p}}} \otimes \mathbb{M}_{\mathbf{q}}$ such that p_0 and q_0 are Murray–von Neumann equivalent in $1_{\mathbb{M}_{\mathbf{p}}} \otimes \mathbb{M}_{\mathbf{q}}$, for every $\theta \in \mathbb{R}$, and for every $\varepsilon > 0$, there exist $v \in H$ and $0 < \delta < \varepsilon$ such that:*

(i) *$v(1) = \mathrm{e}^{\mathrm{i}\theta} p_0 + \mathrm{e}^{-\mathrm{i}\theta} q_0 + (1 - (p_0 + q_0))$;*

(ii) *$v(t) = 1$ for $t \in [0, 1 - \delta]$;*

(iii) *$\|v(t) - (\mathrm{e}^{\mathrm{i}\theta((1/\delta)t+1-1/\delta)} p_0 + \mathrm{e}^{-\mathrm{i}\theta((1/\delta)t+1-1/\delta)} q_0 + (1 - (p_0 + q_0)))\| < \varepsilon$ for $t \in [1 - \delta, 1]$.*

(We can actually choose $\delta > 0$ to be arbitrarily small.)

The proof is similar to the proof of Lemma 3.2.

LEMMA 3.4. *Let $\mathbf{p}, \mathbf{q}$ be supernatural numbers. Let H be a closed normal subgroup of $CU(\mathcal{Z}_{\mathbf{p},\mathbf{q}})$ such that H contains a unitary u with $u(0)$ being a nonscalar unitary. Let $P, Q \geqslant 2$ be positive integers that divide $\mathbf{p}, \mathbf{q}$ respectively. Let $\mathcal{Z}_{P,Q}$ be the corresponding unital C*-subalgebra of $\mathcal{Z}_{\mathbf{p},\mathbf{q}}$. (Note that $\mathcal{Z}_{P,Q}$ is a dimension drop algebra.)*

Suppose that $w \in CU(\mathcal{Z}_{P,Q})$ is a unitary such that

$$\|w(0) - 1\| < r < 1/10.$$

Then for every $\varepsilon > 0$, there exists $v \in H$ such that:

(i) *$\|v(t) - 1\| < r + \varepsilon$ for all $t \in [0, 1]$;*

(ii) *$v(t) = 1$ for $t \in [1/2, 1]$;*

(iii) $v(0) = w(0)$.

Proof. There exist real numbers $\phi_1, \phi_2, \ldots, \phi_P$ and pairwise orthogonal projections $p_1, p_2, \ldots, p_P \in \mathbb{M}_P \otimes 1_{\mathbb{M}_Q}$ such that the following hold:

$$
\begin{aligned}
&\text{(i) } w(0) = \sum_{j=1}^{P} \mathrm{e}^{\mathrm{i}2\pi\phi_j t} p_j;\\
&\text{(ii) } \phi_j \in [-1/12, 1/12] \text{ for } 1 \leqslant j \leqslant P;\\
&\text{(iii) } \sum_{j=1}^{P} \phi_j = 0 \text{ (so } \prod_{j=1}^{P} \mathrm{e}^{\mathrm{i}2\pi\phi_j} = 1 \text{).}
\end{aligned}
\tag{3.2}
$$

We now construct continuous functions $\theta_1, \theta_2, \ldots, \theta_{P-1} : [0,1] \to \mathbb{R}$ as follows:

$$
\begin{aligned}
&\text{(i) } \theta_1 = \phi_1;\\
&\text{(ii) } \theta_j - \theta_{j-1} = \phi_j \text{ for } 2 \leqslant j \leqslant P-1.
\end{aligned}
\tag{3.3}
$$

(Hence, $\mathrm{e}^{\mathrm{i}2\pi\theta_1} = \mathrm{e}^{\mathrm{i}2\pi\phi_1}$ and $\mathrm{e}^{\mathrm{i}2\pi\theta_j}\mathrm{e}^{-\mathrm{i}2\pi\theta_{j-1}} = \mathrm{e}^{\mathrm{i}2\pi\phi_j}$ for $2 \leqslant j \leqslant P-1$.)

By Lemma 3.2, let $\delta > 0$ with $\delta < 1/2$ be such that for $1 \leqslant j \leqslant P-1$, there exists $v_j \in H$ such that the following hold:

$$
\begin{aligned}
&\text{(i) } v_j(0) = \mathrm{e}^{\mathrm{i}2\pi\theta_j} p_j + \mathrm{e}^{-\mathrm{i}2\pi\theta_j} p_{j+1} + (1 - (p_j + p_{j+1}));\\
&\text{(ii) } v_j(t) = 1 \text{ for } t \in [\delta, 1];\\
&\text{(iii) } \|v_j(t) - (\mathrm{e}^{\mathrm{i}2\pi\theta_j(-(1/\delta)t+1)} p_j + \mathrm{e}^{-\mathrm{i}2\pi\theta_j(-(1/\delta)t+1)} p_{j+1}\\
&\quad + (1 - (p_j + p_{j+1})))\| < \varepsilon/(100P(100P^2+1)) \text{ for } t \in [0,\delta].
\end{aligned}
\tag{3.4}
$$

Hence, $v =_{\mathrm{df}} v_1 v_2 \cdots v_P$ is an element of H. By (3.2) statement (i), (3.3) and (3.4) statement (i), we have that

$$v(0) = w(0).$$

By (3.4) statement (ii), we have that $v(t) = 1$ for $t \in [\delta, 1]$.

By (3.2) and (3.4), for $t \in [0, \delta]$,

$$
\begin{aligned}
&\|v(t) - 1\|\\
&\leqslant \Big\| v(t) - \sum_{j=1}^{P} \mathrm{e}^{\mathrm{i}2\pi\phi_j(-(1/\delta)t+1)} p_j \Big\| + \Big\| \sum_{j=1}^{P} \mathrm{e}^{\mathrm{i}2\pi\phi_j(-(1/\delta)t+1)} p_j - 1 \Big\|\\
&< \Big\| v_1(t) v_2(t) \cdots v_P(t) - \prod_{j=1}^{P-1} (\mathrm{e}^{\mathrm{i}2\pi\theta_j(-(1/\delta)t+1)} p_j + \mathrm{e}^{-\mathrm{i}2\pi\theta_j(-(1/\delta)t+1)} p_{j+1}\\
&\qquad + (1 - (p_j + p_{j+1}))) \Big\| + r < \varepsilon/(100(100P^2+1)) + r < \varepsilon + r
\end{aligned}
$$

as required. ∎

LEMMA 3.5. *Let $\mathbf{p}, \mathbf{q}$ be supernatural numbers. Let H be a closed normal subgroup of $CU(\mathcal{Z}_{\mathbf{p},\mathbf{q}})$ such that H contains a unitary u with $u(0)$ and $u(1)$ being non-scalar unitaries. Let $P, Q \geqslant 2$ be positive integers that divide $\mathbf{p}, \mathbf{q}$ respectively. Let $\mathcal{Z}_{P,Q}$ be the corresponding unital C^*-subalgebra of $\mathcal{Z}_{\mathbf{p},\mathbf{q}}$. (Note that $\mathcal{Z}_{P,Q}$ is a dimension drop algebra.)*

Suppose that $w \in CU(\mathcal{Z}_{P,Q})$ is a unitary such that

$$\|w(0) - 1\| < r < 1/10 \quad \text{and} \quad \|w(1) - 1\| < r < 1/10.$$

Then for every $\varepsilon > 0$, there exists $v \in H$ such that:

(i) $\|v(t) - 1\| < r + \varepsilon$ *for all* $t \in [0, 1]$;

(ii) $v(0) = w(0)$;

(iii) $v(1) = w(1)$.

The argument is a modification of the argument of Lemma 3.4 to include the right endpoint 1.

LEMMA 3.6. *Let $h : [0, 1] \to \mathbb{R}$ be a continuous map such that $h(0) = h(1) = 0$. Then $\begin{pmatrix} e^{ih} & 0 \\ 0 & e^{-ih} \end{pmatrix}$ and $\begin{pmatrix} \cos(h) & \sin(h) \\ -\sin(h) & \cos(h) \end{pmatrix}$ are both elements of $CU(\mathbb{M}_n(C[0, 1]))_e$.*

Proof. The proof involves modifying the equations in (2.1) and (2.2).

Let $\varepsilon > 0$. We first want to show that $\mathrm{diag}(e^{ih}, e^{-ih})$ can be norm approximated within ε by an element of $CU(\mathbb{M}_n(C[0, 1]))_e$.

Now since $h(0) = h(1) = 0$ and since h and the exponential map are continuous, there is $\delta > 0$ such that if $0 \leqslant t \leqslant \delta$ or $1 - \delta \leqslant t \leqslant 1$ then

$$|e^{ih(t)} - 1| < \varepsilon/100 \quad \text{and} \quad |e^{ih(t)/2} - 1| < \varepsilon/100. \tag{3.5}$$

Now let $v \in U(\mathbb{M}_n(C[0, 1]))$ be a continuous path of unitaries such that:

(i) $v(0) = v(1) = 1_{\mathbb{M}_n}$;

(ii) $v(t) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ for $t \in [\delta/2, 1 - \delta/2]$.

Hence,

$$w =_{\mathrm{df}} \begin{pmatrix} e^{ih/2} & 0 \\ 0 & e^{-ih/2} \end{pmatrix} v \begin{pmatrix} e^{-ih/2} & 0 \\ 0 & e^{ih/2} \end{pmatrix} v^*$$

is an element of $CU(\mathbb{M}_n(C[0, 1]))_e$. Moreover, by the computations in (2.1), (2.2), (3.5), and the definition of w and v, we have that for $t \in [\delta/2, 1-\delta/2]$, $\mathrm{diag}(e^{ih(t)}, e^{-ih(t)}) = w(t)$. And for $t \in [0, \delta/2] \cup [1 - \delta/2, 1]$,

$$\begin{aligned}
&\|w(t) - 1_{\mathbb{M}_n}\| \\
&= \left\| \begin{pmatrix} e^{ih(t)/2} & 0 \\ 0 & e^{-ih(t)/2} \end{pmatrix} v(t) \begin{pmatrix} e^{-ih(t)/2} & 0 \\ 0 & e^{ih(t)/2} \end{pmatrix} v(t)^* - 1_{\mathbb{M}_n} \right\| \\
&= \left\| \begin{pmatrix} e^{ih(t)/2} & 0 \\ 0 & e^{-ih(t)/2} \end{pmatrix} v(t) \begin{pmatrix} e^{-ih(t)/2} & 0 \\ 0 & e^{ih(t)/2} \end{pmatrix} v(t)^* - 1_{\mathbb{M}_n} v(t) 1_{\mathbb{M}_n} v(t)^* \right\| \\
&\leqslant \left\| \begin{pmatrix} e^{ih(t)/2} & 0 \\ 0 & e^{-ih(t)/2} \end{pmatrix} v(t) \begin{pmatrix} e^{-ih(t)/2} & 0 \\ 0 & e^{ih(t)/2} \end{pmatrix} v(t)^* - 1_{\mathbb{M}_n} v(t) \begin{pmatrix} e^{-ih(t)/2} & 0 \\ 0 & e^{ih(t)/2} \end{pmatrix} v(t)^* \right\| \\
&\qquad + \left\| 1_{\mathbb{M}_n} v(t) \begin{pmatrix} e^{-ih(t)/2} & 0 \\ 0 & e^{ih(t)/2} \end{pmatrix} v(t)^* - 1_{\mathbb{M}_n} v(t) 1_{\mathbb{M}_n} v(t)^* \right\|
\end{aligned}$$

$$\leqslant \left\| \begin{pmatrix} e^{ih(t)/2} & 0 \\ 0 & e^{-ih(t)/2} \end{pmatrix} - 1_{\mathbb{M}_n} \right\| + \left\| \begin{pmatrix} e^{-ih(t)/2} & 0 \\ 0 & e^{ih(t)/2} \end{pmatrix} - 1_{\mathbb{M}_n} \right\| < \varepsilon/100 + \varepsilon/100 = \varepsilon/50.$$

From the above and (3.5), we see that w is within ε of $\mathrm{diag}(e^{ih}, e^{-ih})$. Hence, $\mathrm{diag}(e^{ih}, e^{-ih})$ is within ε of an element of $CU(\mathbb{M}_n(C[0,1]))_e$. Since $\varepsilon > 0$ is arbitrary, it follows that $\mathrm{diag}(e^{ih}, e^{-ih}) \in CU(\mathbb{M}_n(C[0,1]))_e$ as required.

