

FELL BUNDLES OVER INVERSE SEMIGROUPS AND TWISTED ÉTALE GROUPOIDS

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ABSTRACT. Given a saturated Fell bundle \mathcal{A} over an inverse semigroup S which is semi-abelian in the sense that the fibers over the idempotents of S are commutative, we construct a twisted étale groupoid (\mathcal{G}, Σ) such that \mathcal{A} can be recovered from (\mathcal{G}, Σ) in a canonical way. As an application we recover most of Renault's recent result on the classification of Cartan subalgebras of C^* -algebras through twisted étale groupoids.

KEYWORDS: *Fell bundle, inverse semigroup, twisted étale groupoid, upper-semicontinuous Banach bundle, crossed product, Cartan subalgebra.*

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1. INTRODUCTION

Among the most interesting examples of C^* -algebras one finds the “dynamical C^* -algebras”, meaning C^* -algebras constructed out of some dynamical system. The interest in their study lies in the fact that they are algebraic representations of their accompanying systems, sometimes revealing features which are not immediately seen with a naked eye.

The classical notion of a group action on a topological space, the most basic form of a dynamical system, leads to the *crossed product* or *covariance C^* -algebra* which, over the years, has proven to be an invaluable tool in the study of group actions.

Dynamical systems often take slightly more sophisticated forms, such as semigroup actions, pseudogroups, or topological groupoids and, in most cases, crossed-product-like constructions may be performed providing C^* -algebras mirroring dynamical features algebraically.

The huge variety of dynamical C^* -algebras has prompted some to reverse the order of things and to look for dynamical data attached to C^* -algebras which are not necessarily born from a dynamical system. This point of view has been enormously successful, so much so that it is now a standard tool in the study

of C^* -algebras. When available it often gives important information on simplicity, the structure of ideals, faithfulness of representations and K-theory, among others.

One of the earliest attempts at uncovering dynamical data beneath unsuspecting algebraic systems is Feldman and Moore's [9] description of Cartan subalgebras of von Neumann algebras via twisted measured equivalence relations. Kumjian [13] and Renault [22] have later made these ideas to work in the context of C^* -algebras and, thanks to them, we now know that Cartan subalgebras of C^* -algebras may be described via twisted, essentially principal, étale groupoids.

Motivated by these developments, the second named author has recently found a generalized notion of (non-commutative) Cartan subalgebras, for which a similar characterization may be given [6]. The *dynamical* object underneath this characterization comes in the form of a *Fell bundle over an inverse semigroup*, a concept introduced by Sieben in a talk given at the Groupoid Fest in 1998 (see [23], [6]). But, perhaps due to the fact that this concept is still deeply rooted in algebra, it may not immediately appear to deserve the label of a *dynamical system*.

The present work intends to bridge this gap, clarifying the relationship between such Fell bundles and dynamics proper.

By definition a Fell bundle over an inverse semigroup S consists of a family $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$ of Banach spaces, equipped with bilinear *multiplication* operations $\mathcal{A}_s \times \mathcal{A}_t \rightarrow \mathcal{A}_{st}$, conjugate-linear *involution* operations $\mathcal{A}_s \rightarrow \mathcal{A}_{s^*}$, and *inclusion* maps $\mathcal{A}_s \hookrightarrow \mathcal{A}_t$, whenever $s \leq t$. All these data are required to satisfy certain natural axioms (see Definition 2.8 below).

The expression *Fell bundle* has its roots in Fell's pioneering work [10] and is used in this work primarily to refer to Fell bundles over inverse semigroups, as briefly defined above, but the concept of Fell bundles over groupoids, as introduced by Kumjian and Yamagami [25], [14], also plays a crucial role since the latter provides examples of the former: given a Fell bundle \mathcal{B} over an étale groupoid \mathcal{G} , let S be any inverse semigroup consisting of bisections of \mathcal{G} , and for each $U \in S$, let \mathcal{A}_U be the space of all continuous sections of \mathcal{B} over U vanishing at infinity. The operations on \mathcal{B} may be used to give the collection $\mathcal{A} = \{\mathcal{A}_U\}_{U \in S}$ the structure of a Fell bundle over S . Under mild conditions, we prove that the cross sectional C^* -algebras of \mathcal{B} and of \mathcal{A} are isomorphic (see Theorem 2.13). In case the groupoid \mathcal{G} is Hausdorff, we follow a partition-of-unit argument appearing in Theorem 7.1 of [20]. This idea also appears in the unpublished preprint ([23], Proposition 3.5), by Nánbor Sieben. We also deal with the non-Hausdorff case by using some ideas borrowed from [5], [17].

A special subcase of this construction is obtained when, starting from a twisted groupoid (\mathcal{G}, Σ) , we form the associated Fell line bundle \mathcal{B} over \mathcal{G} .

