J. OPERATOR THEORY
65:1(2011), 187–195

NOTES ABOUT WEAKLY HYPERCYCLIC OPERATORS

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Communicated by Şerban Strătilă

ABSTRACT. The present article gives a brief discussion about operators which are weakly hypercyclic and answers the following three questions:
(i) Must $T \oplus T$ be weakly hypercyclic whenever T is?
(ii) Is T^n weakly hypercyclic for every $n \in \mathbb{N}$ whenever T is?
(iii) Is λT weakly hypercyclic for all $|\lambda| = 1$ whenever T is?
Question (i) was explicitly posed by Chan and Sanders.

KEYWORDS: *Hypercyclic operators, weakly hypercyclic operators, direct sums of weakly hypercyclic operators, rotations of weakly hypercyclic operators.*

MSC (2000): Primary 47A16; Secondary 47A15, 47A05, 47A11, 46A03.

INTRODUCTION

The set of integers is denoted by $\mathbb{Z}$. The set of positive integers without the element zero is denoted by $\mathbb{N}$, when this set does contain the element zero, then it is denoted by $\mathbb{N}_0$. The fields of rational numbers and complex numbers are denoted by $\mathbb{Q}$ and $\mathbb{C}$ respectively. Let $(X, \|\cdot\|)$ be an infinite-dimensional normed space over $\mathbb{C}$. A continuous linear operator $T : X \to X$ is *hypercyclic* if there exists a vector $x \in X$ such that its *orbit*, that is the set $\mathrm{Orb}(T, x) = \{x, T(x), T^2(x), T^3(x), \ldots\}$, is a dense set in X. An operator T is called *supercyclic* if there exists a vector $x \in X$ such that $\{\alpha T^n x : \alpha \in \mathbb{C}, n \in \mathbb{N}_0\}$ is a dense set in X. In each case, such a vector x is called a *hypercyclic vector* for T and a *supercyclic vector* for T. An operator $T : X \to X$ is called *weakly hypercyclic* if there exists a vector x such that its orbit is weakly dense. Similarly, an operator $T : X \to X$ is called *weakly supercyclic* if there exists a vector x such that $\{\alpha T^n x : \alpha \in \mathbb{C}, n \in \mathbb{N}_0\}$ is weakly dense. The concept weakly dense means dense with respect to the weak topology. Recall that the *weak topology of* X, denoted by $\sigma(X, X^*)$, is the smallest topology for the space such that every member of the dual space X^* is continuous with respect to that topology. Finally, an operator $T : X \to X$ is *cyclic* if there exists a vector x such that the linear span of its orbit, denoted by $\mathrm{span}\big(\mathrm{Orb}(T, x)\big)$ is dense; such a vector x is called a *cyclic vector* for T.

We could say that an operator $T : X \to X$ is *weakly cyclic* if there exists a vector x such that the linear span of its orbit is weakly dense. However S. Mazur [13] in 1933 proved that the closure and the weak closure agree on convex subsets of a norm space, thus weak cyclicity and cyclicity are exactly the same. It is obvious from the definition that for either the norm topology or the weak topology, hypercyclicity implies supercyclicity, which in turn implies cyclicity.

The study of weak orbits was introduced by J. van Neerven [15] in 1996. Important contributions to weak hypercyclicity are due to K. Chan and R. Sanders [6], [17]. Also V. Müller [14], S. Dilworth, V. Troitsky [8] and G. Prăjitură [16], have shown significant results in this area.

This paper is divided in three sections. Section 1 gives a small survey of some properties that weakly hypercyclic operators share with hypercyclic ones.

Section 2 presents the three main results of this paper: in Theorem 2.1 it is shown that there exists an operator T on $\ell^p(\mathbb{N})$ $(1 \leqslant p < \infty)$ whose direct sum $T \oplus T$ acting on $\ell^p(\mathbb{N}) \oplus \ell^p(\mathbb{N})$ is not weakly hypercyclic, answering in the negative a question posed by Chan and Sanders [6]. Also it is proved in Theorems 2.4 and 2.8 that the operators T^n for all $n > 1$ and λT for all complex numbers λ with $|\lambda| = 1$ are weakly hypercyclic provided the operator T is weakly hypercyclic.

