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AUTOMATIC CONTINUITY AND $C_0(\Omega)$-LINEARITY OF LINEAR MAPS BETWEEN $C_0(\Omega)$-MODULES

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ABSTRACT. If Ω is a locally compact Hausdorff space, we show that any local $\mathbb{C}$-linear map between Banach $C_0(\Omega)$-modules is "nearly $C_0(\Omega)$-linear" and "nearly bounded". Thus, any local $\mathbb{C}$-linear map θ between Hilbert $C_0(\Omega)$-modules is $C_0(\Omega)$-linear. If, in addition, Ω contains no isolated point, any $C_0(\Omega)$-linear map between Hilbert $C_0(\Omega)$-modules is bounded. Moreover, if θ is a bijective "biseparating" map from a full essential Banach $C_0(\Omega)$-module E to a full Hilbert $C_0(\Delta)$-module F, then θ is "nearly bounded" and there is a homeomorphism $\sigma : \Delta \to \Omega$ with $\theta(e \cdot \varphi) = \theta(e) \cdot \varphi \circ \sigma$ ($e \in E, \varphi \in C_0(\Omega)$).

KEYWORDS: *Banach modules, Banach bundles, local mappings, separating mappings, automatic continuity, $C_0(\Omega)$-linearity.*

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1. INTRODUCTION

A linear map θ between the spaces of continuous sections of two bundle spaces over the same locally compact Hausdorff base space Ω is said to be *local* if for any continuous section f, one has $\operatorname{supp}\theta(f) \subseteq \operatorname{supp} f$, or equivalently, for each $\psi \in C_0(\Omega)$,

$$f\psi = 0 \quad \Rightarrow \quad \theta(f)\psi = 0.$$

Consequently, local property is weaker than $C_0(\Omega)$-linearity. In the case when the domain and the range bundles are over different base spaces, a more general notion is defined; namely, *disjointness preserving*, or *separating* (see Section 5).

Local and disjointness preserving linear maps are found in many researches in analysis. For example, a theorem of Peetre [19] states that local linear maps of the space of smooth functions defined on a manifold modelled on $\mathbb{R}^n$ are exactly linear differential operators (see, e.g., [17]). This is further extended to the case of vector-valued differentiable functions defined on a finite dimensional manifold by Kantrowitz and Neumann [16] and Araujo [3].

In the topological setting, similar results have been obtained. Local linear maps of the space of continuous functions over a locally compact Hausdorff space are multiplication operators, while disjointness preserving (separating) linear maps between two such spaces over possibly different base spaces are weighted composition operators (see, e.g., [1], [5], [18], [14], [12], [15]). Among many interesting questions arising from these two notions, quite a few efforts has been put on the automatic continuity of such maps. See, e.g., [2], [7], [14], [15] for the scalar case, and [13], [4], [3], [6] for the vector-valued case.

In this paper, we extend this context to local or separating linear maps between spaces of continuous sections of vector bundles. Note that similar to the correspondence developed by Swan [20] between finite dimensional vector bundles over a locally compact Hausdorff space Ω and certain $C_0(\Omega)$-modules, the spaces of continuous sections of "Banach bundles" are certain Banach $C_0(\Omega)$-modules (see, e.g., [10], and Section 2 below).

One of the original motivations behind this work is to investigate up to what extent will a local linear map between two Banach $C_0(\Omega)$-modules be $C_0(\Omega)$-linear. Surprisingly, on top of finding that such maps are "nearly $C_0(\Omega)$-linear", we find that they are also "nearly bounded". In fact, it is well known that there are many unbounded $\mathbb{C}$-linear maps from an infinite dimensional Banach space to another Banach space and so, if S is a finite set, there are many unbounded $C(S)$-module maps from certain Banach $C(S)$-module to another Banach $C(S)$-module. The interesting thing we discovered is that the above is, in many cases, the "only obstruction" to the automatic boundedness of $C_0(\Omega)$-module maps (see Proposition 3.5 as well as Theorems 3.7 and 4.2).

More precisely, if θ is a local $\mathbb{C}$-linear map (not assumed to be bounded) from an essential Banach $C_0(\Omega)$-module E to another such module F, then θ is "nearly $C_0(\Omega)$-linear", in the sense that the induced map $\widetilde{\theta} : E \to \widetilde{F}$ is a $C_0(\Omega)$-module map (where $\widetilde{F}$ is the image of F in the space of C_0-sections on the canonical "(H)-Banach bundle" associated with F; see Section 2). Moreover, θ is "nearly bounded" in the sense that there exists a finite subset $S \subseteq \Omega$ and a positive number κ such that

$$\sup_{\omega\in\Omega\setminus S} \|\widetilde{\theta}(e)(\omega)\| \leqslant \kappa\|e\| \quad (e \in E).$$

Furthermore, if F is "$C_0(\Omega)$-normed" (in particular, if F is a Hilbert $C_0(\Omega)$-module), then the finite set S consists of isolated points in Ω, and

$$\theta = \theta_0 \oplus \bigoplus_{\omega\in S} \theta_\omega$$

where $\theta_0 : E_{\Omega\setminus S} \to F_{\Omega\setminus S}$ is a bounded $C_0(\Omega \setminus S)$-linear map (where $E_{\Omega\setminus S}$ and $F_{\Omega\setminus S}$ are the canonical essential Banach $C_0(\Omega \setminus S)$-modules induced from E and F respectively) and θ_ω are (possibly unbounded) $\mathbb{C}$-linear maps (see Theorems 4.2 and 3.7). Consequently, if Ω contains no isolated point and F is $C_0(\Omega)$-normed, then θ is automatically bounded. As another application, if X and Y are Banach spaces, and $\theta : \ell^\infty(X) \to \ell^\infty(Y)$ is a linear map preserving "zero ultra-limits" (see Corollary 4.7 for the precise definition), then θ induces a linear map $\theta_{\mathcal{F}}$ from the ultrapower $X^{\mathcal{F}}$ to $Y^{\mathcal{F}}$ for each free ultrafilter $\mathcal{F}$ on $\mathbb{N}$, and there exist free ultrafilters $\mathcal{F}_1, \dots, \mathcal{F}_n$ on $\mathbb{N}$ such that $\sup_{\mathcal{F} \neq \mathcal{F}_1, \dots, \mathcal{F}_n} \|\theta_{\mathcal{F}}\| < \infty$.

On the other hand, we will also study $\mathbb{C}$-linear maps between two Banach modules over two different base spaces. In this case, we will consider "separating" maps instead of local maps. More precisely, if Ω and Δ are two locally compact Hausdorff spaces, E is a "full" essential Banach $C_0(\Omega)$-module (see Remark 3.2(ii)), and F is a "full" Banach $C_0(\Delta)$-normed module, then for any bijective linear map $\theta : E \to F$ (not assumed to be bounded) with both θ and θ^{-1} being separating, there exists a homeomorphism $\sigma : \Delta \to \Omega$ such that $\theta(e \cdot \varphi) = \theta(e) \cdot \varphi \circ \sigma$ ($e \in E, \varphi \in C_0(\Omega)$), and there exists a finite set S consisting of isolated points of Δ such that the restriction of θ from $E_{\Omega \setminus \sigma(S)}$ to $F_{\Delta \setminus S}$ is bounded.

This paper is organised as follows. In Section 2, we will first collect some basic facts about the correspondence between Banach bundles and Banach $C_0(\Omega)$-modules. In Section 3, we will show two technical lemmas concerning "near $C_0(\Omega)$-linearity" and "near boundedness" of certain mappings. Section 4 is devoted to automatic $C_0(\Omega)$-linearity and automatic boundedness of local linear mappings, while Section 5 is devoted to the automatic boundedness of bijective biseparating linear mappings between Banach modules over different base spaces. Finally, as an attempt to a further generalisation, we show in the Appendix that for an arbitrary C^*-algebra A, every *bounded* local linear map from a Banach A-module into a Hilbert A-module is A-linear. The boundedness assumption can be removed in the case when A is finite dimensional (Corollary 4.9).

2. PRELIMINARIES AND NOTATIONS

Let us first recall (mainly from [10]) some basic terminologies and results concerning Banach modules and Banach bundles.

NOTATION 2.1. In this article, Ω and Δ are two locally compact Hausdorff spaces, E is an essential Banach $C_0(\Omega)$-module, F is an essential Banach $C_0(\Delta)$-module, and $\theta : E \to F$ is a $\mathbb{C}$-linear map (not assumed to be bounded). Furthermore, Ω_∞ and Δ_∞ are the one-point compactifications of Ω and Δ respectively. We denote by $\mathcal{N}_\Omega(\omega)$ the set of all compact neighbourhoods of an element ω in Ω, and by $\mathrm{Int}_\Omega(S)$ the set of all interior points of a subset S in Ω. Moreover, if $U, V \subseteq \Omega$ such that the closure of V is a compact subset of $\mathrm{Int}_\Omega(U)$, we denote by $\mathcal{U}_\Omega(V, U)$ the collection of all $\lambda \in C_c(\Omega)$ with $0 \leqslant \lambda \leqslant 1$, $\lambda \equiv 1$ on V and the support of λ lies inside $\mathrm{Int}_\Omega(U)$.

