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AN ADDITIVE INVARIANT ON THE VECTOR-VALUED HARDY SPACE OVER THE BALL

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ABSTRACT. This paper concerns an additive invariant on any invariant subspaces of the vector-valued Hardy space over the ball, which is an analogous version of the Arveson's curvature invariant on the symmetric Fock space [2]. Inspired by Fang's work [9], [7], we prove that this invariant equals to the fiber dimension of the invariant subspace.

KEYWORDS: *Hardy space, curvature invariant, Samuel multiplicity, fiber dimension.*

MSC (2000): 47A13, 47B37.

INTRODUCTION

The Arveson's curvature invariant was first introduced in [2]. Since then various applications and generalizations to other function spaces have been made, see [8], [10], [12], [16]. Many known proofs rely on the properties of Nevanlinna-Pick kernels. In 2007, Fang [9] introduced an analogous curvature invariant for the vector-valued Hardy space over the polydisc, where there is no Nevanlinna-Pick kernel, and similar results are obtained by using technical methods. This approach makes it possible to consider this curvature invariant on many other spaces. Motivated by [7], [8], we consider such an invariant on the vector-valued Hardy space over the ball.

Firstly, we introduce some notations. Let $\mathbb{B}_n$ denote the unit ball in $\mathbb{C}^n$, and $H^2(\mathbb{B}_n) \otimes \mathbb{C}^N$ the vector-valued Hardy space over the ball, $N \in \mathbb{N}$. For a closed subspace $\mathcal{M}$ of $H^2(\mathbb{B}_n) \otimes \mathbb{C}^N$, let $P_{\mathcal{M}}$ be the orthogonal projection onto $\mathcal{M}$. An invariant subspace of $H^2(\mathbb{B}_n) \otimes \mathbb{C}^N$ means a closed subspace that is invariant for the coordinate operators M_{z_j} $(1 \leqslant j \leqslant n)$. We use $\widetilde{P}_k$ to denote the projection from $H^2(\mathbb{B}_n)$ onto the space of polynomials of degree at most k, and put $P_k = \widetilde{P}_k \otimes I_N$. For an operator T in the trace class, denote the trace of T by $\mathrm{tr}(T)$.

DEFINITION 0.1. Given an invariant subspace $\mathcal{M} \subset H^2(\mathbb{B}_n) \otimes \mathbb{C}^N$, define the multiplicity invariant $m(\mathcal{M})$ as follows:

$$m(\mathcal{M}) = \lim_{k\to\infty} \frac{\mathrm{tr}(P_{\mathcal{M}} P_k)}{\mathrm{tr}(\widetilde{P}_k)}.$$

Similarly, define $m(\mathcal{M}^\perp)$ by setting $m(\mathcal{M}^\perp) = \lim_{k\to\infty} \frac{\mathrm{tr}(P_{\mathcal{M}^\perp} P_k)}{\mathrm{tr}(\widetilde{P}_k)}$.

In fact, in the case of symmetric Fock space, it was shown in [7] that this invariant is exactly the Arveson's curvature invariant. It is well known that

$$\mathrm{tr}(\widetilde{P}_k) = \mathrm{rank}(\widetilde{P}_k) = \frac{(n+k)\cdots(n+1)}{k!} = \binom{n+k}{n}.$$

Notice also that if $m(\mathcal{M})$ exists, then does $m(\mathcal{M}^\perp)$, and $m(\mathcal{M}) + m(\mathcal{M}^\perp) = N$.

For any $\lambda \in \mathbb{B}_n$, let $\mathcal{M}(\lambda) = \{f(\lambda) : f \in \mathcal{M}\}$. From elementary linear algebraic analysis, $\dim \mathcal{M}(\lambda)$ is a lower semi-continuous function on $\mathbb{B}_n$, and it is almost everywhere constant. This constant is called the *fiber dimension* of $\mathcal{M}$, and denoted by $f-\dim(\mathcal{M})$. Our main result is as follows.

THEOREM 0.2. *For any invariant subspace $\mathcal{M} \subset H^2(\mathbb{B}_n) \otimes \mathbb{C}^N$,*

$$m(\mathcal{M}) = f - \dim(\mathcal{M}).$$

In particular, the multiplicity $m(\mathcal{M})$ is an integer.

Next we illustrate the connection between this invariant and the Samuel multiplicity. R. Douglas and V. Paulsen [4] introduced the Hilbert modules, and [3] is also a good reference for Hilbert modules. Based on their works, many algebraic methods can be used to study multivariate operator theory. For example, Hilbert polynomials not only have been investigated intensely in commutative algebra, but also play an important role in describing numerical invariant in operator theory. For example, Fang [7] studied Hilbert polynomials for certain holomorphic function spaces, and obtained some formulas that revealed the relation between the Arveson's curvature invariant and the Samuel multiplicity.

