

## THE HIGHER-DIMENSIONAL AMENABILITY OF TENSOR PRODUCTS OF BANACH ALGEBRAS

ZINAIDA A. LYKOVA

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ABSTRACT. We investigate the higher-dimensional amenability of tensor products  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  of Banach algebras  $\mathcal{A}$  and  $\mathcal{B}$ . We prove that the weak bidimension  $db_w$  of the tensor product  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  of Banach algebras  $\mathcal{A}$  and  $\mathcal{B}$  with bounded approximate identities satisfies

$$db_w \mathcal{A} \widehat{\otimes} \mathcal{B} = db_w \mathcal{A} + db_w \mathcal{B}.$$

We show that it cannot be extended to arbitrary Banach algebras. For example, for a biflat Banach algebra  $\mathcal{A}$  which has a left or right, but not two-sided, bounded approximate identity, we have  $db_w \mathcal{A} \widehat{\otimes} \mathcal{A} \leq 1$  and  $db_w \mathcal{A} + db_w \mathcal{A} = 2$ . We describe explicitly the continuous Hochschild cohomology  $\mathcal{H}^n(\mathcal{A} \widehat{\otimes} \mathcal{B}, (X \widehat{\otimes} Y)^*)$  and the cyclic cohomology  $\mathcal{H}C^n(\mathcal{A} \widehat{\otimes} \mathcal{B})$  of certain tensor products  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  of Banach algebras  $\mathcal{A}$  and  $\mathcal{B}$  with bounded approximate identities; here  $(X \widehat{\otimes} Y)^*$  is the dual bimodule of the tensor product of essential Banach bimodules  $X$  and  $Y$  over  $\mathcal{A}$  and  $\mathcal{B}$  respectively.

KEYWORDS: *Hochschild cohomology, cyclic cohomology,  $C^*$ -algebra, semigroup algebra.*

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### 1. INTRODUCTION

Hochschild cohomology groups of Banach and  $C^*$ -algebras play an important role in K-theory [6] and in noncommutative geometry [5]. However, it is very difficult to describe explicitly non-trivial higher-dimensional cohomology for Banach algebras. In this paper we consider Hochschild cohomology groups of tensor products  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  of Banach algebras  $\mathcal{A}$  and  $\mathcal{B}$  with bounded approximate identities.

A Banach algebra  $\mathcal{A}$  such that  $\mathcal{H}^1(\mathcal{A}, X^*) = \{0\}$  for all Banach  $\mathcal{A}$ -bimodules  $X$  is called *amenable* [16]. B.E. Johnson proved in Proposition 5.4 of [16] that  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  is amenable if the Banach algebras  $\mathcal{A}$  and  $\mathcal{B}$  are amenable. We determine relations between the higher-dimensional amenability of Banach algebras  $\mathcal{A}$  and  $\mathcal{B}$ ,

and the higher-dimensional amenability of their tensor product algebra  $\mathcal{A} \widehat{\otimes} \mathcal{B}$ . Virtual diagonals and higher-dimensional amenability of Banach algebras were investigated in a paper of E.G. Effros and A. Kishimoto [10] for unital algebras and in papers of A.L.T. Paterson and R.R. Smith [23], [24] in the non-unital case. Recall that, for any  $n \geq 1$ , a Banach algebra  $\mathcal{A}$  is called *n-amenable* if the continuous Hochschild cohomology  $\mathcal{H}^n(\mathcal{A}, X^*) = \{0\}$  for every Banach  $\mathcal{A}$ -bimodule  $X$ . The *weak bidimension* of a Banach algebra  $\mathcal{A}$  is

$$db_w \mathcal{A} = \inf \{n : \mathcal{H}^{n+1}(\mathcal{A}, X^*) = \{0\} \text{ for all Banach } \mathcal{A}\text{-bimodule } X\}$$

(see [26]). It is clear that a Banach algebra  $\mathcal{A}$  is *n-amenable* but not  $(n-1)$ -amenable if and only if  $db_w \mathcal{A} = n-1$ , and that  $\mathcal{A}$  is amenable if and only if  $db_w \mathcal{A} = 0$ .

In 1996 Yu.V. Selivanov announced in Remark 4 of [27] without proof that the weak bidimension  $db_w$  of the tensor product  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  of Banach algebras  $\mathcal{A}$  and  $\mathcal{B}$  with bounded approximate identities satisfies

$$db_w \mathcal{A} \widehat{\otimes} \mathcal{B} = db_w \mathcal{A} + db_w \mathcal{B}.$$

In 2002 he gave in Theorem 4.6.8 of [28] a proof of the formula in the particular case of algebras with identities, and his proof depends heavily on the existence of identities. In Theorem 4.16 of this paper we prove that the formula is correct for algebras with bounded approximate identities (b.a.i.). We show further that the formula does *not* hold for algebras with no b.a.i, nor for algebras with only 1-sided b.a.i. To this end we need to develop homological tools for algebras with bounded approximate identities and flat essential modules. The well-known trick of adjoining an identity to the algebra does not work for the tensor product of algebras. The homological properties of the tensor product algebras  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  and  $\mathcal{A}_+ \widehat{\otimes} \mathcal{B}_+$  are different; here  $\mathcal{A}_+$  is the Banach algebra obtained by adjoining an identity to  $\mathcal{A}$ . In Theorem 4.17 we show that, for a biflat Banach algebra  $\mathcal{A}$  which has a left or right, but not two-sided, bounded approximate identity, we have  $db_w \mathcal{A} \widehat{\otimes} \mathcal{A} \leq 1$  and  $db_w \mathcal{A}_+ \widehat{\otimes} \mathcal{A}_+ = 2db_w \mathcal{A} = 2$ . For example, for the algebra  $\mathcal{K}(\ell_2 \widehat{\otimes} \ell_2)$  of compact operators on  $\ell_2 \widehat{\otimes} \ell_2$  and any integer  $n \geq 1$ ,

$$db_w[\mathcal{K}(\ell_2 \widehat{\otimes} \ell_2)]^{\widehat{\otimes} n} \leq 1 \quad \text{and} \quad db_w[\mathcal{K}(\ell_2 \widehat{\otimes} \ell_2)_+]^{\widehat{\otimes} n} = n.$$

We prove that, for the tensor product  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  of Banach algebras  $\mathcal{A}$  and  $\mathcal{B}$  with bounded approximate identities, for the tensor product  $X \widehat{\otimes} Y$  of essential Banach bimodules  $X$  and  $Y$  over  $\mathcal{A}$  and  $\mathcal{B}$  respectively and for  $n \geq 0$ , up to topological isomorphism,

$$\mathcal{H}^n(\mathcal{A} \widehat{\otimes} \mathcal{B}, (X \widehat{\otimes} Y)^*) = H^n((C_\sim(\mathcal{A}, X) \widehat{\otimes} C_\sim(\mathcal{B}, Y))^*),$$

where  $C_\sim(\mathcal{A}, X)$  and  $C_\sim(\mathcal{B}, Y)$  are the standard homological chain complexes.

We describe explicitly the continuous Hochschild cohomology  $\mathcal{H}^n(\mathcal{A} \widehat{\otimes} \mathcal{C}, (X \widehat{\otimes} Y)^*)$  and the cyclic cohomology  $\mathcal{HC}^n(\mathcal{A} \widehat{\otimes} \mathcal{C})$  of certain tensor

products of Banach algebras. For example, in Corollary 5.6 we prove that, for an amenable Banach algebra  $\mathcal{C}$ ,

$$\mathcal{H}^n(L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C}, (L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C})^*) \cong \bigoplus^{\binom{k}{n}} [L^1(\mathbb{R}_+^k) \widehat{\otimes} (\mathcal{C}/[\mathcal{C}, \mathcal{C}])]^*$$

if  $n \leq k$ . In Corollary 5.9 we show that all continuous cyclic type cohomology groups are trivial for a Banach algebra  $\mathcal{D}$  belonging to one of the following classes:

- (i)  $\mathcal{D} = \ell^1(\mathbb{Z}_+^k) \widehat{\otimes} \mathcal{C}$ , where  $\mathcal{C}$  is a  $C^*$ -algebra without non-zero bounded traces; or
- (ii)  $\mathcal{D} = L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C}$ , where  $\mathcal{C}$  is a  $C^*$ -algebra without non-zero bounded traces.

## 2. DEFINITIONS AND NOTATION

We recall some notation and terminology used in homological theory. These can be found in any textbook on homological algebra, for instance, MacLane [21] for the pure algebraic case and Helemskii [14] for the continuous case.

For a Banach algebra  $\mathcal{A}$ , let  $X$  be a Banach  $\mathcal{A}$ -bimodule and let  $S$  be a subset of  $\mathcal{A}$ . Then  $S^2$  is defined to be the linear span of the set  $\{a_1 \cdot a_2 : a_1, a_2 \in S\}$ ,  $SX$  is the linear span of the set  $\{a \cdot x : a \in S, x \in X\}$  and  $\overline{SX}$  is the closure of  $SX$  in  $X$ . Expressions like  $\overline{XS}$  and  $\overline{SXS}$  have similar meanings. A Banach  $\mathcal{A}$ -bimodule  $X$  is called *essential* if  $X = \overline{AXA}$ . Let  $I$  be a closed two-sided ideal with a bounded approximate identity; then, by an extension of the Cohen factorization theorem,  $\overline{IX} = \{b \cdot x : b \in I, x \in X\}$ . These facts may be found in [15], [7]; see also Proposition 1.8 of [16] and Theorem 5.2.2 of [22].

The category of Banach spaces and continuous linear operators is denoted by *Ban*. For a Banach algebra  $\mathcal{A}$ , the category of left [essential] {unital} Banach  $\mathcal{A}$ -modules is denoted by  $\mathcal{A}\text{-mod}$  [ $\mathcal{A}$ -essmod]  $\{\mathcal{A}\text{-unmod}\}$ , the category of right [essential] {unital} Banach  $\mathcal{A}$ -modules is denoted by  $\text{mod-}\mathcal{A}$  [essmod- $\mathcal{A}$ ]  $\{\text{unmod-}\mathcal{A}\}$  and the category of [essential] {unital} Banach  $\mathcal{A}$ -bimodules is denoted by  $\mathcal{A}\text{-mod-}\mathcal{A}$  [ $\mathcal{A}$ -essmod- $\mathcal{A}$ ]  $\{\mathcal{A}\text{-unmod-}\mathcal{A}\}$ .

For a Banach space  $E$ , we will denote by  $E^*$  the dual space of  $E$ . In the case of Banach algebras  $\mathcal{A}$ , for a Banach  $\mathcal{A}$ -bimodule  $X$ ,  $X^*$  is the Banach  $\mathcal{A}$ -bimodule dual to  $X$  with the module multiplications given by

$$(a \cdot f)(x) = f(x \cdot a), \quad (f \cdot a)(x) = f(a \cdot x) \quad (a \in \mathcal{A}, f \in X^*, x \in X).$$

A *chain complex*  $\mathcal{X}$  in *Ban* is a family of Banach spaces  $X_n$  and continuous linear maps  $d_n$  (called *boundary maps*)

$$\dots \xleftarrow{d_{n-2}} X_{n-1} \xleftarrow{d_{n-1}} X_n \xleftarrow{d_n} X_{n+1} \xleftarrow{d_{n+1}} X_{n+2} \xleftarrow{\dots}$$

such that  $\text{Im } d_n \subset \text{Ker } d_{n-1}$ . The subspace  $\text{Im } d_n$  of  $X_n$  is denoted by  $B_n(\mathcal{X})$  and its elements are called *boundaries*. The Banach subspace  $\text{Ker } d_{n-1}$  of  $X_n$  is denoted by  $Z_n(\mathcal{X})$  and its elements are *cycles*. The *homology groups* of  $\mathcal{X}$  are defined by

$H_n(\mathcal{X}) = Z_n(\mathcal{X})/B_n(\mathcal{X})$ . As usual, we will often drop the subscript  $n$  of  $d_n$ . If there is a need to distinguish between various boundary maps on various chain complexes, we will use subscripts, that is, we will denote the boundary maps on  $\mathcal{X}$  by  $d_{\mathcal{X}}$ . A chain complex  $\mathcal{X}$  is called *bounded* if  $X_n = \{0\}$  whenever  $n$  is less than a certain fixed integer  $N \in \mathbb{Z}$ .

Let  $\mathcal{K}$  be one of the above categories of Banach  $\mathcal{A}$ -modules and morphisms. For  $X, Y \in \mathcal{K}$ , the Banach space of morphisms from  $X$  to  $Y$  is denoted by  $h_{\mathcal{K}}(X, Y)$ . We shall abbreviate  $h_{\mathcal{A}\text{-mod}}$  to  $h_{\mathcal{A}}$  and  $h_{\mathcal{A}\text{-mod-}\mathcal{A}}$  to  ${}_{\mathcal{A}}h_{\mathcal{A}}$ .

A complex of Banach  $\mathcal{A}$ -modules and morphisms is called *admissible* if it splits as a complex of Banach spaces ([14], III.3.11). A complex of Banach  $\mathcal{A}$ -modules and morphisms is called *weakly admissible* if its dual complex splits as a complex of Banach spaces ([14], III.3.11).

A module  $P \in \mathcal{K}$  is called *projective in  $\mathcal{K}$*  if, for each module  $Y \in \mathcal{K}$  and each epimorphism of modules  $\varphi \in h_{\mathcal{K}}(Y, P)$  such that  $\varphi$  has a right inverse as a morphism of Banach spaces, there exists a right inverse morphism of Banach modules from  $\mathcal{K}$ .

For  $Y \in \mathcal{K}$ , a complex

$$0 \longleftarrow Y \xleftarrow{\varepsilon} P_0 \xleftarrow{\phi_0} P_1 \xleftarrow{\phi_1} P_2 \longleftarrow \dots \quad (0 \longleftarrow Y \xleftarrow{\varepsilon} (\mathcal{P}, \phi))$$

is called a *projective resolution of  $Y$  in  $\mathcal{K}$*  if it is admissible and all the modules in  $\mathcal{P}$  are projective in  $\mathcal{K}$  ([14], Definition III.2.1).

We shall denote the  $n$ th cohomology of the complex  $h_{\mathcal{K}}(\mathcal{P}, X)$  where  $X \in \mathcal{K}$  by  $\text{Ext}_{\mathcal{K}}^n(Y, X)$ . We shall abbreviate  $\text{Ext}_{\mathcal{A}\text{-mod}}^n$  to  $\text{Ext}_{\mathcal{A}}^n$  and  $\text{Ext}_{\mathcal{A}\text{-mod-}\mathcal{A}}^n$  to  $\text{Ext}_{\mathcal{A}^e}^n$ .

Further  $\widehat{\otimes}$  is the projective tensor product of Banach spaces ([4] or [14], II.4.1)  $\widehat{\otimes}_{\mathcal{A}}$  is the projective tensor product of left and right Banach  $\mathcal{A}$ -modules,  $\widehat{\otimes}_{\mathcal{A}\text{-}\mathcal{A}}$  is the projective tensor product of Banach  $\mathcal{A}$ -bimodules (see [25]). Note that by  $X^{\widehat{\otimes}0} \widehat{\otimes} Y$  we mean  $Y$ , by  $X^{\widehat{\otimes}1}$  we mean  $X$  and by  $X^{\widehat{\otimes}n}$  we mean the  $n$ -fold projective tensor power  $X \widehat{\otimes} \dots \widehat{\otimes} X$  of  $X$ .

For any Banach algebra  $\mathcal{A}$ , not necessarily unital,  $\mathcal{A}_+$  is the Banach algebra obtained by adjoining an identity to  $\mathcal{A}$ . For a Banach algebra  $\mathcal{A}$ , the algebra  $\mathcal{A}^e = \mathcal{A}_+ \widehat{\otimes} \mathcal{A}_+^{\text{op}}$  is called the *enveloping algebra of  $\mathcal{A}$* , where  $\mathcal{A}_+^{\text{op}}$  is the *opposite algebra* of  $\mathcal{A}_+$  with multiplication  $a \cdot b = ba$ .

A module  $Y \in \mathcal{A}\text{-mod}$  is called *flat* if for any admissible complex  $\mathcal{X}$  of right Banach  $\mathcal{A}$ -modules the complex  $\mathcal{X} \widehat{\otimes}_{\mathcal{A}} Y$  is exact. A module  $Y \in \mathcal{A}\text{-mod-}\mathcal{A}$  is called *flat* if for any admissible complex  $\mathcal{X}$  of Banach  $\mathcal{A}$ -bimodules the complex  $\mathcal{X} \widehat{\otimes}_{\mathcal{A}^e} Y$  is exact.