Next, again, let $\varepsilon > 0$ be given. We want to find a unitary $w' \in CU(\mathbb{M}_n(C[0,1]))_e$ such that w' is norm within ε of $\begin{pmatrix} \cos(h) & \sin(h) \\ -\sin(h) & \cos(h) \end{pmatrix}$. The argument is similar to the argument for $\mathrm{diag}(e^{ih}, e^{-ih})$, except that we now replace the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ with the matrix $(1/\sqrt{2}) \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$, and we need to use the identity

$$\begin{pmatrix} \cos(h) & \sin(h) \\ -\sin(h) & \cos(h) \end{pmatrix} = (1/\sqrt{2}) \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{ih} & 0 \\ 0 & e^{-ih} \end{pmatrix} (1/\sqrt{2}) \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

and except that we now have to use that $\mathrm{diag}(e^{ih}, e^{-ih})$ is an element of $CU(\mathbb{M}_n(C[0,1]))_e$ (previous proof). ∎

LEMMA 3.7. (i) *Let $n \geqslant 2$ and let X be a compact Hausdorff topological space. Then the commutator subgroup of $U(\mathbb{M}_n(C(X)))_0$ is exactly*

$$\{u \in U(\mathbb{M}_n(C(X))) : \det(u(x)) = 1 \ \forall x \in X\}.$$

(ii) *Let $p, q \geqslant 2$ be positive integers. Consider the dimension drop algebra $\mathcal{Z}_{p,q}$. Let $\pi_p : \mathcal{Z}_{p,q} \to \mathbb{M}_p$ and $\pi_q : \mathcal{Z}_{p,q} \to \mathbb{M}_q$ be the irreducible representations of $\mathcal{Z}_{p,q}$ corresponding to the endpoints 0, 1 respectively.*

Let $u \in U(\mathcal{Z}_{p,q})$ be a unitary. Suppose that

$$\det(u(t)) = 1$$

for all $t \in [0,1]$, and

$$\det(\pi_p(u)) = \det(\pi_q(u)) = 1.$$

Then $u \in CU(\mathcal{Z}_{p,q})$.

The first statement follows from Proposition 2.4 of [23]. The second statement follows from Proposition 5.3 of [16] (with slight modification).

LEMMA 3.8. *Say that $n \geqslant 2$. Suppose that $u \in U(\mathbb{M}_n(C[0,1]))$ is a unitary. Then for every $\varepsilon > 0$, there exist continuous functions $f_i : [0,1] \to \mathbb{T}$ and pairwise orthogonal projections $p_i \in \mathbb{M}_n(C[0,1])$, $1 \leqslant i \leqslant n$, such that $\sum_{i=1}^{n} f_i p_i \in \mathbb{M}_n(C[0,1])$ and*

$$\left\| \sum_{i=1}^{n} f_i p_i - u \right\| < \varepsilon.$$

Moreover, if $u \in CU(\mathbb{M}_n(C[0,1]))$ and $u(0) = u(1) = 1_{\mathbb{M}_n(C[0,1])}$ then we can require that $f_i(0) = f_i(1) = 1$ for $1 \leqslant i \leqslant n$ and $\prod_{i=1}^{n} f_i(t) = 1_{\mathbb{M}_n}$ for $t \in [0,1]$.

This follows from Lemma 1.9 of [22].

LEMMA 3.9. *Let $\alpha : [0,1] \to \mathbb{T}$ be a continuous function such that for all $t \in [0,1]$, the real and imaginary parts of $\alpha(t)$ and $\alpha(t)^2$ satisfy*

$$\mathrm{Re}(\alpha(t)), \mathrm{Im}(\alpha(t)), \mathrm{Re}((\alpha(t))^2), \mathrm{Im}((\alpha(t))^2) > 0.$$

Let $\theta : [0,1] \to \mathbb{R}$ be a continuous function with $\theta(0) = \theta(1) = 0$.

Then there exist unitaries $u, u_0 \in CU(\mathbb{M}_2(C[0,1]))$ and an integer $N \geqslant 1$ such that the following hold:

(i) $u(0) = u(1) = 1_{\mathbb{M}_2(\mathbb{C})}$*;*

(ii) $\mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta}) = ((u_0)^* u \mathrm{diag}(\alpha, \overline{\alpha}) u^* \mathrm{diag}(\overline{\alpha}, \alpha) u_0)^N$*;*

(iii) *for every $\varepsilon > 0$, there exist unitaries $u', u'' \in CU(\mathbb{M}_2(C[0,1]))_{\mathrm{e}}$ such that*

$$\|\mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta}) - ((u'')^* u' \mathrm{diag}(\alpha, \overline{\alpha})(u')^* \mathrm{diag}(\overline{\alpha}, \alpha) u'')^N\| < \varepsilon.$$

Proof. By hypothesis,

$$0 < \inf\{\mathrm{Re}((\alpha(t))^2) : t \in [0,1]\} \leqslant \sup\{\mathrm{Re}((\alpha(t))^2) : t \in [0,1]\} < 1.$$

Case 1. Suppose that $|\theta(t)|$ is small enough so that

$$\inf\{\cos(\theta(t)) : t \in [0,1]\} > \sup\{\mathrm{Re}((\alpha(t))^2) : t \in [0,1]\}$$

and so that

$$\cos(\theta(t)) > \mathrm{Re}((\alpha(t))^2) = 1 + (\mathrm{Re}((\alpha(t))^2) - 1)$$

for all $t \in [0,1]$. Hence, let $h : [0,1] \to [0,\infty)$ be a nonnegative continous function such that $h(0) = h(1) = 0$ and

$$\mathrm{Re}(\mathrm{e}^{\mathrm{i}\theta(t)}) = \cos(\theta(t)) = 1 + (\mathrm{Re}((\alpha(t))^2) - 1)\sin^2(h(t)) \tag{3.6}$$

for all $t \in [0,1]$. (Note that, necessarily, $\sin^2(h(t)) \neq 1$ for all $t \in [0,1]$.)

Now let $u \in U(\mathbb{M}_n(C[0,1]))$ be the unitary given by

$$u =_{\mathrm{df}} \begin{pmatrix} \cos(h) & \sin(h) \\ -\sin(h) & \cos(h) \end{pmatrix}.$$

We will show that u allows us to verify the statements in Lemma 3.9.

Firstly, we note that by the remarks in (2.2), $u \in CU(\mathbb{M}_2(C[0,1]))$.

Claim 1. $u\mathrm{diag}(\alpha, \overline{\alpha})u^*\mathrm{diag}(\overline{\alpha}, \alpha) = v$

where

$$v =_{\mathrm{df}} \begin{pmatrix} \cos^2(h) + (\overline{\alpha})^2 \sin^2(h) & (1-\alpha^2)\sin(h)\cos(h) \\ -(1-\overline{\alpha}^2)\sin(h)\cos(h) & \alpha^2 \sin^2(h) + \cos^2(h) \end{pmatrix}.$$

Proof of Claim 1:

$$\begin{aligned} u\mathrm{diag}(\alpha, \overline{\alpha})u^*\mathrm{diag}(\overline{\alpha}, \alpha) &= \begin{pmatrix} \cos(h) & \sin(h) \\ -\sin(h) & \cos(h) \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \overline{\alpha} \end{pmatrix} \begin{pmatrix} \cos(h) & -\sin(h) \\ \sin(h) & \cos(h) \end{pmatrix} \begin{pmatrix} \overline{\alpha} & 0 \\ 0 & \alpha \end{pmatrix} \\ &= \begin{pmatrix} \alpha\cos(h) & \overline{\alpha}\sin(h) \\ -\alpha\sin(h) & \overline{\alpha}\cos(h) \end{pmatrix} \begin{pmatrix} \cos(h) & -\sin(h) \\ \sin(h) & \cos(h) \end{pmatrix} \begin{pmatrix} \overline{\alpha} & 0 \\ 0 & \alpha \end{pmatrix} \\ &= \begin{pmatrix} \alpha\cos^2(h) + \overline{\alpha}\sin^2(h) & -\alpha\cos(h)\sin(h) + \overline{\alpha}\cos(h)\sin(h) \\ -\alpha\cos(h)\sin(h) + \overline{\alpha}\cos(h)\sin(h) & \alpha\sin^2(h) + \overline{\alpha}\cos^2(h) \end{pmatrix} \begin{pmatrix} \overline{\alpha} & 0 \\ 0 & \alpha \end{pmatrix} \\ &= \begin{pmatrix} \cos^2(h) + (\overline{\alpha})^2\sin^2(h) & (1-\alpha^2)\sin(h)\cos(h) \\ -(1-\overline{\alpha}^2)\sin(h)\cos(h) & \cos^2(h) + \alpha^2\sin^2(h) \end{pmatrix} \end{aligned}$$

as required. This completes the proof of Claim 1.

Note that the trace of v is $2\cos^2(h) + (\alpha^2 + \overline{\alpha}^2)\sin^2(h)$. Hence, since for all $t \in [0,1]$, $v(t)$ is a unitary in $\mathbb{M}_2(\mathbb{C})$ with determinant one and by the definition of $h(t)$ in (3.6), we know that $\cos^2(h(t)) + (1/2)(\alpha(t)^2 + \overline{\alpha(t)}^2)\sin^2(h(t)) = \cos(\theta(t))$ is the real part of both eigenvalues of $v(t)$. Hence, for $t \in [0,1]$, the eigenvalues of $v(t)$ are

$$\lambda_{\pm}(t) = e^{\pm i\theta(t)} = \cos^2(h(t)) + (1/2)(\alpha^2 + \overline{\alpha}^2)\sin^2(h(t)) \pm i\sin(\theta(t)).$$

Claim 2. For $t \in [0,1]$ such that $\sin(h(t)) \neq 0$, the one by one component of $v(t) - \lambda_{\pm}(t)$ is nonzero.