Now we recall the definition of the Samuel multiplicity with respect to $\mathcal{M}$. Equip $H^2(\mathbb{B}_n) \otimes \mathbb{C}^N$ with module structure over the polynomial ring $A = \mathbb{C}[z_1, \ldots, z_n]$ by

$$\begin{array}{rcl} A \times (H^2(\mathbb{B}_n) \otimes \mathbb{C}^N) & \longrightarrow & H^2(\mathbb{B}_n) \otimes \mathbb{C}^N \\ (p\,,\, f) & \longmapsto & p(M_{z_1}, \ldots, M_{z_n}) f. \end{array}$$

Hence $\mathcal{M}$ is a submodule of $H^2(\mathbb{B}_n) \otimes \mathbb{C}^N$. Let $\mathcal{I} = (z_1, \ldots, z_n)$ be the maximal ideal of A at the origin. When $\dim(\mathcal{M}/\overline{\mathcal{I}\cdot\mathcal{M}}) < \infty$, the graded A-module

$$\mathrm{gr}(\mathcal{M}) = (\mathcal{M}/\overline{\mathcal{I}\cdot\mathcal{M}}) \oplus (\overline{\mathcal{I}\cdot\mathcal{M}}/\overline{\mathcal{I}^2\cdot\mathcal{M}}) \oplus (\overline{\mathcal{I}^2\cdot\mathcal{M}}/\overline{\mathcal{I}^3\cdot\mathcal{M}}) \oplus \cdots$$

is a finitely generated algebraic module over A [5]. By [6], there is a polynomial $P(z) \in \mathbb{Q}[z]$ and a natural number $N \geqslant 1$ such that

$$p(k) = \dim(\mathcal{M} \ominus \mathcal{I}^k \cdot \mathcal{M}) = \dim(\mathcal{M}/\overline{\mathcal{I}^k \cdot \mathcal{M}}), \quad k \geqslant N,$$

where $\mathbb{Q}[z]$ is the ring of polynomials in one variable with rational coefficients. Furthermore, there exist integers $c_0, \ldots, c_n$ such that

$$P(z) = c_0 + c_1 \binom{z}{1} + \cdots + c_n \binom{z}{n}, \quad z \in \mathbb{Z},$$

where $\binom{z}{i} = \frac{z \cdots (z-i+1)}{i!}$ $(1 \leqslant i \leqslant n)$ is the binomial coefficient function. Thus the leading coefficient of $P(z)$ is of the form $\frac{c_n}{n!}$. This c_n is called the *Samuel multiplicity* of $\mathcal{M}$, denoted by $e(\mathcal{M})$. We have the following.

THEOREM 0.3. *If $\mathcal{M} \subset H^2(\mathbb{B}_n) \otimes \mathbb{C}^N$ is generated by polynomials, then*

$$m(\mathcal{M}) = e(\mathcal{M}).$$

The proof is essentially the same as that of Lemma 9 in [7], which is omitted here.

1. PRELIMINARIES

In this paper, we will apply the Tauberian theory, that has played an important role in [9]. Some preliminaries are in order.

Given a series $\sum\limits_{k=0}^{\infty} c_k$, not necessarily convergent, define its 0-th *Cesaro sum* by

$$S_k^{(0)} = c_0 + \cdots + c_k,$$

and in general, define the n-th *Cesaro sum* by

$$S_k^{(n)} = S_0^{(n-1)} + \cdots + S_k^{(n-1)}.$$

The series $\sum\limits_{k=0}^{\infty} c_k$ is said (C, n) summable if

$$\lim_{k \to \infty} M(n, k) = c,$$

where

$$M(n,k) = \frac{S_k^{(n)}}{\binom{n+k}{n}} \tag{1.1}$$

is the n-th *Cesaro sum mean*. In this case, we write $(C, n) \sum\limits_{k=0}^{\infty} c_k = c$.

It is known that if we put

$$\sigma_i = \binom{n+k-i}{n}. \tag{1.2}$$

then $S_k^{(n)} = \sum_{i=0}^{k} \sigma_i c_i$. Now we state the following Tauberian theorem [1].