For  $X \in \mathcal{K}$ , a complex

$$0 \longleftarrow X \xleftarrow{\varepsilon} Q_0 \xleftarrow{\phi_0} Q_1 \xleftarrow{\phi_1} Q_2 \longleftarrow \dots \quad (0 \longleftarrow X \xleftarrow{\varepsilon} (\mathcal{Q}, \phi))$$

is called a *pseudo-resolution of  $X$  in  $\mathcal{K}$*  if it is weakly admissible, and a *flat pseudo-resolution of  $X$  in  $\mathcal{K}$*  if, in addition, all the modules in  $\mathcal{Q}$  are flat in  $\mathcal{K}$ .

For  $X \in \text{mod-}\mathcal{A}$  and  $Y \in \mathcal{A}\text{-mod}$ , we shall denote by  $\text{Tor}_n^{\mathcal{A}}(X, Y)$  the  $n$ th homology of the complex  $X \widehat{\otimes}_{\mathcal{A}} \mathcal{P}$ , where  $(0 \leftarrow Y \leftarrow \mathcal{P})$  is a projective resolution in  $\mathcal{A}\text{-mod}$ , ([14], Definition III.4.23). For  $X, Y \in \mathcal{A}\text{-mod-}\mathcal{A}$ , the Tor-spaces are defined by using the standard identifications  $\mathcal{A}\text{-mod-}\mathcal{A} \cong \mathcal{A}^e\text{-unmod} \cong \text{unmod-}\mathcal{A}^e$ .

Throughout the paper  $1_X : X \rightarrow X$  denotes the identity operator and  $\cong$  denotes an isomorphism of Banach spaces. Given a Banach space  $E$  and a chain complex  $(\mathcal{X}, d)$  in  $\mathcal{B}an$ , we can form the chain complex  $E \widehat{\otimes} \mathcal{X}$  of the Banach spaces  $E \widehat{\otimes} X_n$  and boundary maps  $1_E \otimes d$ .

We recall the definition of the tensor product  $\mathcal{X} \widehat{\otimes} \mathcal{Y}$  of bounded complexes  $\mathcal{X}$  and  $\mathcal{Y}$  in  $\mathcal{B}an$  which can be found in Definition II.5.25 of [14].

DEFINITION 2.1. Let  $\mathcal{X}, \mathcal{Y}$  be chain complexes in  $\mathcal{B}an$ :

$$0 \xleftarrow{\phi_{-1}} X_0 \xleftarrow{\phi_0} X_1 \xleftarrow{\phi_1} X_2 \xleftarrow{\phi_2} X_3 \xleftarrow{\dots}$$

and

$$0 \xleftarrow{\psi_{-1}} Y_0 \xleftarrow{\psi_0} Y_1 \xleftarrow{\psi_1} Y_2 \xleftarrow{\psi_2} Y_3 \xleftarrow{\dots}$$

The tensor product  $\mathcal{X} \widehat{\otimes} \mathcal{Y}$  of bounded complexes  $\mathcal{X}$  and  $\mathcal{Y}$  in  $\mathcal{B}an$  is the chain complex

$$(2.1) \quad 0 \xleftarrow{\delta_{-1}} (\mathcal{X} \widehat{\otimes} \mathcal{Y})_0 \xleftarrow{\delta_0} (\mathcal{X} \widehat{\otimes} \mathcal{Y})_1 \xleftarrow{\delta_1} (\mathcal{X} \widehat{\otimes} \mathcal{Y})_2 \xleftarrow{\delta_2} (\mathcal{X} \widehat{\otimes} \mathcal{Y})_3 \xleftarrow{\dots},$$

where

$$(\mathcal{X} \widehat{\otimes} \mathcal{Y})_n = \bigoplus_{m+q=n} X_m \widehat{\otimes} Y_q$$

and

$$\delta_{n-1}(x \otimes y) = \phi_{m-1}(x) \otimes y + (-1)^m x \otimes \psi_{q-1}(y),$$

$x \in X_m, y \in Y_q$  and  $m + q = n$ .

For Banach spaces  $E$  and  $F$ , let  $\mathfrak{B}(E, F)$  be the Banach space of all continuous linear operators from  $E$  to  $F$ .

### 3. PSEUDO-RESOLUTIONS IN CATEGORIES OF BANACH MODULES

LEMMA 3.1. *Let*

$$(3.1) \quad \dots \xleftarrow{d_{n-2}} X_{n-1} \xleftarrow{d_{n-1}} X_n \xleftarrow{d_n} X_{n+1} \xleftarrow{d_{n+1}} X_{n+2} \xleftarrow{\dots}$$

*be a weakly admissible complex of Banach spaces and continuous linear operators. Then, for every Banach space  $Y$ , the sequence*

$$(3.2) \quad \dots \xleftarrow{1_Y \otimes d_{n-1}} Y \widehat{\otimes} X_{n-1} \xleftarrow{1_Y \otimes d_n} Y \widehat{\otimes} X_n \xleftarrow{1_Y \otimes d_{n+1}} Y \widehat{\otimes} X_{n+1} \xleftarrow{\dots}$$

*is weakly admissible.*

*Proof.* By assumption the complex (3.1) is weakly admissible. Therefore, for every  $n$ , there is a bounded linear operator

$$s_n : X_{n+1}^* \rightarrow X_n^*,$$

such that  $d_{n-1}^* \circ s_{n-1} + s_n \circ d_n^* = 1_{X_n^*}$ , where

$$\cdots \xrightarrow{d_{n-2}^*} X_{n-1}^* \xrightarrow{d_{n-1}^*} X_n^* \xrightarrow{d_n^*} X_{n+1}^* \xrightarrow{d_{n+1}^*} X_{n+2}^* \longrightarrow \cdots$$

Further we will use the well-known isomorphism ([14], Theorem 2.2.17)

$$\mathfrak{B}(Y, X_n^*) \rightarrow (Y \widehat{\otimes} X_n)^* : \phi \mapsto \Phi_\phi$$

where  $\Phi_\phi(y \otimes x) = [\phi(y)](x); y \in Y, x \in X_n$  and

$$(Y \widehat{\otimes} X_n)^* \rightarrow \mathfrak{B}(Y, X_n^*) : f \mapsto \phi_f$$

where  $[\phi_f(y)](x) = f(y \otimes x); y \in Y, x \in X_n$ . One can see that, for  $f \in (Y \widehat{\otimes} X_n)^*$ , we have the following  $\phi_f \in \mathfrak{B}(Y, X_n^*)$ ,  $s_{n-1} \circ \phi_f \in \mathfrak{B}(Y, X_{n-1}^*)$  and  $\Phi_{s_{n-1} \circ \phi_f} \in (Y \widehat{\otimes} X_{n-1})^*$ . We define a map

$$\gamma_{n-1} : (Y \widehat{\otimes} X_n)^* \rightarrow (Y \widehat{\otimes} X_{n-1})^*$$

by  $\gamma_{n-1}(f) = \Phi_{s_{n-1} \circ \phi_f}$  for  $f \in (Y \widehat{\otimes} X_n)^*$ . It is easy to see that  $\gamma_{n-1}$  is a continuous linear operator. One can check that

$$\gamma_n \circ (1_Y \otimes d_n)^* + (1_Y \otimes d_{n-1})^* \circ \gamma_{n-1} = 1_{(Y \widehat{\otimes} X_n)^*},$$

where

$$(3.3) \quad \cdots \xrightarrow{(1_Y \otimes d_{n-2})^*} (Y \widehat{\otimes} X_{n-1})^* \xrightarrow{(1_Y \otimes d_{n-1})^*} (Y \widehat{\otimes} X_n)^* \xrightarrow{(1_Y \otimes d_n)^*} (Y \widehat{\otimes} X_{n+1})^* \longrightarrow \cdots$$

The result follows. ■

We shall denote  $\gamma_{n-1}$  from the previous lemma by  $\gamma_{(s, X_n, Y)}$ .

LEMMA 3.2. *Let*

$$(3.4) \quad \cdots \xleftarrow{d_{n-2}} X_{n-1} \xleftarrow{d_{n-1}} X_n \xleftarrow{d_n} X_{n+1} \xleftarrow{d_{n+1}} X_{n+2} \longleftarrow \cdots$$

be a weakly admissible complex of Banach spaces and continuous operators. Then, for Banach spaces  $Y$  and  $Z$  and a bounded linear operator  $\delta : Y \rightarrow Z$ , the following diagram

$$(3.5) \quad \begin{array}{ccccccc} \cdots & \xleftarrow{\gamma_{(s, X_{n-1}, Z)}} & (Z \widehat{\otimes} X_{n-1})^* & \xleftarrow{\gamma_{(s, X_n, Z)}} & (Z \widehat{\otimes} X_n)^* & \xleftarrow{\gamma_{(s, X_{n+1}, Z)}} & (Z \widehat{\otimes} X_{n+1})^* \cdots \\ \cdots & & (\delta \otimes 1_{X_{n-1}})^* \downarrow & & (\delta \otimes 1_{X_n})^* \downarrow & & (\delta \otimes 1_{X_{n+1}})^* \downarrow \\ \cdots & \xleftarrow{\gamma_{(s, X_{n-1}, Y)}} & (Y \widehat{\otimes} X_{n-1})^* & \xleftarrow{\gamma_{(s, X_n, Y)}} & (Y \widehat{\otimes} X_n)^* & \xleftarrow{\gamma_{(s, X_{n+1}, Y)}} & (Y \widehat{\otimes} X_{n+1})^* \cdots \end{array}$$

is commutative.

*Proof.* We shall show that for every  $f \in (Z \widehat{\otimes} X_n)^*$ ,

$$(\delta \otimes 1_{X_{n-1}})^* \circ \gamma_{(s, X_n, Z)}(f) = \gamma_{(s, X_n, Y)} \circ (\delta \otimes 1_{X_n})^*.$$

We shall use notations from the previous lemma. Note that for every  $x \in X_{n-1}$ ,  $y \in Y$ ,

$$[\phi_f(\delta y)](x) = f(\delta y \otimes x)$$

and

$$[\phi_{f \circ (\delta \otimes 1_{X_{n-1}})}(y)](x) = f \circ (\delta \otimes 1_{X_{n-1}})(y \otimes x) = f(\delta y \otimes x).$$

Thus, for all  $y \in Y$ ,

$$[\phi_f(\delta y)] = [\phi_{f \circ (\delta \otimes 1_{X_{n-1}})}(y)]$$

and

$$[s_{n-1} \circ \phi_f](\delta y) = [s_{n-1} \circ \phi_{f \circ (\delta \otimes 1_{X_{n-1}})}](y) = [s_{n-1} \circ \phi_{(\delta \otimes 1_{X_{n-1}})^* f}](y).$$

Therefore, for all  $x \in X_{n-1}$  and  $y \in Y$ ,

$$\begin{aligned} [(\delta \otimes 1_{X_{n-1}})^* \circ \gamma_{(s, X_n, Z)}(f)](y \otimes x) &= [\gamma_{(s, X_n, Z)}(f)](\delta y \otimes x) \\ &= \Phi_{s_{n-1} \circ \phi_f}(\delta y \otimes x) = [s_{n-1} \circ \phi_f](\delta y)(x) \end{aligned}$$

and

$$\begin{aligned} &[\gamma_{(s, X_n, Y)} \circ (\delta \otimes 1_{X_n})^*(f)](y \otimes x) \\ &= \Phi_{s_{n-1} \circ (\delta \otimes 1_{X_n})^* f}(y \otimes x) = [s_{n-1} \circ \phi_{(\delta \otimes 1_{X_{n-1}})^* f}](y)(x) = [s_{n-1} \circ \phi_f](\delta y)(x). \quad \blacksquare \end{aligned}$$

**PROPOSITION 3.3.** *Suppose that complexes of Banach spaces and continuous linear operators*

$$0 \longleftarrow X \xleftarrow{\varepsilon_1} X_0 \xleftarrow{d_0} X_1 \xleftarrow{d_1} X_2 \longleftarrow \dots \quad (0 \longleftarrow X \xleftarrow{\varepsilon_1} \mathcal{X})$$

and

$$0 \longleftarrow Y \xleftarrow{\varepsilon_2} Y_0 \xleftarrow{\tilde{d}_0} Y_1 \xleftarrow{\tilde{d}_1} Y_2 \longleftarrow \dots \quad (0 \longleftarrow Y \xleftarrow{\varepsilon_2} \mathcal{Y})$$

are weakly admissible. Then the complex  $0 \longleftarrow X \widehat{\otimes} Y \xleftarrow{\varepsilon_1 \otimes \varepsilon_2} \mathcal{X} \widehat{\otimes} \mathcal{Y}$  is also weakly admissible.

*Proof.* By assumption the complexes  $0 \longleftarrow X \xleftarrow{\varepsilon_1} \mathcal{X}$  and  $0 \longleftarrow Y \xleftarrow{\varepsilon_2} \mathcal{Y}$  are weakly admissible. Therefore, for every  $n$ , there is a bounded linear operator

$$s_n : X_{n+1}^* \rightarrow X_n^*$$

such that

$$d_{n-1}^* \circ s_{n-1} + s_n \circ d_n^* = 1_{X_n^*}, \quad \varepsilon_1^* \circ s_{-1} + s_0 \circ d_0^* = 1_{X_0^*} \quad \text{and} \quad s_{-1} \circ \varepsilon_1^* = 1_{X^*}.$$

Similarly, for every  $n$ , there is a bounded linear operator

$$t_n : Y_{n+1}^* \rightarrow Y_n^*,$$

such that

$$\tilde{d}_{n-1}^* \circ t_{n-1} + t_n \circ \tilde{d}_n^* = 1_{Y_n^*}, \quad \varepsilon_2^* \circ t_{-1} + t_0 \circ \tilde{d}_0^* = 1_{Y_0^*} \quad \text{and} \quad t_{-1} \circ \varepsilon_2^* = 1_{Y^*}.$$

Let us show that the complex

$$0 \longleftarrow X \widehat{\otimes} Y \xleftarrow{\varepsilon_1 \otimes \varepsilon_2} (\mathcal{X} \widehat{\otimes} \mathcal{Y})_0 \xleftarrow{\delta_0} (\mathcal{X} \widehat{\otimes} \mathcal{Y})_1 \xleftarrow{\delta_1} (\mathcal{X} \widehat{\otimes} \mathcal{Y})_2 \longleftarrow \dots$$

is weakly admissible too. Recall that

$$(\mathcal{X} \widehat{\otimes} \mathcal{Y})_n = \bigoplus_{m+q=n} X_m \widehat{\otimes} Y_q$$

and for each  $x \otimes y \in X_m \widehat{\otimes} Y_q$ ,

$$\delta_n(x \otimes y) = d_{m-1}(x) \otimes y + (-1)^m x \otimes \widetilde{d}_{q-1}(y).$$

By virtue of Lemma 3.1, for  $Y$  and  $Y_p$ ,  $p \geq 0$ , the sequence

$$0 \longrightarrow (X \widehat{\otimes} Y_p)^* \xrightarrow{(\varepsilon_1 \otimes 1_{Y_p})^*} (X_0 \widehat{\otimes} Y_p)^* \xrightarrow{(d_0 \otimes 1_{Y_p})^*} (X_1 \widehat{\otimes} Y_p)^* \xrightarrow{(d_1 \otimes 1_{Y_p})^*} (X_2 \widehat{\otimes} Y_p)^* \dots$$

splits as a complex of Banach spaces, so that there exist bounded linear operators

$$\gamma_{(s, X_n, Y_p)} : (X_n \widehat{\otimes} Y_p)^* \rightarrow (X_{n-1} \widehat{\otimes} Y_p)^*$$

such that

$$(3.6) \quad \begin{aligned} & \gamma_{(s, X_{n+1}, Y_p)} \circ (d_n \otimes 1_{Y_p})^* + (d_{n-1} \otimes 1_{Y_p})^* \circ \gamma_{(s, X_n, Y_p)} = 1_{(X_n \widehat{\otimes} Y_p)^*}, \\ & (\varepsilon_1 \otimes 1_{Y_p})^* \circ \gamma_{(s, X_0, Y_p)} + \gamma_{(s, X_1, Y_p)} \circ (d_0 \otimes 1_{Y_p})^* = 1_{(X_0 \widehat{\otimes} Y_p)^*} \quad \text{and} \\ & \gamma_{(s, X_0, Y_p)} \circ (\varepsilon_1 \otimes 1_{Y_p})^* = 1_{(X \widehat{\otimes} Y_p)^*}. \end{aligned}$$

