

ON ALGEBRAS GENERATED BY INNER DERIVATIONS

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ABSTRACT. We look for an effective description of the algebra $D_{\text{Lie}}(\mathcal{X}, B)$ of operators on a bimodule \mathcal{X} over an algebra B , generated by all operators $x \rightarrow ax - xa$, $a \in B$. It is shown that in some important examples $D_{\text{Lie}}(\mathcal{X}, B)$ consists of all elementary operators $x \rightarrow \sum_i a_i x b_i$ satisfying the conditions $\sum_i a_i b_i = \sum_i b_i a_i = 0$. The Banach algebraic versions of these results are also obtained and applied to the description of closed Lie ideals in some Banach algebras, and to the proof of a density theorem for Lie algebras of operators on Hilbert space.

KEYWORDS: *Banach bimodule, Lie submodule, inner derivation, projective tensor product, Varopoulos algebra.*

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INTRODUCTION

Let B be an algebra. A subspace C of B is called a Lie ideal of B if $ax - xa \in C$ for all $a \in B$, $x \in C$. The structure of Lie ideals of an associative algebra attracted the attention of algebraists and Banach-algebraists since seminal works of Herstein and Jacobson–Rickart [8], [9]. A more general subject is the structure of *Lie submodules* in an arbitrary B -bimodule \mathcal{X} — the subspaces of \mathcal{X} , defined in formally identical way. For example Lie ideals of each algebra that contains B as a subalgebra are Lie submodules over B . (This example is in fact most general: if \mathcal{X} is a B -bimodule and \mathcal{Y} is a Lie submodule of \mathcal{X} then one can introduce a product $(a_1 \oplus x_1)(a_2 \oplus x_2) = a_1 a_2 \oplus (a_1 x_2 + x_1 a_2)$ in $A = B \oplus \mathcal{X}$ and identify \mathcal{Y} with a Lie ideal $0 \oplus \mathcal{Y}$ of A).

If one denotes by L_a and R_a respectively the operators of the left and right multiplication by a , then one can say that Lie submodules are invariant subspaces for the set $\Xi(\mathcal{X}, B)$ of all *inner derivations* $\delta_a = L_a - R_a$, $a \in B$.

Note that the structure of the algebra \mathfrak{A} , generated by some set \mathcal{P} of operators on a linear space \mathcal{X} , gives useful information about the invariant subspaces

of \mathcal{P} . It suffices to say that all invariant subspaces are sums of cyclic subspaces $\mathfrak{A}x, x \in \mathcal{X}$. Of course, the usefulness of such information depends on the clarity of the description of \mathfrak{A} .

Thus trying to describe the structure of Lie ideals of an algebra B and Lie submodules of bimodules over B , it is natural to consider the algebras $\mathcal{D}_{\text{Lie}}(\mathcal{X}, B)$, generated by the sets $\Xi(\mathcal{X}, B)$ for various algebras B and bimodules \mathcal{X} . We will see that in some important cases these algebras can be described effectively: they coincide with the algebras $\mathcal{M}_{\text{Lie}}(\mathcal{X}, B)$ of all elementary operators $\sum_{k=1}^n L_{a_k} R_{b_k}$ on \mathcal{X} , satisfying the conditions

$$(0.1) \quad \sum a_k b_k = 0,$$

$$(0.2) \quad \sum b_k a_k = 0.$$

The algebra of all elementary operators on a bimodule \mathcal{X} is denoted by $El(\mathcal{X}, B)$. For the most popular case $\mathcal{X} = B$, we write $El(B), \mathcal{D}_{\text{Lie}}(B)$ and $\mathcal{M}_{\text{Lie}}(B)$ instead of $El(B, B), \mathcal{D}_{\text{Lie}}(B, B)$ and $\mathcal{M}_{\text{Lie}}(B, B)$.

It is convenient and interesting to consider the corresponding problems in tensor algebras. Let B be a unital algebra and B^{op} denote the opposite algebra to B (that is the same linear space with the reverse multiplication: $a*b = ba$). Then each bimodule \mathcal{X} over B can be considered as a module over the tensor product $B \otimes B^{\text{op}}$, and the map $a \otimes b \rightarrow L_a R_b$ extends to a surjective homomorphism of $B \otimes B^{\text{op}}$ onto the algebra of all elementary operators in \mathcal{X} . It sends the elements of the form $a \otimes 1 - 1 \otimes a$ to the inner derivations δ_a . Let $\mathcal{T}_{\text{Lie}}(B)$ be the algebra, generated by all elements of the form $a \otimes 1 - 1 \otimes a$, and $\mathcal{N}_{\text{Lie}}(B)$ the algebra of all tensors $\sum_k a_k \otimes b_k$ satisfying (0.1) and (0.2). If one proves that

$$(0.3) \quad \mathcal{T}_{\text{Lie}}(B) = \mathcal{N}_{\text{Lie}}(B)$$

then one obtains the equality

$$(0.4) \quad \mathcal{D}_{\text{Lie}}(\mathcal{X}, B) = \mathcal{M}_{\text{Lie}}(\mathcal{X}, B)$$

for all B -bimodules \mathcal{X} .

We consider also the Banach-algebraic versions of the problems. If B is a Banach algebra and \mathcal{X} is a Banach B -bimodule then all elementary operators are bounded and one can consider the norm closures $\overline{\mathcal{D}_{\text{Lie}}(\mathcal{X}, B)}$ and $\overline{\mathcal{M}_{\text{Lie}}(\mathcal{X}, B)}$ of the algebras $\mathcal{D}_{\text{Lie}}(\mathcal{X}, B)$ and $\mathcal{M}_{\text{Lie}}(\mathcal{X}, B)$ in the algebra $\mathcal{B}(\mathcal{X})$ of all bounded operators on \mathcal{X} . These algebras can coincide even if $\mathcal{D}_{\text{Lie}}(\mathcal{X}, B) \neq \mathcal{M}_{\text{Lie}}(\mathcal{X}, B)$.

Furthermore each Banach B -bimodule can be considered as a Banach module over the projective tensor product $V_B = B \widehat{\otimes} B^{\text{op}}$. In V_B one can consider the closures of the algebras $\mathcal{T}_{\text{Lie}}(B)$ and $\mathcal{N}_{\text{Lie}}(B)$. It is natural to consider also the algebra $\mathfrak{N}_{\text{Lie}}(B)$ of all elements $\sum_{k=1}^{\infty} a_k \otimes b_k \in V_B$ satisfying (0.1) and (0.2) (with the

norm-convergence of series). It is not difficult to see that

$$(0.5) \quad \overline{\mathcal{T}_{\text{Lie}}(B)} \subset \overline{\mathcal{N}_{\text{Lie}}(B)} \subset \mathfrak{N}_{\text{Lie}}(B).$$

We do not know examples for which $\overline{\mathcal{N}_{\text{Lie}}(B)} \neq \mathfrak{N}_{\text{Lie}}(B)$. If B is commutative then the coincidence of these algebras follows from the identity $\sum_{k=1}^{\infty} a_k \otimes b_k = \sum_{k=1}^{\infty} (a_k \otimes b_k - a_k b_k \otimes 1)$.

Note that if B is an algebra of functions on a compact K ($B \subset C(K)$) then $V_B \subset C(K \times K)$ and $\mathfrak{N}_{\text{Lie}}$ consists of all functions $f(x, y) \in V_B$ for which

$$f(x, x) = 0.$$

Of course if

$$(0.6) \quad \overline{\mathcal{T}_{\text{Lie}}(B)} = \overline{\mathcal{N}_{\text{Lie}}(B)}$$

then

$$(0.7) \quad \overline{\mathcal{D}_{\text{Lie}}(\mathcal{X}, B)} = \overline{\mathcal{M}_{\text{Lie}}(\mathcal{X}, B)}$$

for each bimodule \mathcal{X} .

In Section 2 we consider the case that B is an algebra. It is shown that the equality (0.3) holds for algebras with one generator, for semisimple finite-dimensional algebras and for the algebras of finite rank operators on linear spaces. On the other hand, it does not hold for (the algebra of) polynomials of $n > 1$ variables, trigonometrical polynomials, rational functions and for free algebras with $n \geq 2$ generators. The corresponding results are obtained for elementary operators.

In Section 3 we discuss the equality (0.6) for commutative Banach algebras. Some results in this case can be obtained from the purely algebraic results of Section 2 by using the fact that if a commutative Banach algebra B has a dense subalgebra satisfying (0.3) then B satisfies (0.6). But in general the condition (0.6) is more "common" than (0.3): the completion (in a natural norm) of an algebra non-satisfying (0.3) can satisfy (0.6). The main positive result of the section is that (0.6) is true when $B = C(K)$, the algebra of all continuous functions on a compact K . More generally, (0.6) holds for all commutative regular Banach algebras B such that the diagonal is the set of spectral synthesis for $B \hat{\otimes} B$.

Section 4 is devoted to the case that $B = \mathcal{K}(\mathfrak{X})$, the algebra of all compact operators on a Banach space \mathfrak{X} . It should be noted that in the case of non-commutative Banach algebras the validity of (0.6) for a dense subalgebra does not imply its validity for the whole algebra. Hence one cannot deduce (0.6) for $B = \mathcal{K}(\mathfrak{X})$ from the validity of (0.3) for $\mathcal{F}(\mathfrak{X})$.

We establish (0.7) for $\mathfrak{X} = H$, the separable Hilbert space, when elementary operators act on B itself. For general \mathfrak{X} , a somewhat more weak than (0.6) equality is proved:

$$\overline{\mathcal{N}_{\text{Lie}}(B)^2} = \overline{\mathcal{T}_{\text{Lie}}(B)^2}.$$

It implies a weaker version of (0.7):

$$\overline{\mathcal{D}_{\text{Lie}}(\mathcal{X}, B)^2} = \overline{\mathcal{M}_{\text{Lie}}(\mathcal{X}, B)^2},$$

where the B -bimodule \mathcal{X} is arbitrary.

In Section 5 we obtain some applications to the structure of closed Lie ideals in algebras $A = B \widehat{\otimes} F$, where F is a uniformly hyperfinite C^* -algebra, B an arbitrary unital Banach algebra. The main result is that each closed Lie ideal of A is of the form

$$L = I \widehat{\otimes} F_\tau + M \widehat{\otimes} 1$$

where I is a closed ideal of B , M is a closed Lie ideal of B .

In Section 6 the obtained results are applied to the study of invariant subspaces of some Lie algebras of operators on a separable Hilbert space H . In particular we establish a Burnside type theorem for Lie algebras of operators on H that contain maximal abelian selfadjoint subalgebras of the algebra $B(H)$ of all bounded operators on H .

1. ALGEBRAIC RESULTS

The following result shows that $\mathcal{N}_{\text{Lie}}(B)$ and $\mathcal{M}_{\text{Lie}}(\mathcal{X}, B)$ are "semiideals" of $B \otimes B^{\text{op}}$ and $El(\mathcal{X}, B)$, generated respectively by $\mathcal{T}_{\text{Lie}}(B)$ and $\mathcal{D}_{\text{Lie}}(\mathcal{X}, B)$.