DEFINITION 2.2. Let Ξ be a Hausdorff space and $p : \Xi \to \Omega$ be a surjective continuous open map. Suppose that for each $\omega \in \Omega$,

(i) there exists a complex Banach space structure on $\Xi_\omega := p^{-1}(\omega)$ such that its norm topology coincides with the topology on Ξ_ω (as a topological subspace of Ξ);

(ii) $\{W(\varepsilon, U) : \varepsilon > 0, U \in \mathcal{N}_\Omega(\omega)\}$ forms a neighbourhood basis for the zero element $0_\omega \in \Xi_\omega$ where $W(\varepsilon, U) := \{\xi \in p^{-1}(U) : \|\xi\| < \varepsilon\}$;

(iii) the maps $\mathbb{C} \times \Xi \to \Xi$ and $\{(\xi, \eta) \in \Xi \times \Xi : p(\xi) = p(\eta)\} \to \Xi$ given respectively, by the scalar multiplications and the additions are continuous.

Then (Ξ, Ω, p) (or simply, Ξ) is called an *(H)-Banach bundle* (respectively, an *(F)-Banach bundle*) over Ω if $\xi \mapsto \|\xi\|$ is an upper-semicontinuous (respectively, continuous) map from Ξ to $\mathbb{R}_+$. In this case, Ω is called the *base space* of Ξ, the map p is called the *canonical projection* and Ξ_ω is called the *fibre* over $\omega \in \Omega$.

If Ξ is an (H)-Banach bundle over Ω and $\Omega_0 \subseteq \Omega$ is an open set, then

$$\Xi_{\Omega_0} := p^{-1}(\Omega_0)$$

is an (H)-Banach bundle over Ω_0 and is called the *restriction of Ξ to Ω_0*. If Ξ is an (F)-Banach bundle, then so is Ξ_{Ω_0}.

DEFINITION 2.3. If Ξ and Λ are (H)-Banach bundle over Ω and Δ respectively, a map $\rho : \Xi \to \Lambda$ is called a *fibrewise linear map covering a map $\sigma : \Omega \to \Delta$* if $\rho(\Xi_\omega) \subseteq \Lambda_{\sigma(\omega)}$ and the restriction $\rho_\omega : \Xi_\omega \to \Lambda_{\sigma(\omega)}$ is linear. Moreover, a fibrewise linear map ρ covering a continuous map $\sigma : \Omega \to \Delta$ is called a *Banach bundle map* if ρ is continuous. A Banach bundle map ρ is said to be *bounded* if one has $\sup_{\|\xi\| \leqslant 1, \xi \in \Xi} \|\rho(\xi)\| < \infty$.

For any map $e : \Omega \to \Xi$, we denote

$$|e|(\omega) := \|e(\omega)\| \quad (\omega \in \Omega).$$

Such an e is called a *C_0-section* on Ξ if e is continuous, $p(e(\omega)) = \omega$ ($\omega \in \Omega$), and for any $\varepsilon > 0$, there exists a compact set $C \subseteq \Omega$ such that $|e|(\omega) < \varepsilon$ ($\omega \in \Omega \setminus C$). We put

$$\Gamma_0(\Xi) := \{e : \Omega \to \Xi : e \text{ is a } C_0\text{-section on } \Xi\}.$$

Note that $|e|$ is always upper semi-continuous for every $e \in \Gamma_0(\Xi)$ and Ξ is an (F)-Banach bundle if and only if all such $|e|$ are continuous.

Next, we recall some terminologies and properties about essential Banach (right) $C_0(\Omega)$-modules. A Banach $C_0(\Omega)$-module E is said to be *essential* if there exists an approximate unit $\{\psi_i\}_{i \in I}$ in $C_0(\Omega)$ such that $\|x - x\psi_i\| \to 0$ for any $x \in E$. In this case, E can be regarded as a unital Banach $C(\Omega_\infty)$-module in a unique way. Furthermore, for any $\omega \in \Omega_\infty$ and $S \subseteq \Omega_\infty$, we denote

$$K_S := \{\varphi \in C(\Omega_\infty) : \varphi(S) = \{0\}\}, \quad K_S^E := \overline{E \cdot K_S} \quad \text{and} \quad I_\omega^E := \bigcup_{V \in \mathcal{N}_{\Omega_\infty}(\omega)} K_V^E.$$

For simplicity, we set $K^E_\omega := K^E_{\{\omega\}}$. Note that $K^E_\infty = E$ because E is an essential Banach $C_0(\Omega)$-module. By p. 37 of [10], there exists an (H)-Banach bundle $\check{\Xi}^E$ over Ω_∞ with $\check{\Xi}^E_\omega = E/K^E_\omega$ ($\omega \in \Omega_\infty$). Since $\check{\Xi}^E_\infty = \{0\}$, if we set $\Xi^E := p^{-1}(\Omega)$, then $\Gamma_0(\Xi^E) \cong \Gamma_0(\check{\Xi}^E)$ under the canonical identification. Furthermore, there exists a contraction

$$\sim : E \longrightarrow \Gamma_0(\Xi^E)$$

such that $\widetilde{e}(\omega) = e + K^E_\omega$. We put $\widetilde{E}$ to be the closure of the image of $\sim$.

On the other hand, if θ is as in Notation 2.1, we define

$$\widetilde{\theta} : E \to \widetilde{F} \quad \text{by} \quad \widetilde{\theta}(e) = \widetilde{\theta(e)} \quad (e \in E).$$

DEFINITION 2.4. Let E be an essential Banach $C_0(\Omega)$-module.

(a) E is called a *Banach $C_0(\Omega)$-convex module* if for any $\varphi, \psi \in C(\Omega_\infty)_+$ with $\varphi + \psi = 1$, one has $\|x\varphi + y\psi\| \leqslant \max\{\|x\|, \|y\|\}$.

(b) E is called a *Banach $C_0(\Omega)$-normed module* if there exists a map $|\cdot| : E \to C_0(\Omega)_+$ such that for any $x, y \in X$ and $a \in A$,

(i) $|x + y| \leqslant |x| + |y|$;

(ii) $|xa| = |x||a|$;

(iii) $\|x\| = \||x|\|$.

Recall that every Hilbert $C_0(\Omega)$-module is a Banach $C_0(\Omega)$-normed module, and every Banach $C_0(\Omega)$-normed module is $C_0(\Omega)$-convex. On the other hand, an essential Banach $C_0(\Omega)$-module E is $C_0(\Omega)$-convex if and only if $\sim$ is an isometric isomorphism onto $\Gamma_0(\Xi^E)$ (see e.g. Theorem 2.5 of [10]). In this case, we will not distinguish E and $\Gamma_0(\Xi^E)$. Furthermore, E is $C_0(\Omega)$-normed if and only if E is $C_0(\Omega)$-convex and Ξ^E is an (F)-Banach bundle (see e.g. p. 48 of [10]).

For any open subset $\Omega_0 \subseteq \Omega$, we set $E_{\Omega_0} := K^E_{\Omega\setminus\Omega_0}$ and $\widetilde{E}_{\Omega_0} := \Gamma_0(\Xi^E_{\Omega_0})$. One can regard $K^E_{\Omega\setminus\Omega_0}$ as an essential Banach $C_0(\Omega_0)$-module under the identification $C_0(\Omega_0) \cong K_{\Omega\setminus\Omega_0}$. Notice that if E is $C_0(\Omega)$-convex, then $\widetilde{E}_{\Omega_0} = E_{\Omega_0}$.

REMARK 2.5. (i) It is well-known that $\omega \mapsto 0_\omega$ is a continuous map from Ω into Ξ^E. Thus, if $\{\omega_i\}_{i\in I}$ is a net in Ω converging to $\omega_0 \in \Omega$ and $e \in \bigcap_{i\in I} K^E_{\omega_i}$, then $e \in K^E_{\omega_0}$. Consequently, if $e \notin K^E_\omega$, there exists $U \in \mathcal{N}_\Omega(\omega)$ such that $e \notin K^E_\alpha$ for any $\alpha \in U$.

(ii) For each $\omega \in \Omega$ and $e \in K^E_\omega$, there exists a net $\{e_V\}_{V\in\mathcal{N}_\Omega(\omega)}$ such that $e_V \in K^E_V$ and $\|e - e_V\| \to 0$.