Similarly, by virtue of Lemma 3.1, for  $X$  and  $X_p$ ,  $p \geq 0$ , the sequence

$$0 \longrightarrow (X_p \widehat{\otimes} Y)^* \xrightarrow{(1_{X_p} \otimes \varepsilon_2)^*} (X_p \widehat{\otimes} Y_0)^* \xrightarrow{(1_{X_p} \otimes \widetilde{d}_0)^*} (X_p \widehat{\otimes} Y_1)^* \xrightarrow{(1_{X_p} \otimes \widetilde{d}_1)^*} (X_p \widehat{\otimes} Y_2)^* \dots$$

splits as a complex of Banach spaces, so that there exist bounded linear operators

$$\widetilde{\gamma}_{(t, X_p, Y_n)} : (X_p \widehat{\otimes} Y_n)^* \rightarrow (X_p \widehat{\otimes} Y_{n-1})^*$$

such that

$$(3.7) \quad \begin{aligned} & \widetilde{\gamma}_{(t, X_p, Y_{n+1})} \circ (1_{X_p} \otimes \widetilde{d}_n)^* + (1_{X_p} \otimes \widetilde{d}_{n-1})^* \circ \widetilde{\gamma}_{(t, X_p, Y_n)} = 1_{(X_p \widehat{\otimes} Y_n)^*}, \\ & (1_{X_p} \otimes \varepsilon_2)^* \circ \widetilde{\gamma}_{(s, X_p, Y_0)} + \widetilde{\gamma}_{(s, X_p, Y_1)} \circ (1_{X_p} \otimes \widetilde{d}_0)^* = 1_{(X_p \widehat{\otimes} Y_0)^*} \quad \text{and} \\ & \widetilde{\gamma}_{(s, X_p, Y_0)} \circ (1_{X_p} \otimes \varepsilon_2)^* = 1_{(X_p \widehat{\otimes} Y)^*}. \end{aligned}$$

Consider the two morphisms  $1_{(\mathcal{X} \widehat{\otimes} \mathcal{Y})^*}$  and  $(\varepsilon_1 \otimes 1_Y)^* \circ \gamma_{(s, X_0, \mathcal{Y})}$  of the complex  $(\mathcal{X} \widehat{\otimes} \mathcal{Y})^*$ . Here  $(\varepsilon_1 \otimes 1_Y)^* \circ \gamma_{(s, X_0, \mathcal{Y})}$  is trivial on  $(X_m \widehat{\otimes} Y_p)^*$  for  $m \geq 1$  and equal to  $(\varepsilon_1 \otimes 1_{Y_p})^* \circ \gamma_{(s, X_0, Y_p)}$  on  $(X_0 \widehat{\otimes} Y_p)^*$ . By virtue of Lemma 3.2 and equalities (3.6),  $\gamma_{(s, \mathcal{X}, \mathcal{Y})}$  which is equal to  $\gamma_{(s, X_p, Y_n)}$  on  $(X_p \widehat{\otimes} Y_n)^*$  is a homotopy between  $1_{(\mathcal{X} \widehat{\otimes} \mathcal{Y})^*}$  and  $(\varepsilon_1 \otimes 1_Y)^* \circ \gamma_{(s, X_0, \mathcal{Y})}$ , so

$$\gamma_{(s, \mathcal{X}, \mathcal{Y})} : 1_{(\mathcal{X} \widehat{\otimes} \mathcal{Y})^*} \cong (\varepsilon_1 \otimes 1_Y)^* \circ \gamma_{(s, X_0, \mathcal{Y})} : (\mathcal{X} \widehat{\otimes} \mathcal{Y})^* \rightarrow (\mathcal{X} \widehat{\otimes} \mathcal{Y})^*.$$

Similarly, by virtue of Lemma 3.2 and equalities (3.7),  $\widetilde{\gamma}_{(t, \mathcal{X}, \mathcal{Y})}$  which is equal to  $\widetilde{\gamma}_{(t, X_p, Y_n)}$  on  $(X_p \widehat{\otimes} Y_n)^*$  is a homotopy between  $1_{(\mathcal{X} \widehat{\otimes} \mathcal{Y})^*}$  and  $(1_X \otimes \varepsilon_2)^* \circ \widetilde{\gamma}_{(t, X, Y_0)}$ , so

$$\widetilde{\gamma}_{(t, \mathcal{X}, \mathcal{Y})} : 1_{(\mathcal{X} \widehat{\otimes} \mathcal{Y})^*} \cong (1_X \otimes \varepsilon_2)^* \circ \widetilde{\gamma}_{(t, X, Y_0)} : (\mathcal{X} \widehat{\otimes} \mathcal{Y})^* \rightarrow (\mathcal{X} \widehat{\otimes} \mathcal{Y})^*.$$



Here  $(1_{\mathcal{X}} \otimes \varepsilon_2)^* \circ \tilde{\gamma}_{(t, \mathcal{X}, \mathcal{Y}_0)}$  is trivial on  $(X_m \widehat{\otimes} Y_q)^*$  for  $q \geq 1$  and equal to  $(1_{X_m} \otimes \varepsilon_2)^* \circ \tilde{\gamma}_{(t, X_m, Y_0)}$  on  $(X_m \widehat{\otimes} Y_0)^*$ . Therefore, by Proposition II.2.3 of [21], there exists a product homotopy

$$\tilde{\gamma}_{(t, \mathcal{X}, \mathcal{Y})} + \gamma_{(s, \mathcal{X}, \mathcal{Y})} \circ (1_{\mathcal{X}} \otimes \varepsilon_2)^* \circ \tilde{\gamma}_{(t, \mathcal{X}, \mathcal{Y}_0)}$$

between morphisms  $1_{(\mathcal{X} \widehat{\otimes} \mathcal{Y})^*}$  and  $(\varepsilon_1 \otimes 1_{\mathcal{Y}})^* \circ \gamma_{(s, X_0, \mathcal{Y})} \circ (1_{\mathcal{X}} \otimes \varepsilon_2)^* \circ \tilde{\gamma}_{(t, \mathcal{X}, \mathcal{Y}_0)}$  of the complex  $(\mathcal{X} \widehat{\otimes} \mathcal{Y})^*$ . One can check that

$$(\varepsilon_1 \otimes 1_{\mathcal{Y}})^* \circ \gamma_{(s, X_0, \mathcal{Y})} \circ (1_{\mathcal{X}} \otimes \varepsilon_2)^* \circ \tilde{\gamma}_{(t, \mathcal{X}, \mathcal{Y}_0)}$$

is trivial on  $(\mathcal{X} \widehat{\otimes} \mathcal{Y})_n^*$  for  $n \geq 1$  and equal to  $(\varepsilon_1 \otimes \varepsilon_2)^* \circ \gamma_{(s, X_0, \mathcal{Y})} \circ \tilde{\gamma}_{(t, X_0, \mathcal{Y}_0)}$  on  $(X_0 \widehat{\otimes} Y_0)^*$ . Note that

$$\gamma_{(s, X_0, \mathcal{Y})} \circ \tilde{\gamma}_{(t, X_0, \mathcal{Y}_0)} \circ (\varepsilon_1 \otimes \varepsilon_2)^* = 1_{(X \widehat{\otimes} Y)^*}.$$

Thus the dual complex  $0 \rightarrow (X \widehat{\otimes} Y)^* \xrightarrow{(\varepsilon_1 \otimes \varepsilon_2)^*} (\mathcal{X} \widehat{\otimes} \mathcal{Y})^*$  splits as a complex of Banach spaces. ■

The following lemma is essentially ([20], Lemma 3.6).

LEMMA 3.4. *Let  $\mathcal{A}$  be a Banach algebra and let  $I$  be a closed two-sided ideal of  $\mathcal{A}_+$ . Suppose that one of the following conditions is satisfied:*

- (i)  *$I$  is flat in  $\mathcal{A}$ -mod and has a left bounded approximate identity  $(e_\alpha)_{\alpha \in \Lambda}$ ;*
- (ii)  *$I$  is flat in mod- $\mathcal{A}$  and has a right bounded approximate identity  $(e_\alpha)_{\alpha \in \Lambda}$ ;*
- (iii)  *$I$  has a bounded approximate identity  $(e_\alpha)_{\alpha \in \Lambda}$ .*

*Then the sequence*

$$(3.8) \quad 0 \longleftarrow I \xleftarrow{\varepsilon} I \widehat{\otimes} I \xleftarrow{d_0} I \widehat{\otimes} I \widehat{\otimes} I \longleftarrow \dots \longleftarrow I \widehat{\otimes}^{(n+2)} \xleftarrow{d_n} I \widehat{\otimes}^{(n+3)} \longleftarrow \dots,$$

*where  $\varepsilon(b_0 \otimes b_1) = b_0 b_1$  and*

$$d_n(b_0 \otimes b_1 \otimes b_2 \otimes \dots \otimes b_{n+2}) = \sum_{i=0}^{n+1} (-1)^i (b_0 \otimes b_1 \otimes \dots \otimes b_i b_{i+1} \otimes \dots \otimes b_{n+2}),$$

*is a pseudo-resolution of  $I$  in  $\mathcal{A}$ -essmod- $\mathcal{A}$  such that all modules  $I \widehat{\otimes}^{(n)}$ ,  $n \geq 2$ , are flat in  $\mathcal{A}$ -mod- $\mathcal{A}$ .*

*Proof.* It is easy to check that  $d_{n-1} \circ d_n = 0$  for  $n \geq 1$  and  $\varepsilon \circ d_0 = 0$ . Thus (3.8) is a complex. By Theorem VII.1.5 of [14],  $I$  is strictly flat as a left and as a right Banach  $\mathcal{A}$ -module. Hence, by Proposition VII.2.4 of [14], for any  $n \geq 2$ , the Banach  $\mathcal{A}$ -bimodule  $I \widehat{\otimes}^n$  is flat in  $\mathcal{A}$ -mod- $\mathcal{A}$ . Note that  $I \widehat{\otimes}^n$  is an essential Banach  $\mathcal{A}$ -bimodule since  $I$  has a left or right bounded approximate identity.

We consider the case when  $I$  is flat in  $\mathcal{A}$ -mod and has a left bounded approximate identity  $(e_\alpha)_{\alpha \in \Lambda}$ . Now we have to show that the dual complex

$$(3.9) \quad 0 \rightarrow I^* \xrightarrow{\varepsilon^*} (I \widehat{\otimes} I)^* \xrightarrow{d_0^*} (I \widehat{\otimes} I \widehat{\otimes} I)^* \rightarrow \dots \rightarrow (I \widehat{\otimes}^{(n+2)})^* \xrightarrow{d_n^*} (I \widehat{\otimes}^{(n+3)})^* \rightarrow \dots$$

is admissible.

Consider the Fréchet filter  $F$  on  $\Lambda$ , with base  $\{Q_\lambda : \lambda \in \Lambda\}$ , where  $Q_\lambda = \{\alpha \in \Lambda : \alpha \geq \lambda\}$ . Thus

$$F = \{E \subset \Lambda : \text{there is a } \lambda \in \Lambda \text{ such that } Q_\lambda \subset E\}.$$

Let  $U$  be an ultrafilter on  $\Lambda$  which refines  $F$ . One can find information on filters in [2].

For  $n \geq -1$  and  $f \in (I^{\widehat{\otimes}(n+3)})^*$ , we define  $g_f \in (I^{\widehat{\otimes}(n+2)})^*$  by

$$g_f(u) = \lim_{\alpha \rightarrow U} f(e_\alpha \otimes u) \text{ for all } u \in I^{\widehat{\otimes}(n+2)}.$$

One can check the following:  $g_f$  is a bounded linear functional, the operator

$$s_n : (I^{\widehat{\otimes}(n+3)})^* \rightarrow (I^{\widehat{\otimes}(n+2)})^* : f \mapsto g_f$$

is a bounded linear operator,  $d_{n-1}^* \circ s_{n-1} + s_n \circ d_n^* = \text{id}_{(I^{\widehat{\otimes}(n+2)})^*}$  for all  $n \geq 1$  and  $\varepsilon^* \circ s_{-1} + s_0 \circ d_0^* = \text{id}_{(I^{\widehat{\otimes}2})^*}$ . Thus (3.9) is admissible (see III.1.9 of [14]). Therefore, by definition, (3.8) is a pseudo-resolution of  $I$  in  $\mathcal{A}$ -essmod- $\mathcal{A}$  such that all modules  $I^{\widehat{\otimes}(n)}$ ,  $n \geq 2$ , are flat in  $\mathcal{A}$ -mod- $\mathcal{A}$ . ■

#### 4. WEAK BIDIMENSION OF BANACH ALGEBRAS WITH BOUNDED APPROXIMATE IDENTITIES

Let  $\mathcal{A}$  be a Banach algebra and  $X$  be a Banach  $\mathcal{A}$ -bimodule. Let us recall the definition of the standard homological chain complex  $\mathcal{C}_\sim(\mathcal{A}, X)$ . For  $n \geq 0$ , let  $C_n(\mathcal{A}, X)$  denote the projective tensor product  $X \widehat{\otimes} \mathcal{A}^{\widehat{\otimes}n}$ . The elements of  $C_n(\mathcal{A}, X)$  are called  $n$ -chains. Let the differential  $d_n : C_{n+1} \rightarrow C_n$  be given by

$$\begin{aligned} d_n(x \otimes a_1 \otimes \cdots \otimes a_{n+1}) \\ = x \cdot a_1 \otimes \cdots \otimes a_{n+1} + \sum_{k=1}^n (-1)^k (x \otimes a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_{n+1}) \\ + (-1)^{n+1} (a_{n+1} \cdot x \otimes a_1 \otimes \cdots \otimes a_n) \end{aligned}$$

with  $d_{-1}$  the null map. The space of boundaries  $B_n(\mathcal{C}_\sim(\mathcal{A}, X)) = \text{Im } d_n$  is denoted by  $B_n(\mathcal{A}, X)$  and the space of cycles  $Z_n(\mathcal{C}_\sim(\mathcal{A}, X)) = \text{Ker } d_{n-1}$  is denoted by  $Z_n(\mathcal{A}, X)$ . The homology groups of this complex  $H_n(\mathcal{C}_\sim(\mathcal{A}, X)) = Z_n(\mathcal{A}, X) / B_n(\mathcal{A}, X)$  are called the *continuous Hochschild homology groups of  $\mathcal{A}$  with coefficients in  $X$*  and denoted by  $\mathcal{H}_n(\mathcal{A}, X)$  ([14], Definition II.5.28).

The Hochschild cohomology groups  $\mathcal{H}^n(\mathcal{A}, X^*)$  of the Banach algebra  $\mathcal{A}$  with coefficients in the dual  $\mathcal{A}$ -bimodule  $X^*$  are topologically isomorphic to the cohomology groups  $H^n((\mathcal{C}_\sim(\mathcal{A}, X))^*)$  of the dual complex  $(\mathcal{C}_\sim(\mathcal{A}, X))^*$ , see [16] and Definition I.3.2 and Proposition II.5.27 of [14].

Let  $\mathcal{A}$  be a Banach algebra with a bounded approximate identity  $(e_\alpha)_{\alpha \in \Lambda}$ . We put  $\beta_n(\mathcal{A}) = \mathcal{A}^{\widehat{\otimes}^{n+2}}$ ,  $n \geq 0$ , and let  $d_n : \beta_{n+1}(\mathcal{A}) \rightarrow \beta_n(\mathcal{A})$  be given by

$$d_n(a_0 \otimes \cdots \otimes a_{n+2}) = \sum_{k=0}^{n+1} (-1)^k (a_0 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_{n+2}).$$

By Lemma 3.4, the complex

$$0 \leftarrow \mathcal{A} \xleftarrow{\pi} \beta_0(\mathcal{A}) \xleftarrow{d_0} \beta_1(\mathcal{A}) \xleftarrow{d_1} \cdots \leftarrow \beta_n(\mathcal{A}) \xleftarrow{d_n} \beta_{n+1}(\mathcal{A}) \leftarrow \cdots,$$

where  $\pi : \beta_0(\mathcal{A}) \rightarrow \mathcal{A} : a \otimes b \mapsto ab$ , is a flat pseudo-resolution of the  $\mathcal{A}$ -bimodule  $\mathcal{A}$ . We denote it by  $0 \leftarrow \mathcal{A} \xleftarrow{\pi} \beta(\mathcal{A})$ .