LEMMA 1.1. (i) *The intersection of one-sided ideals of $B \otimes B^{\text{op}}$, generated by $\mathcal{T}_{\text{Lie}}(B)$, coincides with $\mathcal{N}_{\text{Lie}}(B)$.*

(ii) *The intersection of one-sided ideals of $El(\mathcal{X}, B)$, generated by $\mathcal{D}_{\text{Lie}}(\mathcal{X}, B)$, coincides with $\mathcal{M}_{\text{Lie}}(\mathcal{X}, B)$.*

Proof. Let J be a left ideal of $B \otimes B^{\text{op}}$, containing $\mathcal{T}_{\text{Lie}}(B)$. Then each element of the form $a \otimes b - 1 \otimes ab$ belongs to J , because it equals to $(1 \otimes b)(a \otimes 1 - 1 \otimes a)$. Hence J contains the set J_1 of all elements $\sum a_i \otimes b_i$ with $\sum a_i b_i = 0$, because each of them can be written in the form $\sum(a_i \otimes b_i - 1 \otimes a_i b_i)$. It follows that J_1 is the left ideal of $B \otimes B^{\text{op}}$, generated by $\mathcal{T}_{\text{Lie}}(B)$. Similarly the right ideal generated by $\mathcal{T}_{\text{Lie}}(B)$ coincides with $J_2 = \{\sum a_i \otimes b_i : \sum b_i a_i = 0\}$. Since $\mathcal{N}_{\text{Lie}}(B) = J_1 \cap J_2$, this proves (i). The proof of (ii) is similar. ■

As a consequence we get the inclusions

$$(1.1) \quad \mathcal{T}_{\text{Lie}}(B)(B \otimes B^{\text{op}})\mathcal{T}_{\text{Lie}}(B) \subset \mathcal{N}_{\text{Lie}}(B),$$

$$(1.2) \quad \mathcal{D}_{\text{Lie}}(\mathcal{X}, B)El(\mathcal{X}, B)\mathcal{D}_{\text{Lie}}(\mathcal{X}, B) \subset \mathcal{M}_{\text{Lie}}(\mathcal{X}, B).$$

We consider the problem of validity of the equality (0.3) for arbitrary associative algebra B . Let \mathfrak{L} denote the class of all algebras for which this equality is true.

LEMMA 1.2. *If B is a commutative unital algebra in \mathfrak{L} , then each quotient of B belongs to \mathfrak{L} .*

Proof. Let I be an ideal in B , $C = B/I$ and $\pi : B \rightarrow C$ the canonical epimorphism.

Let $f = \sum_k u_k \otimes v_k \in \mathcal{N}_{\text{Lie}}(C)$. For each k choose $a_k, b_k \in B$ such that $\pi(a_k) = u_k$, $\pi(b_k) = v_k$ and set $g = \sum_k a_k \otimes b_k$. Since $\sum_k u_k v_k = 0$ we have that $c := \sum_k a_k b_k \in I$.

The element $g - c \otimes 1$ belongs to $\mathcal{N}_{\text{Lie}}(B)$. Hence it belongs to $\mathcal{T}_{\text{Lie}}(B)$. Since $\pi \otimes \pi$ clearly sends $\mathcal{T}_{\text{Lie}}(B)$ to $\mathcal{T}_{\text{Lie}}(C)$, we get that $f = \pi \otimes \pi(g - c \otimes 1) \in \mathcal{T}_{\text{Lie}}(C)$. This is what we need. ■

THEOREM 1.3. *Each algebra B with one generator belongs to \mathfrak{L} .*

Proof. Since each algebra with one generator is a quotient of the algebra of polynomials in one variable it follows from Lemma 1.2 that we may restrict to the case when B is the algebra of polynomials in one variable. Clearly $B \otimes B$ can be identified with the algebra \mathcal{P}_2 of polynomials in two variables. So in this presentation $\mathcal{T}_{\text{Lie}}(B)$ is the subalgebra in \mathcal{P}_2 , generated by all polynomials of the form $p(x) - p(y)$. It can be easily checked that $\mathcal{N}_{\text{Lie}}(B)$ coincides with the algebra J of all polynomials $p(x, y)$ satisfying the equality $p(x, x) = 0$. It is not difficult to see that J is the ideal of \mathcal{P}_2 generated by the polynomial $x - y$. Let J_n be the set of all uniform polynomials of degree n in J ; since $x - y$ is uniform, $J = \sum_n J_n$. It is

clear that a polynomial $p(x, y) = \sum_{i=0}^n a_i x^i y^{n-i}$ belongs to J_n if and only if $\sum_i a_i = 0$.

We will show by induction that $J_n \subset \mathcal{T}_{\text{Lie}}(B)$. For $n = 1$ this is evident.

Suppose this is proved for $n < k$. Let e_1, \dots, e_k be a basis in J_k , and let $e(x, y) = x^k - y^k$. Then for each i there is $\lambda = \lambda_i$ such that $(x - y)^2$ divides $e_i - \lambda_i e$. Indeed setting $u_i(x, y) = \frac{e_i(x, y)}{x - y}$, $u(x, y) = \frac{e(x, y)}{x - y}$ we see that $u \notin J$, or in other words $s(u) \neq 0$, where $s(u)$ is the sum of coefficients of u . So it suffices to take $\lambda_i = \frac{s(u_i)}{s(u)}$.

It follows that the functions $u_i - \lambda_i u$ belong to J_{k-1} . By induction hypothesis, they belong to $\mathcal{T}_{\text{Lie}}(B)$. Hence $e_i - \lambda_i e \in \mathcal{T}_{\text{Lie}}(B)$. Since also $e \in \mathcal{T}_{\text{Lie}}(B)$, we get that all e_i are in $\mathcal{T}_{\text{Lie}}(B)$. Thus $J_k \subset \mathcal{T}_{\text{Lie}}(B)$. ■

Dealing with Lie ideals of associative algebras it is natural to put the question: which algebraic expressions in elements x_1, \dots, x_n of an algebra one can write, being guaranteed that their results are in the given Lie ideal, containing all x_i ? A more correct formulation: which elementary operators preserve Lie ideals? Using Theorem 1.3, we obtain the answer in a simplest case.

COROLLARY 1.4. *Let a matrix $(\lambda_{k,m})_{k,m \in \mathbb{N}}$ be given with only finite number of non-zero entries. Suppose that*

$$(1.3) \quad \sum_{k+m=n} \lambda_{k,m} = 0 \quad \text{for all } n.$$

If \mathcal{L} is a Lie ideal of an associative algebra \mathcal{B} then $\sum_{k,m} \lambda_{k,m} a^k b a^m \in \mathcal{L}$, for all $a \in \mathcal{B}$, $b \in \mathcal{L}$.

Proof. Clearly $\sum_{k,m} \lambda_{k,m} a^k b a^m = P(L_a, R_a)b$ where $P(\alpha, \beta) = \sum_{k,m} \lambda_{k,m} \alpha^k \beta^m$. The condition (1.3) provides that $P(\alpha, \alpha) = 0$. It follows from Theorem 1.3 that P belongs to the subalgebra generated by all polynomials $p(\alpha) - p(\beta)$. Since L is invariant under $p(L_a) - p(R_a) = L_{p(a)} - R_{p(a)}$, it is invariant under $P(L_a, R_a)$. ■

It can also be proved that the condition (1.3) is necessary in some sense. Namely, if $\lambda_{k,m}$ do not satisfy it then there are an algebra \mathcal{A} , its Lie ideal \mathcal{L} and elements $a, b \in \mathcal{L}$ such that $\sum_{k,m} \lambda_{k,m} a^k b a^m \notin \mathcal{L}$.

Indeed, setting $\mu_n = \sum_{k+m=n} \lambda_{k,m}$ we have from the above that $\sum_{k,m} \lambda_{k,m} a^k b a^m - \sum_n \mu_n a^n b \in \mathcal{L}$. So it suffices to construct \mathcal{L} in such a way that $\sum_n \mu_n a^n b \notin \mathcal{L}$. It is easy to show that if p is a polynomial of degree ≥ 1 and \mathcal{A} is commutative algebra then there is a Lie ideal \mathcal{L} of \mathcal{A} (any subspace is a Lie ideal) and elements $a, b \in \mathcal{L}$ with $p(a)b \notin \mathcal{L}$ (take $b = a$, $\mathcal{L} = \mathbb{C}a$).

To demonstrate possible applications of the results of such kind let us consider one of the simplest examples: the algebra M_n of all $n \times n$ matrices as a bimodule over the algebra D_n of all diagonal matrices, with respect to the matrix multiplications.

For a subset K of $\{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$, denote by $Z(K)$ the space of all matrices $a \in M_n$ with $a_{jk} = 0$ for $(j, k) \notin K$.

COROLLARY 1.5. *Each Lie D_n -submodule of M_n is a direct sum $S = G + Z(K)$ where G is a subspace of D_n and K is a subset of $\{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$, non-intersecting the diagonal.*

Proof. Lie submodules are invariant subspaces for the algebra $\mathcal{T}_{\text{Lie}}(D_n)$. Clearly D_n is generated by one element, hence, by Theorem 1.3, $\mathcal{T}_{\text{Lie}}(D_n)$ coincides with $\mathcal{N}_{\text{Lie}}(D_n)$ which can be realized as the algebra of all matrices with zero's on the diagonal (and pointwise multiplication). In this realization the action of $\mathcal{N}_{\text{Lie}}(D_n)$ on M_n is also the pointwise (Hadamard) multiplication. Then in fact we have the action of the algebra $\mathbb{C}^{n(n-1)}$ on the direct sum $D \oplus \mathbb{C}^{n(n-1)}$, and the action on the first summand is trivial. It follows that the invariant subspaces are direct sums of subspaces of D and "coordinate" subspaces of $\mathbb{C}^{n(n-1)}$. This proves our assertion. ■

As a consequence we get a well known (see for example [8] where a similar result was obtained for matrices over arbitrary ring) description of Lie ideals in M_n :

COROLLARY 1.6. *The only Lie ideals in M_n are 0, M_n , $\mathbb{C}1$ and the space M_n^0 of all matrices with zero trace.*

Proof. Any Lie ideal S of M_n is a Lie submodule over D_n . Hence it has the form $S = G + Z(K)$, where $G \subset D_n$. Hence

$$[a, Z(K)] \subset Z(K) + D_n, \quad \text{for all } a \in M_n.$$

The multiplication table of matrix units shows that this condition holds if either $K = \emptyset$ or $K = \{(j, k) : j \neq k\}$. In the first case $S = G \subset D_n$. If $a_{ii} \neq a_{jj}$ for some $a \in S$, then $[e_{ij}, a] \notin D_n$, where e_{ij} is the corresponding matrix unit. Hence S consists of matrices $\lambda 1$, so either $S = 0$ or $S = \mathbb{C}1$.

If $K = \{(j, k) : j \neq k\}$ then $e_{ii} - e_{jj} = [e_{ij}, e_{ji}] \in S$, for all (i, j) with $i \neq j$. Hence G contains the linear span of all $e_{ii} - e_{jj}$ which coincides with the space of all diagonal matrices of zero trace. It follows that either $G = D_n$ or $G = D_n \cap M_n^0$. So $S = M_n$ or $S = M_n^0$. ■

Now let us show that (0.3) does not hold for all (even for all commutative) algebras.

THEOREM 1.7. *The equality (0.3) is not true when:*

- (i) $B = \mathcal{P}_2$, the algebra of polynomials in two variables (x_1, x_2) ;
- (ii) $B = \mathcal{L}$, the algebra of Laurent polynomials;
- (iii) $B = \mathcal{R}_1$, the algebra of rational functions of one variable.

Proof. (i) Clearly we can identify $B \otimes B$ with the algebra \mathcal{P}_4 of polynomials in four variables $(\vec{x}, \vec{y}) = (x_1, x_2, y_1, y_2)$ by the equality $(p \otimes q)(\vec{x}, \vec{y}) = p(\vec{x})q(\vec{y})$. The polynomial $p(\vec{x}, \vec{y}) = (x_1 - y_1)x_2$ belongs to $\mathcal{N}_{\text{Lie}}(B)$ because $p(x_1, x_2, x_1, x_2) = 0$. We will prove that it does not belong to $\mathcal{T}_{\text{Lie}}(B)$.