Given any Fell bundle \mathcal{A} over an inverse semigroup S , and given an element e in the idempotent semilattice $E(S)$, the *fiber over e* , namely \mathcal{A}_e , is always a C^* -algebra. In the special case of the above Fell bundle constructed from a twisted étale groupoid (\mathcal{G}, Σ) , a bisection U is idempotent if and only if it is contained in

the unit space $\mathcal{G}^{(0)}$. In this case the fiber over U is just the algebra $\mathcal{C}_0(U)$ of continuous complex-valued functions (the twist over the unit space is always trivial so it may be disregarded when $U \subseteq \mathcal{G}^{(0)}$) vanishing at infinity on U , which is obviously a commutative C^* -algebra. This suggests the terminology *semi-abelian*, referring to Fell bundles for which \mathcal{A}_e is commutative for every idempotent element e .

Our main result shows that every semi-abelian Fell bundle arises from a twisted étale groupoid in the above fashion. This gives substance to the statement that Fell bundles over inverse semigroups are indeed dynamical objects, and it also supports the claim that general Fell bundles (not necessarily semi-abelian ones) should be considered as *twisted groupoids with non-commutative unit space*.

Our techniques borrow lavishly from Kumjian [13] and Renault [22], especially when constructing a groupoid from a semi-abelian Fell bundle. Our construction of the twist is also heavily inspired by these works, although we have found it more economical to construct the associated line bundle directly, without first passing through the twist itself. Should the twist be needed, it can be easily recovered as the circle bundle associated to our line bundle.

In the last section we apply our result to Cartan subalgebras. Given a (commutative) Cartan subalgebra B of a C^* -algebra A , the results of [6] yield a Fell bundle \mathcal{A} over an inverse semigroup S , and an isomorphism $A \cong C_r^*(\mathcal{A})$, sending B onto $C_r^*(\mathcal{E})$, where \mathcal{E} is the restriction of \mathcal{A} to the idempotent semilattice of S . Since B is commutative, \mathcal{A} must be semi-abelian, so we may apply our results in order to obtain a twisted étale groupoid (\mathcal{G}, Σ) together with a canonical isomorphism $A \cong C_r^*(\mathcal{G}, \Sigma)$, which sends B onto $\mathcal{C}_0(\mathcal{G}^{(0)})$. This proves most of Renault's main result in [22].

It should be stressed that the groupoids that come out of our construction are not necessarily Hausdorff, as opposed to the groupoids considered in [13] and [22]. In fact, starting with Connes' work on foliation groupoids [1], the recent literature on non-Hausdorff groupoids has increased significantly providing efficient techniques which often only require small changes in relation to the Hausdorff case.

Returning to the Hausdorff question, in our proof of Renault's characterization of Cartan subalgebras, we have not seen how to prove, without appealing to Renault's ideas, that the underlying groupoid is Hausdorff.

Since Renault's proof relies on the existence of a conditional expectation, we were led to conjecture that, if \mathcal{A} is a semi-abelian Fell bundle over an inverse semigroup such that there exists a conditional expectation from $C_r^*(\mathcal{A})$ onto $C_r^*(\mathcal{E})$, then the underlying groupoid should be Hausdorff. However this is not true as shown by the example given in Proposition 5.3. We are therefore forced to accept that Hausdorffness is not only a consequence of the existence of the conditional expectation, but that it also depends on maximal abeliannes, as in Renault's result.

2. PRELIMINARIES

2.1. **BANACH BUNDLES.** In this section we will establish basic facts about upper-semicontinuous Banach bundles that we need in the sequel. Even though this is a well known theory, covered in detail in several references (see for instance [3], [10], [16] and references therein), its applications to non-Hausdorff spaces have not been widely considered up to now. Even though large parts of the theory generalize nicely to certain non-Hausdorff spaces, most of the classical texts deal exclusively with the Hausdorff case.

DEFINITION 2.1 ([3], [16]). Let X be a (not necessarily Hausdorff) topological space. An *upper-semicontinuous-Banach bundle* over X is a pair $\mathcal{A} = (A, p)$ consisting of a topological space A together with a continuous, open surjection $p : A \rightarrow X$. It is moreover assumed that for each $x \in X$, the *fiber over x* , namely $\mathcal{A}_x := p^{-1}(x)$, has the structure of a complex Banach space satisfying:

- (i) the map $v \mapsto \|v\|$ is upper-semicontinuous from A to \mathbb{R}^+ ;
- (ii) the map $(v, w) \mapsto v + w$ is continuous from $\{(v, w) \in A \times A : p(v) = p(w)\}$ (seen as a topological subspace of $A \times A$) to A ;
- (iii) for each $\lambda \in \mathbb{C}$, the map $v \mapsto \lambda v$ is continuous from A to A ;
- (iv) if $\{v_i\}_i$ is a net in A such that $p(v_i) \rightarrow x$ and $\|v_i\| \rightarrow 0$, then $\{v_i\}_i$ converges to 0_x , the zero element in \mathcal{A}_x .