Further we will need the following result of the author and M.C. White.

PROPOSITION 4.1 ([20], Proposition 5.2(i)). *Let  $\mathcal{A}$  be a Banach algebra and let  $I$  be a closed two-sided ideal of  $\mathcal{A}$ . Suppose that  $I$  has a bounded approximate identity. Then, for any Banach  $I$ -bimodule  $Z$ ,*

$$\mathcal{H}_n(I, Z) = \mathcal{H}_n(\mathcal{A}, \overline{IZI}) \quad \text{and} \quad \mathcal{H}^n(I, Z^*) = \mathcal{H}^n(\mathcal{A}, (\overline{IZI})^*) \quad \text{for all } n \geq 1.$$

COROLLARY 4.2. *Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras with bounded approximate identities. Then*

$$db_w(\mathcal{A} \widehat{\otimes} \mathcal{B}) \leq db_w(\mathcal{A}_+ \widehat{\otimes} \mathcal{B}_+) = db_w \mathcal{A} + db_w \mathcal{B}.$$

*Proof.* By assumption,  $\mathcal{A}$  and  $\mathcal{B}$  are Banach algebras with bounded approximate identities. Thus  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  has a bounded approximate identity too [8]. For the Banach algebra  $\mathcal{A}_+ \widehat{\otimes} \mathcal{B}_+$  and its closed two-sided ideal  $\mathcal{A} \widehat{\otimes} \mathcal{B}$ , by Proposition 4.1,

$$db_w(\mathcal{A} \widehat{\otimes} \mathcal{B}) \leq db_w(\mathcal{A}_+ \widehat{\otimes} \mathcal{B}_+).$$

By Selivanov's result ([28], Theorem 4.6.8),

$$db_w(\mathcal{A}_+ \widehat{\otimes} \mathcal{B}_+) = db_w \mathcal{A}_+ + db_w \mathcal{B}_+.$$

It is known that for any Banach algebra  $\mathcal{D}$ ,  $db_w \mathcal{D} = db_w \mathcal{D}_+$ . ■

REMARK 4.3. In Theorem 4.17 we prove that the homological properties of the tensor product algebras  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  and  $\mathcal{A}_+ \widehat{\otimes} \mathcal{B}_+$  are different. Thus in proofs of homological properties of  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  one needs to avoid adjoining an identity to the algebra. On the other hand, the previous Proposition 4.1 shows that in the case of Banach algebras  $\mathcal{A}$  with bounded approximate identities we can restrict ourselves to the category of essential Banach modules in questions on  $db_w$  and  $\mathcal{H}^n(\mathcal{A}, X^*)$ . In the next propositions we develop standard homological tools for injective and flat essential Banach modules without adjoining an identity to the algebra. We present results for one of the categories:  $\mathcal{A}$ -mod, mod- $\mathcal{A}$  or  $\mathcal{A}$ -mod- $\mathcal{A}$ ; for the other categories similar results hold. These homological tools will allow us to prove Theorems 4.16 and 4.17 without the use of Selivanov's result that  $db_w(\mathcal{A}_+ \widehat{\otimes} \mathcal{B}_+) = db_w \mathcal{A}_+ + db_w \mathcal{B}_+$ .

Let  $\mathcal{A}$  be a Banach algebra. Recall that, for  $X \in \mathcal{A}\text{-mod}$ , the canonical morphism

$$\pi_+ : \mathcal{A}_+ \widehat{\otimes} X \rightarrow X$$

is defined by

$$\pi_+(a \otimes x) = a \cdot x \quad (a \in \mathcal{A}_+, x \in X).$$

By Chapter VII of [14],  $X^* \in \text{mod-}\mathcal{A}$  is injective if and only if

$$\pi_+^* : X^* \rightarrow (\mathcal{A}_+ \widehat{\otimes} X)^*$$

is a coretraction in  $\text{mod-}\mathcal{A}$ , that is, there is a morphism in  $\text{mod-}\mathcal{A}$

$$\zeta_+ : (\mathcal{A}_+ \widehat{\otimes} X)^* \rightarrow X^*$$

such that  $\zeta_+ \circ \pi_+^* = 1_{X^*}$ .

**PROPOSITION 4.4.** *Let  $\mathcal{A}$  be a Banach algebra and let  $X$  be a left essential Banach  $\mathcal{A}$ -module, that is,  $X = \overline{\mathcal{A}X}$ . Then  $X^* \in \text{mod-}\mathcal{A}$  is injective if and only if*

$$\pi^* : X^* \rightarrow (\mathcal{A} \widehat{\otimes} X)^*$$

is a coretraction in  $\text{mod-}\mathcal{A}$ , that is, there is a morphism in  $\text{mod-}\mathcal{A}$

$$\zeta : (\mathcal{A} \widehat{\otimes} X)^* \rightarrow X^*$$

such that  $\zeta \circ \pi^* = 1_{X^*}$ .

*Proof.* Consider the natural embedding

$$i : \mathcal{A} \widehat{\otimes} X \rightarrow \mathcal{A}_+ \widehat{\otimes} X : a \otimes x \mapsto a \otimes x.$$

Note that  $\pi = \pi_+ \circ i$ , thus  $\pi^* = i^* \circ \pi_+^*$ .

( $\Rightarrow$ ) Suppose  $X^*$  is injective in  $\text{mod-}\mathcal{A}$ . Thus there exists a morphism in  $\text{mod-}\mathcal{A}$

$$\zeta_+ : (\mathcal{A}_+ \widehat{\otimes} X)^* \rightarrow X^*$$

such that  $\zeta_+ \circ \pi_+^* = 1_{X^*}$ .

If  $f \in \text{Ker } i^*$ , then

$$[f \cdot a](c \otimes x) = f(ac \otimes x) = [i^*(f)](ac \otimes x) = 0,$$

for all  $a \in \mathcal{A}$ ,  $c \in \mathcal{A}_+$ ,  $x \in X$ . This implies that, for all  $f \in \text{Ker } i^*$  and  $a \in \mathcal{A}$ ,

$$[\zeta_+(f)] \cdot a = \zeta_+(f \cdot a) = 0.$$

Therefore, for all  $f \in \text{Ker } i^*$ ,  $\zeta_+(f)$  is zero on  $\overline{\mathcal{A}X} = X$ . Thus there is a unique morphism of right  $\mathcal{A}$ -modules  $\zeta : (\mathcal{A} \widehat{\otimes} X)^* \rightarrow X^*$  such that Diagram (4.1) is commutative.

$$(4.1) \quad \begin{array}{ccc} (\mathcal{A}_+ \widehat{\otimes} X)^* & \xrightarrow{\zeta_+} & X^* \\ i^* \downarrow & \nearrow \zeta & \\ (\mathcal{A} \widehat{\otimes} X)^* & & \end{array}$$

One can check that  $\zeta \circ \pi^* = 1_{X^*}$ .

( $\Leftarrow$ ) Suppose that there is a morphism in  $\text{mod-}\mathcal{A}$

$$\zeta : (\mathcal{A} \widehat{\otimes} X)^* \rightarrow X^*$$

such that  $\zeta \circ \pi^* = 1_{X^*}$ . Put  $\zeta_+ = \zeta \circ i^*$ . It is obvious that  $\zeta_+$  is a morphism of right  $\mathcal{A}$ -modules and  $\zeta_+ \circ \pi_+^* = \zeta \circ i^* \circ \pi_+^* = \zeta \circ \pi^* = 1_{X^*}$ . ■

**PROPOSITION 4.5.** (i) *Let  $\mathcal{A}$  be a Banach algebra with a left [right] bounded approximate identity  $(e_\alpha)_{\alpha \in \Lambda}$  and let  $X$  be a left [right] essential Banach  $\mathcal{A}$ -module. Then  $X^*$  is injective in  $\text{mod-}\mathcal{A}$  [ $\mathcal{A}\text{-mod}$ ] if and only if  $X^*$  is injective in  $\text{essmod-}\mathcal{A}$  [ $\mathcal{A}\text{-essmod}$ ].*

(ii) *Let  $\mathcal{A}$  be a Banach algebra with a bounded approximate identity  $(e_\alpha)_{\alpha \in \Lambda}$  and let  $X$  be an essential Banach  $\mathcal{A}$ -bimodule. Then  $X^*$  is injective in  $\mathcal{A}\text{-mod-}\mathcal{A}$  if and only if  $X^*$  is injective in  $\mathcal{A}\text{-essmod-}\mathcal{A}$ .*

*Proof.* (i) It is obvious that the injectivity of  $X^*$  in  $\text{mod-}\mathcal{A}$  implies the injectivity of  $X^*$  in  $\text{essmod-}\mathcal{A}$ .

Suppose that  $X^*$  is injective in  $\text{essmod-}\mathcal{A}$ . As in Lemma 3.4 one can define a bounded linear operator

$$\alpha : (\mathcal{A} \widehat{\otimes} X)^* \rightarrow X^* : f \mapsto g_f,$$

where  $g_f \in X^*$  is given by

$$g_f(x) = \lim_{\alpha \rightarrow U} f(e_\alpha \otimes x) \quad \text{for all } x \in X.$$

One can check that  $\alpha \circ \pi^* = 1_{X^*}$ . Therefore, by Proposition III.1.14(ii) of [14], there is a morphism of right Banach  $\mathcal{A}$ -modules

$$\zeta : (\mathcal{A} \widehat{\otimes} X)^* \rightarrow X^*$$

such that  $\zeta \circ \pi^* = 1_{X^*}$ . By Proposition 4.4,  $X^*$  is injective in  $\text{mod-}\mathcal{A}$ .

A proof of part (ii) is similar to the proof of part (i). ■

**PROPOSITION 4.6.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras, let  $X$  be an essential Banach  $\mathcal{A}$ -bimodule and let  $Y$  be an essential Banach  $\mathcal{B}$ -bimodule. Suppose  $X$  is flat in  $\mathcal{A}\text{-mod-}\mathcal{A}$  and  $Y$  is flat in  $\mathcal{B}\text{-mod-}\mathcal{B}$ . Then  $X \widehat{\otimes} Y$  is flat in  $\mathcal{A} \widehat{\otimes} \mathcal{B}\text{-mod-}\mathcal{A} \widehat{\otimes} \mathcal{B}$ .*

*Proof.* For Banach spaces  $U$  and  $V$ , we will use the well-known isomorphism ([14], Theorem 2.2.17):

$$\mathfrak{B}(U, \mathfrak{B}(V, W)) \cong \mathfrak{B}(U \widehat{\otimes} V, W) : \psi \mapsto \phi,$$

where  $\phi(u \otimes v) = [\psi(u)](v)$ ,  $u \in U$ ,  $v \in V$ .

As in Chapter VII of [14],  $X$  is a flat left Banach  $\mathcal{A} \widehat{\otimes} \mathcal{A}^{\text{op}}$ -module with multiplication

$$(a_1 \otimes a_2) \cdot x = a_1 \cdot x \cdot a_2$$

and  $X^* \in \text{mod-}\mathcal{A} \widehat{\otimes} \mathcal{A}^{\text{op}}$  is injective. By Proposition 4.4, since  $X$  is a left essential Banach  $\mathcal{A} \widehat{\otimes} \mathcal{A}^{\text{op}}$ -module, there is a morphism in  $\text{mod-}\mathcal{A} \widehat{\otimes} \mathcal{A}^{\text{op}}$

$$\zeta_X : ((\mathcal{A} \widehat{\otimes} \mathcal{A}^{\text{op}}) \widehat{\otimes} X)^* \rightarrow X^*$$

such that  $\zeta_X \circ \pi_X^* = 1_{X^*}$ . Here the canonical morphism

$$\pi_X : (\mathcal{A} \widehat{\otimes} \mathcal{A}^{\text{op}}) \widehat{\otimes} X \rightarrow X$$

is defined by

$$\pi_X(u \otimes x) = u \cdot x = a_1 \cdot x \cdot a_2 \quad (u = a_1 \otimes a_2 \in \mathcal{A} \widehat{\otimes} \mathcal{A}^{\text{op}}, x \in X).$$

Therefore we can define

$$\begin{aligned} \widetilde{\zeta}_X &: \mathfrak{B}(Y, ((\mathcal{A} \widehat{\otimes} \mathcal{A}^{\text{op}}) \widehat{\otimes} X)^*) \rightarrow \mathfrak{B}(Y, X^*) : \phi \mapsto \zeta_X \circ \phi; \\ \widetilde{\pi}_X^* &: \mathfrak{B}(Y, X^*) \rightarrow \mathfrak{B}(Y, ((\mathcal{A} \widehat{\otimes} \mathcal{A}^{\text{op}}) \widehat{\otimes} X)^*) : \psi \mapsto \pi_X^* \circ \psi; \end{aligned}$$

and one can see that  $\widetilde{\zeta}_X \circ \widetilde{\pi}_X^* = 1_{\mathfrak{B}(Y, X^*)}$ . Thus the  $\mathcal{A} \widehat{\otimes} \mathcal{B}$ -bimodule  $(X \widehat{\otimes} Y)^* \stackrel{i_1}{\cong} \mathfrak{B}(Y, X^*)$  is a retract of

$$\mathfrak{B}(Y, ((\mathcal{A} \widehat{\otimes} \mathcal{A}^{\text{op}}) \widehat{\otimes} X)^*) \stackrel{i_2}{\cong} \mathfrak{B}((\mathcal{A} \widehat{\otimes} \mathcal{A}^{\text{op}}) \widehat{\otimes} X, Y^*).$$

We can consider  $Y$  as a flat right Banach  $\mathcal{B}^{\text{op}} \widehat{\otimes} \mathcal{B}$ -module with multiplication

$$y \cdot (b_1 \otimes b_2) = b_1 \cdot y \cdot b_2.$$

By Proposition 4.4, since  $Y$  is a right essential Banach  $\mathcal{B}^{\text{op}} \widehat{\otimes} \mathcal{B}$ -module and  $Y^*$  is injective in  $\mathcal{B}^{\text{op}} \widehat{\otimes} \mathcal{B}$ -mod, there is a morphism in  $\mathcal{B}^{\text{op}} \widehat{\otimes} \mathcal{B}$ -mod

$$\zeta_Y : (Y \widehat{\otimes} (\mathcal{B}^{\text{op}} \widehat{\otimes} \mathcal{B}))^* \rightarrow Y^*$$

such that  $\zeta_Y \circ \pi_Y^* = 1_{Y^*}$ . Here the canonical morphism

$$\pi_Y : Y \widehat{\otimes} (\mathcal{B}^{\text{op}} \widehat{\otimes} \mathcal{B}) \rightarrow Y$$

is defined by

$$\pi_Y(y \otimes u) = y \cdot u = b_1 \cdot y \cdot b_2 \quad (u = b_1 \otimes b_2 \in \mathcal{B}^{\text{op}} \widehat{\otimes} \mathcal{B}, y \in Y).$$

Thus we can define

$$\begin{aligned} \widetilde{\zeta}_Y &: \mathfrak{B}((\mathcal{A} \widehat{\otimes} \mathcal{A}^{\text{op}}) \widehat{\otimes} X, (Y \widehat{\otimes} (\mathcal{B}^{\text{op}} \widehat{\otimes} \mathcal{B}))^*) \rightarrow \mathfrak{B}((\mathcal{A} \widehat{\otimes} \mathcal{A}^{\text{op}}) \widehat{\otimes} X, Y^*) : \phi \mapsto \zeta_Y \circ \phi; \\ \widetilde{\pi}_Y^* &: \mathfrak{B}((\mathcal{A} \widehat{\otimes} \mathcal{A}^{\text{op}}) \widehat{\otimes} X, Y^*) \rightarrow \mathfrak{B}((\mathcal{A} \widehat{\otimes} \mathcal{A}^{\text{op}}) \widehat{\otimes} X, (Y \widehat{\otimes} (\mathcal{B}^{\text{op}} \widehat{\otimes} \mathcal{B}))^*) : \psi \mapsto \pi_Y^* \circ \psi; \end{aligned}$$

and  $\widetilde{\zeta}_Y \circ \widetilde{\pi}_Y^* = 1_{\mathfrak{B}((\mathcal{A} \widehat{\otimes} \mathcal{A}^{\text{op}}) \widehat{\otimes} X, Y^*)}$ . Therefore  $(X \widehat{\otimes} Y)^*$  is a retract of

$$\mathfrak{B}((\mathcal{A} \widehat{\otimes} \mathcal{A}^{\text{op}}) \widehat{\otimes} X, (Y \widehat{\otimes} (\mathcal{B}^{\text{op}} \widehat{\otimes} \mathcal{B}))^*) \stackrel{i_3}{\cong} \mathfrak{B}((\mathcal{A} \widehat{\otimes} \mathcal{B}) \widehat{\otimes} (\mathcal{A} \widehat{\otimes} \mathcal{B})^{\text{op}} \widehat{\otimes} (X \widehat{\otimes} Y), \mathbb{C}).$$