Assume the contrary, then

$$(1.4) \quad p(\vec{x}, \vec{y}) = \sum_i q_i(\vec{x}, \vec{y})$$

where each q_i is the product of polynomials of the form $a(\vec{x}) - a(\vec{y})$. It is evident that one can assume that all $a(\vec{x})$ are monomials: $a(\vec{x}) = x_1^k x_2^m$. Hence each q_i is a uniform polynomial. Taking only the uniform polynomials of degree 2 in the right hand side of (1.4), we see that

$$(1.5) \quad p(\vec{x}, \vec{y}) = \lambda_1(x_1 - y_1)^2 + \lambda_2(x_1 - y_1)(x_2 - y_2) + \lambda_3(x_2 - y_2)^2 + \lambda_4(x_1^2 - y_1^2) + \lambda_5(x_1 x_2 - y_1 y_2) + \lambda_6(x_2^2 - y_2^2).$$

Setting $x_1 = y_1$ we get

$$0 = \lambda_3(x_2 - y_2)^2 + \lambda_5 x_1(x_2 - y_2) + \lambda_6(x_2^2 - y_2^2).$$

Hence

$$0 = \lambda_3(x_2 - y_2) + \lambda_5 x_1 + \lambda_6(x_2 + y_2),$$

which easily implies that $\lambda_3 = \lambda_5 = \lambda_6 = 0$. So (1.5) gives:

$$x_2(x_1 - y_1) = \lambda_1(x_1 - y_1)^2 + \lambda_2(x_1 - y_1)(x_2 - y_2) + \lambda_4(x_1^2 - y_1^2).$$

It follows that

$$x_2 = \lambda_1(x_1 - y_1) + \lambda_2(x_2 - y_2) + \lambda_4(x_1 + y_1).$$

Setting $x_1 = y_1, x_2 = y_2$ we get $x_2 = 0$, a contradiction.

To prove (ii) and (iii) we need the following auxiliary statement.

LEMMA 1.8. *Let \mathcal{R}_2 be the algebra of all rational functions in two variables, and \mathcal{E} , the subalgebra of \mathcal{R}_2 , generated by all functions $f(x) - f(y), f \in \mathcal{R}_1$. Then the function $g(x, y) = \frac{x-y}{x}$ does not belong to \mathcal{E} .*

Proof. Assuming the contrary we have the equality

$$(1.6) \quad g(x, y) = f(x) - f(y) + h(x, y),$$

here $h(x, y)$ belongs to the linear span U of all elements of \mathcal{R}_2 of the form

$$(1.7) \quad u(x, y) = \prod_{k=1}^n (f_k(x) - f_k(y)) \quad \text{with } n \geq 2, f_i \in \mathcal{R}_1.$$

We divide both parts of (1.6) by $x - y$ and, fixing x in the domains of definition of all functions f_k in all products of the form (1.7) that participate in $h(x, y)$, take the limit for $y \rightarrow x$.

Note that for $u(x, y)$ of the form (1.7), one has $\lim_{y \rightarrow x} \frac{u(x, y)}{x - y} = 0$. Hence we obtain the equality

$$\frac{1}{x} = f'(x)$$

which is impossible because f is rational. ■

To finish the proof of parts (ii) and (iii) of Theorem 1.7, note that for $B = \mathcal{R}_1$ and $B = \mathcal{L}$, the function $g(x, y) = \frac{x-y}{x}$ belongs to the ideal of B generated by all functions of the form $f(x) - f(y)$. But Lemma 1.8 shows that it does not belong to the subalgebra generated by these functions. ■

Note that the algebra of Laurent polynomials is isomorphic to the algebra of trigonometrical polynomials. Thus the equality (0.3) does not hold for the latter.

REMARK 1.9. In the proof of Theorem 1.7(i) we established that the polynomial $(x_1 - y_1)x_2$ does not belong to $\mathcal{T}_{\text{Lie}}(\mathcal{P}_2)$. In what follows we will need a more strong result: the polynomial $(x_1 - y_1)^2x_2$ also does not belong to $\mathcal{T}_{\text{Lie}}(\mathcal{P}_2)$.

Proof. Suppose that $p(x_1, x_2, y_1, y_2) = (x_1 - y_1)^2x_2$ is in $\mathcal{T}_{\text{Lie}}(\mathcal{P}_2)$ and hence can be represented in the form $\sum_i q_i(x_1, x_2, y_1, y_2)$ where every q_i is a product of polynomials of the form $a(x_1, x_2) - a(y_1, y_2)$. We may assume that every a is a monomial. Since $p(x_1, x_2, x_1, y_2) = 0$, it is not hard to see that the equality must

have the form

$$p(x_1, x_2, y_1, y_2) = \lambda_1(x_1^3 - y_1^3) + \lambda_2(x_1 - y_1)(x_1x_2 - y_1y_2) + \lambda_3(x_1^2 - y_1^2)(x_2 - y_2) \\ + \lambda_4(x_1 - y_1)(x_2^2 - y_2^2) + \lambda_5(x_1 - y_1)^2(x_2 - y_2) \\ + \lambda_6(x_1 - y_1)^3 + \lambda_7(x_1 - y_1)(x_2 - y_2)^2.$$

Dividing the both sides by $x_1 - y_1$ we get

$$(1.8) \quad (x_1 - y_1)x_2 = \lambda_1(x_1^2 + x_1y_1 + y_1^2) + \lambda_2(x_1x_2 - y_1y_2) + \lambda_3(x_1 - y_1)(x_2 - y_2) \\ + \lambda_4(x_1 - y_1)(x_2 + y_2) + \lambda_5(x_1 + y_1)(x_2 - y_2) \\ + \lambda_6(x_1 - y_1)^2 + \lambda_7(x_2 - y_2)^2.$$

For $y_1 = x_1$ we obtain

$$3\lambda_1x_1^2 + \lambda_2x_1x_2 - \lambda_2x_1y_2 + 2\lambda_5x_1x_2 - 2\lambda_5x_1y_2 + \lambda_7x_2^2 - 2\lambda_7x_2y_2 + \lambda_7y_2^2 = 0$$

whence $\lambda_1 = \lambda_7 = 0, \lambda_2 = -2\lambda_5$ and

$$(x_1 - y_1)x_2 = -2\lambda_5(x_1x_2 - y_1y_2) + \lambda_3(x_1 - y_1)(x_2 - y_2) \\ + \lambda_4(x_1 - y_1)(x_2 + y_2) + \lambda_5(x_1 + y_1)(x_2 - y_2) + \lambda_6(x_1 - y_1)^2.$$

For $x_1 = x_2 = x$ and $y_1 = y_2 = y$ it gives us

$$(x - y)x = -2\lambda_5(x^2 - y^2) + \lambda_3(x - y)^2 + \lambda_4(x - y)(x + y) \\ + \lambda_5(x + y)(x - y) + \lambda_6(x - y)^2$$

whence we obtain 3 equations : $\lambda_3 + \lambda_4 - \lambda_5 + \lambda_6 = 0, \lambda_3 - \lambda_4 + \lambda_5 + \lambda_6 = 0, \lambda_3 + \lambda_6 = 0$. This system has the solution of the form $\lambda_3 = \alpha, \lambda_4 = \beta, \lambda_5 = \beta, \lambda_6 = -\alpha$. Substituting into (1.8) we get the equality

$$(x_1 - y_1)x_2 = \alpha((x_1 - y_1)(x_2 - y_2) - (x_1 - y_1)^2)$$

which is a contradiction. ■

COROLLARY 1.10. *The equality (0.3) is not true when $B = \mathcal{F}_2$, the free algebra in two generators a, b . In particular the element $(a \otimes 1 - 1 \otimes a)b \otimes 1(a \otimes 1 - 1 \otimes a)$ does not belong to $\mathcal{T}_{\text{Lie}}(\mathcal{F}_2)$.*

Proof. Let $\pi : \mathcal{F}_2 \rightarrow \mathcal{P}_2$ be the homomorphism which sends the first generator a of \mathcal{F}_2 to x_1 and the second generator b , to x_2 . Denote by $\pi \otimes \pi$ the corresponding homomorphism from $\mathcal{F}_2 \otimes \mathcal{F}_2^{\text{op}}$ to the algebra $\mathcal{P}_2 \otimes \mathcal{P}_2 = \mathcal{P}_4$. Since $\pi \otimes \pi(u \otimes 1 - 1 \otimes u) = \pi(u) \otimes 1 - 1 \otimes \pi(u)$ and π is surjective, we have the equality $\pi \otimes \pi(\mathcal{T}_{\text{Lie}}(\mathcal{F}_2)) = \mathcal{T}_{\text{Lie}}(\mathcal{P}_2)$.

Set $z = (a \otimes 1 - 1 \otimes a)b \otimes 1(a \otimes 1 - 1 \otimes a)$. This element belongs to $\mathcal{N}_{\text{Lie}}(\mathcal{F}_2)$ by (1.1). On the other hand if z belongs to $\mathcal{T}_{\text{Lie}}(\mathcal{F}_2)$ then $\pi \otimes \pi(z) \in \mathcal{T}_{\text{Lie}}(\mathcal{P}_2)$. But this contradicts Remark 1.9, because $\pi \otimes \pi(z) = (x_1 - y_1)^2x_2 \notin \mathcal{T}_{\text{Lie}}(\mathcal{P}_2)$. ■

On the other hand, the equality (0.3) turns out to be true for some important non-commutative examples.

THEOREM 1.11. *The equality (0.3) holds if B is an arbitrary semisimple finite-dimensional algebra.*

Proof. It is easy to check that the class \mathcal{L} is closed under forming direct sums. Thus, by the Wedderburn’s Theorem, it suffices to prove the equality (0.3) for $B = M_n(\mathbb{C})$, the algebra of all complex $n \times n$ -matrices.

Denote, for brevity, $\mathcal{N}_{\text{Lie}}(B)$ by \mathcal{N} and $\mathcal{T}_{\text{Lie}}(B)$ by \mathcal{T} . Let π be the representation of $B \otimes B^{\text{op}}$ on the space B , defined by the equality: $\pi(a \otimes b)(x) = axb$. Then π is irreducible and faithful (because B and $B \otimes B^{\text{op}}$ are simple). So it suffices to show that $\pi(\mathcal{T}) = \pi(\mathcal{N})$.

Set $H_1 = \mathbb{C}1$ and $H_2 = \{x \in M : \text{tr}(x) = 0\}$. These subspaces are invariant for $\pi(\mathcal{N})$ (hence for $\pi(\mathcal{T})$). Moreover

$$H_1 = \text{Ker } \pi(\mathcal{N}) = \text{Ker } \pi(\mathcal{T}) \quad \text{and} \quad H_2 \supseteq \pi(\mathcal{N})B \supseteq \pi(\mathcal{T})B.$$

Indeed it is easy to see that $H_1 \subset \text{Ker } \pi(\mathcal{N}) \subset \text{Ker } \pi(\mathcal{T}) = H_1$. Moreover if $T = \sum_i a_i \otimes b_i \in \mathcal{N}$ then $\text{tr}(\pi(T)x) = \text{tr} \sum_i a_i x b_i = \text{tr} \sum_i b_i a_i x = 0$, for each $x \in B$, so the range of $\pi(\mathcal{N})$ is contained in H_2 .

Let \mathcal{S} denote the restriction of the algebra $\pi(\mathcal{T})$ to H_2 . Then each non-trivial invariant subspace of the algebra \mathcal{S} is a non-zero Lie ideal of B strictly contained in H_2 . By Corollary 1.6, B has no such Lie ideals. So \mathcal{S} has no non-trivial invariant subspaces; by Burnside’s Theorem, \mathcal{S} coincides with the algebra $L(H_2)$ of all operators on H_2 . Hence the restriction of the algebra $\pi(\mathcal{N})$ to H_2 is also $L(H_2)$. Since $H_1 = \text{Ker } \pi(\mathcal{T}) = \text{Ker } \pi(\mathcal{N})$ and the space B is a direct sum of H_1 and H_2 , we conclude that $\pi(\mathcal{N}) = \pi(\mathcal{T})$ and $\mathcal{N} = \mathcal{T}$. ■

Below we always denote by \widetilde{B} the unitization of B .