(iii) Let $\Omega = \{\omega_1, \omega_2, \ldots\}$ be a countable compact Hausdorff space and E be a Banach $C(\Omega)$-module. Then

$$\bigcap_{\omega\in\Omega} K^E_\omega = \{0\},$$

or equivalently, the map $\sim$ is injective. In fact, consider any $e \in \bigcap_{\omega\in\Omega} K^E_\omega$ and any $\varepsilon > 0$. For $k \in \mathbb{N}$, there exists $\overline{\varphi}_k \in K_{\{\omega_k\}}$ with $\|e - e\overline{\varphi}_k\| < \varepsilon/2^{k+1}$. Thus, there exists $\varphi_k \in C(\Omega)$ with φ_k vanishing on an open neighbourhood V_k of ω_k and $\|e - e\varphi_k\| < \varepsilon/2^k$. Now, consider a finite subcover $\{V_1, \dots, V_n\}$ for Ω and a continuous partition of unity $\{\psi_1, \dots, \psi_n\}$ subordinated to $\{V_1, \dots, V_n\}$. Then $\|e\| = \left\| e - e \sum_{k=1}^{n} \varphi_k \psi_k \right\| \leqslant \sum_{k=1}^{n} \|e - e\varphi_k\| < \varepsilon$.

(iv) Suppose that $\sim : E \to \widetilde{E}$ is injective (in particular, if E is $C_0(\Omega)$-convex). If $e \in E$, $\omega \in \Omega_\infty$ and $U \in \mathcal{N}_{\Omega_\infty}(\omega)$ such that $\widetilde{e}(\alpha) = 0$ for all $\alpha \in U$, then $e \in I^E_\omega$. In fact, let $V \in \mathcal{N}_{\Omega_\infty}(\omega)$ with $V \subseteq \mathrm{Int}_{\Omega_\infty}(U)$ and $\varphi \in \mathcal{U}_{\Omega_\infty}(V, U)$, then it is clear that $\widetilde{e(1-\varphi)} = \widetilde{e}$ which implies that $e = e(1-\varphi) \in K^E_V$.

3. SOME TECHNICAL RESULTS

In this section, we will give two technical lemmas (3.3 and 3.6) which are the crucial ingredients for all the results in this paper. Before presenting them, let us give another automatic continuity type lemma that is needed for those two essential lemmas.

LEMMA 3.1. *$\mathfrak{Z}_\theta := \{\nu \in \Delta : \widetilde{\theta}(e)(\nu) = 0$ for all $e \in E\}$ is a closed subset (where $\widetilde{\theta}$ is as in Section 2). Moreover, if $\sigma : \Delta_\theta \to \Omega_\infty$ (where $\Delta_\theta := \Delta \setminus \mathfrak{Z}_\theta$) is a map satisfying $\theta(I^E_{\sigma(\nu)}) \subseteq K^F_\nu$ ($\nu \in \Delta_\theta$), then σ is continuous.*

Proof. It follows from Remark 2.5(i) that $\mathfrak{Z}_\theta$ is closed. Suppose on the contrary, that there exists a net $\{\nu_i\}_{i\in I}$ in Δ_θ that converges to $\nu_0 \in \Delta_\theta$ but $\sigma(\nu_i) \nrightarrow \sigma(\nu_0)$. Then there are $U, W \in \mathcal{N}_{\Omega_\infty}(\sigma(\nu_0))$ with $U \subseteq \mathrm{Int}_{\Omega_\infty}(W)$ and $\{i \in I : \sigma(\nu_i) \notin \mathrm{Int}_{\Omega_\infty}(W)\}$ being cofinal. As Ω_∞ is compact, by passing to a subnet if necessary, we can assume that $\{\sigma(\nu_i)\}$ converges to an element $\omega \in \Omega_\infty$, and there exists $V \in \mathcal{N}_{\Omega_\infty}(\omega)$ with $V \cap U = \emptyset$. Pick any $e \in E$ and $\varphi \in \mathcal{U}_{\Omega_\infty}(V, \Omega_\infty \setminus U)$. Since $\sigma(\nu_i) \to \omega$, we see that $e(1-\varphi) \in I^E_{\sigma(\nu_i)}$ when i is large enough and so eventually,

$$\widetilde{\theta}(e(1-\varphi))(\nu_i) = 0$$

(by the hypothesis). By Remark 2.5(i), we see that $\widetilde{\theta}(e(1-\varphi))(\nu_0) = 0$. On the other hand, we have $\theta(e\varphi) \in K^F_{\nu_0}$ (because $e\varphi \in I^E_{\sigma(\nu_0)}$), and so $\theta(e) \in K^F_{\nu_0}$, which gives the contradiction that $\nu_0 \in \mathfrak{Z}_\theta$. ∎

REMARK 3.2. (i) Note that for any $\nu \in \mathfrak{Z}_\theta$, one has

$$\theta(I^E_\omega) \subseteq \theta(E) \subseteq K^F_\nu \quad (\omega \in \Omega). \tag{3.1}$$

Consequently, if we extend σ in Lemma 3.1 by setting $\sigma(\nu)$ arbitrarily for each $\nu \in \mathfrak{Z}_\theta$, then $\theta(I^E_{\sigma(\nu)}) \subseteq K^F_\nu$ ($\nu \in \Delta$) but one should not expect such σ to be continuous on the whole of Δ.

(ii) θ is said to be *full* if $\mathfrak{Z}_\theta = \emptyset$. Moreover, E is said to be *full* if $\mathrm{id} : E \to E$ is full (or equivalently, $E \neq K^E_\omega$ for any $\omega \in \Omega$). Note that if E is full, then for any $\omega \in \Omega$, there exists $e \in E$ with $\widetilde{e}(\omega) \neq 0$.

LEMMA 3.3. *Let $\sigma : \Delta_\theta \to \Omega$ be a map satisfying $\theta(I^E_{\sigma(\nu)}) \subseteq K^F_\nu$ ($\nu \in \Delta_\theta$).*

(i) *If $\mathfrak{U}_\theta := \Big\{\nu \in \Delta : \sup_{\|e\|\leqslant 1} \|\widetilde{\theta}(e)(\nu)\| = \infty\Big\}$, then $\mathfrak{U}_\theta \subseteq \Delta_\theta$,*

$$\sup_{\nu\in\Delta\setminus\mathfrak{U}_\theta;\|e\|\leqslant 1} \|\widetilde{\theta}(e)(\nu)\| < \infty,$$

(as usual, $\sup\emptyset = -\infty$) and $\sigma(\mathfrak{U}_\theta)$ is a finite set.

(ii) *If $\mathfrak{N}_{\theta,\sigma} := \{\nu \in \Delta_\theta : \theta(K^E_{\sigma(\nu)}) \nsubseteq K^F_\nu\}$, then $\mathfrak{N}_{\theta,\sigma} \subseteq \mathfrak{U}_\theta$ and $\sigma(\mathfrak{N}_{\theta,\sigma})$ consists of non-isolated points in Ω.*

(iii) *If, in addition, σ is an injection sending isolated points in Δ_θ to isolated points in Ω, then $\widetilde{\theta}(e\varphi) = \widetilde{\theta}(e)(\varphi \circ \sigma)$ ($e \in E, \varphi \in C_0(\Omega)$), when we extend σ to a map from Δ to Ω arbitrarily.*

Proof. (i) The first conclusion is clear. We put Y to be the ℓ^∞-direct sum $\bigoplus^{\ell^\infty}_{\nu\in\Delta} \Xi^F_\nu$. For every $\nu \in \Delta \setminus \mathfrak{U}_\theta$, one can regard $e \mapsto \widetilde{\theta}(e)(\nu)$ as a bounded $\mathbb{C}$-linear map from E into Y such that $\|\widetilde{\theta}(e)(\nu)\| \leqslant \|\theta(e)\|$ ($e \in E$), the uniform boundedness principle will give the second conclusion. Assume now that $\sigma(\mathfrak{U}_\theta)$ is infinite. For $n = 1$, we can find $\nu_1 \in \mathfrak{U}_\theta$ as well as $e_1 \in E$ with $\|e_1\| \leqslant 1$ and $\|\widetilde{\theta}(e_1)(\nu_1)\| > 1$. Inductively, we can find $\nu_n \in \mathfrak{U}_\theta$ and $e_n \in E$ such that

$$\sigma(\nu_n) \neq \sigma(\nu_k) \quad (k = 1, \ldots, n-1), \quad \|e_n\| \leqslant 1 \quad \text{and} \quad \|\widetilde{\theta}(e_n)(\nu_n)\| > n^3.$$