One can check that

$$\zeta_{X \widehat{\otimes} Y} = i_1^{-1} \circ \widetilde{\zeta}_X \circ i_2^{-1} \circ \widetilde{\zeta}_Y \circ i_3^{-1} : ((\mathcal{A} \widehat{\otimes} \mathcal{B}) \widehat{\otimes} (\mathcal{A} \widehat{\otimes} \mathcal{B})^{\text{op}} \widehat{\otimes} (X \widehat{\otimes} Y))^* \rightarrow (X \widehat{\otimes} Y)^*$$

is a morphism in  $(\mathcal{A} \widehat{\otimes} \mathcal{B}) \widehat{\otimes} (\mathcal{A} \widehat{\otimes} \mathcal{B})^{\text{op}}$ -mod, that

$$\pi_{X \widehat{\otimes} Y}^* = i_3 \circ \widetilde{\pi}_Y^* \circ i_2 \circ \widetilde{\pi}_X^* \circ i_1 : (X \widehat{\otimes} Y)^* \rightarrow ((\mathcal{A} \widehat{\otimes} \mathcal{B}) \widehat{\otimes} (\mathcal{A} \widehat{\otimes} \mathcal{B})^{\text{op}} \widehat{\otimes} (X \widehat{\otimes} Y))^*,$$

and  $\zeta_{X \widehat{\otimes} Y} \circ \pi_{X \widehat{\otimes} Y}^* = 1_{(X \widehat{\otimes} Y)^*}$ . By Proposition 4.4,  $(X \widehat{\otimes} Y)^*$  is injective in  $\mathcal{A} \widehat{\otimes} \mathcal{B}$ -mod- $\mathcal{A} \widehat{\otimes} \mathcal{B}$  and thus  $X \widehat{\otimes} Y$  is flat in  $\mathcal{A} \widehat{\otimes} \mathcal{B}$ -mod- $\mathcal{A} \widehat{\otimes} \mathcal{B}$ . ■

COROLLARY 4.7. *The tensor product algebra  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  of biflat Banach algebras  $\mathcal{A}$  and  $\mathcal{B}$  is biflat.*

*Proof.* By Proposition VII.2.6 of [14], a biflat Banach algebra is essential. Hence, by Proposition 4.6,  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  is flat in  $\mathcal{A} \widehat{\otimes} \mathcal{B}\text{-mod-}\mathcal{A} \widehat{\otimes} \mathcal{B}$ . ■

PROPOSITION 4.8. *Let  $\mathcal{A}_1$  and  $\mathcal{A}_2$  be Banach algebras. Let  $0 \leftarrow X \xleftarrow{\varepsilon_1} \mathcal{X}$  be a pseudo-resolution of  $X$  in  $\mathcal{A}_1\text{-essmod-}\mathcal{A}_1$  such that all modules in  $\mathcal{X}$  are flat in  $\mathcal{A}_1\text{-mod-}\mathcal{A}_1$  and  $0 \leftarrow Y \xleftarrow{\varepsilon_2} \mathcal{Y}$  be a pseudo-resolution of  $Y$  in  $\mathcal{A}_2\text{-essmod-}\mathcal{A}_2$  such that all modules in  $\mathcal{Y}$  are flat in  $\mathcal{A}_2\text{-mod-}\mathcal{A}_2$ . Then  $0 \leftarrow X \widehat{\otimes} Y \xleftarrow{\varepsilon_1 \otimes \varepsilon_2} \mathcal{X} \widehat{\otimes} \mathcal{Y}$  is a pseudo-resolution of  $X \widehat{\otimes} Y$  in  $\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2\text{-essmod-}\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2$  such that all modules in  $\mathcal{X} \widehat{\otimes} \mathcal{Y}$  are flat in  $\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2\text{-mod-}\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2$ .*

*Proof.* By Definition 2.1, for any  $n \geq 0$ , the Banach  $\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2$ -bimodule

$$(\mathcal{X} \widehat{\otimes} \mathcal{Y})_n = \bigoplus_{m+q=n} X_m \widehat{\otimes} Y_q.$$

By Proposition 4.6,  $(\mathcal{X} \widehat{\otimes} \mathcal{Y})_n$  is flat in  $\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2\text{-mod-}\mathcal{A}_1 \widehat{\otimes} \mathcal{A}_2$  for all  $n \geq 0$ .

By assumption the complexes  $0 \leftarrow X \xleftarrow{\varepsilon_1} \mathcal{X}$  and  $0 \leftarrow Y \xleftarrow{\varepsilon_2} \mathcal{Y}$  are weakly admissible. By Proposition 3.3, the complex  $0 \leftarrow X \widehat{\otimes} Y \xleftarrow{\varepsilon_1 \otimes \varepsilon_2} \mathcal{X} \widehat{\otimes} \mathcal{Y}$  is weakly admissible too. ■

By Theorem III.4.9 of [14], we have for a Banach algebra  $\mathcal{A}$  and a Banach  $\mathcal{A}$ -bimodule  $X$ ,

$$\mathcal{H}^n(\mathcal{A}, X) = \text{Ext}_{\mathcal{A}^e}^n(\mathcal{A}_+, X).$$

For a Banach algebra with a bounded approximate identity and for dual bimodules, we may avoid adjoining identity to the algebra.

PROPOSITION 4.9 ([20], Proposition 3.12). *Let  $\mathcal{A}$  be a Banach algebra with a bounded approximate identity and let  $X$  be an essential Banach  $\mathcal{A}$ -bimodule. Then, for all  $n \geq 0$ ,*

$$\mathcal{H}^n(\mathcal{A}, X^*) = \text{Ext}_{\mathcal{A}^e}^n(\mathcal{A}, X^*).$$

THEOREM 4.10. *Let  $\mathcal{A}$  be a Banach algebra with bounded approximate identity. For each integer  $n \geq 0$  the following properties of a Banach algebra  $\mathcal{A}$  are equivalent:*

- (i)  $db_w \mathcal{A} \leq n$ ;
- (ii)  $\mathcal{H}^{n+1}(\mathcal{A}, X^*) = \{0\}$  for all  $X \in \mathcal{A}\text{-essmod-}\mathcal{A}$ ;
- (iii)  $\mathcal{H}^m(\mathcal{A}, X^*) = \{0\}$  for all  $m \geq n+1$  and for all  $X \in \mathcal{A}\text{-essmod-}\mathcal{A}$ ;
- (iv)  $\mathcal{H}_{n+1}(\mathcal{A}, X) = \{0\}$  and  $\mathcal{H}_n(\mathcal{A}, X)$  is a Hausdorff space for all  $X \in \mathcal{A}\text{-essmod-}\mathcal{A}$ ;
- (v) if

$$0 \longleftarrow \mathcal{A} \xleftarrow{\varepsilon} P_0 \xleftarrow{\phi_0} P_1 \xleftarrow{\phi_1} \dots P_{n-1} \xleftarrow{\phi_{n-1}} Y \longleftarrow 0 \quad (0 \leftarrow \mathcal{A} \leftarrow \mathcal{P})$$

*is a pseudo-resolution of  $\mathcal{A}$  in which all the modules  $P_i$  are flat in  $\mathcal{A}\text{-essmod-}\mathcal{A}$ , then  $Y$  is also flat in  $\mathcal{A}\text{-essmod-}\mathcal{A}$ .*

(vi) the  $\mathcal{A}$ -bimodule  $\mathcal{A}$  has a flat pseudo-resolution of length  $n$  in the category of  $\mathcal{A}$ -essmod- $\mathcal{A}$ .

*Proof.* By definition, (i)  $\Rightarrow$  (ii). By Proposition 4.1, for a Banach algebra  $\mathcal{A}$  with a bounded approximate identity and for any Banach  $\mathcal{A}$ -bimodule  $Z$ ,

$$\mathcal{H}^n(\mathcal{A}, Z^*) = \mathcal{H}^n(\mathcal{A}, (\overline{\mathcal{A}Z\mathcal{A}})^*) \text{ for all } n \geq 1.$$

Thus (ii)  $\Rightarrow$  (i) and therefore (ii)  $\iff$  (i).

By Corollary 1.3 and 1.a Reduction of dimension of [16], (ii)  $\iff$  (iii) and (ii)  $\iff$  (iv).

(vi)  $\Rightarrow$  (ii) By Proposition 4.9, for a Banach algebra  $\mathcal{A}$  with a bounded approximate identity, for an essential  $X$  Banach  $\mathcal{A}$ -bimodule and for all  $n \geq 0$ ,

$$\mathcal{H}^n(\mathcal{A}, X^*) = \text{Ext}_{\mathcal{A}^e}^n(\mathcal{A}, X^*).$$

By VII.1.19 of [14],  $\text{Ext}_{\mathcal{A}^e}^n(\mathcal{A}, X^*)$  is the cohomology of the complex  $h_{\mathcal{A}^e}(\mathcal{P}, X^*)$  where  $(0 \leftarrow \mathcal{A} \leftarrow \mathcal{P})$  is a pseudo-resolution of  $\mathcal{A}$  in  $\mathcal{A}\text{-mod-}\mathcal{A}$ . The rest is clear.

(ii)  $\Rightarrow$  (v) By assumption, the complex  $(0 \leftarrow \mathcal{A} \leftarrow \mathcal{P})$  is weakly admissible, that is, the dual complex

$$0 \longrightarrow \mathcal{A}^* \xrightarrow{\varepsilon^*} P_0^* \xrightarrow{\phi_0^*} P_1 \xrightarrow{\phi_1^*} P_2^* \longrightarrow \cdots \longrightarrow P_{n-1}^* \xrightarrow{\phi_{n-1}^*} Y^* \longrightarrow 0$$

splits as a complex of Banach spaces. Therefore there are the following admissible short exact sequences:

$$\begin{aligned} 0 &\longrightarrow \mathcal{A}^* \xrightarrow{\varepsilon^*} P_0^* \xrightarrow{\phi_0^*} \text{Im } \phi_0^* \longrightarrow 0, \\ 0 &\longrightarrow \text{Im } \phi_{k-1}^* = \text{Ker } \phi_k^* \xrightarrow{i_k} P_k^* \xrightarrow{\phi_k^*} \text{Im } \phi_k^* \longrightarrow 0, \end{aligned}$$

$k = 1, 2, \dots, n-2$ , and

$$0 \longrightarrow \text{Im } \phi_{n-2}^* = \text{Ker } \phi_{n-1}^* \xrightarrow{i_{n-1}} P_{n-1}^* \xrightarrow{\phi_{n-1}^*} Y^* \longrightarrow 0,$$

where  $i_k, k = 1, \dots, n-1$ , are natural inclusions. Thus, by Theorem III.4.4 of [14], for every  $X \in \mathcal{A}\text{-essmod-}\mathcal{A}$ , there are long exact sequences of  $\text{Ext}_{\mathcal{A}^e}(X, \cdot)$  associated with these admissible short exact sequences. By assumption, all the modules  $P_i$  are flat in  $\mathcal{A}\text{-essmod-}\mathcal{A}$  and therefore

$$\text{Ext}_{\mathcal{A}^e}^n(X, P_i^*) = \{0\}$$

for all  $n \geq 1$  and all  $X \in \mathcal{A}\text{-essmod-}\mathcal{A}$ . Hence, for every  $X \in \mathcal{A}\text{-essmod-}\mathcal{A}$ ,

$$\begin{aligned} \text{Ext}_{\mathcal{A}^e}^1(X, Y^*) &\cong \text{Ext}_{\mathcal{A}^e}^2(X, \text{Im } \phi_{n-2}^*) \cong \text{Ext}_{\mathcal{A}^e}^3(X, \text{Im } \phi_{n-3}^*) \cong \cdots \\ &\cong \text{Ext}_{\mathcal{A}^e}^n(X, \text{Im } \phi_0^*) \cong \text{Ext}_{\mathcal{A}^e}^{n+1}(X, \mathcal{A}^*). \end{aligned}$$

By Proposition III.4.13 of [14],

$$\text{Ext}_{\mathcal{A}^e}^{n+1}(X, \mathcal{A}^*) \cong \text{Ext}_{\mathcal{A}^e}^{n+1}(\mathcal{A}, X^*)$$



and, by Proposition 4.9, for the Banach algebra with bounded approximate identity,

$$\text{Ext}_{\mathcal{A}^e}^{n+1}(\mathcal{A}, X^*) \cong \mathcal{H}^{n+1}(\mathcal{A}, X^*).$$

Therefore  $Y^*$  is injective and hence  $Y$  is flat in  $\mathcal{A}$ -essmod- $\mathcal{A}$ .

It is obvious that (v)  $\Rightarrow$  (vi). ■

REMARK 4.11. To get the full picture for an arbitrary Banach algebra  $\mathcal{A}$ , one can see also Theorem 1 of [27] on equivalent conditions to  $db_w \mathcal{A} \leq n$  for an integer  $n \geq 0$ .

REMARK 4.12. It is clear that for Banach algebras with bounded approximate identities  $\mathcal{A}$  and  $\mathcal{B}$ , by Theorem 4.10 and Proposition 4.8,  $db_w(\mathcal{A} \widehat{\otimes} \mathcal{B}) \leq db_w \mathcal{A} + db_w \mathcal{B}$ . To get the equality here we need the following lemma of Yu.V. Selivanov and extensions of his Propositions 4.6.2 and 4.6.5 of [28] to the case of Banach algebras with bounded approximate identities.

Recall that a continuous linear operator  $T : X \rightarrow Y$  between Banach spaces  $X$  and  $Y$  is topologically injective if it is injective and its image is closed, that is,  $T : X \rightarrow \text{Im } T$  is a topological isomorphism.

LEMMA 4.13 ([27], Lemma 1). *Let  $E_0, E, F_0$  and  $F$  be Banach spaces, and let  $S : E_0 \rightarrow E$  and  $T : F_0 \rightarrow F$  be continuous linear operators. Suppose  $S$  and  $T$  are not topologically injective. Then the continuous linear operator*

$$\Delta : E_0 \widehat{\otimes} F_0 \rightarrow (E_0 \widehat{\otimes} F) \oplus (E \widehat{\otimes} F_0)$$

defined by

$$\Delta(x \otimes y) = (x \otimes T(y), S(x) \otimes y) \quad (x \in E_0, y \in F_0).$$

is not topologically injective.

PROPOSITION 4.14. *Let  $\mathcal{A}$  be a Banach algebra with bounded approximate identity and let*

$$(4.2) \quad 0 \longleftarrow \mathcal{A} \xleftarrow{\varepsilon} P_0 \xleftarrow{\phi_0} P_1 \xleftarrow{\phi_1} \cdots P_{n-1} \xleftarrow{\phi_{n-1}} P_n \longleftarrow 0 \quad (0 \leftarrow \mathcal{A} \leftarrow \mathcal{P})$$

be a flat pseudo-resolution of  $\mathcal{A}$  in  $\mathcal{A}$ -essmod- $\mathcal{A}$ . Then  $db_w \mathcal{A} < n$  if and only if, for every  $X$  in  $\mathcal{A}$ -essmod- $\mathcal{A}$ , the operator

$$\phi_{n-1} \otimes_{\mathcal{A}-\mathcal{A}} 1_X : P_n \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} X \rightarrow P_{n-1} \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} X$$

is topologically injective.