We will use the notation "$v_{1,1}$" to denote the one by one corner of v.

Proof of Claim 2:

$$\begin{aligned} v_{1,1} - \lambda_{\pm} &= (\cos^2(h) + \overline{\alpha}^2\sin^2(h)) - (\cos^2(h) + (1/2)(\alpha^2 + \overline{\alpha}^2)\sin^2(h) \pm i\sin(\theta)) \\ &= (1/2)(\overline{\alpha}^2 - \alpha^2)\sin^2(h) \pm i\sin(\theta). \end{aligned}$$

Now suppose, to the contrary, that there is a point $t \in [0,1]$ with $\sin(h(t)) \neq 0$ such that

$$v_{1,1}(t) - \lambda_{\pm}(t) = 0.$$

Therefore,

$$(1/2)(\overline{\alpha}(t)^2 - \alpha(t)^2)\sin^2(h(t)) = \pm i\sin(\theta(t)).$$

Therefore,

$$-i(1/2)(\overline{\alpha}(t)^2 - \alpha(t)^2)\sin^2(h(t)) = \pm\sin(\theta(t)).$$

Therefore,

$$-(1/4)(\overline{\alpha}(t)^2 - \alpha(t)^2)^2\sin^4(h(t)) + \cos^2(\theta(t)) = 1.$$

Applying (3.6), we see that

$$\begin{aligned} 1 &= -(1/4)(\overline{\alpha}(t)^2 - \alpha(t)^2)^2\sin^4(h(t)) + (1 + ((1/2)(\alpha(t)^2 + \overline{\alpha(t)}^2) - 1)\sin^2(h(t)))^2 \\ &= -(1/4)[\overline{\alpha(t)}^4 - 2 + \alpha(t)^4]\sin^4(h(t)) + [1 + (\alpha(t)^2 + \overline{\alpha(t)}^2 - 2)\sin^2(h(t)) \\ &\qquad + ((1/2)(\alpha(t)^2 + \overline{\alpha(t)}^2) - 1)^2\sin^4(h(t))]. \end{aligned}$$

Cancelling out the 1 term from both sides and continuing to expand, we have:

$$\begin{aligned} 0 &= -(1/4)[\overline{\alpha(t)}^4 + \alpha(t)^4 - 2]\sin^4(h(t)) + [\alpha(t)^2 + \overline{\alpha(t)}^2 - 2]\sin^2(h(t)) \\ &\qquad + [(1/4)(\alpha(t)^2 + \overline{\alpha(t)}^2)^2 - (\alpha(t)^2 + \overline{\alpha(t)}^2) + 1]\sin^4(h(t)) \\ &= -(1/4)[\overline{\alpha(t)}^4 + \alpha(t)^4 - 2]\sin^4(h(t)) + [\alpha(t)^2 + \overline{\alpha(t)}^2 - 2]\sin^2(h(t)) \\ &\qquad + [(1/4)(\alpha(t)^4 + \overline{\alpha(t)}^4 + 2) - (\alpha(t)^2 + \overline{\alpha(t)}^2) + 1]\sin^4(h(t)) \\ &= \sin^2(h(t))[\alpha(t)^2 + \overline{\alpha(t)}^2 - 2] + \sin^4(h(t))[2 - \alpha(t)^2 - \overline{\alpha(t)}^2]. \end{aligned}$$

Hence, we have that

$$\sin^2(h(t))[2 - \alpha(t)^2 - \overline{\alpha(t)}^2] = \sin^4(h(t))[2 - \alpha(t)^2 - \overline{\alpha(t)}^2]. \tag{3.7}$$

By (3.6), $\sin^2(h(t)) \neq 1$. By hypothesis of Claim 2, $\sin^2(h(t)) \neq 0$. Finally, by hypothesis of Lemma 3.9, $\alpha(t)^2 + \overline{\alpha(t)}^2 < 2$. Hence, the equation (3.7) is impossible, and we have a contradiction. This completes the proof of Claim 2.

Recall, from the statement before Claim 2, that for $t \in [0,1]$, the eigenvalues of the determinant one unitary $v(t)$ are

$$\begin{aligned}
&\lambda_\pm(t)\\
&= e^{\pm i\theta(t)} = \cos^2(h(t)) + (1/2)(\alpha(t)^2 + \overline{\alpha(t)}^2)\sin^2(h(t)) \pm i\sin(\theta(t))\\
&= 1 + (\mathrm{Re}(\alpha(t)^2) - 1)\sin^2(h(t)) \pm i\sqrt{1 - (1 + (\mathrm{Re}(\alpha(t)^2) - 1)\sin^2(h(t)))^2}\\
&= 1 + (\mathrm{Re}(\alpha(t)^2) - 1)\sin^2(h(t))\\
&\qquad \pm i\sqrt{1 - (1 + 2(\mathrm{Re}(\alpha(t)^2) - 1)\sin^2(h(t)) + (\mathrm{Re}(\alpha(t)^2) - 1)^2\sin^4(h(t))}\\
&= 1 + (\mathrm{Re}(\alpha(t)^2) - 1)\sin^2(h(t))\\
&\qquad \pm i\sqrt{-2(\mathrm{Re}(\alpha(t)^2) - 1)\sin^2(h(t)) - (\mathrm{Re}(\alpha(t)^2) - 1)^2\sin^4(h(t))}\\
&= 1 + (\mathrm{Re}(\alpha(t)^2) - 1)\sin^2(h(t))\\
&\qquad \pm i\sin(h(t))\sqrt{-2(\mathrm{Re}(\alpha(t)^2) - 1) - (\mathrm{Re}(\alpha(t)^2) - 1)^2\sin^2(h(t))}\,.
\end{aligned}$$

Hence, for $t \in [0,1]$,

$$\begin{aligned}
v_{1,1}(t) - \lambda_\pm(t) = (1/2)(\overline{\alpha(t)}^2 - \alpha(t)^2)\sin^2(h(t)) \pm i\sin(h(t))&\\
\sqrt{-2(\mathrm{Re}(\alpha(t)^2) - 1) - (\mathrm{Re}(\alpha(t)^2) - 1)^2\sin^2(h(t))}&\,.
\end{aligned}$$

Now consider the continuous functions $f_\pm : [0,1] \to \mathbb{C}$ that are given, for $t \in [0,1]$, by

$$\begin{aligned}
f_\pm(t) =_{\mathrm{df}} (1/2)(\overline{\alpha(t)}^2 - \alpha(t)^2)\sin(h(t))&\\
\pm i\sqrt{-2(\mathrm{Re}(\alpha(t)^2) - 1) - (\mathrm{Re}(\alpha(t)^2) - 1)^2\sin^2(h(t))}&\,.
\end{aligned}$$

If $\sin(h(t)) = 0$ then $f_\pm(t) = \pm i\sqrt{-2(\mathrm{Re}(\alpha(t)^2) - 1)}$ which is nonzero by our hypothesis on $\alpha(t)^2$ in the statement of Lemma 3.9. On the other hand, if $\sin(h(t)) \neq 0$, then $f_\pm(t) \neq 0$ by Claim 2. Hence, $f_\pm$ are nonzero continuous functions on $[0,1]$.

Now consider the one by two component $v_{1,2} = (1 - \alpha^2)\sin(h)\cos(h)$ of v. We see that for $t \in [0,1]$, the eigenvector of $v(t)$ corresponding to the eigenvalue $\lambda_\pm(t)$ (respectively) is

$$\{(s_1, s_2) \in \mathbb{C}^2 : s_1 = -s_2(1 - \alpha(t)^2)\cos(h(t))/f_\pm(t)\}.$$

(Recall that $f_\pm$ is never zero.) Hence, for $t \in [0,1]$, let $\vec{x}_\pm(t) \in \mathbb{C}^2$ be the eigenvector of $v(t)$ for the eigenvalue $\lambda_\pm(t)$ given by

$$\vec{x}_\pm(t) =_{\text{df}} \begin{pmatrix} -(1-\alpha(t)^2)\cos(h(t))/f_\pm(t) \\ 1 \end{pmatrix}.$$

Then $\vec{x}_\pm$ is continuous and never zero on $[0,1]$. For all $t \in [0,1]$, let $\vec{y}_\pm(t) \in \mathbb{C}^2$ be the unit vector, which is an eigenvector of $v(t)$ for the eigenvalue $\lambda_\pm(t)$, given by

$$\vec{y}_\pm(t) =_{\text{df}} \vec{x}_\pm(t)/\|\vec{x}_\pm(t)\|.$$

Then $\vec{y}_\pm$ is continuous on $[0,1]$.

Note that for $t \in [0,1]$, if $\lambda_+(t) \neq \lambda_-(t)$, then $\vec{y}_+(t)$ is orthogonal to $\vec{y}_-(t)$ in the inner product space $\mathbb{C}^2$.

And, by the definition of $\lambda_\pm$, for $t \in [0,1]$, if $\lambda_+(t) = \lambda_-(t)$ then $\lambda_+(t) = \lambda_-(t) = 1$ and $h(t) = \theta(t) = 0$. Hence, $\sin(h(t)) = 0$, and

$$\vec{x}_\pm(t) = \begin{pmatrix} -(1-\alpha(t)^2)/\pm i\sqrt{-2(\mathrm{Re}(\alpha(t)^2)-1)} \\ 1 \end{pmatrix}.$$

Hence, taking the inner product in $\mathbb{C}^2$, we have that

$$\begin{aligned}\langle \vec{x}_+(t)|\vec{x}_-(t)\rangle &= \frac{-(1-\alpha(t)^2)}{i\sqrt{-2(\mathrm{Re}(\alpha(t)^2)-1)}} \frac{-(1-\overline{\alpha(t)}^2)}{i\sqrt{-2(\mathrm{Re}(\alpha(t)^2)-1)})+1} \\ &= \frac{-(1-\alpha(t)^2)(1-\overline{\alpha(t)}^2)}{2-\alpha(t)^2-\overline{\alpha(t)}^2}+1 = \frac{-(2-\alpha(t)^2-\overline{\alpha(t)}^2)}{2-\alpha(t)^2-\overline{\alpha(t)}^2}+1 = -1+1 = 0.\end{aligned}$$

Hence, $\vec{x}_+(t)$ and $\vec{x}_-(t)$ are orthogonal. Hence, $\vec{y}_+(t)$ and $\vec{y}_-(t)$ are orthogonal.