Let X be a linear space, $\mathcal{L}(X)$ the algebra of all linear operators on X , $F(X)$ the algebra of all finite-rank operators on X . The algebra $\widetilde{F(X)}$ in this case can be realized as $F(X) + \mathbb{C}1 \subset \mathcal{L}(X)$. Our next aim is to show that for the algebra $\widetilde{F(X)}$ the equality (0.3) holds.

LEMMA 1.12. *Let B be a unital algebra. Then:*

- (i) *for any $x \in B$, the element $1 \otimes x^2 - x \otimes x$ belongs to $\mathcal{T}_{\text{Lie}}(B)$;*
- (ii) *if $a, x \in B$ and $ax = xa = 0$ then*

$$(1.9) \quad a \otimes x^3 = (1 \otimes x^2 - x \otimes x)(a \otimes 1 - 1 \otimes a)(x \otimes 1 - 1 \otimes x)$$

whence $a \otimes x^3 \in \mathcal{T}_{\text{Lie}}(B)$.

Proof. Part (i) follows from the equality $2(1 \otimes x^2 - x \otimes x) = (x \otimes 1 - 1 \otimes x)^2 - (x^2 \otimes 1 - 1 \otimes x^2)$. The formula (1.9) can be checked by easy calculation; using (i) this implies part (ii). ■

THEOREM 1.13. *The equality (0.3) holds if $B = \widetilde{F(X)}$.*

Proof. Note first of all that for finite-dimensional X the result immediately follows from Theorem 1.11. So we have to consider only when $\dim(X) = \infty$.

An arbitrary element of $B \otimes B^{\text{op}}$ can be written in the form $R = \lambda + a \otimes 1 + 1 \otimes b + \sum_{i=1}^n a_i \otimes b_i$, with $a, b, a_i, b_i \in F(X)$. If $R \in \mathcal{N}_{\text{Lie}}(B)$ then it is easy to see that $\lambda = 0$.

Let $a \in F(X)$ and let p be a finite-rank projection in $F(X)$ such that $ap = pa = a$. Setting $q = 1 - p$ we get $aq = qa = 0$. By Lemma 1.12, $a \otimes q^3 \in \mathcal{T}_{\text{Lie}}(B)$. Since $q^3 = q$ we see that $a \otimes p - a \otimes 1 \in \mathcal{T}_{\text{Lie}}(B)$. Similarly $1 \otimes b - p \otimes b \in \mathcal{T}_{\text{Lie}}(B)$ for an appropriate projection $p \in F(X)$. It follows that modulo $\mathcal{T}_{\text{Lie}}(B)$ each element of $\mathcal{N}_{\text{Lie}}(B)$ can be written in the following form, with $a_i, b_i \in F(X)$,

$$R = \sum_{i=1}^n a_i \otimes b_i.$$

Let now p be a finite rank projection such that $a_i p = p a_i = a_i$ and $b_i p = p b_i = b_i$ for all i . Then all a_i and b_i can be considered as operators on finite-dimensional space $Y = pX$. By Theorem 1.11, R belongs to the algebra $\mathcal{T}_{\text{Lie}}(\mathcal{L}(Y))$. But the natural imbedding of $\mathcal{L}(Y)$ into $\mathcal{L}(X)$ maps $\mathcal{T}_{\text{Lie}}(\mathcal{L}(Y))$ into $\mathcal{T}_{\text{Lie}}(F(X)) = \mathcal{T}_{\text{Lie}}(B)$. We conclude that $R \in \mathcal{T}_{\text{Lie}}(B)$. ■

Turning to elementary operators we have the problem of the validity of the equality

$$(1.10) \quad \mathcal{D}_{\text{Lie}}(B, \mathcal{X}) = \mathcal{M}_{\text{Lie}}(B, \mathcal{X}).$$

It is straightforward that if for an algebra \widetilde{B} the equality (0.3) holds then (1.10) is also true. As a consequence we obtain

COROLLARY 1.14. *For algebras $M_n, \widetilde{F(X)}$ and for each algebra with one generator, the equality (1.10) holds.*

The next result extends part (ii) of Theorem 1.7.

THEOREM 1.15. $\mathcal{M}_{\text{Lie}}(\mathcal{F}_2) \neq \mathcal{D}_{\text{Lie}}(\mathcal{F}_2)$.

Proof. Let $f : \mathcal{F}_2 \otimes \mathcal{F}_2^{\text{op}} \rightarrow \text{El}(\mathcal{F}_2)$ be the standard representation of $\mathcal{F}_2 \otimes \mathcal{F}_2^{\text{op}}$ by elementary operators on \mathcal{F}_2 . By Theorem 2.3.13 of [5], it is injective (because \mathcal{F}_2 is centrally closed, by Theorem 2.4.4 of [5]); the surjectivity of f is obvious. It follows that $f(\mathcal{T}_{\text{Lie}}(\mathcal{F}_2)) = \mathcal{D}_{\text{Lie}}(\mathcal{F}_2)$ and $f(\mathcal{N}_{\text{Lie}}(\mathcal{F}_2)) = \mathcal{M}_{\text{Lie}}(\mathcal{F}_2)$; using Corollary 1.10, we conclude that $\mathcal{D}_{\text{Lie}}(\mathcal{F}_2) \neq \mathcal{M}_{\text{Lie}}(\mathcal{F}_2)$. ■

2. COMMUTATIVE BANACH ALGEBRAS

Let now B be a Banach algebra. A natural Banach-algebraic analogue of (0.3) is the equality (0.6) which for commutative B is equivalent to

$$(2.1) \quad \overline{\mathcal{T}_{\text{Lie}}(B)} = \mathfrak{N}_{\text{Lie}}(B).$$

We are going to consider the question of the validity of these equalities for different Banach algebras.

Let us firstly list some consequences of Theorem 1.3.

Recall that if B is a function algebra on a compact K , such that $\|f\|_B \geq \|f\|_{C(K)}$ then the natural embedding of $B \widehat{\otimes} B$ into $C(K \times K)$ is injective, so $B \widehat{\otimes} B$ can be considered as a subalgebra of $C(K \times K)$.

COROLLARY 2.1. *Let B be an algebra of functions on a compact $K \subset \mathbb{C}$, supplied with a complete norm in which polynomials are dense in B . Then $\overline{\mathcal{T}_{\text{Lie}}(B)}$ coincides with the ideal $J = \{f(x, y) \in B \widehat{\otimes} B : f(x, x) = 0\}$.*

Proof. Let $f(x, y) = \sum_{i=1}^{\infty} a_i(x)b_i(y) \in J$, then $f(x, y) = \sum_{i=1}^{\infty} (a_i(x) - a_i(y))b_i(y)$ and the series of norms converges. Hence it suffices to show that

$$(2.2) \quad (a(x) - a(y))b(y) \in \overline{\mathcal{T}_{\text{Lie}}(B)}.$$

Let firstly b be a polynomial. The set of all $a(x)$ for which (2.2) holds, is closed in \mathcal{B} . It contains all polynomials by Theorem 1.3. Hence it coincides with \mathcal{B} . Thus (2.2) holds for each $a \in \mathcal{B}$ and each polynomial b . Since the set of all b , for which (2.2) holds with given a , is closed, the condition (2.2) holds for all $a, b \in \mathcal{B}$. ■

As example for \mathcal{B} , one can take $C(0, 1)$, or $C^p(0, 1)$, or the disk algebra or, more generally, the closure of polynomials in $C(K)$, for arbitrary K , or the algebra of absolutely convergent Taylor series on \mathbb{D} .

PROBLEM 2.2. Is the result of Corollary 2.1 true for the algebra $A(K)$ or $R(K)$, where $K \subset \mathbb{C}$ is arbitrary compact? Here $A(K) \subset C(K)$ is the algebra of all functions on K , analytical on $\text{int}(K)$, $R(K)$, the closure of the algebra of rational functions with poles outside K .

Let us look what Corollary 2.1 gives for the case of Lie submodules in Banach bimodules.

Denote by $\mathfrak{T}(\mathbb{D})$ the algebra of all absolutely converging Taylor series on \mathbb{D} , that is all functions $f(z) = \sum_{k=1}^{\infty} \gamma_k z^k$ with $\|f\|_{\mathfrak{T}} := \sum_k |\gamma_k| < \infty$. It is clear that the functions in $\mathfrak{T}(\mathbb{D})$ can be applied to any operator A of norm ≤ 1 , and that $\|f(A)\| \leq \|f\|_{\mathfrak{T}}$. Hence for any function f in the algebra $\mathcal{S} = \mathfrak{T}(\mathbb{D}) \widehat{\otimes} \mathfrak{T}(\mathbb{D})$ and any two commuting operators A, B with norms ≤ 1 , one can calculate $f(A, B)$ and $\|f(A, B)\| \leq \|f\|_{\mathcal{S}}$.

COROLLARY 2.3. *Let \mathcal{X} be a Banach bimodule over a Banach algebra \mathcal{A} . Let \mathcal{L} be a closed Lie submodule in \mathcal{X} . If $a \in \mathcal{A}$, $\|a\| \leq 1$, then for each function $f(\lambda, \mu)$ in $\mathcal{S} = \mathfrak{T}(\mathbb{D}) \widehat{\otimes} \mathfrak{T}(\mathbb{D})$ with $f(\lambda, \lambda) = 0$, the operator $f(L_a, R_a)$ leaves \mathcal{L} invariant.*

Our next aim is to show that for $B = C(K)$ the result of Corollary 2.1 holds without the assumption $K \subset \mathbb{C}$.

Let K be an arbitrary compact. The Banach algebra $V(K) = C(K) \widehat{\otimes} C(K)$ is called the Varopoulos algebra of K . It is naturally realized as a regular symmetric function algebra on $K \times K$. The theory of such algebras and their relations to various branches of analysis was developed in [17].

THEOREM 2.4. *The closed subalgebra in $V(K)$, generated by all functions $f(x) - f(y)$, coincides with the ideal of all functions $F(x, y)$, vanishing on the diagonal:*

$$\overline{\mathcal{T}_{\text{Lie}}(C(K))} = \{F \in V(K) : F(x, x) = 0 \text{ for all } x \in K\}.$$

Proof. The inclusion \subset is evident; we have to prove \supset .

Let us fix two non-intersecting open subsets V_1, V_2 of K . We claim that if $\text{supp}(f) \subset V_1$ and $\text{supp}(g) \subset V_2$ then $f(x)g(y) \in \mathcal{T}_{\text{Lie}}(C(K))$.

To prove the claim, set $J_i = \{f \in C(K) : \text{supp}(f) \subset V_i\}$, for $i = 1, 2$. By Lemma 1.12, $g(x)f(y)^3 \in \mathcal{T}_{\text{Lie}}(C(K))$ for any two functions f, g such that $f(x)g(x) = 0$. Hence $f(x)^3g(y) \in \mathcal{T}_{\text{Lie}}(C(K))$ for all $f \in J_1, g \in J_2$. Furthermore the set $J_1 = \{f \in C(K) : \text{supp}(f) \subset V_1\}$ is an ideal of $C(K)$. Since $C(K)$ is a regular algebra, J_1 coincides with the ideal J_1^3 , the linear span of all products $f_1f_2f_3$, where $f_i \in J_1$ (for each $f \in J_1$ one can find $f_1, f_2 \in J_1$ equal 1 on $\text{supp}(f)$, so $f_1f_2f = f$). On the other hand it is not hard to see that J_1^3 is linearly generated by all functions f^3 with $f \in J_1$: it suffices, for each f_1, f_2, f_3 , to consider the sum

$$\sum_{i=1}^3 (f_1 + \omega_i f_2 + \omega_i^2 f_3)^3$$

where ω_i are the cubic roots of 1. This proves our claim.