*Proof.* By Theorem 4.10,  $db_w \mathcal{A} < n$  if and only if  $\mathcal{H}^n(\mathcal{A}, X^*) = \{0\}$  for every  $X$  in  $\mathcal{A}$ -essmod- $\mathcal{A}$ . By Proposition 4.9, for a Banach algebra  $\mathcal{A}$  with a bounded approximate identity, for an essential  $X$  Banach  $\mathcal{A}$ -bimodule and for all  $n \geq 0$ ,

$$\mathcal{H}^n(\mathcal{A}, X^*) = \text{Ext}_{\mathcal{A}^e}^n(\mathcal{A}, X^*).$$

By assumption,  $\mathcal{A}$  is a Banach algebra with bounded approximate identity. Thus, by Proposition 4.5(ii) and by Theorem VII.1.14 of [14],  $0 \leftarrow \mathcal{A} \leftarrow$

$(\mathcal{P}, \phi)$  is a flat pseudo-resolution of  $\mathcal{A}$  in  $\mathcal{A}\text{-mod-}\mathcal{A}$ . By Exercise VII.1.19 of [14],  $\text{Ext}_{\mathcal{A}^e}^n(\mathcal{A}, X^*)$  is the cohomology of the complex  ${}_{\mathcal{A}}h_{\mathcal{A}}(\mathcal{P}, X^*)$ . Therefore, up to topological isomorphism,  $\text{Ext}_{\mathcal{A}^e}^n(\mathcal{A}, X^*)$  is the  $n$ -th cohomology of the complex  ${}_{\mathcal{A}}h_{\mathcal{A}}(\mathcal{P}, X^*)$ :

$$(4.3) \quad 0 \longrightarrow {}_{\mathcal{A}}h_{\mathcal{A}}(P_0, X^*) \xrightarrow{h(\phi_0)} \cdots \longrightarrow {}_{\mathcal{A}}h_{\mathcal{A}}(P_{n-1}, X^*) \xrightarrow{h(\phi_{n-1})} {}_{\mathcal{A}}h_{\mathcal{A}}(P_n, X^*) \longrightarrow 0,$$

where  $h(\phi_{n-1})$  is the operator defined by  $h(\phi_{n-1})(\eta) = \eta \circ \phi_{n-1}$  for  $\eta \in {}_{\mathcal{A}}h_{\mathcal{A}}(P_{n-1}, X^*)$ . Therefore  $\mathcal{H}^n(\mathcal{A}, X^*) = \{0\}$  for every  $X$  in  $\mathcal{A}\text{-essmod-}\mathcal{A}$  if and only if  $h(\phi_{n-1})$  is surjective for every  $X$  in  $\mathcal{A}\text{-essmod-}\mathcal{A}$ .

By the conjugate associativity law ([14], Theorem II.5.21), there is a natural isomorphism of Banach spaces:

$${}_{\mathcal{A}}h_{\mathcal{A}}(\mathcal{P}, X^*) \cong (\mathcal{P} \widehat{\otimes}_{\mathcal{A}^e} X)^*.$$

It is clear that  $h(\phi_{n-1})$  is the dual to  $\phi_{n-1} \otimes_{\mathcal{A}-\mathcal{A}} 1_X$ . By Corollary 8.6.15 of [9],  $h(\phi_{n-1})$  is surjective if and only if  $\phi_{n-1} \otimes_{\mathcal{A}-\mathcal{A}} 1_X$  is topologically injective. ■

**PROPOSITION 4.15.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras with bounded approximate identities. Then*

$$db_w(\mathcal{A} \widehat{\otimes} \mathcal{B}) \geq \max\{db_w\mathcal{A}, db_w\mathcal{B}\}.$$

*Proof.* By assumption,  $\mathcal{A}$  and  $\mathcal{B}$  are Banach algebras with bounded approximate identities. Thus  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  has a bounded approximate identity too [8]. By Proposition 4.5(ii) and by Theorem VII.1.14 of [14], for a Banach algebra  $\mathcal{D}$  with a bounded approximate identity,  $X$  is flat in  $\mathcal{D}\text{-mod-}\mathcal{D}$  if and only if  $X$  is flat in  $\mathcal{D}\text{-essmod-}\mathcal{D}$ .

Suppose that  $db_w(\mathcal{A} \widehat{\otimes} \mathcal{B}) = n < \infty$ . By Theorem 4.10, the  $\mathcal{A} \widehat{\otimes} \mathcal{B}$ -bimodule  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  has a flat pseudo-resolution  $0 \leftarrow \mathcal{A} \widehat{\otimes} \mathcal{B} \leftarrow \mathcal{P}$  of length  $n$  in the category  $\mathcal{A} \widehat{\otimes} \mathcal{B}\text{-essmod-}\mathcal{A} \widehat{\otimes} \mathcal{B}$ . By Proposition VII.2.2 of [14],  $0 \leftarrow \mathcal{A} \widehat{\otimes} \mathcal{B} \leftarrow \mathcal{P}$  is a flat pseudo-resolution in the category  $\mathcal{A}\text{-essmod-}\mathcal{A}$ . Therefore, for every essential Banach  $\mathcal{A}$ -bimodule  $X$ ,  $\text{Ext}_{\mathcal{A}^e}^{n+1}(\mathcal{A} \widehat{\otimes} \mathcal{B}, X^*) = \{0\}$ .

It is easy to see that  $\mathcal{A}$  is isomorphic to  $\mathcal{A} \widehat{\otimes} \mathbb{C}$  in  $\mathcal{A}\text{-mod-}\mathcal{A}$ . On the other hand  $\mathcal{A} \widehat{\otimes} \mathbb{C}$  is a direct module summand of  $\mathcal{A} \widehat{\otimes} \mathcal{B} \in \mathcal{A}\text{-mod-}\mathcal{A}$ . Therefore, since  $\text{Ext}$  is additive, for every essential Banach  $\mathcal{A}$ -bimodule  $X$ ,

$$\mathcal{H}^{n+1}(\mathcal{A}, X^*) = \text{Ext}_{\mathcal{A}^e}^{n+1}(\mathcal{A}, X^*) = \{0\}.$$

Thus, by Theorem 4.10,  $db_w\mathcal{A} \leq n$  and  $db_w(\mathcal{A} \widehat{\otimes} \mathcal{B}) \geq db_w\mathcal{A}$ . As in the case of  $\mathcal{A}$  one can show that  $db_w(\mathcal{A} \widehat{\otimes} \mathcal{B}) \geq db_w\mathcal{B}$ . ■

**THEOREM 4.16.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras with bounded approximate identities. Then*

$$db_w(\mathcal{A} \widehat{\otimes} \mathcal{B}) = db_w\mathcal{A} + db_w\mathcal{B}.$$

*Proof.* By Theorem 4.10 and Proposition 4.8,

$$db_w(\mathcal{A} \widehat{\otimes} \mathcal{B}) \leq db_w\mathcal{A} + db_w\mathcal{B}.$$

Therefore, by Proposition 4.15,

$$\max\{db_w\mathcal{A}, db_w\mathcal{B}\} \leq db_w(\mathcal{A} \widehat{\otimes} \mathcal{B}) \leq db_w\mathcal{A} + db_w\mathcal{B}.$$

Hence, in the case  $db_w\mathcal{A} [db_w\mathcal{B}]$  is equal to 0 or  $\infty$ ,  $db_w(\mathcal{A} \widehat{\otimes} \mathcal{B}) = db_w\mathcal{A} + db_w\mathcal{B}$ .

Suppose  $db_w\mathcal{A} = m$  and  $db_w\mathcal{B} = q$  where  $0 < m, q < \infty$ . By Theorem 4.10, there is a flat pseudo-resolution  $0 \leftarrow \mathcal{A} \xleftarrow{\varepsilon_1} (\mathcal{P}, \phi)$  of length  $m$  in the category  $\mathcal{A}$ -essmod- $\mathcal{A}$ . By Proposition 4.14, there exists  $X \in \mathcal{A}$ -essmod- $\mathcal{A}$  such that the operator

$$\phi_{m-1} \otimes_{\mathcal{A}-\mathcal{A}} 1_X : P_m \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} X \rightarrow P_{m-1} \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} X$$

is not topologically injective. Similarly,  $db_w\mathcal{B} = q$  implies that there is a flat pseudo-resolution  $0 \leftarrow \mathcal{B} \xleftarrow{\varepsilon_2} (\mathcal{Q}, \psi)$  of length  $q$  in the category  $\mathcal{B}$ -essmod- $\mathcal{B}$  and there exist  $Y \in \mathcal{B}$ -essmod- $\mathcal{B}$  such that the operator

$$\psi_{q-1} \otimes_{\mathcal{B}-\mathcal{B}} 1_Y : Q_q \widehat{\otimes}_{\mathcal{B}-\mathcal{B}} Y \rightarrow Q_{q-1} \widehat{\otimes}_{\mathcal{B}-\mathcal{B}} Y$$

is not topologically injective.

By Proposition 4.8,  $0 \leftarrow \mathcal{A} \widehat{\otimes} \mathcal{B} \xleftarrow{\varepsilon_1 \otimes \varepsilon_2} (\mathcal{P} \widehat{\otimes} \mathcal{Q}, \delta)$  is a flat pseudo-resolution of  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  in  $\mathcal{A} \widehat{\otimes} \mathcal{B}$ -essmod- $\mathcal{A} \widehat{\otimes} \mathcal{B}$  of length  $m + q$ . Take  $Z = X \widehat{\otimes} Y$  in  $\mathcal{A} \widehat{\otimes} \mathcal{B}$ -essmod- $\mathcal{A} \widehat{\otimes} \mathcal{B}$ . By Definition 2.1,  $(\mathcal{P} \widehat{\otimes} \mathcal{Q})_{m+q} = P_m \widehat{\otimes} Q_q$ ,  $(\mathcal{P} \widehat{\otimes} \mathcal{Q})_{m+q-1} = (P_{m-1} \widehat{\otimes} Q_q) \oplus (P_m \widehat{\otimes} Q_{q-1})$  and

$$\delta_{m+q-1} : (\mathcal{P} \widehat{\otimes} \mathcal{Q})_{m+q} \rightarrow (\mathcal{P} \widehat{\otimes} \mathcal{Q})_{m+q-1}$$

is defined by

$$\delta_{m+q-1}(x \otimes y) = \phi_{m-1}(x) \otimes y + (-1)^m x \otimes \psi_{q-1}(y), \quad x \in P_m, y \in Q_q.$$

By Lemma 4.13, the operator

$$\begin{aligned} \Delta : (P_m \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} X) \widehat{\otimes} (Q_q \widehat{\otimes}_{\mathcal{B}-\mathcal{B}} Y) \rightarrow \\ ((P_{m-1} \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} X) \widehat{\otimes} (Q_q \widehat{\otimes}_{\mathcal{B}-\mathcal{B}} Y)) \oplus ((P_m \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} X) \widehat{\otimes} (Q_{q-1} \widehat{\otimes}_{\mathcal{B}-\mathcal{B}} Y)) \end{aligned}$$

which is defined by

$$\Delta(u \otimes v) = (\phi_{m-1} \otimes_{\mathcal{A}-\mathcal{A}} 1_X)(u) \otimes v + u \otimes ((-1)^m \psi_{q-1} \otimes_{\mathcal{B}-\mathcal{B}} 1_Y)(v)$$

is not topologically injective. Note that there are natural isometric isomorphisms of Banach spaces

$$(\mathcal{P} \widehat{\otimes} \mathcal{Q})_{m+q} \widehat{\otimes}_{\mathcal{A} \widehat{\otimes} \mathcal{B} - \mathcal{A} \widehat{\otimes} \mathcal{B}} Z \stackrel{i_1}{\cong} (P_m \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} X) \widehat{\otimes} (Q_q \widehat{\otimes}_{\mathcal{B}-\mathcal{B}} Y)$$

and

$$\begin{aligned} (\mathcal{P} \widehat{\otimes} \mathcal{Q})_{m+q-1} \widehat{\otimes}_{\mathcal{A} \widehat{\otimes} \mathcal{B} - \mathcal{A} \widehat{\otimes} \mathcal{B}} Z \stackrel{i_2}{\cong} \\ ((P_{m-1} \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} X) \widehat{\otimes} (Q_q \widehat{\otimes}_{\mathcal{B}-\mathcal{B}} Y)) \oplus ((P_m \widehat{\otimes}_{\mathcal{A}-\mathcal{A}} X) \widehat{\otimes} (Q_{q-1} \widehat{\otimes}_{\mathcal{B}-\mathcal{B}} Y)). \end{aligned}$$

Thus one can see that the operator

$$\delta_{m+q-1} \otimes_{\mathcal{A} \widehat{\otimes} \mathcal{B} - \mathcal{A} \widehat{\otimes} \mathcal{B}} 1_Z : (\mathcal{P} \widehat{\otimes} \mathcal{Q})_{m+q} \widehat{\otimes}_{\mathcal{A} \widehat{\otimes} \mathcal{B} - \mathcal{A} \widehat{\otimes} \mathcal{B}} Z \rightarrow (\mathcal{P} \widehat{\otimes} \mathcal{Q})_{m+q-1} \widehat{\otimes}_{\mathcal{A} \widehat{\otimes} \mathcal{B} - \mathcal{A} \widehat{\otimes} \mathcal{B}} Z$$

which is equal to the operator  $i_2^{-1} \circ \Delta \circ i_1$  is not topologically injective. Therefore, by Proposition 4.14,  $db_w(\mathcal{A} \widehat{\otimes} \mathcal{B}) = m + q$ . ■

THEOREM 4.17. Let  $\mathcal{A}$  and  $\mathcal{B}$  be biflat Banach algebras. Then:

(i) if  $\mathcal{A}$  and  $\mathcal{B}$  have bounded approximate identities (b.a.i.)

$$db_w(\mathcal{A} \widehat{\otimes} \mathcal{B}) = 0 \quad \text{and} \quad db_w(\mathcal{A}_+ \widehat{\otimes} \mathcal{B}_+) = db_w \mathcal{A} + db_w \mathcal{B} = 0;$$

(ii) if  $\mathcal{A}$  and  $\mathcal{B}$  have left [right], but not two-sided b.a.i.

$$db_w(\mathcal{A} \widehat{\otimes} \mathcal{B}) \leq 1 \quad \text{and} \quad db_w(\mathcal{A}_+ \widehat{\otimes} \mathcal{B}_+) = db_w \mathcal{A} + db_w \mathcal{B} = 2;$$

(iii) if  $\mathcal{A}$  and  $\mathcal{B}$  have neither left nor right b.a.i.

$$db_w(\mathcal{A} \widehat{\otimes} \mathcal{B}) \leq 2 \quad \text{and} \quad db_w(\mathcal{A}_+ \widehat{\otimes} \mathcal{B}_+) = db_w \mathcal{A} + db_w \mathcal{B} = 4.$$

*Proof.* By Theorem 6 of [26], for a biflat Banach algebra  $\mathcal{D}$ ,

$$db_w \mathcal{D} = \begin{cases} 0 & \text{if } \mathcal{D} \text{ has a b.a.i.,} \\ 1 & \text{if } \mathcal{D} \text{ has a left or right, but not two-sided b.a.i.,} \\ 2 & \text{if } \mathcal{D} \text{ has neither a left nor a right b.a.i.} \end{cases}$$

It is known that for any Banach algebra  $\mathcal{D}$ ,  $db_w \mathcal{D} = db_w \mathcal{D}_+$ . Hence, by Theorem 4.16, for unital Banach algebras  $\mathcal{A}_+$  and  $\mathcal{B}_+$ ,

$$db_w(\mathcal{A}_+ \widehat{\otimes} \mathcal{B}_+) = db_w \mathcal{A}_+ + db_w \mathcal{B}_+ = db_w \mathcal{A} + db_w \mathcal{B}.$$

By Corollary 4.7, the tensor product Banach algebra  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  is biflat. Note that  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  has a [left] (right) bounded approximate identity if  $\mathcal{A}$  and  $\mathcal{B}$  have [left] (right) bounded approximate identities [8]. The rest is clear. ■

EXAMPLE 4.18. In Theorem 5.3.2 of [28] Yu.V. Selivanov proved that the algebra  $\mathcal{K}(\ell_2 \widehat{\otimes} \ell_2)$  of compact operators on  $\ell_2 \widehat{\otimes} \ell_2$  is a biflat Banach algebra with a left, but not two-sided bounded approximate identity. Therefore, by Theorem 4.17, for an integer  $n \geq 1$ ,

$$db_w[\mathcal{K}(\ell_2 \widehat{\otimes} \ell_2)]^{\widehat{\otimes} n} \leq 1 \quad \text{and} \quad db_w[\mathcal{K}(\ell_2 \widehat{\otimes} \ell_2)_+]^{\widehat{\otimes} n} = n.$$

EXAMPLE 4.19. Let  $(E, F)$  be a pair of infinite-dimensional Banach spaces endowed with a nondegenerate jointly continuous bilinear form  $\langle \cdot, \cdot \rangle : E \times F \rightarrow \mathbb{C}$  that is not identically zero. The space  $\mathcal{A} = E \widehat{\otimes} F$  is a  $\widehat{\otimes}$ -algebra with respect to the multiplication defined by

$$(x_1 \otimes x_2)(y_1 \otimes y_2) = \langle x_2, y_1 \rangle x_1 \otimes y_2, \quad x_i \in E, y_i \in F.$$

Then  $\mathcal{A} = E \widehat{\otimes} F$  is called the *tensor algebra generated by the duality*  $(E, F, \langle \cdot, \cdot \rangle)$ . In Examples 2.1.13 and 2.1.14 of [28] Yu.V. Selivanov proved that  $\mathcal{A}$  is biprojective, and, by Theorem 2.6.21 and Corollary 2.6.24 of [28], has neither a left nor a right bounded approximate identity.