There exist $n_1 \in \mathbb{N}$ and $U_1 \in \mathfrak{N}_\Omega(\sigma(\nu_{n_1}))$ such that $\{n \in \mathbb{N} : n > n_1 \text{ and } \sigma(\nu_n) \notin U_1\}$ is infinite (because a sequence in Ω can converge to at most one point). Inductively, we can find a subsequence $\{\nu_{n_k}\}$ and $U_k \in \mathfrak{N}_\Omega(\sigma(\nu_{n_k}))$ ($k \in \mathbb{N}$) such that $U_k \cap U_l = \emptyset$ for distinct $k, l \in \mathbb{N}$. Without loss of generality, we assume that $n_k = k$. Pick $V_n \in \mathfrak{N}_\Omega(\sigma(\nu_n))$ such that V_n is subset of $\mathrm{Int}_\Omega(U_n)$. Consider $\lambda_n \in \mathfrak{U}_\Omega(V_n, U_n)$ ($n \in \mathbb{N}$) and notice that $\|e_n\lambda_n^2\| \leqslant 1$. Define $e := \sum_{k=1}^{\infty} e_k\lambda_k^2/k^2 \in E$ and take $n \in \mathbb{N}$. Since

$$n^2 e - e_n\lambda_n^2 = n^2\Big(\sum_{k\neq n} \frac{e_k\lambda_k}{k^2}\Big)\Big(\sum_{k\neq n} \lambda_k\Big) \in K^E_{U_n},$$

we have $n^2\widetilde{\theta}(e)(\nu_n) = \widetilde{\theta}(e_n\lambda_n^2)(\nu_n)$ (by the hypothesis). On the other hand, as $e_n - e_n\lambda_n^2 = e_n(1-\lambda_n^2) \in K^E_{V_n}$, we have $\widetilde{\theta}(e_n)(\nu_n) = \widetilde{\theta}(e_n\lambda_n^2)(\nu_n)$ and

$$\|\widetilde{\theta}(e)\| \geqslant \|\widetilde{\theta}(e)(\nu_n)\| = \frac{1}{n^2}\|\widetilde{\theta}(e_n\lambda_n^2)(\nu_n)\| = \frac{1}{n^2}\|\widetilde{\theta}(e_n)(\nu_n)\| > n,$$

which contradicts the finiteness of $\|\widetilde{\theta}(e)\|$.

(ii) Consider $\nu \in \Delta \setminus \mathfrak{U}_\theta$ and denote $\kappa := \sup_{\|e\|\leqslant 1} \|\widetilde{\theta}(e)(\nu)\| < \infty$. Pick any $e \in K^E_{\sigma(\nu)}$ and $e_V \in K^E_V$ ($V \in \mathcal{N}_\Omega(\sigma(\nu))$) with $\|e_V - e\| \to 0$ (Remark 2.5(ii)). As $\theta(e_V) \in K^F_\nu$, one has

$$\|\widetilde{\theta}(e)(\nu)\| = \|\widetilde{\theta}(e - e_V)(\nu)\| \leqslant \kappa\|e - e_V\|,$$

which shows that $\nu \in \Delta \setminus \mathfrak{N}_{\theta,\sigma}$. Now, if $\nu \in \mathfrak{N}_{\theta,\sigma}$ such that $\sigma(\nu)$ is an isolated point in Ω, then $\{\sigma(\nu)\} \in \mathcal{N}_\Omega(\sigma(\nu))$, and we have the contradiction that $\theta(K^E_{\sigma(\nu)}) \subseteq K^F_\nu$. This gives second statement.

(iii) Fix $\varphi \in C_0(\Omega)$ and $e \in E$. For any $\nu \in \Delta \setminus \mathfrak{N}_{\theta,\sigma}$, we have $e\varphi - e\varphi(\sigma(\nu)) = e(\varphi - \varphi(\sigma(\nu))1) \in K^E_{\sigma(\nu)}$. Thus, for any $\nu \in \Delta \setminus \mathfrak{N}_{\theta,\sigma}$, one has from (3.1),

$$\widetilde{\theta}(e\varphi)(\nu) = \widetilde{\theta}(e)(\nu)\varphi(\sigma(\nu)). \tag{3.2}$$

In particular, (3.2) is true when $\nu \in \Delta \setminus \mathfrak{U}_\theta$ (by part (ii)) or when $\nu \in \mathfrak{U}_\theta$ is an isolated point of Δ_θ (by the hypothesis as well as part (ii)). Suppose that $\nu \in \mathfrak{U}_\theta$ is a non-isolated point of Δ_θ. As σ is injective, part (i) implies that $\mathfrak{U}_\theta$ is a finite set. Hence, there exists a net $\{\nu_i\}$ in $\Delta_\theta \setminus \mathfrak{U}_\theta$ converging to ν. Now, by Lemma 3.1,

$$\widetilde{\theta}(e\varphi)(\nu) = \lim \widetilde{\theta}(e\varphi)(\nu_i) = \lim \widetilde{\theta}(e)(\nu_i)\varphi(\sigma(\nu_i)) = \widetilde{\theta}(e)(\nu)\varphi(\sigma(\nu)). \quad \blacksquare$$

REMARK 3.4. Note that since $\mathfrak{Z}_\theta$ is closed, isolated points in Δ_θ are the same as points in Δ_θ which are isolated points of Δ. Moreover, for any $\nu \in \mathfrak{Z}_\theta$, we have $\sup_{\|e\|\leqslant 1} \|\widetilde{\theta}(e)(\nu)\| = 0$, and (3.1) holds. Therefore, Lemma 3.3 remains valid if we replace all the Δ_θ with Δ (in fact, the current form is stronger as any injection on Δ restricted to an injection on Δ_θ). The same is true for all the remaining results in this section.

If σ is injective, then $\mathfrak{U}_\theta$ is finite and we have, by Lemma 3.3(i), our first nearly automatically boundedness result which states that if θ is a "module map through an injection $\sigma : \Delta \to \Omega$" (one can relax this slightly to an injection on Δ_θ), then θ is "bounded after taking away finite number of points from Δ".

PROPOSITION 3.5. *Let Ω and Δ be two locally compact Hausdorff spaces. Let E and F be an essential Banach $C_0(\Omega)$-module and an essential Banach $C_0(\Delta)$-module respectively, and let $\theta : E \to F$ be a $\mathbb{C}$-linear map (not assumed to be bounded). Suppose that $\sigma : \Delta_\theta \to \Omega$ is an injection satisfying $\widetilde{\theta}(e\varphi)(\nu) = \widetilde{\theta}(e)(\nu)\varphi(\sigma(\nu))$ ($e \in E, \varphi \in C_0(\Omega), \nu \in \Delta_\theta$). Then there exist a finite subset $T \subseteq \Delta$ and a positive number κ such that*

$$\sup_{\nu\in\Delta\setminus T} \|\widetilde{\theta}(e)(\nu)\| \leqslant \kappa\|e\| \quad (e \in E).$$

LEMMA 3.6. *Let $\sigma : \Delta_\theta \to \Omega$ be a map satisfying $\theta(I^E_{\sigma(\nu)}) \subseteq K^F_\nu$ ($\nu \in \Delta_\theta$). Suppose, in addition, that F is a Banach $C_0(\Delta)$-normed module.*

(i) *$\mathfrak{N}_{\theta,\sigma}$ is an open subset of Δ.*

(ii) *If σ is injective, then $\mathfrak{U}_\theta$ is a finite set consisting of isolated points of Δ. If, in addition, $\mathfrak{U}_\theta \neq \Delta$, then $F = F_{\Delta\setminus\mathfrak{U}_\theta} \oplus \bigoplus_{\nu\in\mathfrak{U}_\theta} \Xi^F_\nu$ and*

$$\theta_0 := P_{\theta,\sigma} \circ \theta|_{E_{\Omega\setminus\sigma(\mathfrak{U}_\theta)}} : E_{\Omega\setminus\sigma(\mathfrak{U}_\theta)} \to F_{\Delta\setminus\mathfrak{U}_\theta}$$

is a bounded linear map (where $P_{\theta,\sigma} : F \to F_{\Delta\setminus\mathfrak{U}_\theta}$ is the canonical projection) such that

$$\theta_0(e\varphi) = \theta_0(e)(\varphi\circ\sigma) \quad (e \in E_{\Omega\setminus\sigma(\mathfrak{U}_\theta)}, \varphi \in C_0(\Omega \setminus \sigma(\mathfrak{U}_\theta))), \tag{3.3}$$

when we extends σ to a map from Δ to Ω arbitrarily.

Proof. Notice, first of all, that as F is $C_0(\Omega)$-convex, one may regard $\widetilde{\theta} = \theta$.