If the map $v \mapsto \|v\|$ is continuous from A to \mathbb{R}^+ , we say that \mathcal{A} is a *continuous Banach bundle*.

REMARK 2.2. Although the norm on \mathcal{A} need not be continuous, upper-semicontinuity forces $\|v_i\|$ to converge to 0 whenever v_i converges to 0_x for some $x \in X$. With this in mind, as observed in [16] (see comments after Definition 3.1), the same proof of Proposition II.13.10 in [10] can still be applied to an upper-semicontinuous Banach bundle \mathcal{A} in order to get a stronger version of property (iii):

- (iii') the map $(\lambda, v) \mapsto \lambda v$ is continuous from $\mathbb{C} \times A$ to A .

In what follows we are going to omit the bundle projection p and the total space A from our notation and identify the latter with the bundle \mathcal{A} itself. Moreover, we also usually identify \mathcal{A} with the collection $\{\mathcal{A}_x\}_{x \in X}$ of all fibers and actually write $\mathcal{A} = \{\mathcal{A}_x\}_{x \in X}$ to denote all the data. We believe that this will not cause any confusion.

As already mentioned, we need efficient methods of constructing Banach bundles, and we shall now devote ourselves to generalizing Fell and Doran's main such tool ([10], Theorem II.13.18).

As a first step we suppose we are given a (not necessarily Hausdorff) topological space X and a pairwise disjoint collection of Banach spaces $\{\mathcal{A}_x\}_{x \in X}$. Denote by \mathcal{A} the disjoint union of the \mathcal{A}_x and let $p : \mathcal{A} \rightarrow X$ be the function which assigns x to every element of \mathcal{A}_x .

DEFINITION 2.3. If U is any subset of X , and if $\zeta : U \rightarrow \mathcal{A}$ is a function such that $\zeta(x) \in \mathcal{A}_x$ for every $x \in U$, we say that ζ is a *local section of \mathcal{A} over U* . In this case, we denote the *domain* of ζ by $\text{dom}(\zeta) := U$.

In Theorem II.13.18 of [10] one chooses a collection of globally defined sections satisfying certain assumptions and one constructs a topology on \mathcal{A} with respect to which the given sections are continuous.

However, there are examples of locally Hausdorff spaces X , for which the trivial one-dimensional bundle admits no global continuous compactly supported section (see Example 1.2 of [12]). To remedy this situation we work here with local sections defined on open subsets of X .

Given a subset Γ of local sections of \mathcal{A} , we shall write $\text{span } \Gamma$ for the set of the local sections of the form

$$\sum_{i=1}^n \lambda_i \zeta_i \quad \text{with } \lambda_i \in \mathbb{C}, \zeta_i \in \Gamma \text{ and } n \in \mathbb{N},$$

where by definition,

$$\text{dom} \left(\sum_{i=1}^n \lambda_i \zeta_i \right) := \bigcap_{i=1}^n \text{dom}(\zeta_i) \quad \text{and} \quad \left(\sum_{i=1}^n \lambda_i \zeta_i \right)(x) := \sum_{i=1}^n \lambda_i \zeta_i(x).$$

Note that the set of all local sections of \mathcal{A} is *not* a vector space with respect to the sum and scalar product defined above because the sum fails to have additive inverses. However, it is a so called *semi-vector space*, that is, all the axioms of a vector space are satisfied, except for the existence of additive inverses.

The following result, which is a non-Hausdorff version of Theorem II.13.18 in [10], is certainly well-known to specialists and is essentially the same as Proposition 3.6 in [11]. However we have chosen to include the proof here for the reader's convenience.

PROPOSITION 2.4. *Let Γ be a set of local sections of \mathcal{A} whose domains $\text{dom}(\zeta)$ are open subsets of X for all $\zeta \in \Gamma$. Suppose that:*

- (i) *given $v \in \mathcal{A}$, there exists $\zeta \in \Gamma$ such that $p(v) \in \text{dom}(\zeta)$ and $v = \zeta(p(v))$;*
- (ii) *the map $x \mapsto \|\zeta(x)\|$ is upper-semicontinuous from $\text{dom}(\zeta)$ to \mathbb{R}^+ for all $\zeta \in \text{span } \Gamma$, that is, if $\zeta \in \text{span } \Gamma$ and if α is a positive real number, then the following is open in X :*

$$\{x \in \text{dom}(\zeta) : \|\zeta(x)\| < \alpha\}.$$

Then there exists a unique topology on \mathcal{A} making it an upper-semicontinuous Banach bundle over X and such that all the local sections ζ of $\text{span } \Gamma$, viewed as functions $\zeta : \text{dom}(\zeta) \rightarrow \mathcal{A}$, are continuous. A basis of open sets for this topology is given by the sets of the following form, where $\zeta \in \Gamma$, U is an open subset of $\text{dom}(\zeta)$, and $\varepsilon > 0$:

$$\Omega(U, \zeta, \varepsilon) = \{v \in \mathcal{A} : p(v) \in U, \|v - \zeta(p(v))\| < \varepsilon\}.$$