THEOREM 1.1 (Tauberian). *If a series* $\sum_{k=0}^{\infty} A_k$ *of real numbers is Abel summable to* A *(that is,* $\lim_{r\to 1^-} \sum_{k=0}^{\infty} A_k r^k = A$*). Suppose its* $(n-1)$*-th Cesaro sum is nonnegative, i.e.* $S_k^{(n-1)} \geqslant 0$ *for any* k*, then* $\sum_{k=0}^{\infty} A_k$ *is (C,n) summable to A.*

Next we turn to the reproducing kernel of $H^2(\mathbb{B}_n)$, which will be discussed in Step 3 of the proof of Theorem 0.2. It is well known that the reproducing kernel of $H^2(\mathbb{B}_n)$ is given by $K(z,w) = \frac{1}{(1-\langle z,w\rangle)^n}$, $z, w \in \mathbb{B}_n$. So $H = H^2(\mathbb{B}_n) \otimes \mathbb{C}^N$ has a $B(\mathbb{C}^N)$-valued reproducing kernel $K^H(z,w) = K(z,w) \cdot I_N$, i.e. $K^H(\cdot,w)\xi = K(\cdot,w)\xi$, for any $\xi \in \mathbb{C}^N$.

Let $ev_w : H^2(\mathbb{B}_n) \otimes \mathbb{C}^N \to \mathbb{C}^N$ be the evaluation functional at w; that is, $ev_w(f) = f(w)$, $w \in \mathbb{B}_n$, $f \in H^2(\mathbb{B}_n) \otimes \mathbb{C}^N$. Then $\|ev_w\| = \|K(\cdot,w)\|_{H^2(\mathbb{B}_n)}$. Put $j_w = \frac{ev_w}{\|ev_w\|}$, and it is easy to see that j_w is a partial isometry such that $j_w j_w^* = I_N$ and $j_w^* j_w = P_w$, where P_w denotes the orthogonal projection onto the N dimensional subspace $K(\cdot,w) \otimes \mathbb{C}^N$.

For each invariant subspace $\mathcal{M} \subset H^2(\mathbb{B}_n) \otimes \mathbb{C}^N$, the reproducing kernel $K^{\mathcal{M}}(z,w)$ of $\mathcal{M}$ is given by

$$K^{\mathcal{M}}(\cdot,w)\xi = P_{\mathcal{M}} K(\cdot,w)\xi, \quad w \in \mathbb{B}_n,\ \xi \in C^N.$$

Now define

$$k_{\mathcal{M}}(\lambda) = \operatorname{tr}(P_{\mathcal{M}} P_\lambda), \quad \lambda \in \mathbb{B}_n,$$

which is a vector version of the Berezin transform of $P_{\mathcal{M}}$, as mentioned in [9].

Observe that $\frac{K^{\mathcal{M}}(\lambda,\lambda)}{K(\lambda,\lambda)} = j_\lambda P_{\mathcal{M}} j_\lambda^*$, which has the same trace as $P_{\mathcal{M}} j_\lambda^* j_\lambda = P_{\mathcal{M}} P_\lambda$, and then it follows that

$$k_{\mathcal{M}}(\lambda) = \operatorname{tr}\Big(\frac{K^{\mathcal{M}}(\lambda,\lambda)}{K(\lambda,\lambda)}\Big).$$

Apply the method of the proof of Theorem 6 in [9], we immediately get the following.

THEOREM 1.2. *Let* $\mathcal{M}$ *be an invariant subspace in* $H^2(\mathbb{B}_n) \otimes \mathbb{C}^N$*. Then for almost every* $\lambda \in \partial\mathbb{B}_n$*, the limit function* $\frac{K^{\mathcal{M}}(\lambda,\lambda)}{K(\lambda,\lambda)} = \lim_{r\to 1^-} \frac{K^{\mathcal{M}}(r\lambda,r\lambda)}{K(r\lambda,r\lambda)}$ *exists, and for such* λ*,* $\frac{K^{\mathcal{M}}(\lambda,\lambda)}{K(\lambda,\lambda)}$ *is a projection with constant rank* $f - \dim(\mathcal{M})$*.*

The above treatment of boundary values can be regarded as a vector version of [15].

2. THE PROOF OF THEOREM 0.2

In this section, we will give the proof of Theorem 0.2. Some notations are needed. For any multi-index $I = (i_1, \ldots, i_n) \in \mathbb{Z}_+^n$, let $|I| = i_1 + \cdots + i_n$, $I! = i_1! \cdots i_n!$, $z^I = z_1^{i_1} \cdots z_n^{i_n}$, and $M_z^I = M_{z_1}^{i_1} \cdots M_{z_n}^{i_n}$, where $M_z = (M_{z_1}, \ldots, M_{z_n})$ is the tuple of coordinate operators. Let $\widetilde{E}_i$ be the orthogonal projection from $H^2(\mathbb{B}_n)$ onto the subspace of those polynomials of degree i, and $\widetilde{E}_i^\alpha$ denotes the orthogonal projection from $H^2(\mathbb{B}_n)$ onto the subspace generated by z^α, where α is a multi-index with $|\alpha| = i$. Set $E_i = \widetilde{E}_i \otimes I_N$, and $E_i^\alpha = \widetilde{E}_i^\alpha \otimes I_N$. Clearly, $E_i = \sum\limits_{|\alpha|=i} E_i^\alpha$. Then we have