(i) As Δ_θ is open in Δ and $\mathfrak{N}_{\theta,\sigma} \subseteq \mathfrak{U}_\theta \subseteq \Delta_\theta$, it suffices to show that $\mathfrak{N}_{\theta,\sigma}$ is open in Δ_θ. By Lemma 3.3(i),

$$\kappa := \sup_{\nu\notin\mathfrak{U}_\theta} \sup_{\|e\|\leqslant 1} \|\theta(e)(\nu)\| < \infty.$$

Let $\{\nu_i\}_{i\in I}$ be a net in $\Delta_\theta \setminus \mathfrak{N}_{\theta,\sigma}$ converging to $\nu_0 \in \Delta_\theta$, and e be an arbitrary element in $K^E_{\sigma(\nu_0)}$. By Lemma 3.1, we know that $\sigma(\nu_i) \to \sigma(\nu_0)$. Now, we consider two cases separately. The first case is when $\{\sigma(\nu_i)\}_{i\in I}$ is finite. In this case, by passing to a subnet, we may assume that $\sigma(\nu_i) = \sigma(\nu_0)$ ($i \in I$). As $e \in K^E_{\sigma(\nu_0)} = K^E_{\sigma(\nu_i)}$ and $\nu_i \notin \mathfrak{N}_{\theta,\sigma}$, we have $\theta(e)(\nu_i) = 0$ which gives $\theta(e)(\nu_0) = 0$, and so, $\theta(e) \in K^F_{\nu_0}$. The second case (of $\{\sigma(\nu_i)\}_{i\in I}$ being infinite) can be subdivided into two cases. More precisely, if there exists $i_0 \in I$ such that $\nu_j \in \mathfrak{U}_\theta$ for every $j \geqslant i_0$, then we may assume that $\{\sigma(\nu_i)\}_{i\in I} \subseteq \sigma(\mathfrak{U}_\theta)$ which is a finite set, and the above implies that $\theta(e) \in K^F_{\nu_0}$. Otherwise, $\{i \in I : \nu_i \notin \mathfrak{U}_\theta\}$ is cofinal, and by passing to a subnet, we may assume that $\nu_i \notin \mathfrak{U}_\theta$ ($i \in I$). For any $\varepsilon > 0$, pick $V \in \mathcal{N}_\Omega(\sigma(\nu_0))$ and $e_V \in K^E_V$ with $\|e_V - e\| < \varepsilon$. When i is large enough, $\sigma(\nu_i) \in V$ and $e_V(\sigma(\nu_i)) = 0$. Thus,

$$\|\theta(e)(\nu_i)\| = \|\theta(e - e_V)(\nu_i)\| \leqslant \kappa\varepsilon.$$

By the continuity of the norm function on Ξ^F, we have $\|\theta(e)(\nu_0)\| \leqslant \kappa\varepsilon$ which implies that $\theta(e)(\nu_0) = 0$.

(ii) By the hypothesis and Lemma 3.3(i), one knows that $\mathfrak{U}_\theta$ is finite. Without loss of generality, we assume $\Delta \neq \mathfrak{U}_\theta$. Let

$$\kappa := \sup_{\nu\in\Delta\setminus\mathfrak{U}_\theta} \sup_{\|e\|\leqslant 1} \|\theta(e)(\nu)\| < \infty. \tag{3.4}$$

Suppose on the contrary that there is $\nu_0 \in \mathfrak{U}_\theta$ which is not an isolated point in Δ. As $\mathfrak{U}_\theta$ is finite, there is a net $\{\nu_i\}$ in $\Delta \setminus \mathfrak{U}_\theta$ such that $\nu_i \to \nu_0$. By the definition of $\mathfrak{U}_\theta$, there is $e \in E$ with $\|e\| \leqslant 1$ and $\|\theta(e)(\nu_0)\| > \kappa+1$. However, this will contradict the continuity of $|\theta(e)|$ (because of (3.4)). Now, as $\mathfrak{U}_\theta$ is a finite set consisting of isolated points in Δ and F is the space of C_0-sections on Ξ^F, we see that

$$F = K^F_{\mathfrak{U}_\theta} \oplus \bigoplus_{\nu\in\mathfrak{U}_\theta} \Xi^F_\nu.$$

By the argument of Lemma 3.3(iii) (more precisely, (3.2) is true for $\nu \in \Delta \setminus \mathfrak{U}_\theta$), one easily check that θ_0 will satisfy (3.3). On the other hand, the boundedness of θ_0 follows from (3.4). ∎

Observe that in Lemmas 3.3(iii) and 3.6(ii), one can replace the injectivity of σ with the condition that $\sigma^{-1}(\omega)$ is at most finite for any $\omega \in \Omega$.

The following is our second nearly automatically boundedness result that applies, in particular, when F is a Hilbert $C_0(\Delta)$-module.

THEOREM 3.7. *Let Ω and Δ be two locally compact Hausdorff spaces. Let E be an essential Banach $C_0(\Omega)$-module, let F be an essential Banach $C_0(\Delta)$-normed module, and let $\theta : E \to F$ be a $\mathbb{C}$-linear map (not assumed to be bounded). Suppose that $\sigma : \Delta_\theta \to \Omega$ is an injection satisfying $\theta(I^E_{\sigma(\nu)}) \subseteq K^F_\nu$ ($\nu \in \Delta$).*

(i) *If Δ contains no isolated point, then θ is bounded.*

(ii) *If σ sends isolated points in Δ_θ to isolated points in Ω, then $\mathfrak{N}_{\theta,\sigma} = \emptyset$ and there exist a finite set T consisting of isolated points of Δ, a bounded linear map $\theta_0 : E_{\Omega\setminus\sigma(T)} \to F_{\Delta\setminus T}$ as well as linear maps $\theta_\nu : \Xi^E_{\sigma(\nu)} \to \Xi^F_\nu$ ($\nu \in T$) such that $E = E_{\Omega\setminus\sigma(T)} \oplus \bigoplus_{\nu\in T} \Xi^E_{\sigma(\nu)}$ and $F = F_{\Delta\setminus T} \oplus \bigoplus_{\nu\in T} \Xi^F_\nu$ under which $\theta = \theta_0 \oplus \bigoplus_{\nu\in T} \theta_\nu$.*

Proof. (i) This follows directly from Lemma 3.6(ii).

(ii) The first conclusion follows from Lemmas 3.3(ii) and 3.6(ii), while the second conclusion follows from Lemma 3.6(ii) (notice that we have a sharper conclusion here since $\mathfrak{N}_{\theta,\sigma} = \emptyset$). ∎

COROLLARY 3.8. *Let Ξ be an* (H)*-Banach bundle over Ω, let Λ be an* (F)*-Banach bundle over Δ, and let $\rho : \Xi \to \Lambda$ be a map (not assumed to be bounded nor continuous). Suppose that $\sigma : \Delta \to \Omega$ is an injection sending isolated points in Δ to isolated points in Ω such that $e \mapsto \rho \circ e \circ \sigma$ defines a linear map $\theta : \Gamma_0(\Xi) \to \Gamma_0(\Lambda)$. Then there exists a finite set T consisting of isolated points of Δ such that the restriction of ρ induces a bounded Banach bundle map $\rho_0 : \Xi_{\Omega\setminus\sigma(T)} \to \Lambda_{\Delta\setminus T}$ (covering $\sigma|_{\Delta\setminus T}$). Moreover, σ is continuous on $\Delta \setminus \mathfrak{Z}_{\rho,\sigma}$ where $\mathfrak{Z}_{\rho,\sigma} := \{\nu \in \Delta : \rho(e(\sigma(\nu))) = 0$ for all $e \in E\}$.*

Proof. As θ is linear, we see that $\rho(0_{\sigma(\nu)}) = 0_\nu$ for any $\nu \in \Delta$. Consequently, $\theta(I^E_{\sigma(\nu)}) \subseteq K^F_\nu$ ($\nu \in \Delta$). Now the first conclusion follows from Theorem 3.7, and the second conclusion follows from Lemma 3.1 as well as the fact that $\mathfrak{Z}_{\rho,\sigma} = \mathfrak{Z}_\theta$. ∎

4. APPLICATIONS TO LOCAL LINEAR MAPPINGS

In the section, we will consider the case when $\Delta = \Omega$, $\sigma = \mathrm{id}$, and the $\mathbb{C}$-linear map θ is a *local map* in the sense that $\theta(e)\varphi = 0$ whenever $e \in E$ and $\varphi \in C_0(\Omega)$ satisfying $e\varphi = 0$. It is obvious that any $C_0(\Omega)$-module map is local.

REMARK 4.1. Suppose that θ is local. Let $U, V \subseteq \Omega$ be open sets with the closure of V being a compact subset of U, and consider $\lambda \in \mathcal{U}_\Omega(V, U)$. For any

$e \in K_U^E$ and any $\varepsilon > 0$, there exists $\varphi \in K_U$ with $\|e - e\varphi\| < \varepsilon$. Thus, $e\lambda = 0$ which implies that $\theta(e)\lambda = 0$ and $\theta(e) = \theta(e)(1-\lambda) \in K_V^F$. This shows that $\sigma = \mathrm{id}$ will satisfy the requirements in all the results in Section 3.