Moreover, \mathcal{A} is a continuous Banach bundle with this topology if and only if the maps $\text{dom}(\zeta) \ni x \mapsto \|\zeta(x)\| \in \mathbb{R}^+$ are continuous for all $\zeta \in \Gamma$.

Proof. In order to see that the above sets do indeed form the basis of some topology on \mathcal{A} , let

$$v_0 \in \Omega(U_1, \zeta_1, \varepsilon_1) \cap \Omega(U_2, \zeta_2, \varepsilon_2),$$

where $\varepsilon_i > 0$, $\zeta_i \in \Gamma$, and $U_i \subseteq \text{dom}(\zeta_i)$, for $i = 1, 2$. Put $x_0 = p(v_0)$, so $x_0 \in U_i$, and $\|v_0 - \zeta_i(x_0)\| < \varepsilon_i$. Choose $\varepsilon'_i > 0$ such that

$$\|v_0 - \zeta_i(x_0)\| < \varepsilon'_i < \varepsilon_i,$$

and let $\eta \in \Gamma$, such that $\eta(x_0) = v_0$. Put

$$V_i = U_i \cap \{x \in \text{dom}(\eta) \cap \text{dom}(\zeta_i) : \|\eta(x) - \zeta_i(x)\| < \varepsilon'_i\},$$

and observe that V_i is open by (ii). Notice that $x_0 \in V_i$. Letting $\delta > 0$ be such that $\varepsilon'_i + \delta < \varepsilon_i$, we claim that

$$(2.1) \quad v_0 \in \Omega(V_1 \cap V_2, \eta, \delta) \subseteq \Omega(U_1, \zeta_1, \varepsilon_1) \cap \Omega(U_2, \zeta_2, \varepsilon_2).$$

To show that v_0 is in the indicated set, notice that $p(v_0) = x_0 \in V_i$, and

$$\|v_0 - \eta(p(v_0))\| = 0 < \delta.$$

To show that $\Omega(V_1 \cap V_2, \eta, \delta) \subseteq \Omega(U_i, \zeta_i, \varepsilon_i)$, pick any v belonging to the set in the left-hand-side. So $p(v) \in V_i \subseteq U_i$. Moreover

$$\|v - \zeta_i(p(v))\| \leq \|v - \eta(p(v))\| + \|\eta(p(v)) - \zeta_i(p(v))\| \leq \delta + \varepsilon'_i < \varepsilon_i.$$

This shows that $v \in \Omega(U_i, \zeta_i, \varepsilon_i)$, and hence concludes the proof of equation (2.1). To see that the collection of all the $\Omega(U, \zeta, \varepsilon)$ does indeed form the basis for a topology on \mathcal{A} it is therefore enough to check that its union equals \mathcal{A} , but this is clear from (i) because any $v \in \mathcal{A}$ lies in $\Omega(\text{dom}(\zeta), \zeta, \varepsilon)$, as long as $v = \zeta(p(v))$.

Step 1. p is open. To see this it is enough to show that p sends basic open sets to open sets, but this follows immediately from the fact that $p(\Omega(U, \zeta, \varepsilon)) = U$.

Step 2. The norm is upper-semicontinuous. We need to show that the set

$$N_\alpha = \{v \in \mathcal{A} : \|v\| < \alpha\}$$

is open for every $\alpha > 0$. So let $v_0 \in N_\alpha$, and choose $\zeta \in \Gamma$, such that $\zeta(x_0) = v_0$, where $x_0 = p(v_0)$. Pick α' such that $\|v_0\| < \alpha' < \alpha$, and set

$$U = \{x \in \text{dom}(\zeta) : \|\zeta(x)\| < \alpha'\},$$

so U is open by (ii) and $x_0 \in U$. Choose $\varepsilon > 0$, such that $\alpha' + \varepsilon < \alpha$, and let us show that

$$(2.2) \quad v_0 \in \Omega(U, \zeta, \varepsilon) \subseteq N_\alpha.$$

On the one hand $p(v_0) = x_0 \in U$, and on the other $\|v_0 - \zeta(p(v_0))\| = 0 < \varepsilon$, so $v_0 \in \Omega(U, \zeta, \varepsilon)$. Moreover, given any $v \in \Omega(U, \zeta, \varepsilon)$, we have $p(v) \in U$, and

$$\|v\| \leq \|v - \zeta(p(v))\| + \|\zeta(p(v))\| < \varepsilon + \alpha' < \alpha.$$

This proves equation (2.2), and hence we see that N_α is indeed open.