Whichever the case, we have that for all $t \in [0,1]$, $\vec{y}_+(t)$ and $\vec{y}_-(t)$ are orthogonal unit vectors in the inner product space $\mathbb{C}^2$. Hence, let $u_0 \in \mathbb{M}_2(C[0,1])$ be the unitary that is given by

$$u_0(t) =_{\text{df}} \beta(t)\,(\vec{y}_+(t), \vec{y}_-(t))$$

where $\beta : [0,1] \to \mathbb{T}$ is a continuous function such that the determinant $\det(u_0(t)) = 1$ for all $t \in [0,1]$. Hence, by Lemma 3.7, since every unitary in $U(\mathbb{M}_2(C[0,1]))$ is in the connected component of the identity, we have that $u_0 \in CU(\mathbb{M}_2(C[0,1]))$. Then we have that

$$\mathrm{diag}(e^{i\theta}, e^{-i\theta}) = \mathrm{diag}(\lambda_+, \lambda_-) = (u_0)^* v u_0 = (u_0)^* u \,\mathrm{diag}(\alpha, \overline{\alpha}) u^* \mathrm{diag}(\overline{\alpha}, \alpha) u_0.$$

So statements (i) and (ii) of Lemma 3.9 are satisfied.

We now confirm statement (iii).

(3.8) By Lemma 3.6 and by the definition of u, we have that $u \in CU(\mathbb{M}_2(C[0,1]))_e$.

Now by the definition of $CU(\mathbb{M}_2(C[0,1]))_e$, $u(0) = u(1) = 1_{\mathbb{M}_2}$. Hence, at both $t = 0, 1$, we must have that

$$v(t) =_{\text{df}} u(t)\mathrm{diag}(\alpha(t), \overline{\alpha(t)})(u(t))^* \mathrm{diag}(\overline{\alpha(t)}, \alpha(t)) = 1_{\mathbb{M}_2}.$$

So by continuity, let $\delta > 0$ be such that for all $t \in [0,\delta] \cup [1-\delta, 1]$,

$$\|v(t) - 1_{\mathbb{M}_2}\| < \varepsilon/1000. \tag{3.9}$$

Now since $u_0 \in CU(\mathbb{M}_2(C[0,1]))$, let $n \geqslant 1$ be an integer, and let $x_1, x_2, \ldots, x_n$, $y_1, y_2, \ldots, y_n$ be unitaries in $U(\mathbb{M}_2(C[0,1]))$ such that

$$\Big\| u_0 - \prod_{i=1}^{n} (x_i, y_i) \Big\| < \varepsilon/1000.$$

Since $U(\mathbb{M}_2(\mathbb{C}))$ is path-connected, for $1 \leqslant i \leqslant n$, let x_i', y_i' be unitaries in $U(\mathbb{M}_2(C[0,1]))$ such that the following statements hold:

(i) $x_i'(t) = x_i(t)$ and $y_i'(t) = y_i(t)$ for $t \in [\delta/100, 1-\delta/100]$;

(ii) $x_i'(0) = x_i'(1) = y_i'(0) = y_i'(1) = 1_{\mathbb{M}_2}$.

Let $u' \in CU(\mathbb{M}_2(C[0,1]))_{\mathrm{e}}$ be defined by

$$u' =_{\mathrm{df}} \prod_{i=1}^{n} (x_i', y_i').$$

So by the definition of u' and v, by Lemma 3.9 (ii) and and by (3.8), for $t \in [\delta/100, 1-\delta/100]$, we have that

$$\begin{aligned}
&\|\mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta(t)}, \mathrm{e}^{-\mathrm{i}\theta(t)}) - (u'(t))^* u(t) \mathrm{diag}(\alpha(t), \overline{\alpha(t)})(u(t))^* \\
&\qquad\qquad\qquad\qquad \mathrm{diag}(\overline{\alpha(t)}, \alpha(t)) u'(t)\| \\
&\leqslant \|\mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta(t)}, \mathrm{e}^{-\mathrm{i}\theta(t)}) - (u_0(t))^* u(t) \mathrm{diag}(\alpha(t), \overline{\alpha(t)})(u(t))^* \\
&\qquad\qquad\qquad\qquad \mathrm{diag}(\overline{\alpha(t)}, \alpha(t)) u_0(t)\| \\
&\quad + \|(u_0(t))^* u(t) \mathrm{diag}(\alpha(t), \overline{\alpha(t)})(u(t))^* \mathrm{diag}(\overline{\alpha(t)}, \alpha(t)) u_0(t) \\
&\quad - (u'(t))^* u(t) \mathrm{diag}(\alpha(t), \overline{\alpha(t)})(u(t))^* \mathrm{diag}(\overline{\alpha(t)}, \alpha(t)) u_0(t)\| \\
&\quad + \|(u'(t))^* u(t) \mathrm{diag}(\alpha(t), \overline{\alpha(t)})(u(t))^* \mathrm{diag}(\overline{\alpha(t)}, \alpha(t)) u_0(t) \\
&\quad - (u'(t))^* u(t) \mathrm{diag}(\alpha(t), \overline{\alpha(t)})(u(t))^* \mathrm{diag}(\overline{\alpha(t)}, \alpha(t)) u'(t)\| \\
&\leqslant 0 + \|(u_0(t))^* - (u'(t))^*\| + \|u_0(t) - u'(t)\| < \varepsilon/1000 + \varepsilon/1000 = \varepsilon/500.
\end{aligned} \tag{3.10}$$

Also, by (3.9), (3.8), the definition of v and Lemma 3.9 (ii), we have that for $t \in [0, \delta/100] \cup [1-\delta/100, 1]$,

$$\begin{aligned}
&\|\mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta(t)}, \mathrm{e}^{-\mathrm{i}\theta(t)}) - (u'(t))^* u(t) \mathrm{diag}(\alpha(t), \overline{\alpha(t)})(u(t))^* \mathrm{diag}(\overline{\alpha(t)}, \alpha(t)) u'(t)\| \\
&= \|(u_0(t))^* v(t) u_0(t) - (u'(t))^* v(t) u'(t)\| \\
&\leqslant \|(u_0(t))^* v(t) u_0(t) - 1\| + \|1 - (u'(t))^* v(t) u'(t)\| \\
&= \|(u_0(t))^* v(t) u_0(t) - (u_0(t))^* u_0(t)\| + \|(u'(t))^* u'(t) - (u'(t))^* v(t) u'(t)\| \\
&= \|v(t) - 1\| + \|1 - v(t)\| < \varepsilon/1000 + \varepsilon/1000 = \varepsilon/500.
\end{aligned}$$

From this and (3.10), we have that

$$\|\mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta}) - (u')^* u \mathrm{diag}(\alpha, \overline{\alpha}) u^* \mathrm{diag}(\overline{\alpha}, \alpha) u'\| < \varepsilon$$

which proves Lemma 3.9 statement (iii).

Case 2. Suppose that $\theta : [0,1] \to \mathbb{R}$ is an arbitrary continuous function with $\theta(0) = \theta(1) = 0$. Then choose an integer $N \geqslant 1$ such that $\theta/N \approx 0$ and

$$\inf\{\cos(\theta(t)/N) : t \in [0,1]\} > \sup\{\mathrm{Re}(\alpha(t)^2) : t \in [0,1]\}.$$

Hence, the map $[0,1] \to \mathbb{R}$, given by $t \mapsto \theta(t)/N$, satisfies Case 1. Apply Case 1 to get Lemma 3.9 statements (i), (ii) and (iii) for θ/N (replacing θ). Then take appropriate quantities to the power of N to get (i), (ii) and (iii) of Lemma 3.9 for θ as required.

In more detail: Firstly, note that the proof of Case 1 gives statements (i), (ii) and (iii) of Lemma 3.9 with power being one. (In other words, with notation as in Lemma 3.9, $N = 1$. Note that this "N" is different from the "N" that used in the previous paragraph or the rest of this proof!) Hence, choose unitaries $u, u_0 \in CU(\mathbb{M}_2(C[0,1]))$, and choose unitaries $u', u'' \in CU(\mathbb{M}_2(C[0,1]))_{\mathrm{e}}$ such that the following hold:

$$\begin{aligned} &\text{(i) } u(0) = u(1) = 1_{\mathbb{M}_2};\\ &\text{(ii) } \mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta/N}, \mathrm{e}^{-\mathrm{i}\theta/N}) = (u_0)^* u \mathrm{diag}(\alpha, \overline{\alpha}) u^* \mathrm{diag}(\overline{\alpha}, \alpha) u_0;\\ &\text{(iii) } \|\mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta/N}, \mathrm{e}^{-\mathrm{i}\theta/N}) - (u'')^* u' \mathrm{diag}(\alpha, \overline{\alpha})(u')^* \mathrm{diag}(\overline{\alpha}, \alpha) u''\| < \varepsilon/(100N). \end{aligned} \tag{3.11}$$

(Note that the "N" here is the same "N" as in the previous paragraph; i.e., it is chosen so that $|\theta/N|$ is "small enough".)

From (3.11) statement (ii), we have that

$$\mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta}) = ((u_0)^* u \mathrm{diag}(\alpha, \overline{\alpha}) u^* \mathrm{diag}(\overline{\alpha}, \alpha) u_0)^N.$$

This gives statement (ii) of Lemma 3.9 for θ.

From (3.11) statement (iii), we have that

$$\|\mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-\mathrm{i}\theta}) - ((u'')^* u' \mathrm{diag}(\alpha, \overline{\alpha})(u')^* \mathrm{diag}(\overline{\alpha}, \alpha) u'')^N\| < \varepsilon.$$

This gives us statement (iii) of Lemma 3.9 for θ. ∎

LEMMA 3.10. *For every $\varepsilon > 0$, there exists $\delta > 0$ such that for every unital C^*-algebra $\mathcal{A}$, if*

(i) *$p_1, p_2, \ldots, p_n$ are pairwise orthogonal projections in $\mathcal{A}$;*
(ii) *$q_1, q_2, \ldots, q_n$ are pairwise orthogonal projections in $\mathcal{A}$;*
(iii) *$\alpha_1, \alpha_2, \ldots, \alpha_n$ are scalars (complex numbers) with norm one;*
(iv) *$|\alpha_i - \alpha_j| \geqslant \varepsilon$ for $i \neq j$; and*
(v) *$\|(\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n) - (\alpha_1 q_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n)\| < \delta$;*

then $\|p_i - q_i\| < \varepsilon$ and $p_i \sim q_i$ in $\mathcal{A}$ for $1 \leqslant i \leqslant n$.

Proof. Let $I_1, I_2, \ldots, I_N$, with $N \geqslant 3$, be a collection of pairwise disjoint half-open intervals such that the following hold:

$$\begin{aligned} &\text{(i) For } 1 \leqslant k \leqslant N, \text{ the interval } I_k \text{ has length less than } \varepsilon/100.\\ &\text{(ii) } \mathbb{T} = \bigcup_{k=1}^{N} I_i. \text{ (Here } \mathbb{T} \text{ is the unit circle of the complex plane.)}\\ &\text{(iii) For } 1 \leqslant k \leqslant N, \text{ the immediate neighbours of } I_k \text{ are } I_{k-1} \text{ and } I_{k+1}.\\ &\text{(Here we define } I_{-1} =_{\mathrm{df}} I_N \text{ and } I_{N+1} =_{\mathrm{df}} I_1.) \end{aligned} \tag{3.12}$$

For $1 \leqslant k \leqslant N$, let $f_k : \mathbb{T} \to [0,1]$ be a continuous function such that the following hold:

$$\begin{aligned} &\text{(i) } f_k = 1 \text{ on } I_k.\\ &\text{(ii) } f_k = 0 \text{ on } I_l \text{ if } l \text{ is not an element of } \{k-1, k, k+1\}.\\ &\text{(Here, } I_{-1} \text{ and } I_{N+1} \text{ are defined as in (3.12) statement (iii).)} \end{aligned} \tag{3.13}$$

By the Stone–Weierstrass Theorem, for $1 \leqslant k \leqslant N$, let g'_k be a polynomial in $z, \overline{z}$ (where z is a complex variable and, $\overline{z}$ is the complex conjugate of z) such that

$$|g'_k(z, \overline{z}) - f_k(z)| < \varepsilon/100 \tag{3.14}$$

for $z \in \mathbb{T}$. Now define $g_k(z) =_{\mathrm{df}} g'_k(z, \overline{z})$ for all $z \in \mathbb{T}$.