The results given in Section 1 and 2 (for weakly hypercyclic operators) are all satisfied by hypercyclic operators, so the reader may ask if weak hypercyclicity and hypercyclicity are the same. In Section 3, it is shown that even when the class of hypercyclic operators shares many properties with the class of weakly hypercyclic operators, these two classes do not coincide.

1. SIMILAR RESULTS

It follows immediately from the definitions that every hypercyclic vector for a continuous linear operator T on X is automatically a cyclic vector for T. The same applies to a weakly hypercyclic vector. Hypercyclic operators always have an invariant, norm dense, linear subspace in which every nonzero vector is hypercyclic. The complex scalar case of this result was established by Herrero ([10], Proposition 4.1) and independently by Bourdon [4]. The real scalar case was established by Bès [3]. Chan and Sanders [6] showed that the same result holds for weakly hypercyclic operators.

PROPOSITION 1.1 (Chan and Sanders 2004). *Let T be a continuous linear operator on X. Then T is a weakly hypercyclic operator if and only if there is an invariant, norm dense, linear subspace in which every nonzero vector is a weakly hypercyclic vector for T.*

C. Kitai [11] showed that every component of the spectrum of a hypercyclic operator intersects the unit circle. S.J. Dilworth and V.G. Troitsky [8] showed that every component of the spectrum of a weakly hypercyclic operator also intersects the unit circle.

The spectral properties of the operators are not much different in the case of weak density. In his paper [16], Prăjitură did for weakly hypercyclic operators what Herrero did for hypercyclic operators. He listed some spectral properties of weakly hypercyclic operators and he used them to prove that the set of hypercyclic operators and weakly hypercyclic have the same interior and the same closure (in the norm topology).

In the following, $\sigma(T)$ denotes the spectrum of T and $\sigma_{\mathrm{p}}(T)$ is the *point spectrum* of T, that is, the collection of complex numbers α such that the linear operator $T-\alpha$ is not injective. The semi Fredholm domain of an operator T will be denoted by $\rho_{\mathrm{sF}}(T)$ and for $\lambda \in \rho_{\mathrm{sF}}(T)$, $\mathrm{ind}(\lambda - T)$ will stand for the semi Fredholm index of $\lambda - T$. Lastly, the *Weyl spectrum* of an operator T is the set $\sigma_{\mathrm{W}}(T) = \sigma(T) \setminus \{\lambda \in \rho_{\mathrm{sF}}(T) : \mathrm{ind}(\lambda - T) = 0\}$.

THEOREM 1.2 ([16]). *Let H be a Hilbert space and let $T : H \to H$ be a weakly hypercyclic operator. Then:*

(i) *for every invariant subspace M of T the compression of T to the orthogonal complement of M is weakly hypercyclic on the space $M^{\perp}$;*

(ii) $\sigma_{\mathrm{p}}(T^*) = \emptyset$*;*

(iii) $\mathrm{ind}(\lambda - T) = 0$ *for every* $\lambda \in \rho_{\mathrm{sF}}(T)$*;*

(iv) $\sigma_{\mathrm{W}}(T) = \sigma(T)$*;*

(v) $\sigma_{\mathrm{W}}(T) \cup \mathbb{T}$ *is a connected set.*

COROLLARY 1.3 ([16]). *Every weakly hypercyclic operator is the limit of hypercyclic operators.*

PROPOSITION 1.4 ([16]). *The operators which are not weakly hypercyclic are dense in $B(H)$.*

2. SIMILAR QUESTIONS

A *Fréchet* space is a complete, metrizable, locally convex topological vector space. Given an infinite-dimensional separable Fréchet space X, there exist sufficient conditions that guarantee a linear operator $T : X \to X$ to be hypercyclic. These conditions are contained in the Hypercyclicity Criterion (see Section 3). An interesting and long-standing problem in the hypercyclicity theory was to know if every hypercyclic operator satisfies such conditions. After many attempts trying to solve this problem, it turned out that such a problem is equivalent to a question posed by D. Herrero: must $T \oplus T$ on $X \oplus X$ be hypercyclic whenever T is hypercyclic? Motivated by this question, Chan and Sanders ([6], Question 5.2) posed a similar problem but for weakly hypercyclic operators. It will be shown in the following theorem that the answer is in the negative.