$$\frac{\operatorname{tr}(P_{\mathcal{M}} P_k)}{\operatorname{tr}(\widetilde{P}_k)} = \frac{\sum\limits_{i=0}^{k} \operatorname{tr}(P_{\mathcal{M}} E_i)}{\binom{n+k}{n}} = \frac{\sum\limits_{i=0}^{k} \Big(\sum\limits_{|\alpha|=i} \operatorname{tr}(P_{\mathcal{M}} E_i^\alpha) \Big)}{\binom{n+k}{n}}. \tag{2.1}$$

Define two maps $\phi(\cdot)$ and $\phi_*(\cdot)$ on $H^2(\mathbb{B}_n) \otimes \mathbb{C}^N$ as follows:

$$\phi(X) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \Big(\sum_{|I|=k} \frac{k!}{I!} M_z^I X M_z^{I*} \Big),$$

and

$$\phi_*(X) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \Big(\sum_{|I|=k} \frac{k!}{I!} M_z^{I*} X M_z^I \Big),$$

where $X \in B(H^2(\mathbb{B}_n) \otimes \mathbb{C}^N)$, and the multi-index $I \neq (0, \ldots, 0)$.

Since $\phi^t(\cdot) E_i^\alpha = 0$ if $t > i$, then

$$\operatorname{tr}(P_{\mathcal{M}} E_i^\alpha) = \operatorname{tr}\Big(\sum_{t=0}^{i} (\phi^t - \phi^{t+1})(P_{\mathcal{M}}) E_i^\alpha \Big) = \operatorname{tr}\Big((P_{\mathcal{M}} - \phi(P_{\mathcal{M}})) \sum_{t=0}^{i} \phi_*^t(E_i^\alpha) \Big). \tag{2.2}$$

For simplicity, we write $\Phi(X) = X - \phi(X)$.

REMARK 2.1. Later in Step 3, it will be shown that $\Phi(P_{\mathcal{M}})$ is the integral operator given by $\Phi(P_{\mathcal{M}}) f(w) = \int\limits_{\partial \mathbb{B}_n} \frac{K^{\mathcal{M}}(w,z)}{K(w,z)} f(z) \mathrm{d}z$. So the form of $\Phi(\cdot)$ is related to that of the reproducing kernel $K(z,w)$. In fact, $\Phi(P_{\mathcal{M}})$ is the defect operator of $\mathcal{M}$ as introduced in [13]. For the symmetric Fock space, see [14].

Step 1. We first calculate $\operatorname{tr}(P_{\mathcal{M}} E_i^\alpha)$.

Observe that $M_z^{I*} z^\alpha = \frac{\|z^\alpha\|^2}{\|z^{\alpha-I}\|^2} z^{\alpha-I}$, where $\alpha - I = (\alpha_1 - i_1, \ldots, \alpha_n - i_n)$. Here $z^{\alpha-I}$ is understood to be zero if some component of $\alpha - I$ is less than 0. So $M_z^{I*} E_i^\alpha M_z^I = \frac{\|z^\alpha\|^2}{\|z^{\alpha-I}\|^2} E_{i-|I|}^{\alpha-I}$. Thus

$$\phi_*(E_i^\alpha) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \Big(\sum_{|I|=k} \frac{k!}{I!} M_z^{I*} E_i^\alpha M_z^I \Big) = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \Big(\sum_{|I|=k} \frac{k!}{I!} \frac{\|z^\alpha\|^2}{\|z^{\alpha-I}\|^2} E_{i-|I|}^{\alpha-I} \Big),$$

and hence

$$\phi_*^t(E_i^\alpha) = \sum_{k_1=1,\ldots,k_t=1}^{n} \sum_{|I_1|=k_1,\ldots,|I_t|=k_t} (-1)^{k_1-1}\cdots(-1)^{k_t-1}\binom{n}{k_1}\cdots\binom{n}{k_1}\frac{k_1!}{I_1!}\cdots\frac{k_t!}{I_t!}$$
$$\cdot \frac{\|z^\alpha\|^2}{\|z^{\alpha-I_1-\cdots-I_t}\|^2} E_{i-|I_1|-\cdots-|I_t|}^{\alpha-I_1-\cdots-I_t}. \tag{2.3}$$