Let $F(x, y) = \sum_{n=1}^{\infty} a_n(x)b_n(y) \in V(K)$ and $\text{supp}(F) \subset K_1 \times K_2$ where K_i are disjoint compacts. We claim that $F \in \overline{\mathcal{T}_{\text{Lie}}(C(K))}$. Indeed let U_i ($i = 1, 2$) be disjoint open sets containing K_i , and $p_i \in C(X)$ be such that $\text{supp}(p_i) \subset U_i, p_i(x) = 1$ for $x \in K_i$. Then

$$F(x, y) = p_1(x)p_2(y)F(x, y) = \sum_{n=1}^{\infty} a_n(x)p_1(x)b_n(y)p_2(y).$$

Since $\text{supp}(a_n(x)p_1(x)) \subset U_1, \text{supp}(b_n(x)p_2(x)) \subset U_2$, we have, by the above, that

$$a_n(x)p_1(x)b_n(y)p_2(y) \in \mathcal{T}_{\text{Lie}}(C(K)).$$

Hence $F \in \overline{\mathcal{T}_{\text{Lie}}(C(K))}$.

Denote by Δ the diagonal of $K \times K$: $\Delta = \{(x, x) : x \in K\}$. Let F be an arbitrary function in $V(K)$ with $\text{supp}F \cap \Delta = \emptyset$. One can choose a finite covering of $\text{supp}(F)$ by rectangular open sets $U_1^n \times U_2^n$ with $\overline{U_1^n} \times \overline{U_2^n} \cap \Delta = \emptyset$ and $\overline{U_1^n} \cap \overline{U_2^n} = \emptyset$. Let φ_n be the partition of unity corresponding to this covering. Then each function $F(x, y)\varphi_n(x, y)$ belongs to $\overline{\mathcal{T}_{\text{Lie}}(C(K))}$ by the above. Hence $F(x, y) = \sum_n F(x, y)\varphi_n(x, y) \in \overline{\mathcal{T}_{\text{Lie}}(C(K))}$.

Suppose now that $F \in V(K)$ is an arbitrary function vanishing on Δ . Since Δ is a set of spectral synthesis in $V(K)$ (see [17]), there is a sequence $(F_n)_{n=1}^\infty$ of elements of $V(K)$ such that $F_n \rightarrow F$ (by norm of $V(K)$) and $\text{supp } F_n \cap \Delta = \emptyset$. Since all F_n belong to $\overline{\mathcal{T}_{\text{Lie}}(C(K))}$, $F \in \overline{\mathcal{T}_{\text{Lie}}(C(K))}$. ■

REMARK 2.5. The above proof literally extends to the class (SD) of all regular commutative Banach algebras B , such that the diagonal is the set of spectral synthesis for $B \widehat{\otimes} B$. It was proved in [15] that this class is quite wide: it contains any regular Banach algebra generated by all its bounded subgroups. In particular (SD) contains the group algebras and moreover all regular quotient algebras of measures on locally convex abelian groups. Thus we obtain

COROLLARY 2.6. *The equality (2.1) holds for the group algebras of discrete abelian groups.*

Note that even for the Wiener–Fourier algebra $WF(\mathbb{T})$ of periodical functions with absolutely summing Fourier series (the group algebra of \mathbb{Z}) the equality (2.1) cannot be deduced directly from purely algebraic results, because $WF(\mathbb{T})$ does not have one generator. It has the dense subalgebra of trigonometrical polynomials, but this subalgebra does not possess the property (0.3).

An example of a Banach algebra for which (2.1) fails, can be constructed by modifying one of our algebraic counterexamples:

THEOREM 2.7. *If B is the algebra of absolutely summing Taylor series on \mathbb{D}^2 then the equality (2.1) is not true.*

Proof. An element of B is a function of the form $f(\vec{x}) = \sum_{n=0}^\infty f^{(n)}$, where $\vec{x} = (x_1, x_2)$, $f^{(n)}(\vec{x})$ is the uniform component of degree n : $f^{(n)}(\vec{x}) = \sum_{i+j=n} a_{ij} x_1^i x_2^j$,

and $\sum_{i,j=1}^\infty |a_{ij}| < \infty$. Denote by P the projection onto the space of polynomials of degree ≤ 2 : $Pf = f^{(0)} + f^{(1)} + f^{(2)}$. It is continuous on B ; as a consequence the projection $Q = P \otimes P$ is continuous on $B \widehat{\otimes} B$.

It is clear that the polynomial $p(\vec{x}, \vec{y}) = (x_1 - y_1)x_2$ (or in tensor form $x_1 x_2 \otimes 1 - x_2 \otimes x_1$) belongs to $\mathcal{N}_{\text{Lie}}(B)$. We claim that it does not belong to $\overline{\mathcal{T}_{\text{Lie}}(B)}$.

Suppose the contrary: $p = \lim_{k \rightarrow \infty} F_k$ with all F_k in $\mathcal{T}_{\text{Lie}}(B)$. This means that each F_k is a sum of products of functions of the form $a(\vec{x}) \otimes 1 - 1 \otimes a(\vec{x})$. Non restricting generality we may consider only the case that $a(\vec{x}) = x_1^i x_2^j$.

Since Q is continuous, $p = \lim QF_k$. But $QF_k \neq 0$ only if F_k has a summand of the form

$$\begin{aligned} &\lambda_1(x_1 - y_1)^2 + \lambda_2(x_1 - y_1)(x_2 - y_2) + \lambda_3(x_2 - y_2)^2 + \lambda_4(x_1^2 - y_1^2) \\ &\quad + \lambda_5(x_1 x_2 - y_1 y_2) + \lambda_6(x_2^2 - y_2^2) + g_1 + g_3 + g_4, \end{aligned}$$

where g_i are polynomials of order i , and in this case QF_k is of this form. Hence we obtain that p is a limit of such functions. Since the space of these functions is finite-dimensional, p belongs to it. Hence

$$p = \lambda_1(x_1 - y_1)^2 + \lambda_2(x_1 - y_1)(x_2 - y_2) + \lambda_3(x_2 - y_2)^2 + \lambda_4(x_1^2 - y_1^2) + \lambda_5(x_1x_2 - y_1y_2) + \lambda_6(x_2^2 - y_2^2).$$

As the proof of Theorem 1.7 shows, this is impossible. ■

3. ALGEBRAS OF COMPACT OPERATORS

Let us now consider the algebra $\mathcal{K}(\mathfrak{X})$ of all compact operators on a Banach space \mathfrak{X} . We assume that \mathfrak{X} has the approximation property, the compact operators are norm-limits of finite-rank operators. The next result shows that the equality (0.6) "almost holds" for $\widetilde{\mathcal{K}(\mathfrak{X})}$.

THEOREM 3.1. *If $B = \widetilde{\mathcal{K}(\mathfrak{X})}$ then $\mathcal{N}_{\text{Lie}}(B)^2 \subseteq \overline{\mathcal{T}_{\text{Lie}}(B)}$.*

Proof. Let $x, y \in \mathcal{N}_{\text{Lie}}(B)$. By Lemma 1.1, $x = \sum_{i=1}^n d_i x_i$, $y = \sum_{j=1}^m x'_j d'_j$ for some $d_i, d'_j \in \mathcal{T}_{\text{Lie}}(B)$, $x_i, x'_j \in B \otimes B^{\text{op}}$. Hence $xy = \sum_{i,j} d_i x_i x'_j d'_j \in \mathcal{T}_{\text{Lie}}(B)(B \otimes B^{\text{op}})\mathcal{T}_{\text{Lie}}(B)$ and we get

$$(3.1) \quad \mathcal{N}_{\text{Lie}}(B)^2 \subseteq \mathcal{T}_{\text{Lie}}(B)(B \otimes B^{\text{op}})\mathcal{T}_{\text{Lie}}(B).$$

Let $a, b, y, z \in B$, $a_n, b_n, y_n, z_n \in \widetilde{F(\mathfrak{X})}$ with $a_n \rightarrow a$, $b_n \rightarrow b$, $y_n \rightarrow y$, $z_n \rightarrow z$, where as usual $F(\mathfrak{X})$ is the algebra of all finite-rank operators on \mathfrak{X} . Then

$$(a \otimes 1 - 1 \otimes a)(y \otimes z)(b \otimes 1 - 1 \otimes b) = \lim (a_n \otimes 1 - 1 \otimes a_n)(y_n \otimes z_n)(b_n \otimes 1 - 1 \otimes b_n)$$

and $(a_n \otimes 1 - 1 \otimes a_n)(y_n \otimes z_n)(b_n \otimes 1 - 1 \otimes b_n) \in \mathcal{T}_{\text{Lie}}(\widetilde{F(\mathfrak{X})})\widetilde{F(\mathfrak{X})}\widetilde{F(\mathfrak{X})}^{\text{op}}\mathcal{T}_{\text{Lie}}(\widetilde{F(\mathfrak{X})})$. By Lemma 1.1 $\mathcal{T}_{\text{Lie}}(\widetilde{F(\mathfrak{X})})\widetilde{F(\mathfrak{X})} \otimes \widetilde{F(\mathfrak{X})}^{\text{op}}\mathcal{T}_{\text{Lie}}(\widetilde{F(\mathfrak{X})}) \subseteq \mathcal{N}_{\text{Lie}}(\widetilde{F(\mathfrak{X})})$ which in its turn coincides with $\mathcal{T}_{\text{Lie}}(\widetilde{F(\mathfrak{X})})$ by Theorem 1.13. Thus $(a \otimes 1 - 1 \otimes a)(y \otimes z)(b \otimes 1 - 1 \otimes b) \in \mathcal{T}_{\text{Lie}}(\widetilde{F(\mathfrak{X})}) \subseteq \overline{\mathcal{T}_{\text{Lie}}(B)}$, for any $a, b, y, z \in B$, whence $\mathcal{T}_{\text{Lie}}(B)B \otimes B^{\text{op}}\mathcal{T}_{\text{Lie}}(B) \subseteq \overline{\mathcal{T}_{\text{Lie}}(B)}$. By (3.1)

$$\mathcal{N}_{\text{Lie}}(B)^2 \subseteq \overline{\mathcal{T}_{\text{Lie}}(B)}. \quad \blacksquare$$

Our next goal is to establish the equality

$$(3.2) \quad \overline{\mathcal{D}_{\text{Lie}}(B)} = \overline{\mathcal{M}_{\text{Lie}}(B)}$$

for $B = \mathcal{K}(H)$, the algebra of all compact operators on a Hilbert space H .

It follows easily from Lomonosov's Theorem (see [10]) that in a reflexive Banach space \mathfrak{X} with the approximation property each transitive algebra of compact

operators is norm dense in the algebra $\mathcal{K}(\mathfrak{X})$ of all compact operators. We need the following extension of this result.

LEMMA 3.2. *Let \mathfrak{X} be a Banach space with the approximation property, $W \subset \mathcal{K}(\mathfrak{X})$ a closed algebra without (closed) invariant subspaces, Y a closed complemented subspace of \mathfrak{X}^* . Suppose that the following conditions are fulfilled:*

(i) $W^*\mathfrak{X}^* \subset Y$.

(ii) *There is no proper non-zero closed subspace of Y invariant for W^* .*

Then $W = \{T \in \mathcal{K}(\mathfrak{X}) : T^*\mathfrak{X}^* \subset Y\}$.