Step 3. The sum is continuous. Let $v_0, w_0 \in \mathcal{A}$, with $p(v_0) = p(w_0)$, and suppose that $v_0 + w_0$ lies in some basic open set $\Omega(U, \zeta, \varepsilon)$. We need to provide

an open subset Δ of $\mathcal{A} \times \mathcal{A}$, containing (v_0, w_0) , and such that the sum operation sends Δ into $\Omega(U, \xi, \varepsilon)$, meaning that for every $(v, w) \in \Delta$ such that $p(v) = p(w)$, one has $v + w \in \Omega(U, \xi, \varepsilon)$.

Set $x_0 = p(v_0) = p(w_0)$, so we have $x_0 \in U$, and we may pick $\varepsilon' > 0$ such that

$$\|v_0 + w_0 - \xi(x_0)\| < \varepsilon' < \varepsilon.$$

Choose $\eta, \zeta \in \Gamma$, such that $v_0 = \eta(x_0)$, and $w_0 = \zeta(x_0)$, and put

$$V = U \cap \{x \in \text{dom}(\eta) \cap \text{dom}(\zeta) \cap \text{dom}(\xi) : \|\eta(x) + \zeta(x) - \xi(x)\| < \varepsilon'\},$$

and notice that V is open by (ii) and $x_0 \in V$. Let $\delta > 0$ be such that $\varepsilon' + 2\delta < \varepsilon$, and observe that

$$(v_0, w_0) \in \Omega(V, \eta, \delta) \times \Omega(V, \zeta, \delta).$$

We claim that the set in the right-hand side fulfills the task assigned to Δ , above.

In fact, pick $(v, w) \in \Omega(V, \eta, \delta) \times \Omega(V, \zeta, \delta)$, with $p(v) = p(w)$. In order to show that $v + w \in \Omega(U, \xi, \varepsilon)$, notice that $x := p(v + w) \in V \subseteq U$, and moreover $\|v + w - \xi(x)\| = \|v - \eta(x)\| + \|w - \zeta(x)\| + \|\eta(x) + \zeta(x) - \xi(x)\| < 2\delta + \varepsilon' < \varepsilon$.

This proves the continuity of the sum operation.

Step 4. Scalar multiplication is continuous. Given $\lambda \in \mathbb{C}$, and $v_0 \in \mathcal{A}$ such that λv_0 lies in some basic open set $\Omega(U, \xi, \varepsilon)$, we have $x_0 := p(\lambda v_0) = p(v_0) \in U$, and there exists ε' such that

$$\|\lambda v_0 - \xi(x_0)\| < \varepsilon' < \varepsilon.$$

Choose $\eta \in \Gamma$ such that $\eta(x_0) = v_0$, and consider the open set

$$V = U \cap \{x \in \text{dom}(\eta) \cap \text{dom}(\xi) : \|\lambda \eta(x) - \xi(x)\| < \varepsilon'\},$$

which clearly contains x_0 . Choosing $\delta > 0$ such that

$$\varepsilon' + |\lambda| \delta < \varepsilon,$$

we claim that if $w \in \Omega(V, \eta, \delta)$, then $\lambda w \in \Omega(U, \xi, \varepsilon)$. In fact, by assumption we have $p(\lambda w) = p(w) \in V \subseteq U$, and

$$\begin{aligned} \|\lambda w - \xi(p(w))\| &\leq \|\lambda w - \lambda \eta(p(w))\| + \|\lambda \eta(p(w)) - \xi(p(w))\| \\ &< |\lambda| \|w - \eta(p(w))\| + \varepsilon' < |\lambda| \delta + \varepsilon' < \varepsilon, \end{aligned}$$

so indeed $\lambda w \in \Omega(U, \xi, \varepsilon)$.

Step 5. Proof of axiom from Definition 2.1(iv). Let $\{v_i\}_i$ and x be as in Definition 2.1(iv). Denoting by 0_x the zero element of \mathcal{A}_x , let $\Omega(U, \xi, \varepsilon)$ be a basic neighborhood of 0_x , so $x \in U$, and $\|\xi(x)\| < \varepsilon$. Pick ε' such that

$$\|\xi(x)\| < \varepsilon' < \varepsilon,$$

and let

$$V = U \cap \{y \in \text{dom}(\xi) : \|\xi(y)\| < \varepsilon'\}.$$

Clearly V is open and $x \in V$. Choose $\delta > 0$, such that $\varepsilon' + \delta < \varepsilon$. For all i bigger than or equal to some i_0 , one therefore has $p(v_i) \in V$, and $\|v_i\| < \delta$. For $i \geq i_0$ one then has $p(v_i) \in U$, and