For any power series $f(x) = \sum\limits_{I\geqslant 0} a_I x^I$, write $\mathrm{coe}(f,\ x^I) = a_I$ for the coefficient of x^I. Notice that for $x = (x_1,\ldots,x_n),\ I = (i_1,\ldots,i_n)$ and $|I| = k$ ($k \leqslant n$), one has

$$\mathrm{coe}\Big(1-\Big(1-\sum_{j=1}^n x_j\Big)^n,\ x^I\Big) = \mathrm{coe}\Big((-1)^{k-1}\binom{n}{k}\Big(\sum_{j=1}^n x_j\Big)^k,\ x^I\Big) = (-1)^{k-1}\binom{n}{k}\frac{k!}{I!}.$$

Then it follows from (2.3) that

$$\phi_*^t(E_i^\alpha) = \sum_{\beta\leqslant\alpha} \mathrm{coe}\Big(\Big(1-\Big(1-\sum_{j=1}^n x_j\Big)^n\Big)^t,\ x^\beta\Big)\frac{\|z^\alpha\|^2}{\|z^{\alpha-\beta}\|^2}E_{i-|\beta|}^{\alpha-\beta},$$

where $\beta \leqslant \alpha$ means $\beta_j \leqslant \alpha_j$ ($1 \leqslant j \leqslant n$). Now

$$\begin{aligned}
\sum_{t=0}^{i}\phi_*^t(E_i^\alpha) &= \sum_{t=0}^{i}\sum_{\beta\leqslant\alpha} \mathrm{coe}\Big(\Big(1-\Big(1-\sum_{j=1}^n x_j\Big)^n\Big)^t,\ x^\beta\Big)\frac{\|z^\alpha\|^2}{\|z^{\alpha-\beta}\|^2}E_{i-|\beta|}^{\alpha-\beta} \\
&= \sum_{\beta\leqslant\alpha} \mathrm{coe}\Big(\sum_{t=0}^{\infty}\Big(1-\Big(1-\sum_{j=1}^n x_j\Big)^n\Big)^t,\ x^\beta\Big)\frac{\|z^\alpha\|^2}{\|z^{\alpha-\beta}\|^2}E_{i-|\beta|}^{\alpha-\beta} \\
&= \sum_{\beta\leqslant\alpha} \mathrm{coe}\Big(\frac{1}{(1-\sum_{j=1}^n x_j)^n},\ x^\beta\Big)\frac{\|z^\alpha\|^2}{\|z^{\alpha-\beta}\|^2}E_{i-|\beta|}^{\alpha-\beta} \\
&= \sum_{\beta\leqslant\alpha} \frac{(n-1+|\beta|)!}{(n-1)!\beta!}\frac{\|z^\alpha\|^2}{\|z^{\alpha-\beta}\|^2}E_{i-|\beta|}^{\alpha-\beta} = \sum_{\beta\leqslant\alpha}\frac{\|z^\alpha\|^2}{\|z^\beta\|^2\|z^{\alpha-\beta}\|^2}E_{i-|\beta|}^{\alpha-\beta}.
\end{aligned}$$

The second identity follows from $\mathrm{coe}\Big(\Big(1-\Big(1-\sum\limits_{j=1}^n x_j\Big)^n\Big)^t,\ x^\beta\Big) = 0$ when $|\beta| \leqslant i < t$. The fourth identity follows from $\mathrm{coe}\Big(\frac{1}{(1-\sum_{j=1}^n x_j)^n},\ x^\beta\Big) = \frac{(n-1+|\beta|)!}{(n-1)!\beta!}$, which is easy to verify. And the last equation follows from $\int\limits_{\partial\mathbb{B}_n}|\xi^\alpha|^2\mathrm{d}\sigma(\xi) = \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!}$ [17].

Therefore by (2.2), we have

$$\mathrm{tr}(P_{\mathcal{M}}E_i^\alpha)) = \mathrm{tr}\Big((P_{\mathcal{M}}-\phi(P_{\mathcal{M}}))\sum_{\beta\leqslant\alpha}\frac{\|z^\alpha\|^2}{\|z^\beta\|^2\|z^{\alpha-\beta}\|^2}E_{i-|\beta|}^{\alpha-\beta}\Big). \tag{2.4}$$

Step 2. Next we will calculate $\mathrm{tr}(P_{\mathcal{M}}E_i)$.