In particular, if  $E$  is a Banach space with the approximation property, then the algebra  $\mathcal{A} = E \widehat{\otimes} E^*$  is isomorphic to the algebra  $\mathcal{N}(E)$  of nuclear operators on  $E$  ([14], II.2.5).

Therefore, by Theorem 4.17, for an integer  $n \geq 1$ ,

$$db_w[E \widehat{\otimes} F]^{\widehat{\otimes} n} \leq 2 \quad \text{and} \quad db_w[(E \widehat{\otimes} F)_+]^{\widehat{\otimes} n} = 2n.$$

EXAMPLE 4.20. Let  $\mathcal{B}$  be the algebra of  $2 \times 2$ -complex matrices of the form

$$\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$$

with matrix multiplication and norm. It is known that  $\mathcal{B}$  is 2-amenable, biprojective, has a left, but not right identity [23]. Therefore, by Theorem 4.17, for an integer  $n \geq 1$ ,

$$db_w[\mathcal{B}]^{\widehat{\otimes} n} = 1, \quad \text{and} \quad db_w[\mathcal{B}_+]^{\widehat{\otimes} n} = n;$$

$$db_w[\mathcal{B} \widehat{\otimes} \mathcal{K}(\ell_2 \widehat{\otimes} \ell_2)]^{\widehat{\otimes} n} = 1, \quad \text{and} \quad db_w[\mathcal{B}_+ \widehat{\otimes} \mathcal{K}(\ell_2 \widehat{\otimes} \ell_2)_+]^{\widehat{\otimes} n} = 2n.$$

## 5. EXTERNAL PRODUCTS OF HOCHSCHILD COHOMOLOGY OF BANACH ALGEBRAS WITH BOUNDED APPROXIMATE IDENTITIES

THEOREM 5.1. *Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras with bounded approximate identities, let  $X$  be an essential Banach  $\mathcal{A}$ -bimodule and let  $Y$  be an essential Banach  $\mathcal{B}$ -bimodule. Then for  $n \geq 0$ , up to topological isomorphism,*

$$\mathcal{H}^n(\mathcal{A} \widehat{\otimes} \mathcal{B}, (X \widehat{\otimes} Y)^*) = H^n((\mathcal{C}_\sim(\mathcal{A}, X) \widehat{\otimes} \mathcal{C}_\sim(\mathcal{B}, Y))^*).$$

*Proof.* Consider the flat pseudo-resolutions  $0 \leftarrow \mathcal{A} \xleftarrow{\pi_{\mathcal{A}}} \beta(\mathcal{A})$  and  $0 \leftarrow \mathcal{B} \xleftarrow{\pi_{\mathcal{B}}} \beta(\mathcal{B})$  of  $\mathcal{A}$  and  $\mathcal{B}$  in the categories of bimodules. By Proposition 4.8 their projective tensor product  $\beta(\mathcal{A}) \widehat{\otimes} \beta(\mathcal{B})$  is an  $\mathcal{A} \widehat{\otimes} \mathcal{B}$ -flat pseudo-resolution of  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  in  $\mathcal{A} \widehat{\otimes} \mathcal{B}\text{-mod-}\mathcal{A} \widehat{\otimes} \mathcal{B}$ .

By Proposition 3.7 of [20], since the Banach algebra  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  has a bounded approximate identity, for  $n \geq 0$ , up to topological isomorphism,

$$\mathcal{H}^n(\mathcal{A} \widehat{\otimes} \mathcal{B}, (X \widehat{\otimes} Y)^*) = \text{Ext}_{(\mathcal{A} \widehat{\otimes} \mathcal{B})^e}^n(\mathcal{A} \widehat{\otimes} \mathcal{B}, (X \widehat{\otimes} Y)^*).$$

By Exercise VII.1.19 of [14],  $\text{Ext}_{(\mathcal{A} \widehat{\otimes} \mathcal{B})^e}^n(\mathcal{A} \widehat{\otimes} \mathcal{B}, (X \widehat{\otimes} Y)^*)$  is the cohomology of the complex  ${}_{\mathcal{A} \widehat{\otimes} \mathcal{B}} h_{\mathcal{A} \widehat{\otimes} \mathcal{B}}(\mathcal{F}, (X \widehat{\otimes} Y)^*)$ , where  $0 \leftarrow \mathcal{A} \widehat{\otimes} \mathcal{B} \leftarrow \mathcal{F}$  is a flat pseudo-resolution of  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  in  $\mathcal{A} \widehat{\otimes} \mathcal{B}\text{-mod-}\mathcal{A} \widehat{\otimes} \mathcal{B}$ . Thus,  $\text{Ext}_{(\mathcal{A} \widehat{\otimes} \mathcal{B})^e}^n(\mathcal{A} \widehat{\otimes} \mathcal{B}, (X \widehat{\otimes} Y)^*)$  can be computed by use of the flat pseudo-resolution  $\beta(\mathcal{A}) \widehat{\otimes} \beta(\mathcal{B})$ . Hence  $\text{Ext}_{(\mathcal{A} \widehat{\otimes} \mathcal{B})^e}^n(\mathcal{A} \widehat{\otimes} \mathcal{B}, (X \widehat{\otimes} Y)^*)$  is the cohomology of the complex  ${}_{\mathcal{A} \widehat{\otimes} \mathcal{B}} h_{\mathcal{A} \widehat{\otimes} \mathcal{B}}(\beta(\mathcal{A}) \widehat{\otimes} \beta(\mathcal{B}), (X \widehat{\otimes} Y)^*)$ . By the conjugate associativity law ([14], Theorem II.5.21), there is a natural isomorphism

of Banach spaces:

$$\begin{aligned} {}_{\mathcal{A}\widehat{\otimes}\mathcal{B}}h_{{}_{\mathcal{A}\widehat{\otimes}\mathcal{B}}}(\beta(\mathcal{A})\widehat{\otimes}\beta(\mathcal{B}), (X\widehat{\otimes}Y)^*) &\cong ((\beta(\mathcal{A})\widehat{\otimes}\beta(\mathcal{B}))\widehat{\otimes}_{{}_{\mathcal{A}\widehat{\otimes}\mathcal{B}}e}(X\widehat{\otimes}Y))^* \\ &\cong ((X\widehat{\otimes}Y)\widehat{\otimes}_{{}_{\mathcal{A}\widehat{\otimes}\mathcal{B}}e}(\beta(\mathcal{A})\widehat{\otimes}\beta(\mathcal{B})))^*. \end{aligned}$$

By Proposition II.3.12 of [14], one can see that the following chain complexes are topologically isomorphic:

$$\begin{aligned} (X\widehat{\otimes}Y)\widehat{\otimes}_{{}_{\mathcal{A}\widehat{\otimes}\mathcal{B}}e}(\beta(\mathcal{A})\widehat{\otimes}\beta(\mathcal{B})) &\cong (X\widehat{\otimes}_{{}_{\mathcal{A}}e}\beta(\mathcal{A}))\widehat{\otimes}(Y\widehat{\otimes}_{{}_{\mathcal{B}}e}\beta(\mathcal{B})) \\ &\cong \mathcal{C}_\sim(\mathcal{A}, X)\widehat{\otimes}\mathcal{C}_\sim(\mathcal{B}, Y). \end{aligned}$$

Thus, up to topological isomorphism,

$$\begin{aligned} \mathcal{H}^n(\mathcal{A}\widehat{\otimes}\mathcal{B}, (X\widehat{\otimes}Y)^*) &= H^n(((X\widehat{\otimes}Y)\widehat{\otimes}_{{}_{\mathcal{A}\widehat{\otimes}\mathcal{B}}e}(\beta(\mathcal{A})\widehat{\otimes}\beta(\mathcal{B})))^*) \\ &= H^n((\mathcal{C}_\sim(\mathcal{A}, X)\widehat{\otimes}\mathcal{C}_\sim(\mathcal{B}, Y))^*). \quad \blacksquare \end{aligned}$$

Corollary 5.2 to Theorem 5.1 gives another proof of the Künneth formula for Hochschild cohomology groups of Banach algebras with bounded approximate identities (see Theorem 5.5 of [11]).

**COROLLARY 5.2.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras with bounded approximate identities, let  $X$  be an essential Banach  $\mathcal{A}$ -bimodule and let  $Y$  be an essential Banach  $\mathcal{B}$ -bimodule. Suppose that all boundary maps of the standard homology complexes  $\mathcal{C}_\sim(\mathcal{A}, X)$  and  $\mathcal{C}_\sim(\mathcal{B}, Y)$  have closed range and, for all  $n$ ,  $B_n(\mathcal{A}, X)$  and  $\mathcal{H}_n(\mathcal{A}, X)$  are strictly flat. Then, for  $n \geq 0$ , up to topological isomorphism,*

$$\mathcal{H}^n(\mathcal{A}\widehat{\otimes}\mathcal{B}, (X\widehat{\otimes}Y)^*) = \bigoplus_{m+q=n} [\mathcal{H}_m(\mathcal{A}, X)\widehat{\otimes}\mathcal{H}_q(\mathcal{B}, Y)]^*.$$

*Proof.* By Theorem 5.1, up to topological isomorphism,

$$\mathcal{H}^n(\mathcal{A}\widehat{\otimes}\mathcal{B}, (X\widehat{\otimes}Y)^*) = H^n((\mathcal{C}_\sim(\mathcal{A}, X)\widehat{\otimes}\mathcal{C}_\sim(\mathcal{B}, Y))^*).$$

By Corollary 5.4 of [12], up to topological isomorphism,

$$H_n(\mathcal{C}_\sim(\mathcal{A}, X)\widehat{\otimes}\mathcal{C}_\sim(\mathcal{B}, Y)) = \bigoplus_{m+q=n} [H_m(\mathcal{C}_\sim(\mathcal{A}, X))\widehat{\otimes}H_q(\mathcal{C}_\sim(\mathcal{B}, Y))].$$

Hence, by Corollary 4.9 of [12], since the  $H_n(\mathcal{C}_\sim(\mathcal{A}, X)\widehat{\otimes}\mathcal{C}_\sim(\mathcal{B}, Y))$  are Banach spaces,

$$H^n((\mathcal{C}_\sim(\mathcal{A}, X)\widehat{\otimes}\mathcal{C}_\sim(\mathcal{B}, Y))^*) \cong [H_n(\mathcal{C}_\sim(\mathcal{A}, X)\widehat{\otimes}\mathcal{C}_\sim(\mathcal{B}, Y))]^*.$$

Therefore

$$\mathcal{H}^n(\mathcal{A}\widehat{\otimes}\mathcal{B}, (X\widehat{\otimes}Y)^*) = \bigoplus_{m+q=n} [\mathcal{H}_m(\mathcal{A}, X)\widehat{\otimes}\mathcal{H}_q(\mathcal{B}, Y)]^*. \quad \blacksquare$$

**REMARK 5.3.** Under the conditions of Corollary 5.2,  $\mathcal{H}^n(\mathcal{A}\widehat{\otimes}\mathcal{B}, (X\widehat{\otimes}Y)^*)$  is a Banach space, and by Corollary 4.9 of [12],  $\mathcal{H}_n(\mathcal{A}\widehat{\otimes}\mathcal{B}, X\widehat{\otimes}Y)$  is a Banach space too.

The closure in  $X$  of the linear span of elements of the form  $\{a \cdot x - x \cdot a : a \in \mathcal{A}, x \in X\}$  is denoted by  $[X, \mathcal{A}]$ .

**THEOREM 5.4.** *Let  $\mathcal{A}$  and  $\mathcal{B}$  be Banach algebras with bounded approximate identities, let  $X$  be an essential Banach  $\mathcal{A}$ -bimodule and let  $Y$  be an essential Banach  $\mathcal{B}$ -bimodule. Suppose  $\mathcal{A}$  is amenable. Then, for  $n \geq 0$ , up to topological isomorphism,*

$$\mathcal{H}^n(\mathcal{A} \widehat{\otimes} \mathcal{B}, (X \widehat{\otimes} Y)^*) = \mathcal{H}^n(\mathcal{B}, (X/[X, \mathcal{A}] \widehat{\otimes} Y)^*),$$

where  $b \cdot (\bar{x} \otimes y) = (\bar{x} \otimes b \cdot y)$  and  $(\bar{x} \otimes y) \cdot b = (\bar{x} \otimes y \cdot b)$  for  $\bar{x} \in X/[X, \mathcal{A}]$ ,  $y \in Y$  and  $b \in \mathcal{B}$ .

*Proof.* By assumption  $\mathcal{A}$  is biflat, so  $0 \leftarrow \mathcal{A} \xleftarrow{\text{id}_{\mathcal{A}}} \mathcal{A} \leftarrow 0$  is a flat pseudo-resolution of  $\mathcal{A}$  in the category of  $\mathcal{A}$ -bimodules. As in Theorem 5.1 we consider the flat pseudo-resolutions  $0 \leftarrow \mathcal{A} \xleftarrow{\text{id}_{\mathcal{A}}} \mathcal{A} \leftarrow 0$  and  $0 \leftarrow \mathcal{B} \xleftarrow{\pi_{\mathcal{B}}} \beta(\mathcal{B})$  of  $\mathcal{A}$  and  $\mathcal{B}$  in the categories of bimodules. By Proposition 4.8, their projective tensor product  $\mathcal{A} \widehat{\otimes} \beta(\mathcal{B})$  is an  $\mathcal{A} \widehat{\otimes} \mathcal{B}$ -flat pseudo-resolution of  $\mathcal{A} \widehat{\otimes} \mathcal{B}$  in  $\mathcal{A} \widehat{\otimes} \mathcal{B}\text{-mod-}\mathcal{A} \widehat{\otimes} \mathcal{B}$ .

Similar to Theorem 5.1, one can see that, up to topological isomorphism,

$$\mathcal{H}^n(\mathcal{A} \widehat{\otimes} \mathcal{B}, (X \widehat{\otimes} Y)^*) = H^n(((X \widehat{\otimes} Y) \widehat{\otimes}_{(\mathcal{A} \widehat{\otimes} \mathcal{B})^e} (\mathcal{A} \widehat{\otimes} \beta(\mathcal{B})))^*).$$

One can check that, for an amenable Banach algebra  $\mathcal{A}$  and for an essential module  $X$ , up to topological isomorphism,  $X \widehat{\otimes}_{\mathcal{A}^e} \mathcal{A} = X/[X, \mathcal{A}]$ ; see Proposition VII.2.17 of [14]. Therefore, by Proposition II.3.12 of [14], one can see that the following chain complexes are topologically isomorphic:

$$(X \widehat{\otimes} Y) \widehat{\otimes}_{(\mathcal{A} \widehat{\otimes} \mathcal{B})^e} (\mathcal{A} \widehat{\otimes} \beta(\mathcal{B})) \cong (X \widehat{\otimes}_{\mathcal{A}^e} \mathcal{A}) \widehat{\otimes} (Y \widehat{\otimes}_{\mathcal{B}^e} \beta(\mathcal{B})) \cong X/[X, \mathcal{A}] \widehat{\otimes} \mathcal{C}_{\sim}(\mathcal{B}, Y).$$

Thus, up to topological isomorphism,

$$\begin{aligned} \mathcal{H}^n(\mathcal{A} \widehat{\otimes} \mathcal{B}, (X \widehat{\otimes} Y)^*) &= H^n(((X \widehat{\otimes} Y) \widehat{\otimes}_{(\mathcal{A} \widehat{\otimes} \mathcal{B})^e} (\mathcal{A} \widehat{\otimes} \beta(\mathcal{B})))^*) \\ &= H^n(\mathcal{C}_{\sim}(\mathcal{B}, X/[X, \mathcal{A}] \widehat{\otimes} Y)^*) = \mathcal{H}^n(\mathcal{B}, (X/[X, \mathcal{A}] \widehat{\otimes} Y)^*), \end{aligned}$$

where  $b \cdot (\bar{x} \otimes y) = (\bar{x} \otimes b \cdot y)$  and  $(\bar{x} \otimes y) \cdot b = (\bar{x} \otimes y \cdot b)$  for  $\bar{x} \in X/[X, \mathcal{A}]$ ,  $y \in Y$  and  $b \in \mathcal{B}$ . ■

**EXAMPLE 5.5.** Let  $\mathcal{A}$  be the Banach algebra  $L^1(\mathbb{R}_+)$  of complex-valued, Lebesgue measurable functions  $f$  on  $\mathbb{R}_+$  with finite  $L^1$ -norm and convolution multiplication. In Theorem 4.6 of [11] we showed that all boundary maps of the standard homology complex  $\mathcal{C}_{\sim}(\mathcal{A}, \mathcal{A})$  have closed ranges and that  $\mathcal{H}_n(\mathcal{A}, \mathcal{A})$  and  $B_n(\mathcal{A}, \mathcal{A})$  are strictly flat in  $\mathcal{Ban}$ . In Theorem 6.4 of [11] we describe explicitly the simplicial homology groups  $\mathcal{H}_n(L^1(\mathbb{R}_+^k), L^1(\mathbb{R}_+^k))$  and cohomology groups  $\mathcal{H}^n(L^1(\mathbb{R}_+^k), (L^1(\mathbb{R}_+^k))^*)$  of the semigroup algebra  $L^1(\mathbb{R}_+^k)$ .