THEOREM 2.1. *Let X be either $\ell^p(\mathbb{N})$ $(1 \leqslant p < \infty)$ or c_0. Then there exists a weakly hypercyclic operator T on X whose direct sum $T \oplus T$ is not weakly hypercyclic.*

Proof. De la Rosa and Read [7] constructed a Banach space X and hypercyclic operators on X (called maximal operators) whose direct sum is not hypercyclic. Based on this construction Matheron and Bayart [2] showed that such operators also exist on classical Banach $\ell^p(\mathbb{N})$ spaces.

A few years earlier, when Sophie Grivaux [9], like many others, was in the quest for an answer to the great problem (or equivalent Herrero's problem), she found that given a hypercyclic operator T, $T \oplus T$ is hypercyclic if and only if $T \oplus T$ is cyclic. We therefore have that our operator [7], in particular, is a counterexample for cyclic, supercyclic, weakly supercyclic and weakly hypercyclic operators. That is, T belongs to all these classes since it is hypercyclic but its direct sum does not. ∎

Another great result in hypercyclicity is the fact that when T is a hypercyclic operator, then T^n is hypercyclic for all $n \in \mathbb{N}$. This result was proved by Ansari [1] in 1995. A few years later, Bourdon and Feldman [5] proved that somewhere dense orbits are dense orbits, and as a corollary they obtained the result of Ansari.

THEOREM 2.2 (Bourdon and Feldman, 2003). *Let X be a locally convex space and let T be a continuous linear operator on X. Suppose that $x \in X$ is such that* $\operatorname{Orb}(T, x)$ *is somewhere dense, then* $\operatorname{Orb}(T, x)$ *is dense in X.*

COROLLARY 2.3 (Bourdon and Feldman, 2003). *Let X be a locally convex space and let T be a continuous linear operator on X. T^n is hypercyclic for all $n \in \mathbb{N}$ whenever T is hypercyclic.*

THEOREM 2.4. *Let X be a normed space, and let T be a continuous linear operator on X. Then T^n is weakly hypercyclic for all $n \in \mathbb{N}$ whenever T is weakly hypercyclic.*

Proof. Fix $n \in \mathbb{N}$ and let T be a continuous linear operator on X. We know that T is norm to norm continuous if and only if it is weak to weak continuous, so T is a continuous operator on the locally convex space $(Y, \mathscr{T}) = (X, \sigma(X, X^*))$, where $\sigma(X, X^*)$ is the weak topology on X. Suppose that T is weakly hypercyclic on X, or equivalently that T is hypercyclic on $(Y, \mathscr{T})$.

Observe that

$$\operatorname{Orb}(T, x) = \bigcup_{k=1}^{n} \operatorname{Orb}(T^n, T^k x) = \bigcup_{k=1}^{n} \overline{\operatorname{Orb}(T^n, T^k x)} = Y.$$

Therefore, by a basic result of general topology, there exists $k \in \mathbb{N}$ with $1 \leqslant k \leqslant n$ such that $\operatorname{Orb}(T^n, T^k x)$ is somewhere dense, so it must be dense in Y. That is, T^n is weakly hypercyclic on X and the operator T^n has the same weakly hypercyclic vectors for T since T^{n-k} is a continuous function with dense range on Y and

$$T^{n-k}(\operatorname{Orb}(T^n, T^k x)) = \{T^n x, T^{2n} x, T^{3n} x, \dots\} \subset \operatorname{Orb}(T^n, x). \quad ∎$$

León-Saavedra and Müller [12] showed that λT hypercyclic for all complex number λ with $|\lambda| = 1$, whenever T is hypercyclic. In order to translate this result to the class of weakly hypercyclic operators, it is necessary to prove that the result of León-Saavedra and Müller holds not only for Banach spaces, but also for locally convex spaces.