Since $E_i = \sum\limits_{|\alpha|=i} E_i^{\alpha}$, (2.4) shows that

$$\begin{aligned}\operatorname{tr}(P_{\mathcal{M}}E_i) &= \sum_{|\alpha|=i}\operatorname{tr}(P_{\mathcal{M}}E_i^{\alpha}) = \operatorname{tr}\Big((P_{\mathcal{M}}-\phi(P_{\mathcal{M}}))\sum_{|\alpha|=i}\sum_{\beta\leqslant\alpha}\frac{\|z^{\alpha}\|^2}{\|z^{\beta}\|^2\|z^{\alpha-\beta}\|^2}E_{i-|\beta|}^{\alpha-\beta}\Big)\\ &= \operatorname{tr}\Big((P_{\mathcal{M}}-\phi(P_{\mathcal{M}}))\sum_{s=0}^{i}\Big(\sum_{\beta\leqslant\alpha,\ |\beta|=s}\ \sum_{|\alpha|=i}\frac{\|z^{\alpha}\|^2}{\|z^{\beta}\|^2\|z^{\alpha-\beta}\|^2}E_{i-|\beta|}^{\alpha-\beta}\Big)\Big).\end{aligned}$$

Before continuing, we have the following claim.

Claim. The sum $\sum\limits_{|\alpha|=i}\frac{\|z^{\alpha}\|^2}{\|z^{\gamma}\|^2\|z^{\alpha-\gamma}\|^2}$ is independent on the choice of the multi-index γ with $|\gamma| = i-s$.

In fact, for any fixed γ ($\gamma\leqslant\alpha,\ |\gamma| = i-s$), by the formula $\int\limits_{\partial\mathbb{B}_n}|\xi^{\alpha}|^2\mathrm{d}\sigma(\xi) = \frac{(n-1)!\alpha!}{(n-1+|\alpha|)!}$, one has

$$\sum_{|\alpha|=i}\frac{\|z^{\alpha}\|^2}{\|z^{\gamma}\|^2\|z^{\alpha-\gamma}\|^2} = \sum_{|\alpha|=i}\frac{\alpha!(n-1+s)!(n-1+i-s)!}{\gamma!(\alpha-\gamma)!(n-1+i)!(n-1)!}. \tag{2.5}$$

By the Lemma ??? (see Appendix), we have $\sum\limits_{|\alpha|=i}\frac{\alpha!}{\gamma!(\alpha-\gamma)!} = \binom{n-1+i}{s}$, and then (2.5) gives that

$$\sum_{|\alpha|=i}\frac{\|z^{\alpha}\|^2}{\|z^{\gamma}\|^2\|z^{\alpha-\gamma}\|^2} = \binom{n+s-1}{n-1}. \tag{2.6}$$

The proof of the claim is complete.

Now the coefficient of $E_{i-|s|}^{\gamma}$ in the sum $\sum\limits_{\beta\leqslant\alpha,\,|\beta|=s}\ \sum\limits_{|\alpha|=i}\frac{\|z^{\alpha}\|^2}{\|z^{\beta}\|^2\|z^{\alpha-\beta}\|^2}E_{i-|\beta|}^{\alpha-\beta}$ is

$$\sum_{|\alpha|=i,\,|\beta|=s,\,\alpha-\beta=\gamma}\frac{\|z^{\alpha}\|^2}{\|z^{\beta}\|^2\|z^{\gamma}\|^2} = \sum_{|\alpha|=i}\frac{\|z^{\alpha}\|^2}{\|z^{\gamma}\|^2\|z^{\alpha-\gamma}\|^2},$$

which equals to $\binom{n+s-1}{n-1}$ by (2.6). Since $E_{i-s} = \sum\limits_{|\gamma|=i-s}E_{i-s}^{\gamma}$,

$$\sum_{\beta\leqslant\alpha,\,|\beta|=s}\ \sum_{|\alpha|=i}\frac{\|z^{\alpha}\|^2}{\|z^{\beta}\|^2\|z^{\alpha-\beta}\|^2}E_{i-|\beta|}^{\alpha-\beta} = \binom{n+s-1}{n-1}E_{i-s}.$$

Consequently,

$$\operatorname{tr}(P_{\mathcal{M}}E_i) = \sum_{s=0}^{i}\binom{n+s-1}{n-1}\operatorname{tr}((P_{\mathcal{M}}-\phi(P_{\mathcal{M}}))E_{i-s}) \tag{2.7}$$
$$= \sum_{s=0}^{i}\binom{n+i-s-1}{n-1}\operatorname{tr}((P_{\mathcal{M}}-\phi(P_{\mathcal{M}}))E_s)$$
$$= \sum_{s=0}^{i}\binom{n+i-s-1}{n-1}\operatorname{tr}(\Phi(P_{\mathcal{M}})E_s).$$

REMARK 2.2. In the case of polydisc, since $M_z^{I_*}M_z^I = \mathrm{id}$, the similar identity in [9] as (2.7) can be obtained directly by substituting E_i for E_i^α in Step 1.