For $1 \leqslant k \leqslant N$,

$$\begin{aligned} &\text{since } g'_k \text{ is a polynomial and by the definition of } g_k,\\ &\text{choose } \delta_k > 0 \text{ such that whenever } a, b \in \mathcal{A}\\ &\text{are normal elements with norm less than or}\\ &\text{equal to one such that } \|a-b\| < \delta_k \text{ then } \|g_k(a) - g_k(b)\| < \varepsilon/100.\\ &\text{Now let } \delta =_{\mathrm{df}} \min\{\delta_1, \delta_2, \ldots, \delta_N\} > 0. \end{aligned} \tag{3.15}$$

Now suppose that $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n, \alpha_1, \alpha_2, \ldots, \alpha_n$ satisfy the hypothesis of Lemma 3.10. Note that by (3.12), (3.13) and since $|\alpha_i - \alpha_j| \geqslant \varepsilon$ for $i \neq j$, we have that for each i, with $1 \leqslant i \leqslant n$, there exists $k(i)$, with $1 \leqslant k(i) \leqslant N$ such that whenever $r_1, r_2, \ldots, r_n$ are pairwise orthogonal projections in $\mathcal{A}$ then

$$f_{k(i)}(\alpha_1 r_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n) = r_i.$$

From this, (3.14) and (3.15), we have that for $1 \leqslant i \leqslant n$,

$$\begin{aligned} \|p_i - q_i\| &= \|f_{k(i)}(\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n) - f_{k(i)}(\alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_n q_n)\|\\ &\leqslant \|f_{k(i)}(\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n) - g_{k(i)}(\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n)\|\\ &\quad + \|g_{k(i)}(\alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_n p_n) - g_{k(i)}(\alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_n q_n)\|\\ &\quad + \|g_{k(i)}(\alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_n q_n) - f_{k(i)}(\alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_n q_n)\|\\ &< \varepsilon/100 + \varepsilon/100 + \varepsilon/100 < \varepsilon. \quad \blacksquare \end{aligned}$$

The following follows from standard spectral theory arguments:

LEMMA 3.11. *Let $\mathcal{A}$ be a unital C^*-algebra. Let $\delta > 0$ and let $u, v \in U(\mathcal{A})$ unitaries in $\mathcal{A}$ be such that*

$$\|u - v\| < \delta.$$

Then every element of $\sigma(u)$ is within δ of an element of $\sigma(v)$ and vice versa.

LEMMA 3.12. *Let $\mathbf{p}, \mathbf{q}$ be supernatural numbers whch are relatively prime. Let $u \in CU(\mathcal{Z}_{\mathbf{p},\mathbf{q}})$ be a unitary such that every tracial state on $\mathcal{Z}_{\mathbf{p},\mathbf{q}}$ induces the same tracial state on $C^*(u)$.*

Then for all $s, t \in [0,1]$, $\sigma(u(s)) = \sigma(u(t))$.

Proof. Firstly, we show that for all $s, t \in (0,1)$, $\sigma(u(s)) = \sigma(u(t))$. Let τ be the unique tracial state on the UHF-algebra $\mathbb{M}_{\mathbf{p}} \otimes \mathbb{M}_{\mathbf{q}}$.

Fix $s, t \in (0,1)$. Let $\phi_s, \phi_t : C^*(u) \to \mathbb{M}_{\mathbf{p}} \otimes \mathbb{M}_{\mathbf{q}}$ be the two unital $*$-homomorphisms that satisfy $\phi_s(u) = u(s)$ and $\phi_t(u) = u(t)$. By hypothesis, we have that $\tau \circ \phi_s = \tau \circ \phi_t$. Among other things, this implies that $\ker(\phi_s) = \ker(\phi_t)$. Note that since the K-theory of the UHF-algebra $\mathbb{M}_{\mathbf{p}} \otimes \mathbb{M}_{\mathbf{q}}$ is completely determined by τ, since $\sigma(u) \subseteq \mathbb{T}$, by 23.1.1 of [2], and since ϕ_s and ϕ_t are unital, $KL(\phi_s) = KL(\phi_t)$. Hence, by [10], ϕ_s and ϕ_t are approximately unitarily equivalent. Hence, $u(s) = \phi_s(u)$ and $u(t) = \phi_t(u)$ have the same spectrum. So we have shown that for $s, t \in (0,1)$, $\sigma(u(s)) = \sigma(u(t))$.

Now note that $u(0) \in \mathbb{M}_{\mathbf{p}} \otimes 1_{\mathbb{M}_{\mathbf{q}}} \subseteq \mathbb{M}_{\mathbf{p}} \otimes \mathbb{M}_{\mathbf{q}}$. Hence, viewing $u(0)$ as being a unitary in $\mathbb{M}_{\mathbf{p}} \otimes \mathbb{M}_{\mathbf{q}}$, we have that $\|u(0) - u(1/n)\| \to 0$ as $n \to \infty$. Hence, by Lemma 3.11 and by the work of the previous paragraph, we have that $\sigma(u(0)) = \sigma(u(t))$ for all $t \in (0,1)$. Similarly, $\sigma(u(1)) = \sigma(u(t))$ for all $t \in (0,1)$. From this and the work of the previous paragraph, $\sigma(u(s)) = \sigma(u(t))$ for all $s, t \in [0,1]$. ∎

LEMMA 3.13. *Let $\mathbf{p}, \mathbf{q}$ be infinite type supernatural numbers which are relatively prime. Let $\{P_k\}_{k=1}^{\infty}$ and $\{Q_k\}_{k=1}^{\infty}$ be sequences of natural numbers which exponentially generate $\mathbf{p}$, $\mathbf{q}$ respectively. Let $\mathcal{Z}_{\mathbf{p},\mathbf{q}} = \varinjlim \mathcal{Z}_{P_k,Q_k}$ be the corresponding inductive limit decomposition of $\mathcal{Z}_{\mathbf{p},\mathbf{q}}$. (See* [30]*.)*

Let $u \in CU(\mathcal{Z}_{\mathbf{p},\mathbf{q}})$ be a nonscalar unitary such that every tracial state on $\mathcal{Z}_{\mathbf{p},\mathbf{q}}$ induces the same tracial state on $C^*(u)$.

Then for every $\varepsilon > 0$, there exists $K \geqslant 1$ such that for all $k \geqslant K$, there is a unitary $u_k \in CU(\mathcal{Z}_{P_k,Q_k})$ with the following properties:

(i) *$\|u_k - u\| \to 0$ as $k \to \infty$.*

(ii) *For all $k \geqslant K$, there exist continuous functions $\gamma_1, \gamma_2, \ldots, \gamma_m : [0,1] \to \mathbb{T}$ (with $P_kQ_k = m$), and a unitary $w \in \mathbb{M}_{P_kQ_k}(C[0,1])$ such that*

$$\|u_k - w\mathrm{diag}(\gamma_1, \gamma_2, \ldots, \gamma_m)w^*\| < \varepsilon \quad \textit{and} \quad 0 < |\gamma_{i-1}(t) - \gamma_i(t)| < \varepsilon$$

for all $t \in [0,1]$, and for $2 \leqslant i \leqslant m$.

Proof. Since $\mathcal{Z}_{\mathbf{p},\mathbf{q}}$ is projectionless,

$$
\begin{aligned}
&\text{we know that } \sigma(u) \text{ must be a connected (and hence path-connected) set.}\\
&\text{Since } u \text{ is not a scalar unitary, } \sigma(u) \text{ must be a nontrivial arc or}\\
&\text{the whole circle. Hence, by Lemma 3.12, we see that } \sigma(u(t)) \text{ is either}\\
&\text{a nontrivial arc or the whole circle, for all } t \in [0,1]. \qquad (3.16)
\end{aligned}
$$

For simplicity, let us assume that $\sigma(u)$ is a nontrivial arc and that $\mathbb{T} - \sigma(u)$ (is a nontrivial arc that) corresponds to the interval (r,s) (i.e., the map $(r,s) \to \mathbb{T} - \sigma(u) : \phi \mapsto e^{i\phi}$ is a homeomorphism).

Since $u \in CU(\mathcal{Z}_{\mathbf{p},\mathbf{q}})$, choose $K \geqslant 1$ such that for all $k \geqslant K$, there exists a unitary in $CU(\mathcal{Z}_{P_k,Q_k})$ which is within $\min\{\varepsilon/100, (s-r)/100\}$ of u. Hence, for all $k \geqslant K$, let $u_k \in CU(\mathcal{Z}_{P_k,Q_k})$ be a unitary such that $\|u_k - u\| < \min\{\varepsilon/100, (s-r)/100\}$ and, as $k \to \infty$,

$$\|u_k - u\| \to 0. \tag{3.17}$$

Now consider $k \geqslant K$. By Lemma 3.8, let $\theta_1, \theta_2, \dots, \theta_m : [0,1] \to \mathbb{R}$ be continuous functions with $\theta_i(t) \leqslant \theta_{i+1}(t)$ for $1 \leqslant i \leqslant m-1$, and $\theta_1(t) \leqslant \theta_m(t) \leqslant \theta_1(t) + 2\pi$ for all $t \in [0,1]$; and let $w \in \mathbb{M}_m(C[0,1])$ be a unitary such that

$$\|u_k - w\mathrm{diag}(e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_m})w^*\| < \min\{\varepsilon/100, (s-r)/100\}. \tag{3.18}$$

(In the above, $m = P_kQ_k$.)