The reader should remember that the most common way of defining locally convex topologies on vector spaces is in terms of seminorms on X. If X is a vector space, I an index set and $\{p_\beta\}_{\beta\in I}$ is a family of seminorms on X, then the convex sets

$$U_{x,\beta,\varepsilon} = \{y \in X : p_\beta(x-y) < \varepsilon\} \quad (x \in X, \beta \in I, \varepsilon > 0)$$

generate a topology on X with respect to which X is a locally convex topological vector space. Notice that for each $x \in X$, the finite intersections of the sets $U_{x,\beta,\varepsilon}$ $(\beta \in I, \varepsilon > 0)$ form a neighborhood base at x; a net $\{x_\alpha\}_\alpha$ in a locally convex space X converges to $x \in X$ if and only if $p_\beta(x_\alpha) \to p_\beta(x)$ for all $\beta \in I$; and lastly an operator $T : X \to X$ is continuous if and only if for each β, there exist $\beta_1, \beta_2, \ldots, \beta_m$ in I and $C > 0$ such that

$$p_\beta(Tx) \leqslant C \sum_{k=1}^{m} p_{\beta_k}(x) \quad \text{for all } x \in X.$$

The proof of the following proposition is a slight modification of the original proof given by León-Saavedra and Müller. Indeed, only the two first parts (of the original version) were modified here in order to extend the result to locally convex spaces. Due to that fact, the author presents only these two parts. The reader can find the rest of the proof in [12].

In the following proposition, $\mathbb{D} = \{z \in \mathbb{C} : |z| \leqslant 1\}$ is the *closed unit disc* and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the *unit circle*. The *cardinality of a set* A is denoted by $\#A$, the *closure of a set* A is denoted by $\overline{A}$ and its *interior* by $\mathrm{Int}(A)$.

PROPOSITION 2.5. *Let X be a locally convex space, let $\mathfrak{F}$ be a semigroup of continuous linear operators on X and let $x \in X$ be such that the set*

$$\{\mu S x : S \in \mathfrak{F}, \mu \in \mathbb{C}, |\mu| = 1\}$$

is dense in X. Suppose that there is a continuous linear operator T on X with $\sigma_{\mathrm{p}}(T^*) = \emptyset$ satisfying $TS = ST$ for each $S \in \mathfrak{F}$. Then the set $\{Sx : S \in \mathfrak{F}\}$ is dense in X.

Proof. For each $x \in X$, set

$$M_x = \overline{\{Sx : S \in \mathfrak{F}\}}.$$

For x and y in X set

$$F_{x,y} = \{\mu \in \mathbb{T} : \mu y \in M_x\}.$$

Note that $F_{x,y}$ is a closed subset of the unit circle. Let X_0 be the set of all vectors x such that $\{\mu S x : S \in \mathfrak{F}, \mu \in \mathbb{C}, |\mu| = 1\}$ is dense in X.

The proof will be done in several steps.

Step 1. *If x is in X_0, then $F_{x,y} \neq \emptyset$ for all $y \in X$.*

Since the set $\{\mu Sx : S \in \mathfrak{F}, \mu \in \mathbb{T}\}$ is dense, there exists a net $\{\mu_\alpha S_\alpha\}_\alpha$ with $\mu_\alpha \in \mathbb{T}$ and $S_\alpha \in \mathfrak{F}$ such that $\mu_\alpha S_\alpha \to y$. Passing to a subnet if necessary, we can suppose that $\mu_\alpha \to \mu$ for some $\mu \in \mathbb{T}$.

Let $\{p_\beta\}_{\beta \in I}$ be a family of seminorms which generates the locally convex topology on X. Then for all $\beta \in I$ we have

$$p_\beta(S_\alpha x - \mu^{-1}y) \leqslant p_\beta(S_\alpha x - \mu_\alpha^{-1}y) + p_\beta((\mu_\alpha^{-1} - \mu^{-1})y) \to 0.$$

Therefore $\mu^{-1} \in F_{x,y}$.

Step 2. *If x, y and w are in X, $\mu_1 \in F_{x,y}$ and $\mu_2 \in F_{y,w}$, then $\mu_1\mu_2 \in F_{x,w}$.*

Note that $\mu \in F_{x,y}$ if and only if $\mu y \in M_x = \overline{\{Sx : S \in \mathfrak{F}\}}$. That is, if and only if for every $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n > 0$ and $\beta_1, \beta_2, \ldots, \beta_n \in I$ there exists $S \in \mathfrak{F}$ such that

$$p_{\beta_i}(Sx - \mu y) < \varepsilon_i \quad \text{for all } i = 1, 2, \ldots, n.$$

Let $\mu_1 \in F_{x,y}$, $\mu_2 \in F_{y,w}$, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n > 0$ and let $\beta_1, \beta_2, \ldots, \beta_n \in I$.