$$\|v_i - \zeta(p(v_i))\| \leq \|v_i\| + \|\zeta(p(v_i))\| < \delta + \varepsilon' < \varepsilon,$$

so $v_i \in \Omega(U, \zeta, \varepsilon)$, proving that $v_i \rightarrow 0_x$. We conclude that \mathcal{A} is an upper-semicontinuous Banach bundle with the topology which has the sets $\Omega(U, \zeta, \varepsilon)$ as basis. Now, to see that each $\zeta \in \text{span } \Gamma$ is continuous as a function $\zeta : \text{dom}(\zeta) \rightarrow \mathcal{A}$, take any $\eta \in \Gamma$, suppose that $U \subseteq \text{dom}(\eta)$ is an open subset, and take $\varepsilon > 0$. Then we have

$$\begin{aligned} \{x \in \text{dom}(\zeta) : \zeta(x) \in \Omega(U, \eta, \varepsilon)\} &= \{x \in \text{dom}(\zeta) : x \in U \text{ and } \|\zeta(x) - \eta(x)\| < 0\} \\ &= U \cap \{x \in \text{dom}(\zeta) \cap \text{dom}(\eta) : \|\zeta(x) - \eta(x)\| < \varepsilon\}, \end{aligned}$$

which is open by (ii) since $\zeta - \eta \in \text{span } \Gamma$. This says that $\zeta : \text{dom}(\zeta) \rightarrow \mathcal{A}$ is continuous.

To see that the topology on \mathcal{A} is uniquely determined, assume that \mathcal{A} has a topology making it an upper-semicontinuous Banach bundle and such that all the local sections in $\text{span } \Gamma$ are continuous. Since the norm is upper-semicontinuous, and since the map $p^{-1}(U) \ni v \mapsto v - \zeta(p(v)) \in \mathcal{A}$ is continuous for every $\zeta \in \Gamma$ and every open subset $U \subseteq \text{dom}(\zeta)$, the sets $\Omega(U, \zeta, \varepsilon)$ must be open in \mathcal{A} . Moreover, given any open subset $\mathcal{V} \subseteq \mathcal{A}$ and $v_0 \in \mathcal{V}$, we claim that there are $\zeta \in \Gamma$, $U \subseteq \text{dom}(\zeta)$ open and $\varepsilon > 0$ such that $v_0 \in \Omega(U, \zeta, \varepsilon) \subseteq \mathcal{V}$. By (i), there is a local section $\zeta \in \Gamma$ such that $x_0 := p(v_0) \in \text{dom}(\zeta)$ and $\zeta(x_0) = v_0$. Suppose, by contradiction, that for any open subset $U \subseteq \text{dom}(\zeta)$ containing x_0 and for any $\varepsilon > 0$, the set $\Omega(U, \zeta, \varepsilon)$ is not contained in \mathcal{V} . This yields an element $v_{U, \varepsilon} \in \mathcal{A}$ which is not contained in \mathcal{V} and satisfies $p(v_{U, \varepsilon}) \in U$ and $\|v_{U, \varepsilon} - \zeta(p(v_{U, \varepsilon}))\| < \varepsilon$. Then the net $\{p(v_{U, \varepsilon})\}_{U, \varepsilon}$ converges to x_0 (where the pairs U, ε are directed in the canonical way) and the net $w_{U, \varepsilon} := v_{U, \varepsilon} - \zeta(p(v_{U, \varepsilon}))$ converges in norm to 0. By Definition 2.1(iv), the net $w_{U, \varepsilon}$ converges to 0_{x_0} in \mathcal{A} . Since ζ and p are continuous, $\zeta(p(v_{U, \varepsilon}))$ converges to $\zeta(p(x_0)) = v_0$, and hence $v_{U, \varepsilon}$ converges to v_0 . This is a contradiction because \mathcal{V} is open, $v_0 \in \mathcal{V}$ and $v_{U, \varepsilon} \notin \mathcal{V}$ for all U, ε . This proves our claim. As a consequence, the sets $\Omega(U, \zeta, \varepsilon)$ form a basis for the topology on \mathcal{A} , and therefore it must be the same topology we constructed above.