From (3.17), (3.18) and Lemma 3.11, we know that for all $t \in [0,1]$, every element in the spectrum of $w(t)\mathrm{diag}(e^{i\theta_1(t)}, e^{i\theta_2(t)}, \dots, e^{i\theta_m(t)})w(t)^*$ is within $\min\{\varepsilon/50, (s-r)/50\}$ of an element of $\sigma(u(t))$, and vice versa. From this and (3.16) and (3.18), rearranging indices to go in a different direction (say counterclockwise instead of clockwise) and with a different starting point if necessary, we may assume that $|e^{i\theta_j(t)} - e^{i\theta_{j+1}(t)}| < \varepsilon/2$ for all $t \in [0,1]$ and for $1 \leqslant j \leqslant m-1$. Now for $1 \leqslant j \leqslant m$, let $\gamma_j : [0,1] \to \mathbb{T}$ be a continuous map such that $|e^{i\theta_j(t)} - \gamma_j(t)| < \varepsilon/100$ and $\gamma_j(t) \neq \gamma_{j'}(t)$ for all $t \in [0,1]$ if $j \neq j'$. Then $\gamma_1, \gamma_2, \dots, \gamma_m$ have the required properties. ∎

LEMMA 3.14. *Let $\mathbf{p}, \mathbf{q}$ be infinite type supernatural numbers which are relatively prime. Let $u \in CU(\mathcal{Z}_{\mathbf{p},\mathbf{q}})$ be a nonscalar unitary such that every tracial state on $\mathcal{Z}_{\mathbf{p},\mathbf{q}}$ induces the same tracial state on $C^*(u)$.*

Suppose that $\mathcal{A}$ is a unital separable C^*-algebra which contains $\mathcal{Z}_{\mathbf{p},\mathbf{q}}$ as a unital C^*-subalgebra. Suppose that H is a closed subgroup of $CU(\mathcal{A})$ which is normalized by $CU(\mathcal{Z}_{\mathbf{p},\mathbf{q}})$, such that $u \in H$. Then H contains $CU(\mathcal{Z}_{\mathbf{p},\mathbf{q}})$.

Proof. Let $\{P_k\}_{k=1}^\infty$, $\{Q_k\}_{k=1}^\infty$ be sequences of natural numbers which exponentially generate $\mathbf{p}$, $\mathbf{q}$ respectively. Let $\mathcal{Z}_{\mathbf{p},\mathbf{q}} = \varinjlim \mathcal{Z}_{P_k,Q_k}$ be the corresponding inductive limit decomposition of $\mathcal{Z}_{\mathbf{p},\mathbf{q}}$. (See [30].)

It suffices to show that for all $\varepsilon > 0$, for all $v \in \bigcup_{k=1}^{\infty} CU(\mathcal{Z}_{P_k,Q_k})$ such that $\|v-1\| < 1/100$, v is within ε of an element of H. So let $v \in \bigcup_{k=1}^{\infty} CU(\mathcal{Z}_{P_k,Q_k})$ be an arbitrary unitary with $\|v-1\| < 1/100$, and let $\varepsilon > 0$ be given. We may assume that $\varepsilon < 1/100$. We want to show that v is within ε of an element of H.

Suppose that $K \geqslant 1$ is such that $v \in \mathcal{Z}_{P_K,Q_K}$.

Note that since u is nonscalar and by the hypothesis on $C^*(u)$,

we must have that u is noncentral;

in particular, $u(0)$ and $u(1)$ are nonscalar unitaries in

$\mathbb{M}_{\mathbf{p}} \otimes 1_{\mathbb{M}_{\mathbf{q}}}$ and $1_{\mathbb{M}_{\mathbf{p}}} \otimes \mathbb{M}_{\mathbf{q}}$ respectively.

(3.19) Hence, by Lemma 3.5 and since $u \in H$, let $w \in H \cap \mathbb{M}_{P_K,Q_K}$

be such that $\|w-1\| < 1/50, v(0) = w(0)$ and $v(1) = w(1)$.

Hence, $v(0)w(0)^* = 1, v(1)w(1)^* = 1$, and $\|vw^* - 1\| < 1/25$.

Let $v' =_{\mathrm{df}} vw^*$. It suffices to show that v' is within ε of an element of H.

Let $m =_{\mathrm{df}} P_K Q_K$. By Lemma 3.8, let $\gamma_1, \gamma_2, \ldots, \gamma_m : [0,1] \to \mathbb{T}$ be continuous functions, and let $p_1, p_2, \ldots, p_m$ be nonzero pairwise orthogonal projections in $\mathbb{M}_m(C[0,1])$ such that the following hold:

$$\text{(3.20)} \qquad \begin{gathered} \text{(i) } \prod_{i=1}^{m} \gamma_i(t) = 1 \text{ for all } t \in [0,1]; \\ \text{(ii) } \gamma_i(0) = \gamma_i(1) = 1 \text{ for } 1 \leqslant i \leqslant m; \\ \text{(iii) } \left\| \sum_{i=1}^{m} \gamma_i p_i - v' \right\| < \varepsilon/100; \\ \text{(iv) } \left\| \sum_{i=1}^{m} \gamma_i p_i - 1 \right\| < \varepsilon/100 + 1/25. \end{gathered}$$

Using (3.20) statement (i), we construct a finite sequence $\chi_1, \chi_2, \ldots, \chi_{m-1} : [0,1] \to \mathbb{T}$ of continuous functions, with $\chi_i(0) = \chi_i(1) = 1$ for $1 \leqslant i \leqslant m-1$, inductively as follows:

(i) $\chi_1 =_{\mathrm{df}} \gamma_1$.

(ii) For $2 \leqslant i \leqslant m-1$, χ_i is the unique function which satisfies $\gamma_i = \overline{\chi_{i-1}} \chi_i$.

By (3.20) statement (i), we see that $\gamma_m = \overline{\chi_{m-1}}$ and

$$\sum_{i=1}^{m} \gamma_i p_i = \chi_1 p_1 + \sum_{i=2}^{m-1} \overline{\chi_{i-1}} \chi_i p_i + \overline{\chi_{m-1}} p_m.$$

Hence, by (3.19) and (3.20), it suffices to show that

$$\text{(3.21)} \qquad \begin{gathered} \text{for } 1 \leqslant i \leqslant m-1, \chi_i p_i + \overline{\chi_i} p_{i+1} + (1 - (p_i + p_{i+1})) \\ \text{is norm within } \varepsilon/(100m^2 + 1) \text{ of an element of } H. \end{gathered}$$

By Lemmas 3.13 and 3.7, let $L \geqslant K \geqslant 1$, $u_L \in CU(\mathcal{Z}_{P_L,Q_L})$, $\lambda_1, \lambda_2, \ldots, \lambda_n : [0,1] \to \mathbb{T}$ be continuous functions (where $n = P_L Q_L$) and $w' \in CU(\mathbb{M}_{P_L Q_L}(C[0,1]))$ be such that the following hold:

$$\begin{array}{ll} & \text{(i) } \|u - u_L\| < \varepsilon/(100000m^2+1). \\ & \text{(ii) } \prod_{i=1}^{n} \lambda_i(t) = 1 \text{ for } t \in [0,1]. \\ (3.22) & \text{(iii) } \|u_L - w'\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)(w')^*\| < \varepsilon/(100000m^2+1). \\ & \text{(iv) } 0 < |\lambda_{2j-1}(t) - \lambda_{2j}(t)| < 1/1000 \text{ for all } t \in [0,1], \text{ and for } 2 \leqslant 2j \leqslant 2n/m. \\ & \text{(Recall that } m \geqslant 2; \text{ so } n = m(n/m) \geqslant 2(n/m).) \\ & \text{(v) } u_L(0) \text{ and } u_L(1) \text{ are all nonscalar unitaries.} \\ & \text{(Note that } v' \in \mathcal{Z}_{P_L,Q_L} \text{ since } L \geqslant K.) \end{array}$$

Recall that $n = P_L Q_L$. Note that the image of

$$\sum_{i=1}^{m} \gamma_i p_i = \chi_1 p_1 + \sum_{i=2}^{m-1} \overline{\chi_{i-1}} \chi_i p_i + \overline{\chi_{m-1}} p_m$$

in $\mathcal{Z}_{P_L,Q_L}$ is

$$\bigoplus_{j=1}^{n/m} \Big(\sum_{i=1}^{m} \gamma_i p_i \Big) = \bigoplus_{j=1}^{n/m} \Big(\chi_1 p_1 + \sum_{i=2}^{m-1} \overline{\chi_{i-1}} \chi_i p_i + \overline{\chi_{m-1}} p_m \Big).$$

To simplify notation, we rewrite the above expression as

$$\sum_{j=1}^{n/m} \Big(\sum_{i=1}^{m} \gamma_i p_{i,j} \Big) = \sum_{j=1}^{n/m} \Big(\chi_1 p_{1,j} + \sum_{i=2}^{m-1} \overline{\chi_{i-1}} \chi_i p_{i,j} + \overline{\chi_{m-1}} p_{m,j} \Big).$$

Hence, by (3.21), it suffices to show that for $1 \leqslant i \leqslant m-1$,

$$\sum_{j=1}^{n/m} (\chi_i p_{i,j} + \overline{\chi_i} p_{i+1,j}) + (1 - (p_i + p_{i+1})) \tag{3.23}$$

is norm within $\varepsilon/(100m^2+1)$ of an element of H. Without loss of generality, we will prove this statement for the case $i = 1$.

By (3.20) statement (iv), let $\theta : [0,1] \to \mathbb{R}$ be a continuous function such that $\chi_1 = e^{i\theta}$ and $\theta(0) = \theta(1) = 0$. For $2 \leqslant 2j \leqslant 2n/m$, let $\alpha_{2j} : [0,1] \to \mathbb{T}$ be the continuous function that is given by

$$\alpha_{2j} =_{\mathrm{df}} \lambda_{2j-1} \overline{\lambda_{2j}}.$$

By (3.22) statement (iv), we have that for $2 \leqslant 2j \leqslant 2n/m$, for all $t \in [0,1]$,

$$0 < |\alpha_{2j}(t) - 1| < 1/1000.$$

For simplicity, we assume that $\mathrm{Im}(\alpha_{2j}(t)) > 0$ for $t \in [0,1]$ and for $2 \leqslant 2j \leqslant 2n/m$ (otherwise, in the arguments which follow, we can replace α_{2j} by $\overline{\alpha_{2j}}$). Hence, we have that for $1 \leqslant j \leqslant n/m$, for $t \in [0,1]$,

$$\mathrm{Re}(\alpha_{2j}(t)), \mathrm{Im}(\alpha_{2j}(t)), \mathrm{Re}((\alpha_{2j}(t))^2), \mathrm{Im}((\alpha_{2j}(t))^2) > 0. \tag{3.24}$$

Let $\{e_{i,j}\}_{1 \leqslant i,j \leqslant n}$ be a system of matrix units for $\mathbb{M}_n$. By Lemma 3.7, let $x \in CU(\mathbb{M}_n(C[0,1]))$ be the unitary that is given by

$$x = w' \Big(\sum_{2j=2}^{2n/m} (e_{2j-1,2j} + e_{2j,2j-1}) + \Big(1 - \sum_{l=1}^{2n/m} e_l\Big)\Big)(w')^*.$$

Then by (3.22) statements (i) and (iii), we have that

$$\begin{aligned} &\|uxu^*x^* - w'\mathrm{diag}(\alpha_2, \overline{\alpha_2}, \alpha_4, \overline{\alpha_4}, \ldots, \alpha_{2n/m}, \overline{\alpha_{2n/m}}, 1, 1, \ldots, 1)(w')^*\| \\ &\leqslant \|uxu^*x^* - u_L x (u_L)^* x^*\| \\ &\quad + \|u_L x (u_L)^* x^* - w'\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)(w')^* x w' \\ &\qquad\qquad \mathrm{diag}(\overline{\lambda_1}, \overline{\lambda_2}, \ldots, \overline{\lambda_n})(w')^* x^*\| \\ &\quad + \|w'\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)(w')^* x w' \mathrm{diag}(\overline{\lambda_1}, \overline{\lambda_2}, \ldots, \overline{\lambda_n})(w')^* x^* \\ &\quad - w'\mathrm{diag}(\alpha_2, \overline{\alpha_2}, \alpha_4, \overline{\alpha_4}, \ldots, \alpha_{2n/m}, \overline{\alpha_{2n/m}}, 1, 1, \ldots, 1)(w')^*\| \\ &< 2\varepsilon/(100000m^2+1) + 2\varepsilon/(100000m^2+1) + 0 = 4\varepsilon/(100000m^2+1). \end{aligned} \tag{3.25}$$