Since $\mu_2 \in F_{y,w}$, there exists $S_1 \in \mathfrak{F}$ such that

$$p_{\beta_i}(S_1 y - \mu_2 w) < \frac{\varepsilon_i}{2} \quad \text{for all } i = 1, 2, \ldots, n.$$

The continuity of S implies that for each β_i $(1 \leqslant i \leqslant n)$, there exist $\beta_1^i, \beta_2^i, \ldots, \beta_{m_i}^i$ in I and $C_i > 0$ such that

$$p_{\beta_i}(Sx) \leqslant C_i \sum_{k=1}^{m_i} p_{\beta_k^i}(x) \quad \text{for all } x \in X.$$

Since $\mu_1 \in F_{x,y}$, there exists $S_2 \in \mathfrak{F}$ such that for every $k = 1, 2, \ldots, m_i$, we have

$$p_{\beta_k^i}(S_2 x - \mu_1 y) < \frac{\varepsilon_i}{2C_i m_i}.$$

Then for $i = 1, 2, \ldots, n$ we have

$$\begin{aligned} p_{\beta_i}(S_1 S_2 x - \mu_1\mu_2 w) &\leqslant p_{\beta_i}(S_1(S_2 x - \mu_1 y)) + p_{\beta_i}(\mu_1(S_1 y - \mu_2 w)) \\ &\leqslant C_i \sum_{k=1}^{m_i} p_{\beta_k^i}(S_2 x - \mu_1 y) + p_{\beta_i}(S_1 y - \mu_2 w) \\ &\leqslant C_i \sum_{k=1}^{m_i} \frac{\varepsilon_i}{2C_i m_i} + \frac{\varepsilon_i}{2} = \varepsilon_i. \end{aligned}$$

Hence $\mu_1\mu_2 \in F_{x,w}$.

The reader can find the the rest of the proof in [12]. ∎

COROLLARY 2.6. *Let X be a locally convex space and let T be a continuous linear operator on X. A vector $x \in X$ is hypercyclic for T if and only if $\{\mu T^n x : \mu \in \mathbb{T}, n \in \mathbb{N}_0\}$ is dense in X.*

Proof. If $x \in X$ is hypercyclic for T, then $\{T^n x : n \in \mathbb{N}_0\}$ is dense in X and

$$\{T^n x : n \in \mathbb{N}_0\} \subset \{\mu T^n x : \mu \in \mathbb{T}, n \in \mathbb{N}_0\}.$$

On the other hand, let $x \in X$ be such that the set $\{\mu T^n x : \mu \in \mathbb{T}, n \in \mathbb{N}_0\}$ is dense in X. Set $\mathfrak{F} = \{T^n : n \in \mathbb{N}_0\}$, by previous proposition, it is sufficient to prove that $\sigma_{\mathrm{p}}(T^*) = \emptyset$.

Assume towards a contradiction that $\alpha \in \sigma(T^*)$ and let $f \in X^*$ be the corresponding eigenvector. Then

$$\begin{aligned}\mathbb{C} &= \overline{\{\langle \mu T^n x, f\rangle : \mu \in \mathbb{T}, n \in \mathbb{N}_0\}} = \overline{\{\langle \mu x, \alpha^n f\rangle : \mu \in \mathbb{T}, n \in \mathbb{N}_0\}}\\ &= \langle x, f\rangle \overline{\{\mu\alpha^n : \mu \in \mathbb{T}, n \in \mathbb{N}_0\}},\end{aligned}$$

which is a contradiction since if $|\alpha| < 1$ or $\langle x, f\rangle = 0$, then the last set is bounded and therefore nondense in $\mathbb{C}$. If $|\alpha| \geqslant 1$ and $\langle x, f\rangle \neq 0$, then the last set is bounded below, and therefore nondense in $\mathbb{C}$. Thus $\sigma_{\mathrm{p}}(T^*) = \emptyset$. ∎

COROLLARY 2.7. *Let X be a locally convex space and let T be a hypercyclic operator on X. If $\mu \in \mathbb{T}$, then μT is hypercyclic and has the same set of hypercyclic vectors as T.*