**COROLLARY 5.6.** *Let  $\mathcal{C}$  be an amenable Banach algebra. Then*

$$\begin{aligned} \mathcal{H}_n(L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C}, L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C}) &\cong \{0\} \quad \text{if } n > k, \\ \mathcal{H}^n(L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C}, (L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C})^*) &\cong \{0\} \quad \text{if } n > k, \end{aligned}$$

up to topological isomorphism,

$$\mathcal{H}_n(L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C}, L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C}) \cong \bigoplus^{(k)} L^1(\mathbb{R}_+^k) \widehat{\otimes} (\mathcal{C}/[\mathcal{C}, \mathcal{C}]) \text{ if } n \leq k, \text{ and}$$

$$\mathcal{H}^n(L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C}, (L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C})^*) \cong \bigoplus^{(k)} [L^1(\mathbb{R}_+^k) \widehat{\otimes} (\mathcal{C}/[\mathcal{C}, \mathcal{C}])]^*$$

if  $n \leq k$ .

*Proof.* Let  $\mathcal{A} = L^1(\mathbb{R}_+)$ . Note that  $\mathcal{A}$  and  $\mathcal{C}$  have bounded approximate identities. By Theorem 5.4 of [19], for an amenable Banach algebra  $\mathcal{C}$ ,  $\mathcal{H}_n(\mathcal{C}, \mathcal{C}) \cong \{0\}$  for all  $n \geq 1$ ,  $\mathcal{H}_0(\mathcal{C}, \mathcal{C}) \cong \mathcal{C}/[\mathcal{C}, \mathcal{C}]$ . Therefore all boundary maps of the standard homology complex  $\mathcal{C}_\sim(\mathcal{C}, \mathcal{C})$  have closed ranges.

In Theorem 4.6 of [11] we showed that all boundary maps of the standard homology complex  $\mathcal{C}_\sim(\mathcal{A}, \mathcal{A})$  have closed ranges and that  $\mathcal{H}_n(\mathcal{A}, \mathcal{A})$  and  $B_n(\mathcal{A}, \mathcal{A})$  are strictly flat in  $\mathcal{Ban}$ . By Theorem 4.6 of [11], up to topological isomorphism, the simplicial homology groups  $\mathcal{H}_n(\mathcal{A}, \mathcal{A})$  are given by  $\mathcal{H}_0(\mathcal{A}, \mathcal{A}) \cong \mathcal{H}_1(\mathcal{A}, \mathcal{A}) \cong \mathcal{A} = L^1(\mathbb{R}_+)$  and  $\mathcal{H}_n(\mathcal{A}, \mathcal{A}) \cong \{0\}$  for  $n \geq 2$ .

Note that  $L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C} \cong \mathcal{A} \widehat{\otimes} \mathcal{B}$  where  $\mathcal{B} = L^1(\mathbb{R}_+^{k-1}) \widehat{\otimes} \mathcal{C}$ . We use induction on  $k$  to prove the corollary for homology groups. For  $k = 1$ , the result follows from Theorem 5.5 of [11]. The simplicial homology groups  $\mathcal{H}_n(\mathcal{A} \widehat{\otimes} \mathcal{C}, \mathcal{A} \widehat{\otimes} \mathcal{C})$  are given, up to topological isomorphism, by

$$\mathcal{H}_0(\mathcal{A} \widehat{\otimes} \mathcal{C}, \mathcal{A} \widehat{\otimes} \mathcal{C}) \cong \mathcal{H}_1(\mathcal{A} \widehat{\otimes} \mathcal{C}, \mathcal{A} \widehat{\otimes} \mathcal{C}) \cong \mathcal{A} \widehat{\otimes} (\mathcal{C}/[\mathcal{C}, \mathcal{C}])$$

and

$$\mathcal{H}_n(\mathcal{A} \widehat{\otimes} \mathcal{C}, \mathcal{A} \widehat{\otimes} \mathcal{C}) \cong \{0\}$$

for  $n \geq 2$ .

Let  $k > 1$  and suppose that the result for homology holds for  $k - 1$ . As  $L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C} \cong \mathcal{A} \widehat{\otimes} \mathcal{B}$  where  $\mathcal{B} = L^1(\mathbb{R}_+^{k-1}) \widehat{\otimes} \mathcal{C}$ , we have

$$\mathcal{H}_n(L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C}, L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C}) \cong \mathcal{H}_n(\mathcal{A} \widehat{\otimes} \mathcal{B}, \mathcal{A} \widehat{\otimes} \mathcal{B}).$$

It also follows from the inductive hypothesis that, for all  $n$ , the  $\mathcal{H}_n(\mathcal{B}, \mathcal{B})$  are Banach spaces and hence the  $B_n(\mathcal{B}, \mathcal{B})$  are closed. We can therefore apply Theorem 5.5 of [11] for  $\mathcal{A}$  and  $\mathcal{B} = L^1(\mathbb{R}_+^{k-1}) \widehat{\otimes} \mathcal{C}$ , where  $\mathcal{C}$  is an amenable Banach algebra, to get

$$\mathcal{H}_n(\mathcal{A} \widehat{\otimes} \mathcal{B}, \mathcal{A} \widehat{\otimes} \mathcal{B}) \cong \bigoplus_{m+q=n} [\mathcal{H}_m(\mathcal{A}, \mathcal{A}) \widehat{\otimes} \mathcal{H}_q(\mathcal{B}, \mathcal{B})].$$

The terms in this direct sum vanish for  $m \geq 2$ , and thus we only need to consider

$$(\mathcal{H}_0(\mathcal{A}, \mathcal{A}) \widehat{\otimes} \mathcal{H}_n(\mathcal{B}, \mathcal{B})) \oplus (\mathcal{H}_1(\mathcal{A}, \mathcal{A}) \widehat{\otimes} \mathcal{H}_{n-1}(\mathcal{B}, \mathcal{B})).$$

For cohomology groups, by Corollary 4.9 of [11],

$$\begin{aligned} \mathcal{H}^n(L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C}, (L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C})^*) &\cong \mathcal{H}^n(\mathcal{C}_\sim(L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C}, L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C})^*) \\ &\cong \bigoplus^{(k)} [L^1(\mathbb{R}_+^k) \widehat{\otimes} (\mathcal{C}/[\mathcal{C}, \mathcal{C}])]^* \end{aligned}$$



if  $n \leq k$ . ■

EXAMPLE 5.7. Some examples of  $C^*$ -algebras without non-zero bounded traces are:

(i) The  $C^*$ -algebra  $\mathcal{K}(H)$  of compact operators on an infinite-dimensional Hilbert space  $H$ ; see Theorem 2 of [1]. We can also show that  $C(\Omega, \mathcal{K}(H))^{\text{tr}} = 0$ , where  $\Omega$  is a compact space.

(ii) Properly infinite von Neumann algebras  $\mathcal{U}$ ; see Example 4.6 of [17]. This class includes the  $C^*$ -algebra  $\mathcal{B}(H)$  of all bounded operators on an infinite-dimensional Hilbert space  $H$ ; see also [13] for the statement  $\mathcal{B}(H)^{\text{tr}} = 0$ .

EXAMPLE 5.8. Let  $\mathcal{A} = \ell^1(\mathbb{Z}_+)$  where

$$\ell^1(\mathbb{Z}_+) = \left\{ (a_n)_{n=0}^\infty : \sum_{n=0}^\infty |a_n| < \infty \right\}$$

be the unital semigroup Banach algebra of  $\mathbb{Z}_+$  with convolution multiplication and norm  $\|(a_n)_{n=0}^\infty\| = \sum_{n=0}^\infty |a_n|$ . In Theorem 7.4 of [12] we showed that all boundary maps of the standard homology complex  $\mathcal{C}_\sim(\mathcal{A}, \mathcal{A})$  have closed ranges and that  $\mathcal{H}_n(\mathcal{A}, \mathcal{A})$  and  $B_n(\mathcal{A}, \mathcal{A})$  are strictly flat in  $\mathcal{Ban}$ . In Theorem 7.5 of [12] we describe explicitly the simplicial homology groups  $\mathcal{H}_n(\ell^1(\mathbb{Z}_+^k), \ell^1(\mathbb{Z}_+^k))$  and cohomology groups  $\mathcal{H}^n(\ell^1(\mathbb{Z}_+^k), (\ell^1(\mathbb{Z}_+^k))^*)$  of the semigroup algebra  $\ell^1(\mathbb{Z}_+^k)$ .

One can find definitions of the continuous cyclic  $\mathcal{HC}$  and periodic cyclic  $\mathcal{HP}$  homology and cohomology groups of Banach algebras in [5].

COROLLARY 5.9. Let  $\mathcal{D}$  be a Banach algebra belonging to one of the following classes:

- (i)  $\mathcal{D} = \ell^1(\mathbb{Z}_+^k) \widehat{\otimes} \mathcal{C}$ , where  $\mathcal{C}$  is a  $C^*$ -algebra without non-zero bounded traces;
- (ii)  $\mathcal{D} = L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C}$ , where  $\mathcal{C}$  is a  $C^*$ -algebra without non-zero bounded traces.

Then  $\mathcal{H}_n(\mathcal{D}, \mathcal{D}) \cong \{0\}$  and  $\mathcal{H}^n(\mathcal{D}, \mathcal{D}) \cong \{0\}$  for all  $n \geq 0$ ,

$$\mathcal{HH}_n(\mathcal{D}) \cong \mathcal{HH}^n(\mathcal{D}) \cong \{0\} \quad \text{for all } n \geq 0,$$

$$\mathcal{HC}_n(\mathcal{D}) \cong \mathcal{HC}^n(\mathcal{D}) \cong \{0\} \quad \text{for all } n \geq 0, \text{ and}$$

$$\mathcal{HP}_m(\mathcal{D}) \cong \mathcal{HP}^m(\mathcal{D}) \cong \{0\} \quad \text{for } m = 0, 1.$$

*Proof.* Here we consider the case with  $\mathcal{A} = L^1(\mathbb{R}_+)$  and prove the statement for  $\mathcal{D} = L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C}$ . By Theorem 4.1 and Corollary 3.3 of [3], for a  $C^*$ -algebra  $\mathcal{C}$  without non-zero bounded traces  $\mathcal{H}^n(\mathcal{C}, \mathcal{C}^*) \cong \{0\}$  for all  $n \geq 0$ . By Corollary 1.3 of [16],  $\mathcal{H}_n(\mathcal{C}, \mathcal{C}) \cong \{0\}$  for all  $n \geq 0$ . Therefore all boundary maps of the standard homology complex  $\mathcal{C}_\sim(\mathcal{C}, \mathcal{C})$  have closed ranges.

In Theorem 4.6 of [11] we showed that all boundary maps of the standard homology complex  $\mathcal{C}_\sim(\mathcal{A}, \mathcal{A})$  have closed ranges and that  $\mathcal{H}_n(\mathcal{A}, \mathcal{A})$  and  $B_n(\mathcal{A}, \mathcal{A})$  are strictly flat in  $\mathcal{Ban}$ . By Theorem 4.6 of [11], up to topological isomorphism, the simplicial homology groups  $\mathcal{H}_n(\mathcal{A}, \mathcal{A})$  are given by  $\mathcal{H}_0(\mathcal{A}, \mathcal{A}) \cong \mathcal{H}_1(\mathcal{A}, \mathcal{A}) \cong \mathcal{A} = L^1(\mathbb{R}_+)$  and  $\mathcal{H}_n(\mathcal{A}, \mathcal{A}) \cong \{0\}$  for  $n \geq 2$ .

Note that  $\mathcal{D} = L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C} \cong \mathcal{A} \widehat{\otimes} \mathcal{B}$  where  $\mathcal{B} = L^1(\mathbb{R}_+^{k-1}) \widehat{\otimes} \mathcal{C}$ . We use induction on  $k$  to prove the corollary for homology groups. For  $k = 1$ , note that  $\mathcal{A}$  and  $\mathcal{C}$  have bounded approximate identities. By Theorem 5.5 of [11], the simplicial homology groups  $\mathcal{H}_n(\mathcal{A} \widehat{\otimes} \mathcal{C}, \mathcal{A} \widehat{\otimes} \mathcal{C})$  are given, up to topological isomorphism, by

$$\mathcal{H}_n(\mathcal{A} \widehat{\otimes} \mathcal{C}, \mathcal{A} \widehat{\otimes} \mathcal{C}) \cong \bigoplus_{m+q=n} [\mathcal{H}_m(\mathcal{A}, \mathcal{A}) \widehat{\otimes} \mathcal{H}_q(\mathcal{C}, \mathcal{C})] \cong \{0\}$$

for  $n \geq 0$ .

Let  $k > 1$  and suppose that the result for homology holds for  $k - 1$ . As  $L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C} \cong \mathcal{A} \widehat{\otimes} \mathcal{B}$  where  $\mathcal{B} = L^1(\mathbb{R}_+^{k-1}) \widehat{\otimes} \mathcal{C}$ , we have

$$\mathcal{H}_n(L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C}, L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C}) \cong \mathcal{H}_n(\mathcal{A} \widehat{\otimes} \mathcal{B}, \mathcal{A} \widehat{\otimes} \mathcal{B}).$$

Note that  $\mathcal{A}$  and  $\mathcal{B}$  have bounded approximate identities. Further, it follows from the inductive hypothesis that  $\mathcal{H}_n(\mathcal{B}, \mathcal{B}) \cong \{0\}$  for all  $n \geq 0$  and hence the  $B_n(\mathcal{B}, \mathcal{B})$  are closed. We can therefore apply Theorem 5.5 of [11] to get

$$\mathcal{H}_n(\mathcal{A} \widehat{\otimes} \mathcal{B}, \mathcal{A} \widehat{\otimes} \mathcal{B}) \cong \bigoplus_{m+q=n} [\mathcal{H}_m(\mathcal{A}, \mathcal{A}) \widehat{\otimes} \mathcal{H}_q(\mathcal{B}, \mathcal{B})] \cong \{0\}$$

for all  $n \geq 0$ .

For cohomology groups, by Corollary 1.3 of [16],

$$\mathcal{H}^n(L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C}, (L^1(\mathbb{R}_+^k) \widehat{\otimes} \mathcal{C})^*) \cong \{0\}$$

for all  $n \geq 0$ .

The triviality of the continuous cyclic  $\mathcal{H}\mathcal{C}$  and periodic cyclic  $\mathcal{H}\mathcal{P}$  homology and cohomology groups follows from Corollary 4.7 of [18]. ■

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ZINAIDA A. LYKOVA, SCHOOL OF MATHEMATICS AND STATISTICS, NEWCASTLE UNIVERSITY, NEWCASTLE UPON TYNE, NE1 7RU, U.K.

*E-mail address:* Z.A.Lykova@newcastle.ac.uk

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