Hence, $y =_{\mathrm{df}} (w')^* uxu^*x^* w'$ is within $4\varepsilon/(100000m^2+1)$ of

$$\mathrm{diag}(\alpha_2, \overline{\alpha_2}, \alpha_4, \overline{\alpha_4}, \ldots, \alpha_{2n/m}, \overline{\alpha_{2n/m}}, 1, 1, \ldots, 1). \tag{3.26}$$

From (3.26) and Lemma 3.9, let $u', u'' \in CU(\mathbb{M}_n(C[0,1]))_{\mathrm{e}}$ and let $N \geqslant 1$ be an integer such that

$$\begin{aligned} &\Big\| \sum_{2j=2}^{2n/m} \Big(\chi_1 e_{2j-1,2j-1} + \overline{\chi_1} e_{2j,2j}\Big) + \Big(1 - \sum_{l=1}^{2n/m} e_{l,l}\Big) - ((u'')^* u' y (u')^* y^* u'')^N \Big\| \\ &\qquad < 9\varepsilon/(100000m^2+1). \end{aligned} \tag{3.27}$$

(Note that from the proof of Lemma 3.9, we can choose the same N for all the α_{2j}.)

Now by Lemma 3.7, let $w'' \in CU(\mathbb{M}_n(C[0,1]))$ be a unitary such that

$$\begin{aligned} &\sum_{j=1}^{n/m} (\chi_1 p_{1,j} + \overline{\chi_1} p_{2,j}) + (1 - (p_1 + p_2)) \\ &\qquad = w'' \Big(\sum_{2j=2}^{2n/m} (\chi_1 e_{2j-1,2j-1} + \overline{\chi_1} e_{2j,2j}) + \Big(1 - \sum_{l=1}^{2n/m} e_{l,l}\Big)\Big)(w'')^*. \end{aligned}$$

From this and (3.27), we have that

$$\left\| \sum_{j=1}^{n/m} (\chi_1 p_{1,j} + \overline{\chi_1} p_{2,j}) + (1-(p_1+p_2)) - w''((u'')^* u' y (u')^* y^* u'')^N (w'')^* \right\| < 9\varepsilon/(100000m^2+1). \tag{3.28}$$

Since $u'(0) = u'(1) = u''(0) = u''(1) = 1$ and $\chi_1(0) = \chi_1(1) = 1$, let $\delta > 0$ be such that for all $t \in [0,\delta] \cup [1-\delta, 1]$,

$$\begin{aligned} \|u'(t) - 1_{\mathbb{M}_n}\| &< \varepsilon/(100N(1000m^2+1000)), \\ \|u''(t) - 1_{\mathbb{M}_N}\| &< \varepsilon/(100N(1000m^2+1000)), \text{ and} \\ |\chi_1(t) - 1| &< \varepsilon/(100N(1000m^2+1000)). \end{aligned} \tag{3.29}$$

From (3.29), it follows that for $a \in U(\mathbb{M}_n(C[0,1]))$ and for $t \in [0,\delta] \cup [1-\delta, 1]$, we have that

$$\begin{aligned} &\|w''((u'')^* u' a (u')^* a^* u'')^N (w'')^*(t) - 1\| \\ &\leqslant \|w''((u'')^* u' a (u')^* a^* u'')^N (w'')^*(t) - w''(u' a (u')^* a^* u'')^N (w'')^*(t)\| \\ &\quad + \|w''(u' a (u')^* a^* u'')^N (w'')^*(t) - w''(u' a (u')^* a^*)^N (w'')^*(t)\| \\ &\quad + \|w''(u' a (u')^* a^*)^N (w'')^*(t) - w''(a (u')^* a^*)^N (w'')^*(t)\| \\ &\quad + \|w''(a (u')^* a^*)^N (w'')^*(t) - w''(a a^*)^N (w'')^*(t)\| + \|w''(a a^*)^N (w'')^*(t) - 1\| \\ &\leqslant N\|(u'')^*(t) - 1\| + N\|u''(t) - 1\| + N\|u'(t) - 1\| + N\|(u')^*(t) - 1\| + 0 \\ &< 4\varepsilon/(100(1000m^2+1000)). \end{aligned} \tag{3.30}$$

Since $x, w', w'' \in CU(\mathbb{M}_n(C[0,1]))$ and by Lemma 3.7, let $r, s, q \geqslant 1$ be integers and let $x_1, x_2, \ldots, x_r, x_1', x_2', \ldots, x_r', y_1, y_2, \ldots, y_s, y_1', y_2', \ldots, y_s', z_1, z_2, \ldots, z_q, z_1', z_2', \ldots, z_q'$ be elements of $U(\mathbb{M}_n(C[0,1]))$ such that

$$x = \prod_{i=1}^{r} (x_i, x_i'), \quad w' = \prod_{j=1}^{s} (y_j, y_j'), \quad \text{and} \quad w'' = \prod_{k=1}^{q} (z_k, z_k').$$

(Recall that for invertible elements g, g', (g, g') is the commutator $(g, g') =_{\mathrm{df}} g g' g^{-1} (g')^{-1}$.)

Now for $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant s$ and $1 \leqslant k \leqslant q$, let $x_i'', x_i''', y_j'', y_j''', z_k''$ and z_k''' be elements of $U(\mathbb{M}_n(C[0,1]))$ such that:

$$\begin{aligned} &\text{(i) } x_i''(t) = x_i(t) \text{ for } t \in [\delta/100, 1-\delta/100];\\ &\text{(ii) } x_i'''(t) = x_i'(t) \text{ for } t \in [\delta/100, 1-\delta/100];\\ &\text{(iii) } y_j''(t) = y_j(t) \text{ for } t \in [\delta/100, 1-\delta/100];\\ &\text{(iv) } y_j'''(t) = y_j'(t) \text{ for } t \in [\delta/100, 1-\delta/100];\\ &\text{(v) } z_k''(t) = z_k(t) \text{ for } t \in [\delta/100, 1-\delta/100];\\ &\text{(vi) } z_k'''(t) = z_k'(t) \text{ for } t \in [\delta/100, 1-\delta/100];\\ &\text{(vii) } x_i''(0) = x_i''(1) = x_i'''(0) = x_i'''(1) = 1;\\ &\text{(viii) } y_j''(0) = y_j''(1) = y_i'''(0) = y_i'''(1) = 1;\\ &\text{(ix) } z_k''(0) = z_k''(1) = z_k'''(0) = z_k'''(1) = 1. \end{aligned} \tag{3.31}$$

Now let x', w''', w'''' be elements of $CU(\mathbb{M}_n(C[0,1]))_{\mathrm{e}} \subseteq CU(\mathcal{Z}_{P_L,Q_L})$ that are given by

$$x' =_{\mathrm{df}} \prod_{i=1}^{r}(x_i'', x_i'''), \quad w''' =_{\mathrm{df}} \prod_{j=1}^{s}(y_j'', y_j'''), \quad \text{and} \quad w'''' =_{\mathrm{df}} \prod_{k=1}^{q}(z_k'', z_k''').$$

Hence,

$$y' =_{\mathrm{df}} (w''')^* u x' u^* (x')^* w'''$$

is an element of H (since H is normalized by the elements of $CU(\mathcal{Z}_{\mathbf{p},\mathbf{q}})$). For the same reasons the following is an element of H:

$$w''''((u'')^* u' y' (u')^* (y')^* u'')^N (w'''')^*.$$

By reasoning similar to that of (3.30), we have that for all $a \in U(\mathbb{M}_n(C[0,1]))$ and for $t \in [0,\delta] \cup [1-\delta, 1]$,

$$\|w''''((u'')^* u' a (u')^* a^* u'')^N (w'''')^*(t) - 1\| < 4\varepsilon/(100(1000m^2 + 1000)). \tag{3.32}$$

Now consider the quantity $w''''((u'')^* u' y' (u')^* y^* u'')^N (w'''')^*$. For $t \in [\delta/100, 1-\delta/100]$,

$$w''''(t)((u'')^* u' y' (u')^* (y')^* u'')^N(t)(w''''(t))^* = w''(t)((u'')^* u' y (u')^* y^* u'')^N(t)(w''(t))^*.$$

But for $t \in [0, \delta/100] \cup [1-\delta/100, 1]$, we have, from (3.30) and (3.32), that:

$$\begin{aligned} &\|w''''(t)((u''(t))^* u'(t) y'(t)(u'(t))^* y'(t)^* u''(t))^N (w''''(t))^* \\ &\qquad - w''(t)((u''(t))^* u'(t) y(t)(u'(t))^* y(t)^* u''(t))^N (w''(t))^*\| \\ &\leqslant \|w''''(t)((u''(t))^* u'(t) y'(t)(u'(t))^* y'(t)^* u''(t))^N (w''''(t))^* - 1\| \\ &\qquad \|1 - w''(t)((u''(t))^* u'(t) y(t)(u'(t))^* y(t)^* u''(t))^N (w''(t))^*\| \\ &< 8\varepsilon/(100(1000m^2 + 1000)). \end{aligned}$$

Hence, we must have that

$$\left\| w''''((u'')^*u'y'(u')^*(y')^*u'')^N(w'''')^* - w''((u'')^*u'y(u')^*y^*u'')^N(w'')^* \right\| < 2\varepsilon/(25(1000m^2+1000)).$$

From this and (3.28), we have that

$$\left\| \sum_{j=1}^{n/m}(\chi_1 p_{1,j} + \overline{\chi_1} p_{2,j}) + (1-(p_1+p_2)) - w''''((u'')^*u'y'(u')^*(y')^*u'')^N(w'''')^* \right\| < \varepsilon/(100m^2+1).$$

Hence, $\sum_{j=1}^{n/m}(\chi_1 p_{1,j} + \overline{\chi_1} p_{2,j}) + (1-(p_1+p_2))$ is within $\varepsilon/(100m^2+1)$ of an element of H. From this and (3.23), we have completed the proof. Hence, H contains $CU(\mathcal{Z}_{\mathbf{p},\mathbf{q}})$. ∎

4. SIMPLICITY OF $U(\mathcal{Z})/\mathbb{T}$

LEMMA 4.1. *Let $\mathcal{Z}$ be the Jiang–Su algebra. Let $u \in U(\mathcal{Z})$ be a unitary and let $\mathcal{F} \subseteq \mathcal{Z}$ be a finite set. Then for every $\varepsilon > 0$, there exists $v \in CU(\mathcal{Z})$ such that, for all $f \in \mathcal{F}$,*

$$\|ufu^* - vfv^*\| < \varepsilon.$$

This follows from Lemma 1.4 and Remark 1.5 of [5].