Step 3. In this step, we will apply Theorem 1.1 to finish the proof.

First put $A_s = \operatorname{tr}(\Phi(P_{\mathcal{M}})E_s)$. Combining (1.1), (1.2), (2.1) with (2.7), we get

$$\frac{\operatorname{tr}(P_{\mathcal{M}}P_k)}{\operatorname{tr}(\widetilde{P}_k)} = \frac{\sum\limits_{i=0}^{k}\Big(\sum\limits_{s=0}^{i}\binom{n+i-s-1}{n-1}A_s\Big)}{\binom{n+k}{n}} = \frac{\sum\limits_{s=0}^{k}\binom{n+k-s}{n}A_s}{\binom{n+k}{n}} = \frac{S_k^{(n)}}{\binom{n+k}{n}},$$

where $S_k^{(n)}$ is the n-th *Cesaro sum* of the series $\sum\limits_{s=0}^{\infty}A_s$. The $(n-1)$-th *Cesaro sum* is nonnegative because it equals to $\operatorname{tr}(P_{\mathcal{M}}E_k)$ by (1.2) and (2.7). So by Theorem 1.1, it remains to show that the series $\sum\limits_{s=0}^{\infty}A_s$ is Abel summable to $f-\dim(\mathcal{M})$. That is, we must show

$$\lim_{r\to 1^-}\sum_{s=0}^{\infty}A_s r^s = f-\dim(\mathcal{M}).$$

Let $K^{\mathcal{M}}(z,w)$ be the reproducing kernel of the invariant subspace $\mathcal{M}$. Recall that

$$\Phi(P_{\mathcal{M}}) = P_{\mathcal{M}} - \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\Big(\sum_{|I|=k}\frac{k!}{I!}M_z^I P_{\mathcal{M}}M_z^{I_*}\Big),$$

and hence for any $f \in H^2(\mathbb{B}_n)\otimes\mathbb{C}^N$, $w \in \mathbb{B}_n$ and $\xi \in \mathbb{C}$, we have

$$\langle(\Phi(P_{\mathcal{M}})f)(w),\xi\rangle_{\mathbb{C}^N} = \Big\langle f(\cdot),\frac{K^{\mathcal{M}}(\cdot,w)}{K(\cdot,w)}\xi\Big\rangle_{H^2(\mathbb{B}_n)\otimes\mathbb{C}^N} = \int_{\partial\mathbb{B}_n}\Big\langle f(z),\frac{K^{\mathcal{M}}(z,w)}{K(z,w)}\xi\Big\rangle_{\mathbb{C}^N}\mathrm{d}z$$
$$= \int_{\partial\mathbb{B}_n}\Big\langle\frac{K^{\mathcal{M}}(w,z)}{K(w,z)}f(z),\xi\Big\rangle_{\mathbb{C}^N}\mathrm{d}z = \Big\langle\int_{\partial\mathbb{B}_n}\frac{K^{\mathcal{M}}(w,z)}{K(w,z)}f(z)\mathrm{d}z,\xi\Big\rangle_{\mathbb{C}^N}.$$

Here the first equality follows from $K(z,w) = \frac{1}{(1-\langle z,w\rangle)^n}$. So

$$\Phi(P_{\mathcal{M}})f(w) = \int_{\partial\mathbb{B}_n}\frac{K^{\mathcal{M}}(w,z)}{K(w,z)}f(z)\mathrm{d}z. \tag{2.8}$$

Let $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ be the unit vector in $\mathbb{C}^N$, where 1 is at the j-th coordinate. Write $\frac{K^{\mathcal{M}}(w,z)}{K(w,z)} = \sum a_{K,J} w^K \bar{z}^J$, where $a_{K,J}$ are $N \times N$ matrices, then by (2.8) one has

$$\begin{aligned}
A_s &= \operatorname{tr}(\Phi(P_{\mathcal{M}})E_s) \\
&= \sum_{|I|=s, 1\leqslant j\leqslant N} \Big\langle \Big(\Phi(P_{\mathcal{M}}) \frac{z^I \otimes e_j}{\|z\|^I}\Big)(w),\ \frac{w^I \otimes e_j}{\|z\|^I} \Big\rangle_{H^2(\mathbb{B}_n)\otimes\mathbb{C}^N} \\
&= \sum_{|I|=s, 1\leqslant j\leqslant N} \frac{1}{\|z^I\|^2} \Big\langle \int_{\partial\mathbb{B}_n} \sum_{K,J} a_{K,J} w^K \bar{z}^J (z^I \otimes e_j) \mathrm{d}z,\ w^I \otimes e_j \Big\rangle_{H^2(\mathbb{B}_n)\otimes\mathbb{C}^N} \\
&= \sum_{|I|=s, 1\leqslant j\leqslant N} \langle a_{I,I}(w^I \otimes e_j),\ w^I \otimes e_j \rangle_{H^2(\mathbb{B}_n)\otimes\mathbb{C}^N} = \sum_{|I|=s} \operatorname{tr}(a_{I,I}) \|z^I\|^2. \qquad (2.9)
\end{aligned}$$