The following theorem (which follows directly from the results in Section 3 as well as Remark 4.1) is our main result concerning local linear maps.

THEOREM 4.2. *Let Ω be a locally compact Hausdorff space. Suppose that E and F are essential Banach $C_0(\Omega)$-modules, and $\theta : E \to F$ is a local $\mathbb{C}$-linear map (not assumed to be bounded).*

(i) *$\widetilde{\theta}$ is a $C_0(\Omega)$-module map and there exist a finite subset $T \subseteq \Omega$ as well as a positive number κ such that $\sup_{\nu\in\Omega\setminus T} \|\widetilde{\theta}(e)(\nu)\| \leqslant \kappa\|e\|$ ($e \in E$).*

(ii) *If, in addition, F is a Banach $C_0(\Omega)$-normed module, then θ is a $C_0(\Omega)$-module map and there exist a finite set T consisting of isolated points of Ω, a bounded linear map $\theta_0 : E_{\Omega\setminus T} \to F_{\Omega\setminus T}$ as well as a linear map $\theta_\nu : \Xi_\nu^E \to \Xi_\nu^F$ for each $\nu \in T$ such that $E = E_{\Omega\setminus T} \oplus \bigoplus_{\nu\in T} \Xi_\nu^E$, $F = F_{\Omega\setminus T} \oplus \bigoplus_{\nu\in T} \Xi_\nu^F$ and $\theta = \theta_0 \oplus \bigoplus_{\nu\in T} \theta_\nu$.*

It is natural to ask if one can relax the assumption of F being $C_0(\Omega)$-normed to $C_0(\Omega)$-convex in the second statement of Theorem 4.2 (in particular, whether it is true that every $C_0(\Omega)$-module map from an essential Banach $C_0(\Omega)$-module to an essential Banach $C_0(\Omega)$-convex module is automatically bounded provided that Ω contains no isolated point). Unfortunately, it is not the case as can be seen by the following simple example.

EXAMPLE 4.3. Let $E := C([0,1]) \oplus_\infty X$ and $F := C([0,1]) \oplus_\infty Y$, where X and Y are two infinite dimensional Banach spaces. Then E is an essential Banach $C([0,1])$-convex module under the multiplication: $(e,x)\varphi = (e\varphi, x\varphi(0))$ ($e, \varphi \in C([0,1]); x \in X$). In the same way, F is an essential Banach $C([0,1])$-convex module. Suppose that $R : X \to Y$ is an unbounded linear map and $\theta : E \to F$ is given by $\theta(e,x) = (e, R(x))$ ($e \in C([0,1]); x \in X$). Then θ is a $C([0,1])$-module map which is not bounded (as its restriction on X is R). In this case, we have $\mathfrak{U}_\theta = \{0\}$.

COROLLARY 4.4. *Let Ω be a locally compact Hausdorff space. Any local $\mathbb{C}$-linear map θ from an essential Banach $C_0(\Omega)$-module to a Hilbert $C_0(\Omega)$-module is a $C_0(\Omega)$-module map. Moreover, if Ω contains no isolated point, then any such θ is automatically bounded.*

REMARK 4.5. Let $L_{C_0(\Omega)}(E; C_0(\Omega))$ and $\mathcal{B}_{C_0(\Omega)}(E; C_0(\Omega))$ be the set of all $C_0(\Omega)$-module maps and the set of all bounded $C_0(\Omega)$-module maps, respectively, from E into $C_0(\Omega)$ (one may regards them as the "algebraic dual" and the "topological dual" of E respectively). An application of Corollary 4.4 is that the algebraic dual and the topological dual of E coincide in many cases:

If Ω is a locally compact Hausdorff space having no isolated point and E is an essential Banach $C_0(\Omega)$-module, then $\mathcal{B}_{C_0(\Omega)}(E;C_0(\Omega)) = L_{C_0(\Omega)}(E;C_0(\Omega))$.

COROLLARY 4.6. *Let Ξ and Λ be respectively an* (H)*-Banach bundle and an* (F)*-Banach bundle over the same base space Ω. If $\rho : \Xi \to \Lambda$ is a fibrewise linear map covering* id *(without any boundedness nor continuity assumption) such that $\rho \circ e \in \Gamma_0(\Lambda)$ for every $e \in \Gamma_0(\Xi)$, then there exists a finite subset $S \subseteq \Omega$ consisting of isolated points such that ρ restricts to a bounded Banach bundle map $\rho_0 : \Xi_{\Omega\setminus S} \to \Lambda_{\Omega\setminus S}$.*

Let X and Y be Banach spaces. If $E := \ell^\infty(X)$ and $F := \ell^\infty(Y)$ be the sets of all bounded sequences in X and Y respectively, then E and F are unital Banach $C(\beta\mathbb{N})$-modules (where $\beta\mathbb{N}$ is the Stone–Čech compactification of $\mathbb{N}$, which can be identified with the set of all ultrafilters on $\mathbb{N}$). If $x \in E$, we denote by x_k the kth-coordinate of x (i.e. $x = [x_k]_{k\in\mathbb{N}}$). Suppose that $\theta : E \to F$ is a linear map (not assumed to be bounded) such that $\lim_{\mathcal{F}} \theta(x)_k = 0$ for every ultrafilter $\mathcal{F} \in \beta\mathbb{N}$ and $x \in E$ with $\lim_{\mathcal{F}} x_k = 0$. Since

$$K^E_{\mathcal{F}} = \left\{x \in E : \lim_{\mathcal{F}} x_k = 0\right\} \quad \text{and} \quad K^F_{\mathcal{F}} = \left\{y \in F : \lim_{\mathcal{F}} y_k = 0\right\} \quad (\mathcal{F} \in \beta\mathbb{N}),$$

the map θ induces a linear map $\theta_{\mathcal{F}} : \Xi^E_{\mathcal{F}} \to \Xi^F_{\mathcal{F}}$ for each $\mathcal{F} \in \beta\mathbb{N}$. In particular, we obtain $\theta_k : X \to Y$ such that $\theta(x)_k = \theta_k(x_k)$ ($k \in \mathbb{N}; x \in E$). Moreover, by Theorem 4.2, we have the following corollary.

COROLLARY 4.7. *Let X and Y be Banach spaces. Suppose that $\theta : \ell^\infty(X) \to \ell^\infty(Y)$ is a linear map such that $\lim_{\mathcal{F}} \theta(x)_k = 0$ for every ultrafilter $\mathcal{F}$ on $\mathbb{N}$ and every $x \in \ell^\infty(X)$ with $\lim_{\mathcal{F}} x_k = 0$. There exist $\mathcal{F}_1, \dots, \mathcal{F}_n \in \beta\mathbb{N}$ with $\sup_{\mathcal{F}\in\beta\mathbb{N}\setminus\{\mathcal{F}_1,\dots,\mathcal{F}_n\}} \|\theta_{\mathcal{F}}\| < \infty$.*

REMARK 4.8. (i) Note that if $\mathcal{F}$ is a free ultrafilter on $\mathbb{N}$, then $\Xi^{\ell^\infty(X)}_{\mathcal{F}}$ and $\Xi^{\ell^\infty(Y)}_{\mathcal{F}}$ can be identified with the ultrapowers $X^{\mathcal{F}}$ and $Y^{\mathcal{F}}$ of X and Y (over $\mathcal{F}$) respectively. Thus, Corollary 4.7 implies that under the assumption of this corollary, θ induces a bounded linear map $\theta_{\mathcal{F}} : X^{\mathcal{F}} \to Y^{\mathcal{F}}$ for all but a finite number of free ultrafilter $\mathcal{F}$ and they have a common bound.

(ii) In our original version of Corollary 4.7, we also included results concerning linear maps from $c_0(X)$ to $c_0(Y)$ and from $c(X)$ to $c(Y)$. However, the referee has kindly shown us the following more general result with a very simple argument:

If $\theta_k : X \to Y$ is a sequence of linear maps such that the induced map θ from the space of sequences in X to the space of sequences in Y sends $c_0(X)$ to $\ell^\infty(Y)$, there is $n_0 \in \mathbb{N}$ such that $\sup_{n\geqslant n_0} \|\theta_n\| < \infty$.

The simple argument of the referee is as follows: if for each $k \in \mathbb{N}$, there exist $n_k > n_{k-1}$ and $x^0_{n_k} \in X$ with $\|x^0_{n_k}\| = 1$ and $\|\theta_{n_k}(x^0_{n_k})\| \geqslant k^2$, and one sets $x_n :=$

$x^0_{n_k}/k$ when $n = n_k$, as well as $x_n := 0$ when $n \notin \{n_k : k \in \mathbb{N}\}$, then one obtains the contradiction that $\theta(x) \notin \ell^\infty(Y)$.