Finally, suppose that \mathcal{A} is a continuous Banach bundle with the above topology. Since the $\zeta \in \Gamma$ are continuous and the norm on \mathcal{A} is continuous, the maps $\text{dom}(\zeta) \ni x \mapsto \|\zeta(x)\| \in \mathbb{R}^+$ must be continuous for all $\zeta \in \Gamma$. Conversely, suppose these maps are continuous. To show that the map $\mathcal{A} \ni v \mapsto \|v\| \in \mathbb{R}^+$ is continuous, it is enough to prove lower semi-continuity since we already know it is upper-semicontinuous. Thus we have to show that the sets $M_\alpha := \{v \in \mathcal{A} : \|v\| > \alpha\}$ are open in \mathcal{A} for all $\alpha > 0$. Take $v_0 \in M_\alpha$ and choose $\zeta \in \Gamma$ with $x_0 = p(v_0) \in \text{dom}(\zeta)$ and $\zeta(x_0) = v_0$. Pick α' satisfying $\alpha < \alpha' < \|v_0\|$ and define $U := \{x \in \text{dom}(\zeta) : \|\zeta(x)\| > \alpha'\}$. If $0 < \varepsilon < \alpha' - \alpha$, then every $v \in \Omega(U, \zeta, \varepsilon)$ satisfies

$$\|v\| \geq \|\zeta(p(v))\| - \|\zeta(p(v)) - v\| > \alpha' - \varepsilon > \alpha,$$

so that $v_0 \in \Omega(U, \xi, \varepsilon) \subseteq M_{\mathcal{K}}$. This concludes the proof. ■

2.2. FELL BUNDLES OVER GROUPOIDS AND INVERSE SEMIGROUPS. Let us begin by precisely identifying the class of groupoids that will interest us.

DEFINITION 2.5 ([21]). A groupoid \mathcal{G} is said to be a *topological groupoid* if it is equipped with a (not necessarily Hausdorff) topology relative to which the multiplication and inversion operations are continuous. We say that \mathcal{G} is *étale* if, in addition, the unit space $\mathcal{G}^{(0)}$ is locally compact and Hausdorff in the relative topology, and the range map $r : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$ (and consequently also the domain map $d : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$) is a local homeomorphism.

The following definition of Fell bundles over groupoids is the same given by Alex Kumjian in [14] (see also [25]), except that Kumjian only works with Hausdorff groupoids and continuous Banach bundles. Continuous Banach bundles would be enough for us, but our main results require non-Hausdorff groupoids. Given our interest in étale groupoids, we shall restrict ourselves to this situation although the concept below actually makes sense for any topological groupoid.

From now on we fix an étale groupoid \mathcal{G} .

DEFINITION 2.6. A *Fell bundle* over \mathcal{G} is an upper-semicontinuous Banach bundle $\mathcal{A} = \{\mathcal{A}_\gamma\}_{\gamma \in \mathcal{G}}$ over \mathcal{G} together with a *multiplication*

$$\cdot : \mathcal{A}^{(2)} = \{(a, b) \in \mathcal{A} \times \mathcal{A} : (p(a), p(b)) \in \mathcal{G}^{(2)}\} \rightarrow \mathcal{A},$$

where $p : \mathcal{A} \rightarrow \mathcal{G}$ is the bundle projection, and an *involution*

$$* : \mathcal{A} \rightarrow \mathcal{A}, \quad a \mapsto a^*$$

satisfying the following properties:

(i) $p(\gamma_1 \gamma_2) = p(\gamma_1) p(\gamma_2)$, that is, $\mathcal{A}_{\gamma_1} \cdot \mathcal{A}_{\gamma_2} \subseteq \mathcal{A}_{\gamma_1 \gamma_2}$ whenever $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$, and the multiplication map $\mathcal{A}_{\gamma_1} \times \mathcal{A}_{\gamma_2} \rightarrow \mathcal{A}_{\gamma_1 \gamma_2}$, $(a, b) \mapsto a \cdot b$ is bilinear;

(ii) the multiplication on \mathcal{A} is associative, that is, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ whenever $a, b, c \in \mathcal{A}$ and this makes sense;

(iii) the multiplication $\cdot : \mathcal{A}^{(2)} \rightarrow \mathcal{A}$ is continuous, where $\mathcal{A}^{(2)}$ carries the topology induced from the product topology on $\mathcal{A} \times \mathcal{A}$;

(iv) $\|a \cdot b\| \leq \|a\| \|b\|$ for all $a, b \in \mathcal{A}$;

(v) $p(a^*) = p(a)$ for all $a \in \mathcal{A}$, that is, $\mathcal{A}_\gamma^* \subseteq \mathcal{A}_{\gamma^{-1}}$ for all $\gamma \in \mathcal{G}$, and the involution map $* : \mathcal{A}_\gamma \rightarrow \mathcal{A}_{\gamma^{-1}}$ is conjugate linear;

(vi) $(a^*)^* = a$, $\|a^*\| = \|a\|$ and $(a \cdot b)^* = b^* \cdot a^*$ for all $a, b \in \mathcal{A}$;

(vii) the involution $* : \mathcal{A} \rightarrow \mathcal{A}$ is continuous;

(viii) $\|a^* a\| = \|a\|^2$ for all $a \in \mathcal{A}$; and

(ix) $a^* a$ is a positive element of the C^* -algebra $\mathcal{A}_{d(\gamma)}$ for all $\gamma \in \mathcal{G}$ and $a \in \mathcal{A}_\gamma$.