Proof. Given $\mu \in \mathbb{T}$, the operator μT is hypercyclic if and only if

$$\{\lambda(\mu T)^n x : \lambda \in \mathbb{T}, n \in \mathbb{N}_0\}$$

is dense, but $\{\lambda(\mu T)^n x : \lambda \in \mathbb{T}, n \in \mathbb{N}_0\} = \{\lambda \mu^n T^n x : \lambda \in \mathbb{T}, n \in \mathbb{N}_0\}$ which is equal to $\{\lambda T^n x : \lambda \in \mathbb{T}, n \in \mathbb{N}_0\}$, and the operator T is hypercyclic if and only if $\{\lambda T^n x : \lambda \in \mathbb{T}, n \in \mathbb{N}_0\}$ is dense. ∎

THEOREM 2.8. *Let X be a normed space and let T be a weakly hypercyclic operator on X. Then μT is weakly hypercyclic for all $\mu \in \mathbb{T}$ and has the same set of weakly hypercyclic vectors as T.*

Proof. Let $\mu \in \mathbb{T}$ and let T be a weakly hypercyclic operator on X. The operator T is norm continuous if and only if it is weakly continuous, so T is a hypercyclic operator on the locally convex space $(Y, \mathscr{T}) = (X, \sigma(X, X^*))$. By previous corollary μT is hypercyclic on Y and has the same set of hypercyclic vectors as T. However μT is hypercyclic on Y if and only if μT is weakly hypercyclic on X. ∎

3. SIMILAR BUT NOT EQUAL

In this section *all the examples and the observations were made by Chan and Sanders* and they can be found in their article "A weakly hypercyclic operator that is not norm hypercyclic" [6]. The author includes this section for completeness of these notes.

As we have seen, the class of weakly hypercyclic operators shares many properties with the class of hypercyclic operators. However these two classes are not the same, indeed, there exist weakly hypercyclic operators on $\ell^p(\mathbb{Z})$ with

$2 \leqslant p < \infty$ that fail to be norm hypercyclic. The operator $T : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ given by

$$Te_\alpha = \begin{cases} e_{\alpha-1} & \text{if } \alpha \leqslant 0, \\ 2e_{\alpha-1} & \text{if } \alpha \geqslant 1, \end{cases}$$

is one of those operators.

THE HYPERCYCLICITY CRITERION. *Let X be a separable Banach space and let T be a continuous linear operator on X. If there are dense subsets X_0 and Y_0 of X and an increasing sequence of natural numbers $(n_k)_k$ and maps $S_k : Y_0 \to X$, $k \in \mathbb{N}$ such that:*
(i) $T^{n_k} S_k y \to y \ \forall y \in Y_0$,
(ii) $S_k y \to 0 \ \forall y \in Y_0$,
(iii) $T^{n_k} x \to 0 \ \forall x \in X_0$,
then T is hypercyclic.

It makes sense to ask what would happen if we replace in the Hypercyclicity Criterion the norm topology by the weak topology, and the answer is that, unfortunately, the resulting statement fails to hold.

EXAMPLE 3.1. Let $X = \ell^2$ and define $T : X \to X$ by

$$T(x_0, x_1, x_2, \ldots) = (x_1, x_2, x_3, \ldots).$$

If we take $X_0 = Y_0 = \operatorname{span}\{e_n : n \in \mathbb{N}_0\}$ where $\{e_n\}$ is the canonical basis on X and we define $S : Y_0 \to Y_0$ by $S(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, x_2, \ldots)$. Then:
(i) $TS = I$ on Y_0,
(ii) $T^n x \to 0$ weakly for all $x \in X_0$ and
(iii) $S_n y \to 0$ weakly for all $y \in Y_0$.
However, since $\|T\| = 1$, T cannot be weakly hypercyclic.

Another property that weakly hypercyclic operators do not share with hypercyclic operators is the following: given an invertible operator T, T^{-1} is hypercyclic if and only if T is hypercyclic. However this is not the case for weakly hypercyclic operators since there exists an invertible weakly hypercyclic operator whose inverse fails to be weakly hypercyclic (see Corollary 3.6 of [6]).

Acknowledgements. The author would like to thank the referee for helpful suggestions. Supported by CONACYT-MEXICO, Grant number 196679.

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Received June 10, 2008.