Another important point in Theorem 4.2 is the automatic $C_0(\Omega)$-linearity. In fact, it can be shown that for every C^*-algebra A, any bounded local linear map from a Banach right A-module into a Hilbert A-module is automatically A-linear (see Proposition A.1 in the Appendix). Theorem 4.2 tells us that if A is commutative, then one can relax the assumption of the range space to a Banach A-convex module and one can remove the boundedness assumption. Another application of Theorem 4.2 is that if A is a finite dimensional C^*-algebras, then every local linear map between any two Banach right A-modules is A-linear.

COROLLARY 4.9. *Let A be a finite dimensional C^*-algebra. Suppose that E and F are unital Banach right A-modules. If $\theta : E \to F$ is a local $\mathbb{C}$-linear map in the sense of Proposition* A.1 *(not assumed to be bounded), then θ is an A-module map.*

Proof. Pick any $x \in E$ and $a \in A_{sa}$. Let $A_a := C^*(a, 1)$. Theorem 4.2 tell us that $\widetilde{\theta}$ is an A_a-module map. By Remark 2.5(iii), we see that θ is also an A_a-module map. In particular, $\theta(xa) = \theta(x)a$. ∎

REMARK 4.10. (i) Suppose that A is a unital C^*-algebra and F is a unital Banach right A-convex module in the sense $\|xa + y(1-a)\| \leqslant \max\{\|x\|, \|y\|\}$ for $x, y \in F$ and $a \in A_+$ with $a \leqslant 1$. Then, by the argument of Corollary 4.9, all local linear maps from any unital Banach right A-module into F are automatically A-linear.

(ii) If one can show that for every compact subset $\Omega \subseteq \mathbb{R}$ and every essential Banach $C(\Omega)$-module F, the map $\sim: F \to \widetilde{F}$ is injective, then using the argument of Corollary 4.9, one can show that for each C^*-algebra A, all local linear maps between any two Banach right A-modules are A-module maps (without assuming that θ is bounded). However, we do not know if it is true.

5. APPLICATIONS TO SEPARATING MAPPINGS

In this section, we consider Ω and Δ to be possibly different spaces. In this case, one cannot define local property any more, but one has a weaker natural property called separating. More precisely, θ is said to be *separating* if

$$|\widetilde{\theta}(e)||\widetilde{\theta}(g)| = 0, \quad \text{whenever } e, g \in E \text{ satisfying } |\widetilde{e}||\widetilde{g}| = 0.$$

In the case when $E = C_0(\Omega)$ and $F = C_0(\Delta)$, this coincides with the well-known notion of disjointness preserving (see e.g. [1], [5], [18], [14], [12], [15]).

LEMMA 5.1. *If $\sim: F \to \widetilde{F}$ is injective and θ is separating, there is a continuous map $\sigma : \Delta_\theta \to \Omega_\infty$ such that $\theta(I^E_{\sigma(\nu)}) \subseteq I^F_\nu$ ($\nu \in \Delta_\theta$).*

Proof. Set
$$S_\nu := \{\omega \in \Omega_\infty : \theta(I^E_\omega) \subseteq I^F_\nu\} \quad (\nu \in \Delta_\theta).$$
Suppose there is $\nu \in \Delta_\theta$ with $S_\nu = \emptyset$. Then for each $\omega \in \Omega_\infty$, there exist $U_\omega \in \mathcal{N}_{\Omega_\infty}(\omega)$ and $e_\omega \in K^E_{U_\omega}$ with $\theta(e_\omega) \notin I^F_\nu$. Let $\{U_{\omega_i}\}_{i=1}^n$ be a finite subcover of $\{U_\omega\}_{\omega\in\Omega_\infty}$ and $\{\varphi_i\}_{i=1}^n$ be a partition of unity subordinate to $\{U_{\omega_i}\}_{i=1}^n$. Take any $g \in E$. From $|\widetilde{g\varphi_i}||\widetilde{e_{\omega_i}}| = 0$, we obtain $|\widetilde{\theta}(g\varphi_i)||\widetilde{\theta}(e_{\omega_i})| = 0$, which implies that $\widetilde{\theta}(g\varphi_i)(\nu) = 0$ (otherwise, by Remark 2.5(i), one can find $U \in \mathcal{N}_{\Delta_\infty}(\nu)$ with $\widetilde{\theta}(g\varphi_i)(\mu) \neq 0$ for any $\mu \in U$, and the above as well as Remark 2.5(iv) will give the contradiction that $\theta(e_{\omega_i}) \in I^F_\nu$). Consequently,
$$\widetilde{\theta}(g)(\nu) = \sum_{i=1}^n \widetilde{\theta}(g\varphi_i)(\nu) = 0 \quad (g \in E),$$
which contradicts $\nu \notin \mathfrak{Z}_\theta$. Suppose there is $\nu \in \Delta_\theta$ with S_ν containing two distinct points ω_1 and ω_2. Let $U, V \in \mathcal{N}_{\Omega_\infty}(\omega_1)$ with $V \subseteq \mathrm{Int}_{\Omega_\infty}(U)$ and $\omega_2 \notin U$. For any $\varphi \in \mathcal{U}_{\Omega_\infty}(V, U)$ and $e \in E$, we have $e(1-\varphi) \in I^E_{\omega_1}$ and $e\varphi \in I^E_{\omega_2}$ which implies that
$$\theta(e) = \theta(e(1-\varphi)) + \theta(e\varphi) \in I^F_\nu.$$
This gives the contradiction that $\nu \in \mathfrak{Z}_\theta$. Therefore, we can define $\sigma(\nu)$ to be the unique point in S_ν, and it is clear that $\theta(I^E_{\sigma(\nu)}) \subseteq I^F_\nu$. Now, the continuity of σ follows from Lemma 3.1 (because $I^F_\nu \subseteq K^F_\nu$). ∎

THEOREM 5.2. *Let Ω and Δ be two locally compact Hausdorff spaces, and let E be a full essential Banach $C_0(\Omega)$-module (see Remark* 3.2(ii)*) and F be a full essential Banach $C_0(\Delta)$-normed module. Suppose that $\theta : E \to F$ is a bijective $\mathbb{C}$-linear map (not assumed to be bounded) such that it is biseparating in the sense that both θ and θ^{-1} are separating.*

(i) *There exists a homeomorphism $\sigma : \Delta \to \Omega$ satisfying*
$$\theta(e\varphi) = \theta(e)(\varphi \circ \sigma) \quad (e \in E; \varphi \in C_0(\Omega)).$$

(ii) *There exists a possibly empty finite subset $\{\nu_1, \ldots, \nu_n\} \subseteq \Delta$ consisting of isolated points such that the restriction of θ induces a Banach space isomorphism $\theta_0 : E_{\Omega'} \to F_{\Delta'}$ where $\Delta' := \Delta \setminus \{\nu_1, \ldots, \nu_n\}$ and $\Omega' := \sigma(\Delta_\theta)$.*

Proof. (i) If $e \in E$ with $\widetilde{e} = 0$, then $\theta(e) = \widetilde{\theta}(e) = 0$ (as θ is separating and F is $C_0(\Delta)$-convex), which gives $e = 0$ (as θ is injective). Hence, one can identify $\widetilde{e}$ with e and $\widetilde{f}$ with f ($e \in E, f \in F$) as well as regard $\widetilde{\theta} = \theta$ and $\widetilde{\theta^{-1}} = \theta^{-1}$. The fullness of E and F as well as the surjectivity of θ and θ^{-1} ensure that $\mathfrak{Z}_\theta = \emptyset$ and $\mathfrak{Z}_{\theta^{-1}} = \emptyset$. Therefore, by Lemma 5.1, we have two continuous maps
$$\tau : \Omega \to \Delta_\infty \quad \text{and} \quad \sigma : \Delta \to \Omega_\infty$$
such that $\theta^{-1}(I^F_{\tau(\omega)}) \subseteq I^E_\omega$ ($\omega \in \Omega$) and $\theta(I^E_{\sigma(\nu)}) \subseteq I^F_\nu$ ($\nu \in \Delta$). Consequently, for any $\nu \in \Delta_0 := \sigma^{-1}(\Omega)$ and $\omega \in \Omega_0 := \tau^{-1}(\Delta)$, we have
$$\sigma(\tau(\omega)) = \omega \quad \text{and} \quad \tau(\sigma(\nu)) = \nu$$