Proof. Prove first of all that W contains a rank one operator with nonzero trace. By Lomonosov’s Lemma (see [10]) there is such operator $T \in W$ that $\sigma(T) \neq \{0\}$. Let $0 \neq \lambda \in \sigma(T)$ and P be the corresponding spectral projection. Then P is finite-dimensional and belongs to W . Let $W_0 = PWP|_{P\mathfrak{X}}$. Since this algebra has no invariant subspaces it coincides with $B(P\mathfrak{X})$ by Burnside’s Theorem. Hence it contains rank one operator T such that $\text{tr}T \neq 0$. Since we can identify W_0 with a subalgebra of W we have $T \in W$. Writing $T = x_0 \otimes f_0$ where $x_0 \in \mathfrak{X}, f_0 \in \mathfrak{X}^*$, we have that $0 \neq \text{tr}T = f_0(x_0)$. Since $\text{Im } T^* = \mathbb{C}f_0$ we have $f_0 \in Y$.

Consider the set $\{x \in \mathfrak{X} : x \otimes f_0 \in W\} \subset \mathfrak{X}$. It is a closed invariant subspace for W and hence it coincides with \mathfrak{X} . Similarly the set $\{f \in \mathfrak{X}^* : x_0 \otimes f \in W\} \subset \mathfrak{X}^*$ is closed invariant for W^* subspace of Y and hence it coincides with Y . So for any $x \in \mathfrak{X}, f \in Y$ we have $x \otimes f_0 \in W, x_0 \otimes f \in W$ whence $x \otimes f = \frac{1}{f_0(x_0)}(x \otimes f_0)(x_0 \otimes f) \in W$. Hence W contains any finite rank operator T such that $T^*\mathfrak{X}^* \subset Y$. If we denote by W_1 the algebra of all compact operators T such that $T^*\mathfrak{X}^* \subset Y$, then it can be said that $W_1 \cap \mathcal{F}(H) \subset W$. So to show that $W = W_1$ we have only to establish that $W_1 \cap \mathcal{F}(H)$ is norm dense in W_1 .

Let $P : \mathfrak{X}^* \rightarrow \mathfrak{X}^*$ be a projection on Y (it exists because $Y \subset \mathfrak{X}^*$ is assumed to be complemented). Denote by $j : \mathfrak{X} \rightarrow \mathfrak{X}^{**}$ the canonical inclusion. We claim that for any compact operator $T \in \mathcal{K}(\mathfrak{X})$, the subspace $j(\mathfrak{X})$ of \mathfrak{X}^{**} is invariant for the operator $T^{**}P^*$. Indeed for any finite rank operator

$$T = \sum_{i=1}^N x_i \otimes f_i,$$

we have

$$T^* = \sum_{i=1}^N f_i \otimes j(x_i)$$

and for any $z \in \mathfrak{X}, g \in \mathfrak{X}^*$

$$\begin{aligned} (T^{**}P^*j(z))(g) &= (P^*j(z))(T^*g) = (P^*j(z))\left(\sum_{i=1}^N g(x_i)f_i\right) \\ &= \sum_{i=1}^N g(x_i)(Pf_i)(z) = j\left(\sum_{i=1}^N (Pf_i)(z)x_i\right)(g) \end{aligned}$$

whence

$$T^{**}P^*j(z) = j\left(\sum_{i=1}^N(Pf_i)(z)x_i\right)$$

so that $j(\mathfrak{X})$ is invariant for $T^{**}P^*$. Let $T = \lim_{n \rightarrow \infty} T_n$, where T_n are of finite rank. Then $\|T^{**}P^* - T_n^{**}P^*\| \leq \|T^{**} - T_n^{**}\| \rightarrow 0$ and since the closed subspace $j(\mathfrak{X})$ is invariant for all $T_n^{**}P^*$ we get that it is invariant for $T^{**}P^*$.

Now we can define for each $T \in \mathcal{K}(\mathfrak{X})$, an operator $\widehat{T} \in \mathcal{K}(\mathfrak{X})$ by the equality

$$(3.3) \quad j\widehat{T} = T^{**}P^*j.$$

It is easy to see from (3.3) that the map $T \rightarrow \widehat{T}$ is linear and continuous: $\|\widehat{T}\| \leq \|P\|\|T\|$.

If $T = x \otimes f$ then $\widehat{T} = x \otimes Pf$ while $(\widehat{T})^*(Y^*) \subset \mathbb{C}Pf \subset Y$. Thus $\widehat{T} \in W_1$ for any rank one operator T . By linearity and continuity of the map $T \rightarrow \widehat{T}$ we conclude that this is true for all $T \in \mathcal{K}(\mathfrak{X})$.

Now let us show that $\widehat{T} = T$ for each $T \in W_1$. For any $x \in \mathfrak{X}, g \in X^*$, we have $g(\widehat{T}x) = (T^{**}P^*j(x))(g) = j(x)(PT^*g) = (PT^*g)(x)$ and since $T^*g \in Y$ we have $PT^*g = T^*g$ whence $g(\widehat{T}x) = (T^*g)(x) = g(Tx)$. Thus we get $\widehat{T} = T$.

Now we can finish the proof. Let $K \in W_1$. Since \mathfrak{X} has the approximation property, one can choose finite rank operators K_n such that $K_n \rightarrow K$. Since the map $T \mapsto \widehat{T}$ is continuous we have $\widehat{K}_n \rightarrow \widehat{K}$. Then $K = \widehat{K} = \lim \widehat{K}_n$. But $\widehat{K}_n \in W_1 \cap \mathcal{F}(\mathfrak{X}) \subset W$. Since W is closed we get $K \in W$. ■

Let $B = \mathcal{K}(H)$, the algebra of all compact operators on a Hilbert space H . We will apply Lemma 3.2 in the case $\mathfrak{X} = B$. The dual Banach space B^* of B is $C_1(H)$, the ideal of all trace class operators. We denote by $C_1^0(H)$ the subspace in $C_1(H)$ consisting of all operators with zero trace.

COROLLARY 3.3. $\overline{\mathcal{D}_{\text{Lie}}(B)} \cap \mathcal{K}(B) = \{T \in \mathcal{K}(B) : T^*B^* \subset C_1^0(H)\}$.

Proof. Note first of all that $B = \mathcal{K}(H)$ has the approximation property. Indeed if P_n is a sequence of projections increasing to 1_H , then the compact maps $T \rightarrow P_nTP_n$ strongly tend to $1_{\mathcal{K}(H)}$.

Set $W = \overline{\mathcal{D}_{\text{Lie}}(B)} \cap \mathcal{K}(B), Y = C_1^0(H)$. Obviously Y has codimension 1 in $C_1(H)$, so it is complemented. For any $a \in \mathcal{A}, y \in C_1(H), x \in B$, we have

$$((L_a - R_a)^*(y))(x) = y((L_a - R_a)(x)) = y([a, x]) = \text{tr } y[a, x] = -\text{tr } [a, y]x = ([y, a])(x),$$

so $(L_a - R_a)^*(y) = [y, a]$ and we see that $(L_a - R_a)^*$ maps $C_1(H)$ to Y . It follows that all operators in $\overline{\mathcal{D}_{\text{Lie}}(B)}^*$ map $C_1(H)$ to Y . Hence $W^*C_1(H) \subset Y$. We proved that the condition (i) of Lemma 3.2 is fulfilled.

To establish the condition (ii) let us prove firstly that for each pair p, q of finite rank projections with $pq = 0$, the operator L_pR_q belongs to W . It is clear that this operator is compact so we have to show only that it belongs to $\mathcal{D}_{\text{Lie}}(B)$. The operator $T = L_pR_q + L_qR_p = -(L_p - R_p)(L_q - R_q)$ belongs to $\mathcal{D}_{\text{Lie}}(B)$, hence

$S = (L_p - R_p)T = L_pR_q - L_qR_p$ also belongs to $\mathcal{D}_{\text{Lie}}(B)$. It follows that $L_pR_q = \frac{T+S}{2} \in \mathcal{D}_{\text{Lie}}(B)$.

Let $Y_0 \subset Y$ be a closed subspace invariant for W^* . Denote by Z the annihilator of Y_0 in $C_1(H)^* = B(H)$. This subspace is invariant with respect to all operators $T^{**} : T \in W$. It is not difficult to see that the second adjoint of a multiplication operator on $K(H)$ is the "same" multiplication operator on $B(H)$. Thus $pZq \subset Z$ if p, q are finite-rank projections with $pq = 0$.

Suppose now that q is an arbitrary projection satisfying the condition $pq = 0$. Then there is a sequence of finite-rank projections $q_n \leq q$ which tends to q in strong operator topology. Since $pZq_n \subset Z$ and Z is $*$ -weakly closed we obtain that $pZq = 0$. Dealing in the same way with p we conclude that pZq for any projections p, q satisfying the condition $pq = 0$. In particular $pz(1 - p) \in Z$ and $(1 - p)zp \in Z$ for each projection p and each $z \in Z$. Subtracting we get that $pz - zp \in Z$. Using the fact that $B(H)$ is linearly generated by projections (or just Spectral Theorem and the closeness of Z) we obtain that $az - za \in Z$ for all $a \in B(H)$ and $z \in Z$. Thus Z is a Lie ideal of $B(H)$. Since Z is $*$ -weakly closed and non-zero we have, by [12] (see also [13], [7]), that $Z = B(H)$ or $Z = \mathbb{C}1$. Thus $Y_0 = 0$ or $Y_0 = Y$.

We proved that the condition (ii) of Lemma 3.2 is fulfilled. Now applying Lemma 3.2 we get that $\overline{\mathcal{D}_{\text{Lie}}(B)} \cap \mathcal{K}(B) = \{T \in \mathcal{K}(B) : T^*C_1(H) \subset C_1^0(H)\}$. ■

LEMMA 3.4. *Let H be a Hilbert space, $A, B \in \mathcal{K}(H)$. Then there exist $K, X, Y \in \mathcal{K}(H)$ such that $A + K = X^2$ and $B - K = Y^2$.*

Proof. Write $A + B$ in the form $M + iN$ where M, N are hermitian. Then both components are normal and compact hence $M = X^2, N = Y^2$ with compact X, Y . So $A + B = X^2 + Y^2$. Set $K = B - Y^2$. Then $A + K = X^2, B - K = Y^2$. ■

THEOREM 3.5. $\overline{\mathcal{D}_{\text{Lie}}(\mathcal{K}(H))} = \overline{\mathcal{M}_{\text{Lie}}(\mathcal{K}(H))}$.

Proof. Denote $\mathcal{K}(H) = \mathcal{A}$. Let $T = \sum L_{a_i}R_{b_i} \in \mathcal{M}_{\text{Lie}}(\mathcal{A})$, where $a_i, b_i \in \mathcal{A}$. This is a compact operator on \mathcal{A} . For any $u \in C_1^0(H)$ and $x \in \mathcal{A}$ we have

$$(T^*u)(x) = u(Tx) = \text{tr } uTx = \text{tr } u \sum a_i x b_i = \text{tr } \sum b_i u a_i x = \left(\sum b_i u a_i \right)(x)$$

(we use the same notation for a nuclear operator and the corresponding functional on the space of operators). So $T^*u = \sum b_i u a_i$ and $\text{tr } T^*u = \text{tr } \sum a_i b_i u = 0$. Hence $T^* \mathcal{A}^* \subset C_1^0(H)$. By Lemma 3.2, $T \in \overline{\mathcal{D}_{\text{Lie}}(\mathcal{A})}$.