THEOREM 4.2. *Let $\mathcal{Z}$ be the Jiang–Su algebra. Then $CU(\mathcal{Z})/\mathbb{T}$ is a simple topological group.*

Proof. Let τ be the unique tracial state on the Jiang–Su algebra $\mathcal{Z}$.

Let $G \subseteq CU(\mathcal{Z})$ be a closed normal subgroup which properly contains the scalar unitaries. We will show that $CU(\mathcal{Z}) \subseteq G$.

Let $\mathbf{p}, \mathbf{q}$ be relatively prime supernatural numbers of infinite type. By Theorem 3.4 of [21], we have the following:

(4.1)
(i) There is a unital injective $*$-homomorphism $\phi : \mathcal{Z}_{\mathbf{p},\mathbf{q}} \to \mathcal{Z}_{\mathbf{p},\mathbf{q}}$ which is trace collapsing (i.e.,$\tau \circ \phi = \tau' \circ \phi$ for all tracial states $\tau, \tau' \in T(\mathcal{Z}_{\mathbf{p},\mathbf{q}})$).
(ii) $\mathcal{Z}$ can be realized as the C^*-inductive limit $\mathcal{Z} = \varinjlim(\mathcal{A}_n, \phi_{n,n+1})$, where for all $n \geqslant 1$, $\mathcal{A}_n \cong \mathcal{Z}_{\mathbf{p},\mathbf{q}}$ and the connecting map $\phi_{n,n+1} = \phi$ (i.e., the connecting map is always the trace-collapsing map ϕ from the previous item).

For all $n \geqslant 1$, let $\phi_n : \mathcal{A}_n \to \mathcal{Z}$ be the map induced by the connecting maps.

Fix $n \geqslant 1$.

We will show that $CU(\phi_n(\mathcal{A}_n)) \subseteq G$. Let $u \in G$ be a nonscalar unitary.
By Corollary 3.2 of [21] (see also [30]), there is a unital
$*$-homomorphism $\psi : \mathcal{Z} \to \mathcal{A}_n$. Hence, $\psi(u)$ is a nonscalar unitary in $\mathcal{A}_n$
(4.2) such that every tracial state on $\mathcal{A}_n$ induces the same
itracial state on $C^*(\psi(u))$. Hence, $\phi_n \circ \psi(u)$ is a nonscalar
unitary in $\phi_n(\mathcal{A}_n) \cong \mathcal{A}_n$ such that every tracial state on $\phi_n(\mathcal{A}_n)$
induces the same tracial state on $C^*(\phi_n \circ \psi(u))$.

Now consider the two unital $*$-endomorphisms (of $\mathcal{Z}$):

$$\mathrm{id}_{\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z} \quad \text{and} \quad \phi_n \circ \psi : \mathcal{Z} \to \mathcal{Z}.$$

It follows, by Theorem 3 of [14], that $\mathrm{id}_{\mathcal{Z}}$ and $\phi_n \circ \psi$ are approximately unitarily equivalent. Hence, let $\{u_k\}_{k=1}^{\infty}$ be a sequence of unitaries in $\mathcal{Z}$ such that for all $a \in \mathcal{Z}$,

$$\|u_k a (u_k)^* - \phi_n \circ \psi(a)\| \to 0.$$

In particular,

$$\|u_k u (u_k)^* - \phi_n \circ \psi(u)\| \to 0.$$

By Lemma 4.1, we may assume that $u_k \in CU(\mathcal{Z})$ for all $k \geqslant 1$. Hence, since $u \in G$ and since G is a normal subgroup of $CU(\mathcal{Z})$, we have that $u_k u (u_k)^* \in G$ for all $k \geqslant 1$. Hence, since G is a closed subgroup of $CU(\mathcal{Z})$, we have that $\phi_n \circ \psi(u) \in G$.

Since $\phi_n \circ \psi(u) \in G$ (which is normalized by $CU(\mathcal{Z})$) and by (4.2) and Lemma 3.14, we see that $CU(\phi_n(\mathcal{A}_n)) \subseteq G$. But $n \geqslant 1$ is arbitrary and G is closed. Hence, $CU(\mathcal{Z}) \subseteq G$ as required. ∎

5. SIMPLICITY OF $\mathrm{Aut}(\mathcal{Z}) \cong \overline{\mathrm{Inn}(\mathcal{Z})}$

THEOREM 5.1. *Let $\mathcal{Z}$ be the Jiang–Su algebra. Then $\overline{\mathrm{Inn}(\mathcal{Z})}$ is a simple topological group.*

Proof. Firstly, by Lemma 4.1, every automorphism in $\overline{\mathrm{Inn}(\mathcal{Z})}$ can be realized as an approximately inner automorphism where the unitaries are in $CU(\mathcal{Z})$.

The rest of the proof follows the argument of Corollary 2.5 of [8]. Let G be a nontrivial closed normal subgroup of $\overline{\mathrm{Inn}(\mathcal{Z})}$. Let

$$H =_{\mathrm{df}} \{u \in CU(\mathcal{Z}) : \mathrm{Ad}(u) \in G\}.$$

Then H is a closed normal subgroup of $CU(\mathcal{Z})$ such that H contains all the scalar unitaries. Since G is nontrivial, let $\beta \in G$ be different from the identity automorphism. Hence, there exists $v \in CU(\mathcal{Z})$ such that $v^* \alpha(v) \notin \mathbb{C}1$. Since

$$(\mathrm{Ad}(v))^{-1} \alpha (\mathrm{Ad}(v)) \alpha^{-1} = \mathrm{Ad}(v^* \alpha(v))$$

it follows that $v^*\alpha(v) \in H$. Therefore, by Theorem 4.2, we have that $H = CU(\mathcal{Z})$ and $G = \overline{\mathrm{Inn}(\mathcal{Z})}$. ∎

6. THE STRUCTURES OF $U(\mathcal{Z})$ AND $\mathrm{Aut}(\mathcal{Z})$

We require a result that is due to Thomsen:

THEOREM 6.1. *Let $\mathcal{A}$ be a unital exact C^*-algebra. Suppose that the natural map*

$$\pi_1(U(\mathcal{A})) \to \pi_1(U_\infty(\mathcal{A})) = K_0(\mathcal{A})$$

is surjective. Then

$$U(\mathcal{A})/CU(\mathcal{A}) \cong K_1(\mathcal{A}) \oplus \mathrm{Aff}(T(\mathcal{A}))/\overline{\rho(K_0(\mathcal{A}))}.$$

In the above, π_1 is the first homotopy group (i.e., the fundamental group), $U_\infty(\mathcal{A}) = \varinjlim U(\mathbb{M}_n(\mathcal{A}))$ is the infinite unitary group of $\mathcal{A}$, $\rho : K_0(\mathcal{A}) \to \mathrm{Aff}(T(\mathcal{A}))$ is the natural map, and $\overline{\rho(K_0(\mathcal{A}))}$ is the closure of $\rho(K_0(\mathcal{A}))$ in $\mathrm{Aff}(T(\mathcal{A}))$.

This follows from Corollary 3.3 and Corollary 3.4 of [24].

Next we need a lemma about the fundamental group of the unitary group of the Jiang–Su algebra:

LEMMA 6.2. *Let $\mathcal{Z}$ be the Jiang–Su algebra. Then the natural map $\pi_1(U(\mathcal{Z})) \to K_0(\mathcal{Z})$ is surjective. Here, π_1 is the first homotopy group (fundamental group).*

Proof. This lemma actually follows from [13]. However, since [13] is not published, we present a short proof for the convenience of the reader.

By [14], $\mathcal{Z}$ can be decomposed as a C^*-inductive limit $\mathcal{Z} = \varinjlim(\mathcal{A}_n, \phi_{n,n+1})$ such that:

$$\begin{aligned} &\text{(i) each } \mathcal{A}_n \text{ is a prime dimension drop algebra}\\ \text{(6.1)}\quad &\text{(and hence, } K_0(\mathcal{A}_n) = \mathbb{Z}\text{);}\\ &\text{(ii) each map } \phi_{n,n+1} \text{ induces a unital injective map } K_0(\phi_{n,n+1}) : K_0(\mathcal{A}_n)\\ &\to K_0(\mathcal{A}_{n+1}) \text{ (and hence, } K_0(\phi_{n,n+1}) \text{ is an automorphism of } \mathbb{Z}\text{).} \end{aligned}$$

For each $n \geqslant 1$, by (6.1) statement (i), the natural map $\pi_1(U(\mathcal{A}_n)) \to K_0(\mathcal{A}_n)$ is surjective. From this, the naturality of the map $\pi_1(U(\mathcal{A}_n)) \to K_0(\mathcal{A}_n)$, and (6.1) statement (i), we see that the natural map $\pi_1(U(\mathcal{Z})) \to K_0(\mathcal{Z})$ is surjective. ∎

THEOREM 6.3. *Let $\mathcal{Z}$ be the Jiang–Su algebra. Then we have the following:*

(i) *$CU(\mathcal{Z})/\mathbb{T}$ is a simple topological group.*

(Here, $\mathbb{T}$ is the scalar multiples of the identity, where the scalars have absolute value one. By Lemma 2.1*, $\mathbb{T}$ is a subgroup of $CU(\mathcal{Z})$.)*

(ii) *By* [24], *there is an exact sequence*

$$\{1\} \to CU(\mathcal{Z}) \to U(\mathcal{Z}) \to \mathbb{T} \to \{1\}.$$

This, together with (i)*, give the topological normal subgroup structure of* $U(\mathcal{Z})$*. (We note that the copy of the circle* $\mathbb{T}$ *in* (ii) *is different from the copy of* $\mathbb{T}$ *in* (i)*.)*

(iii) $\mathrm{Aut}(\mathcal{Z}) = \overline{\mathrm{Inn}(\mathcal{Z})}$ *is a simple topological group.*

Proof. Proof of (i). This follows from Theorem 4.2.

Proof of (ii). By Lemma 6.2, the natural map $\pi_1(U(\mathcal{Z})) \to K_0(\mathcal{Z})$ is surjective. Hence, by Theorem 6.1, we have an exact sequence

$$\{1\} \to CU(\mathcal{Z}) \to U(\mathcal{Z}) \to \mathrm{Aff}(T(\mathcal{Z}))/\overline{K_0(\mathcal{Z})} \to \{1\}.$$

(Note that $K_1(\mathcal{Z}) = \{0\}$.) Since $\mathcal{Z}$ has unique trace and since $K_0(\mathcal{Z}) = \mathbb{Z}$, we have that $\mathrm{Aff}(T(\mathcal{Z}))/\overline{K_0(\mathcal{Z})} \cong \mathbb{T}$.

Proof of (iii). That $\mathrm{Aut}(\mathcal{Z}) = \overline{\mathrm{Inn}(\mathcal{Z})}$ follows from Theorem 7.6 of [14]. That $\overline{\mathrm{Inn}(\mathcal{Z})}$ is simple follows from Theorem 5.1. ∎

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