(because E and F are full, and we have $I^E_{\sigma(\tau(\omega))} \subseteq I^E_\omega$ as well as $I^F_{\tau(\sigma(\nu))} \subseteq I^F_\nu$; see also Remark 2.5(i) and (iv)). If there exists $\nu \in \Delta \setminus \mathfrak{N}_{\theta,\sigma}$ ($\mathfrak{N}_{\theta,\sigma}$ as in Lemma 3.3(ii)) with $\sigma(\nu) = \infty$, then $F = \theta(K^E_\infty) \subseteq K^F_\nu$, which contradicts the fullness of F. Thus,

$$\Delta \setminus \mathfrak{N}_{\theta,\sigma} \subseteq \Delta_0.$$

On the other hand, as $\Delta_0 \cap \mathfrak{N}_{\theta,\sigma}$ is a finite set (by Lemma 3.3(i) and (ii) and the fact that σ is injective on Δ_0) and is open in Δ (by Lemma 3.6(i)), we see that $\Delta_0 \cap \mathfrak{N}_{\theta,\sigma}$ consists of isolated points of Δ. Thus, $\sigma(\Delta_0 \cap \mathfrak{N}_{\theta,\sigma})$ consists of isolated points of Ω_0 (as σ restricts to a homeomorphism from Δ_0 to Ω_0). We want to show that

$$\Delta_0 \cap \mathfrak{N}_{\theta,\sigma} = \emptyset.$$

Suppose on the contrary that there is $\nu \in \Delta_0 \cap \mathfrak{N}_{\theta,\sigma}$. We know that $\sigma(\nu)$ ($\neq \infty$) is a non-isolated point of Ω_∞ (by Lemma 3.3(ii)). Therefore, there exists a net $\{\omega_i\}_{i\in I}$ in $\Omega \setminus \{\sigma(\nu)\}$ converging to $\sigma(\nu)$. If $\{i \in I : \omega_i \in \Omega_0\}$ is cofinal, then there is a net in $\Omega_0 \setminus \{\sigma(\nu)\}$ converging to $\sigma(\nu)$, which contradicts $\sigma(\nu)$ being an isolated point in Ω_0. Otherwise, $\omega_i \in \tau^{-1}(\infty)$ eventually, which gives the contradiction that $\nu = \infty$ (note that $\tau(\omega_i) \to \tau(\sigma(\nu)) = \nu$ as $\nu \in \Delta_0$).

Consequently,

$$\Delta \setminus \mathfrak{N}_{\theta,\sigma} = \Delta_0.$$

Suppose that $\mathfrak{N}_{\theta,\sigma} \neq \emptyset$ and $\nu \in \mathfrak{N}_{\theta,\sigma}$. Since $\mathfrak{N}_{\theta,\sigma}$ is an open subset of Δ (by Lemma 3.6(i)), there exists $V \in \mathcal{N}_\Delta(\nu)$ with $V \subseteq \mathfrak{N}_{\theta,\sigma}$. Take any $f \in F$ such that $f(\nu) \neq 0$ (by the fullness of F) and f vanishes outside V. Thus, $f \in I^F_\infty$ (as V is compact) and so, $\theta^{-1}(f)(\omega) = 0$ for any $\omega \in \tau^{-1}(\infty)$. On the other hand, for any $\omega \in \Omega_0$, one has $\tau(\omega) \in \Delta_0$ and so, $f \in I^F_{\tau(\omega)}$ (as f vanishes on the open set Δ_0 containing $\tau(\omega)$) which implies that $\theta^{-1}(f)(\omega) = 0$. Hence $\theta^{-1}(f) = 0$ which contradicts the injectivity of θ^{-1}. Therefore, $\mathfrak{N}_{\theta,\sigma} = \emptyset$. This shows that $\sigma : \Delta \to \Omega$ is a homeomorphism and part (i) follows from Lemma 3.3(iii).

(ii) This follows directly from Theorem 3.7(ii). ∎

One can apply the above to the case when F is a full Hilbert $C_0(\Delta)$-module. Another direct application of Theorem 5.2 is the following theorem which extends and enriches a result of Chan [8] (by removing the boundedness assumption on θ), as well as results concerning the product bundle cases discussed in [4], [13]. Notice that if $(\Omega, \{\Xi_x\}, E)$ is a continuous fields of Banach spaces over a locally compact Hausdorff space Ω (as defined in [9], [11]), then E is a full essential Banach $C_0(\Omega)$-normed module.

THEOREM 5.3. *Let $(\Omega, \{\Xi_x\}, E)$ and $(\Delta, \{\Lambda_y\}, F)$ be continuous fields of Banach spaces over locally compact Hausdorff spaces Ω and Δ respectively. Let $\theta : E \to F$ be a bijective linear map such that both θ and its inverse θ^{-1} are separating. Then there is a homeomorphism $\sigma : \Delta \to \Omega$ and a bijective linear operator $H_\nu : \Xi_{\sigma(\nu)} \to \Lambda_\nu$ such that*

$$\theta(f)(\nu) = H_\nu(f(\sigma(\nu))) \quad (f \in E, \nu \in \Delta).$$

Moreover, at most finitely many H_ν are unbounded, and this can happen only when ν is an isolated point in Δ. In particular, if Ω (or Δ) contains no isolated point, then θ is automatically bounded.

Appendix A. BOUNDED LOCAL LINEAR MAPS ARE A-LINEAR

PROPOSITION A.1. *Let A be a C^*-algebra, and let θ be a bounded linear map from a Banach right A-modules E into a Hilbert A-module F. Then θ is a right A-module map if and only if θ is local (in the sense that $\theta(e)a = 0$ whenever $e \in E$ and $a \in A$ with $ea = 0$).*

Proof. Suppose θ is local. Observe, first of all, that E^{**} and F^{**} are unital Banach A^{**}-modules, and the bidual map $\theta^{**} : E^{**} \to F^{**}$ is a bounded weak*-weak*-continuous linear map. Fix $x \in E$ and $a \in A_+$, and let

$$\Phi : C(\sigma(a))^{**} \to A^{**}$$

be the map induced by the canonical normal $*$-homomorphism $\Psi : M(A)^{**} \to A^{**}$. Pick $\alpha, \beta \in \mathbb{R}_+$ with $\alpha < \beta$, and define $p := \Phi(\chi_{\sigma(a)\cap(\alpha,\beta)})$. Let $\{f_n\}$ and $\{g_n\}$ be two bounded sequences in $C(\sigma(a))_+$ such that $f_n g_n = 0$, as well as

$$f_n \uparrow \chi_{\sigma(a)\cap(\alpha,\beta)} \quad \text{and} \quad g_n \downarrow \chi_{\sigma(a)\setminus(\alpha,\beta)} \quad \text{pointwisely.}$$

Note that as $\Psi(A) \subseteq A$, we have $a_n := \Phi(f_n) \in A$ (observe that $f_n(0) = 0$ if $0 \in \sigma(a)$), and we can write $b_n := \Phi(g_n)$ as $c_n + \gamma_n 1$ (where $c_n \in A$ and $\gamma_n \in \mathbb{C}$). Fix $n \in \mathbb{N}$. Since a_n and c_n commute, there is a locally compact Hausdorff space Ω with $C^*(a_n, c_n) \cong C_0(\Omega)$. By considering $b_n \in C_0(\Omega)_+ + \mathbb{R}_+ 1 \subseteq C_\mathrm{b}(\Omega)_+$, one can find a net $\{d_i\}_{i\in I}$ in $C_0(\Omega)_+ \subseteq A_+$ such that $d_i \leqslant b_n$ ($i \in I$) and $d_i \to b_n$ pointwisely. As $0 \leqslant d_i \leqslant b_n$ and $a_n b_n = 0$ in $C_\mathrm{b}(\Omega)$, one knows that $a_n d_i = 0$. Now, the relations $\theta(xa_n)d_i = 0$ and $\theta(xd_i)a_n = 0$ imply that $\theta^{**}(xa_n)b_n = 0$ and $\theta^{**}(xb_n)a_n = 0$. Since the multiplication in the bidual of the linking algebra of F is jointly weak*-continuous on bounded subsets, we see that $\theta^{**}(xp)(1-p) = 0$ and $\theta^{**}(x(1-p))p = 0$, which implies that $\theta^{**}(xp) = \theta^{**}(x)p$. Finally, there exists $r_k \in \mathbb{R}$ and $\alpha_k, \beta_k \in \mathbb{R}_+$ such that $\alpha_k \leqslant \beta_k$ and

$$\sup_{t\in\sigma(a)} \left| t - \sum_{k=1}^{M} r_k \chi_{\sigma(a)\cap(\alpha_k,\beta_k)}(t) \right| \to 0.$$

Thus, by the weak*-continuity again, we get $\theta^{**}(xa) = \theta^{**}(x)a$ as required. ∎

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