Note that $\mathcal{M}_{\text{Lie}}(\mathcal{A})$ is generated by elements of the form $T = \sum L_{a_i}R_{b_i} + R_c + L_d$, where $a_i, b_i, c, d \in \mathcal{A}, \sum_i a_i b_i + c + d = \sum_i b_i a_i + c + d = 0$. So it suffices to prove that any such operator belongs to $\overline{\mathcal{D}_{\text{Lie}}(\mathcal{A})}$. By Lemma 3.4, there exist elements $k, x, y \in \mathcal{A}$ with $c - k = x^2, d + k = y^2$. Then $T = \sum L_{a_i}R_{b_i} + R_{x^2} + L_{y^2}$. Let us consider an operator $S = \sum L_{a_i}R_{b_i} + L_x R_x + L_y R_y$. Then S is compact and belongs to $\mathcal{M}_{\text{Lie}}(\mathcal{A})$ because $\sum_i a_i b_i + x^2 + y^2 = \sum_i b_i a_i + x^2 + y^2 = 0$. By what

we proved above, $S \in \overline{\mathcal{D}_{\text{Lie}}(\mathcal{A})}$. Let us show that $T - S \in \overline{\mathcal{D}_{\text{Lie}}(\mathcal{A})}$. For this we should prove that $R_{x^2} - L_x R_x \in \overline{\mathcal{D}_{\text{Lie}}(\mathcal{A})}$ and $L_{y^2} - L_y R_y \in \overline{\mathcal{D}_{\text{Lie}}(\mathcal{A})}$.

Consider the homomorphism ϕ from the free algebra \mathcal{F} with one generator a to \mathcal{A} , which sends a to x . Correspondingly we have a homomorphism γ from $\mathcal{F} \otimes \mathcal{F}$ to $\text{El}(\mathcal{A})$: $\gamma(u \otimes v) = L_{\phi(u)} R_{\phi(v)}$. Clearly $\gamma(\mathcal{N}_{\text{Lie}}(\mathcal{F})) \subset \mathcal{M}_{\text{Lie}}(\mathcal{A})$, $\gamma(\mathcal{T}_{\text{Lie}}(\mathcal{F})) \subset \mathcal{D}_{\text{Lie}}(\mathcal{A})$, $\gamma(1 \otimes a^2 - a \otimes a) = R_{x^2} - L_x R_x$. By Theorem 1.3, $\mathcal{N}_{\text{Lie}}(\mathcal{F}) = \mathcal{T}_{\text{Lie}}(\mathcal{F})$; since $1 \otimes a^2 - a \otimes a \in \mathcal{N}_{\text{Lie}}(\mathcal{F})$ we get $R_{x^2} - L_x R_x \in \overline{\mathcal{D}_{\text{Lie}}(\mathcal{A})}$. Using the same arguments we obtain $L_{y^2} - L_y R_y \in \overline{\mathcal{D}_{\text{Lie}}(\mathcal{A})}$.

So S and $T - S$ are in $\overline{\mathcal{D}_{\text{Lie}}(\mathcal{A})}$ whence $T \in \overline{\mathcal{D}_{\text{Lie}}(\mathcal{A})}$. ■

4. APPLICATIONS TO LIE IDEALS

Recall that a C^* -algebra F is called uniformly hyperfinite (UHF) if it is the closure of the union of an increasing sequence of C^* -subalgebras F_n isomorphic to full matrix algebras.

Let us denote by F_∞ the union of the algebras F_n .

It is known that F has a unique normalized trace τ . We set $F_\tau = \text{Ker } \tau$.

Let \mathcal{I} be the identity operator on F . We will denote by $\overline{\mathcal{D}_{\text{Lie}}(F) + \mathbb{C}\mathcal{I}}^s$ the closure of the unitalization of the algebra $\mathcal{D}_{\text{Lie}}(F)$ in the strong operator topology.

LEMMA 4.1. *Let F be a uniformly hyperfinite algebra. Then there is a sequence P_n of finite rank norm-one projections in $\overline{\mathcal{D}_{\text{Lie}}(F) + \mathbb{C}\mathcal{I}}^s$ which strongly tends to \mathcal{I} .*

Proof. It is well known that F can be realized as the infinite C^* -tensor product $M_1 \otimes M_2 \otimes \dots$ of matrix algebras $M_i = M(n_i, \mathbb{C})$. We denote by F_n the product of the first n factors. Then F'_n , the commutant of F_n in F , is the product of all factors from $n + 1$ to ∞ .

Let K_n be the operator,

$$(4.1) \quad x \rightarrow \int_{U(F_n)} x u u^* du,$$

where $U(F_n)$ is the unitary group of F_n .

Claim 1. $\|K_n x - \tau(x)1\| \rightarrow 0$ for each $x \in F$.

Indeed since $\|K_n\| = 1$, it suffices to check this for $x \in F_m$. But for $n = m$, it is well known that $K_n(x) = \tau(x)1$. Hence the same equality holds for $n > m$.

Let us denote $\overline{\mathcal{D}_{\text{Lie}}(F)}^s$ by \mathcal{E} , for brevity.

Claim 2. Let P_0 be the operator $x \rightarrow \tau(x)1$. Then $P_0 - \mathcal{I} \in \mathcal{E}$.

Indeed it suffices to show that $K_n - \mathcal{I} \in \mathcal{E}$. Let $\varepsilon > 0$. Let W_1, \dots, W_N be measurable subsets of U_n with $m(W_k) = \frac{1}{N}$ and $\text{diam}(W_k) \leq \varepsilon$. Choose $u_k \in W_k$ and set $T_n(x) = \frac{1}{N} \sum_k u_k x u_k^*$. Then $\|T_n - K_n\| \leq 2\varepsilon$. On the other hand $T_n - \mathcal{I} =$

$\frac{1}{N} \sum_k (L_{u_k} R_{u_k^*} - L_1 R_1) \in \mathcal{E}$. Indeed all operators $L_{u_k} R_{u_k^*} - L_1 R_1$ clearly belong to $\mathcal{M}_{\text{Lie}}(F_n)$. Since F_n is a full matrix algebra they belong to $\mathcal{D}_{\text{Lie}}(F_n) \subset \mathcal{D}_{\text{Lie}}(F) \subset \mathcal{E}$, by Theorem 1.11.

Since ε is arbitrary, $K_n - \mathcal{I} \in \mathcal{E}$.

Let us denote by P_k the expectation onto the subalgebra F_k . It can be calculated as the limit of operators $K_{n,k'}$ which are defined by the formula, similar to (4.1) but with integration by the unitary group of the algebra $M_{k+1} \otimes M_{k+2} \otimes \dots \otimes M_{k+n} \subset F'_k$. The arguments similar to the above show that $P_k - \mathcal{I} \in \mathcal{E}$.

Since P_k are projections onto F_k and $\|P_k\| = 1$, we conclude that $P_k \rightarrow \mathcal{I}$ in the strong operator topology. ■

The lemma implies a localization result for Lie ideals in the projective tensor products with uniformly hyperfinite algebras:

COROLLARY 4.2. *Let B be an arbitrary unital Banach algebra, $A = B \widehat{\otimes} F$, where F is a UHF algebra. Then each closed Lie ideal L in A is the closure of $L_\infty := L \cap (B \otimes F_\infty)$.*

Proof. Let $a = \sum_j b_j \otimes x_j \in L$. Then

$$(4.2) \quad a = \sum_j b_j \otimes P_k x_j + \sum_j b_j \otimes (x_j - P_k x_j)$$

where P_k are the projections constructed in Lemma 4.1. Clearly $1 \otimes \overline{\mathcal{D}_{\text{Lie}}(F)}^s \subset \overline{\mathcal{D}_{\text{Lie}}(B \widehat{\otimes} F)}^s$ and this implies that operators in $1 \otimes \overline{\mathcal{D}_{\text{Lie}}(F)}^s$ preserve Lie ideals of the algebra $B \widehat{\otimes} F$. Since, by Lemma 4.1, $1 - P_k \in \overline{\mathcal{D}_{\text{Lie}}(F)}^s$, the second term in (4.2) belongs to L . Hence the first one belongs to L . Moreover it belongs to the tensor product of B and F_k which is contained in $B \otimes F_\infty$. Therefore it belongs to L_∞ . The second term tends to 0 when $k \rightarrow \infty$. Hence the first one tends to a . We obtain that L_∞ is dense in L . ■

Applying Theorem 4.14 of [4] we obtain the description of closed Lie ideals of $B \widehat{\otimes} F$ in terms of Lie ideals of B .

COROLLARY 4.3. *For each closed Lie ideal L of $A = B \widehat{\otimes} F$, there is a closed ideal I of B and a closed Lie ideal M of B such that $L = I \widehat{\otimes} F_\tau + M \widehat{\otimes} 1$.*

5. APPLICATIONS TO INVARIANT SUBSPACES OF OPERATOR LIE ALGEBRAS

A well known result of Arveson [2] states that if an algebra of operators in a Hilbert space contains a maximal abelian selfadjoint algebra (masa, for short) then either it has a non-trivial invariant subspace or it is dense in $B(H)$ with respect to the ultra-weak topology. We now extend this result to Lie algebras.

Let \mathcal{D}_i be masas in $\mathcal{B}(H_i)$, $i = 1, 2$. All Hilbert spaces will be assumed separable so \mathcal{D}_i can be realized in coordinate way: $\mathcal{D}_1 = L^\infty(X, \mu)$, $\mathcal{D}_2 = L^\infty(Y, \nu)$

acting on $H_1 = L^2(X, \mu)$ and respectively $H_2 = L^2(Y, \nu)$ by multiplications. Here X, Y are metrizable locally compact spaces, μ, ν are regular measures.

We identify a subset K of X with a projection in \mathcal{D}_1 (multiplication by χ_K) and with the range of this projection (the space of all functions in $L^2(X, \mu)$, vanishing almost everywhere outside K).

A set $\kappa \subset X \times Y$ is called marginally null (m.n.) if it is contained in $(P \times Y) \cup (X \times Q)$, where P and Q have zero measure. For brevity, we write $\kappa = 0$ ($\kappa \neq 0$) if κ is (respectively is not) marginally null.

A set is called ω -open if up to a m.n. set, it coincides with the union of a countable family of "rectangulars" $P \times Q$. The complements to ω -open sets are called ω -closed.

We have to define the projections of a subset in $X \times Y$ to the components. Let firstly for any family $\mathcal{P} = P_\lambda$ of measurable subsets of X , define its supremum and infimum. Namely, $\bigvee(\mathcal{P})$ is the subset of X that corresponds to the closed linear span of all subspaces $P_\lambda H$, while $\bigwedge(\mathcal{P})$ corresponds to their intersection. In other words we take infimum and supremum in the measure algebra of (X, μ) (or in the lattice of projections of $L^\infty(X, \mu)$).

We set now

$$\begin{aligned} \pi_1(\kappa) &= X \setminus \bigvee\{P : (P \times Y) \cap \kappa \text{ is m.n.}\}, \\ \pi_2(\kappa) &= Y \setminus \bigvee\{P : (X \times P) \cap \kappa \text{ is m.n.}\}. \end{aligned}$$

Let us call a rectangular $P \times Q$ *non-essential* for a family $\mathcal{L} \subset B(H_1, H_2)$, if

$$(5.1) \quad Q\mathcal{L}P = 0.$$

We say that a set $\kappa \subset X \times Y$ supports \mathcal{L} , if any rectangular non-intersecting κ is non-essential for \mathcal{L} . It is known that among all ω -closed sets supporting \mathcal{L} , there is the smallest one (up to a m.n. set it is contained in each supporting \mathcal{L} set). It is called the *support* of \mathcal{L} and denoted $\text{supp } \mathcal{L}$.

For any ω -closed set κ , we denote by $\mathcal{M}_{\max}(\kappa)$ the set of all operators supported by κ . It is well known (see for example [16]) that it is a \mathcal{D} -bimodule and $\text{supp } (\mathcal{M}_{\max}(\kappa)) = \kappa$.

In what follows $X = Y, \mu = \nu, \mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}, H_1 = H_2 = H$.