Moreover,

$$\begin{aligned}
\int_{\partial(\sqrt{r}\mathbb{B}_n)} \operatorname{tr}\Big(\frac{K^{\mathcal{M}}(z,z)}{K(z,z)}\Big) \mathrm{d}z &= \sum_{K,J} \int_{\partial(\sqrt{r}\mathbb{B}_n)} \operatorname{tr}(a_{K,J} z^K \bar{z}^J) \mathrm{d}z = \sum_I \operatorname{tr}(a_{I,I}) \int_{\partial(\sqrt{r}\mathbb{B}_n)} |z^I|^2 \mathrm{d}z \\
&= \sum_I \operatorname{tr}(a_{I,I}) (\sqrt{r})^{2n-1} (\sqrt{r})^{2|I|} \|z^I\|^2 \\
&= \sqrt{r}^{2n-1} \sum_{s=0}^{\infty} A_s r^s \quad (0 < r < 1). \qquad (2.10)
\end{aligned}$$

The last identity in (2.10) follows from (2.9).

Theorem 1.2 shows that the limit

$$\frac{K^{\mathcal{M}}(\lambda,\lambda)}{K(\lambda,\lambda)} = \lim_{r\to 1^-} \frac{K^{\mathcal{M}}(r\lambda, r\lambda)}{K(r\lambda, r\lambda)}$$

exists for almost every $\lambda \in \partial\mathbb{B}_n$, and for such λ, it is a projection with constant rank $f-\dim(\mathcal{M})$. By (2.10), we have $\lim_{r\to 1^-} \sum_{s=0}^{\infty} A_s r^s = f-\dim(\mathcal{M})$. The proof is completed.

3. APPENDIX

The following lemma for combinatorial number may be of independent interest. We cannot locate a reference, and a proof is included here.

LEMMA 3.1. *For each multi-index $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{Z}_+^n$,*

$$\sum_{|\alpha|=i, \alpha\geqslant\gamma} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} = \binom{n-1+i}{i-|\gamma|},$$

where i is a fixed natural number. Here $\alpha \geqslant \gamma$ means $\alpha_j \geqslant \gamma_j$ $(1 \leqslant j \leqslant n)$.

Proof. We will finish the proof by induction on n. When $n = 1$, the identity is trivial.

Now assume the identity holds for $n = k - 1$. In the case of $n = k$,

$$\sum_{|\alpha|=i,\,\alpha\geqslant\gamma} \frac{\alpha!}{\gamma!(\alpha-\gamma)!} = \sum_{|\alpha|=i,\,\alpha\geqslant\gamma} \frac{\alpha'!\alpha_k!}{\gamma'!(\alpha'-\gamma')!\gamma_k!(\alpha_k-\gamma_k)!}$$
$$= \sum_{\alpha_k=\gamma_k}^{\gamma_k+i-|\gamma|} \binom{k-2+i-\alpha_k}{i-\alpha_k-|\gamma'|}\binom{\alpha_k}{\gamma_k},$$

where $\alpha' = (\alpha_1, \ldots, \alpha_{k-1})$ and $\gamma' = (\gamma_1, \ldots, \gamma_{k-1})$. The second identity follows by induction hypothesis. Write $t = \alpha_k - \gamma_k$, and then

$$\sum_{\alpha_k=\gamma_k}^{\gamma_k+i-|\gamma|} \binom{k-2+i-\alpha_k}{i-\alpha_k-|\gamma'|}\binom{\alpha_k}{\gamma_k} = \sum_{t=0}^{i-|\gamma|} \binom{k-2+i-\gamma_k-t}{i-|\gamma|-t}\binom{\gamma_k+t}{\gamma_k},$$
$$= \sum_{t=0}^{i-|\gamma|} \binom{k-2+i-\gamma_k-t}{i-|\gamma|-t}\binom{\gamma_k+t}{t} = \binom{k-1+i}{i-|\gamma|}.$$

The last identity follows from the combinatorial identity [11]:

$$\sum_{k=0}^{m} \binom{y+m-k}{m-k}\binom{x+k}{k} = \binom{x+y+m+1}{m}.$$

The proof is completed. ∎

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