LEMMA 5.1. *Let \mathcal{E} be an uw-closed \mathcal{D} -bimodule with $\text{supp } (\mathcal{E}) = \kappa$. Then:*

- (i) $\overline{\mathcal{E}H} = \pi_2(\kappa)$;
- (ii) $\ker \mathcal{E} = X \setminus \pi_1(\kappa)$.

Proof. (i) Since \mathcal{E} is an uw-closed \mathcal{D} -bimodule, $\overline{\mathcal{E}H}$ is closed under multiplication by functions from $L^\infty(X, \mu)$ and hence, as is well known, consists of all functions from $L^2(X, \mu)$ vanishing on some subset of X . Thus we may consider $\overline{\mathcal{E}H}$ as a subset of X .

Let P be a projection in $L^\infty(X)$; it can be considered as a subset of X . If $P \cap \overline{\mathcal{E}H} \neq \emptyset$ then $P\overline{\mathcal{E}H} \neq \emptyset$, that is $P\mathcal{E} \neq \emptyset$, or, equivalently, $(X \times P) \cap \kappa \neq \emptyset$, that means $P \cap \pi_2(\kappa) \neq \emptyset$. Thus $\overline{\mathcal{E}H} \subset \pi_2(\kappa)$.

Let $P \cap \pi_2(\kappa) = \emptyset$. It is equivalent to $(X \times P) \cap \kappa = \emptyset$, whence $P\mathcal{E} = 0$ and $P \cap \mathcal{E}H = \emptyset$. Thus $\pi_2(\kappa) \subset \mathcal{E}H$ and hence $\pi_2(\kappa) = \overline{\mathcal{E}H}$.

(ii) Let $\tilde{\kappa}$ be the support of \mathcal{E}^* . It is easy to see that $\tilde{\kappa} = \{(y, x) : (x, y) \in \kappa\}$. Hence we have $\ker \mathcal{E} = (\mathcal{E}^*H)^\perp = (\pi_2(\tilde{\kappa}))^\perp = (\pi_1(\kappa))^\perp = X \setminus \pi_1(\kappa)$. ■

LEMMA 5.2. *If \mathcal{E} is a family of operators, P, Q , projections in \mathcal{D} then:*

- (i) $\text{supp}(Q\mathcal{E}) = (X \times Q) \cap \text{supp}(E)$;
- (ii) $\text{supp}(\mathcal{E}P) = (P \times Y) \cap \text{supp}(\mathcal{E})$.

Proof. Since both sides of (i) are ω -closed, it suffices to prove that for all projections $R, S \in \mathcal{D}$, the condition $(R \times S) \cap \text{supp}(Q\mathcal{E}) = 0$ holds if and only if $(R \times S) \cap (X \times Q) \cap \text{supp}(E) = 0$. It is easy to see that the both conditions are equivalent to $SQ\mathcal{E}R = 0$.

The proof of (ii) is similar. ■

Let us say that a set $\kappa \subset X \times X$ is *the graph of an order* if for each rectangular $P \times Q$ non-intersecting κ ,

$$(5.2) \quad \pi_2(\kappa \cap (P \times X)) \wedge \pi_1(\kappa \cap (X \times Q)) = \emptyset.$$

After this preliminary work we turn to the consideration of uw-closed Lie subalgebras of $B(H)$ that contain masa.

PROPOSITION 5.3. *If an uw-closed Lie subalgebra $\mathcal{L} \subset B(H)$ contains a masa \mathcal{D} then it is a \mathcal{D} -bimodule.*

Proof. Let us prove firstly that if $A, B \in \mathcal{D}$ and $AB = 0$ then $A\mathcal{L}B \subset \mathcal{L}$.

Indeed let us choose a dense separable subalgebra \mathcal{D}_0 of D which contains A, B . Then we may assume that X is the character space of \mathcal{D}_0 and $\mathcal{D}_0 = C(X)$. Now \mathcal{L} is a Lie submodule of a \mathcal{D}_0 -bimodule; by Theorem 2.4, it is stable under multiplying on elements of $\mathcal{D}_0 \hat{\otimes} \mathcal{D}_0$ which vanish on the diagonal. The product $A \otimes B$ is such an element. Hence $A\mathcal{L}B \subset \mathcal{L}$.

Let now \mathcal{P} be a decomposition $X = P_1 \cup P_2 \cup \dots \cup P_n$ of X . Set $\mathcal{E}_{\mathcal{P}}(T) = \sum_{i=1}^n P_i T P_i$, for any $T \in B(H)$. Then, for each $T \in \mathcal{L}$, $T - \mathcal{E}_{\mathcal{P}}(T) := \mathcal{F}_{\mathcal{P}}(T) = \sum_{i \neq j} P_i T P_j \in \mathcal{L}$, by the above (because $P_i P_j = 0$). If $A \in \mathcal{D}$ then $AT = A\mathcal{E}_{\mathcal{P}}(T) + A\mathcal{F}_{\mathcal{P}}(T)$. Again $A\mathcal{F}_{\mathcal{P}}(T) \in \mathcal{L}$ because $(AP_i)P_j = 0$. Thus $AT - A\mathcal{E}_{\mathcal{P}}(T) \in \mathcal{L}$, for any decomposition \mathcal{P} . Taking a decreasing sequence \mathcal{P}_n of the decompositions, we may say that $AT - S \in \mathcal{L}$, for any limit point S of operators $A\mathcal{E}_{\mathcal{P}_n}(T)$.

It is easy to see that $S \in \mathcal{D}$. Indeed $A\mathcal{E}_{\mathcal{P}}(T) = \mathcal{E}_{\mathcal{P}}(AT)$ commutes with all projections $P_i \in \mathcal{P}$, so it commutes with the algebra $\mathcal{D}(\mathcal{P})$, generated by \mathcal{P} . Since the union of the algebras $\mathcal{D}(\mathcal{P}_n)$, for an appropriate decreasing sequence \mathcal{P}_n of decomposition, generates D , each limit point of the sequence $A\mathcal{E}_{\mathcal{P}_n}(T)$ commutes with D and hence belongs to D .

Since $D \subset \mathcal{L}$, we get that $AT = (AT - S) + S \in \mathcal{L}$; similarly $TA \in \mathcal{L}$. ■

THEOREM 5.4. *The following conditions on an ω -closed set $\kappa \subset X$ are equivalent:*

- (i) $\mathcal{M}_{\max}(\kappa)$ is a unital algebra;
- (ii) κ is the support of an algebra that contains \mathcal{D} ;
- (iii) κ is the support of a Lie algebra that contains \mathcal{D} ;
- (iv) κ is the graph of an order and contains the diagonal $\Delta = \{(x, x) : x \in X\}$ of $X \times X$.

Proof. We will denote $\mathcal{M}_{\max}(\kappa)$ by \mathcal{M} , for brevity.

(i) \Rightarrow (ii) Since $\mathcal{M}_{\max}(\kappa)$ is a \mathcal{D} -bimodule we have that $\mathcal{D} \subset \mathcal{M}_{\max}(\kappa)$ if (i) holds.

(ii) \Rightarrow (iii) is evident.

(iii) \Rightarrow (iv) Let \mathcal{L} be a Lie algebra, $\mathcal{D} \subset \mathcal{L}$ and $\text{supp}(\mathcal{L}) = \kappa$. Non restricting generality we may assume that \mathcal{L} is uw-closed. Hence, by Proposition 5.3, it is a \mathcal{D} -bimodule.

It follows from the inclusion $\mathcal{D} \subset \mathcal{L}$ that $\text{supp } \mathcal{D} \subset \text{supp } \mathcal{L}$ that is $\Delta \subset \kappa$.

Let $\kappa \cap (P \times Q) = 0$. Since $\Delta \subset \kappa$, $P \cap Q = 0$. Then $Q\mathcal{L}\mathcal{L}P = [Q\mathcal{L}, \mathcal{L}P] \subset \mathcal{L}$, hence $Q\mathcal{L}\mathcal{L}P \subset Q\mathcal{L}P = 0$. It follows that $\mathcal{L}P\mathcal{H} \subset \ker(Q\mathcal{L})$. By Lemma 5.1, $\pi_2(\text{supp}(\mathcal{L}P)) \cap \pi_1(\text{supp}(Q\mathcal{L})) = 0$. This exactly means (if one takes into account Lemma 5.2) that (5.2) holds. Thus κ is the graph of an order.

(iv) \Rightarrow (i) If $P \times Q$ does not intersect κ then (5.2) holds. Set $P_1 = \pi_1(\kappa \cap (X \times Q))$, $Q_1 = \pi_2(\kappa \cup (P \times X))$. By Lemma 5.2, $\text{supp}(\mathcal{M}P) = (P \times X) \cap \kappa$, $\text{supp } Q\mathcal{M} = (X \times Q) \cap \kappa$. By Lemma 5.1, $\overline{MP\mathcal{H}} = \pi_2(\text{supp } \mathcal{M}P) = \pi_2(\kappa \cup (P \times X)) = Q_1$, $\ker Q\mathcal{M} = X \setminus \pi_1(\text{supp } Q\mathcal{M}) = \pi_1(\kappa \cap (X \times Q)) = X \setminus P_1$. But by (5.2), $Q_1 \subset X \setminus P_1$. This means that $Q\mathcal{M}\mathcal{M}P = 0$, that is $P \times Q$ is non-essential for \mathcal{M}^2 . Hence $\text{supp}(\mathcal{M}^2) \subset \kappa$, $\mathcal{M}^2 \subset \mathcal{M}$.

We proved that \mathcal{M} is an algebra; since $\Delta \subset \kappa$ it is unital. ■

Let us say, for brevity, that a subspace \mathcal{M} of $B(H)$ is *irreducible* if it has no non-trivial closed invariant subspaces. Furthermore \mathcal{M} is *transitive* if $\overline{\mathcal{M}x} = H$ for each non-zero vector $x \in H$. It is easy to see that if \mathcal{M} is a unital algebra then these conditions are equivalent.

It was proved by Arveson [2] that a transitive bimodule over a masa is uw-dense in $B(H)$ (for non-separable H it was proved in [14]). As a consequence each irreducible operator algebra containing a masa is uw-dense in $B(H)$ (the density in the weak operator topology was previously established in [1]). Now we extend this result to Lie algebras.

COROLLARY 5.5. *A Lie algebra \mathcal{L} of operators, containing a masa \mathcal{D} , either has a non-trivial invariant subspace or is uw-dense in $B(H)$.*

Proof. Suppose that \mathcal{L} is irreducible. Let $\kappa = \text{supp}(\mathcal{L})$ and $\mathcal{A} = \mathcal{M}_{\max}(\kappa)$. By Theorem 5.4 (the equivalence (i) \Leftrightarrow (iii)), \mathcal{A} is an algebra. Since \mathcal{L} has no invariant subspaces, so does \mathcal{A} . By [1], $\mathcal{A} = B(H)$ and, consequently, $\kappa = X \times X$.

Non-restricting generality we may assume that \mathcal{L} is uw-closed. By Proposition 5.3, \mathcal{L} is a masa-bimodule. Let us prove that it is transitive. Suppose that $\mathcal{L}x$

is not dense in H , for some $x \in H$. Let P be the projection on the subspace $(\mathcal{L}x)^\perp$, Q be the projection on $\mathcal{D}x$. Then $P\mathcal{L}Q = 0$ and, since the subspaces $(\mathcal{L}x)^\perp$ and $\mathcal{D}x$ are invariant for \mathcal{D} , the projections P, Q belongs to \mathcal{D} . This is a contradiction with the equality $\kappa = X \times X$. Thus \mathcal{L} is a transitive masa-bimodule. Applying [2] we conclude that \mathcal{L} is uw-dense in $B(H)$, $\mathcal{L} = B(H)$. ■

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