We say that \mathcal{A} is *saturated* if the closed linear span of $\mathcal{A}_{\gamma_1} \cdot \mathcal{A}_{\gamma_2}$ equals $\mathcal{A}_{\gamma_1 \gamma_2}$ for all $(\gamma_1, \gamma_2) \in \mathcal{G}^{(2)}$.

Note that for a unit $x \in \mathcal{G}^{(0)}$, the fiber \mathcal{A}_x is in fact a C^* -algebra with respect to the restricted multiplication and involution from \mathcal{A} . This follows from (i)–(viii) above, so that (ix) makes sense. Moreover, the restriction $\mathcal{A}^{(0)} = \mathcal{A}|_{\mathcal{G}^{(0)}}$ is an upper-semicontinuous C^* -bundle [18]. Conversely, if X is any locally compact Hausdorff space, and if \mathcal{B} is an upper-semicontinuous C^* -bundle over X , then \mathcal{B} is a Fell bundle in the sense above over X considered as a groupoid in the trivial way.

PROPOSITION 2.7. *Let \mathcal{A} be an upper-semicontinuous Banach bundle over \mathcal{G} , and let Γ be a set of local sections of \mathcal{A} satisfying the properties (i) and (ii) in Proposition 2.4. Suppose that $\cdot : \mathcal{A}^{(2)} \rightarrow \mathcal{A}$ is a multiplication and $*$: $\mathcal{A} \rightarrow \mathcal{A}$ is an involution on \mathcal{A} satisfying all the axioms (i)–(ix) in Definition 2.6 except, possibly, for (iii) and (vii). Then:*

(i) *the multiplication $\cdot : \mathcal{A}^{(2)} \rightarrow \mathcal{A}$ is continuous if and only if the following map is continuous for all $\xi, \eta \in \Gamma$:*

$$\mathcal{G}^{(2)} \cap (\text{dom}(\xi) \times \text{dom}(\eta)) \ni (\gamma_1, \gamma_2) \mapsto \xi(\gamma_1) \cdot \eta(\gamma_2) \in \mathcal{A};$$

(ii) *the involution $*$: $\mathcal{A} \rightarrow \mathcal{A}$ is continuous if and only if the following map is continuous for all $\xi \in \Gamma$:*

$$\mathcal{G} \supseteq \text{dom}(\xi) \ni \gamma \mapsto \xi(\gamma)^* \in \mathcal{A}.$$

Proof. Although we do not have the same set of hypothesis, the same proof of VIII.2.4 in [10] can be applied to our situation in order to prove (i). And (ii) is also proved in a similar way (see also VIII.3.2 of [10]). ■

Next, we recall the definition of Fell bundles over inverse semigroups and compare with the one over groupoids defined above. This concept was first introduced by Nánbor Sieben in 1998 (see [23]), and we borrow some of his ideas. Unfortunately, his results were not published, but the main ideas are contained in the preprint [6] by the second named author.

DEFINITION 2.8. Let S be an inverse semigroup. A Fell bundle over S is a collection $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$ of Banach spaces \mathcal{A}_s together with a multiplication $\cdot : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$, an involution $*$: $\mathcal{A} \rightarrow \mathcal{A}$, and linear maps $j_{t,s} : \mathcal{A}_s \rightarrow \mathcal{A}_t$ whenever $s \leq t$, satisfying the following properties:

(i) $\mathcal{A}_s \cdot \mathcal{A}_t \subseteq \mathcal{A}_{st}$ and the multiplication is bilinear from $\mathcal{A}_s \times \mathcal{A}_t$ to \mathcal{A}_{st} for all $s, t \in S$;

(ii) the multiplication is associative, that is, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in \mathcal{A}$;

(iii) $\|a \cdot b\| \leq \|a\| \|b\|$ for all $a, b \in \mathcal{A}$;

(iv) $\mathcal{A}_s^* \subseteq \mathcal{A}_{s^*}$ and the involution is conjugate linear from \mathcal{A}_s to \mathcal{A}_{s^*} ;

(v) $(a^*)^* = a$, $\|a^*\| = \|a\|$ and $(a \cdot b)^* = b^* \cdot a^*$;

(vi) $\|a^*a\| = \|a\|^2$ and a^*a is a positive element of the C^* -algebra \mathcal{A}_{s^*s} for all $s \in S$ and $a \in \mathcal{A}_s$;

(vii) $j_{t,s} : \mathcal{A}_s \rightarrow \mathcal{A}_t$ is an isometric linear map for all $s \leq t$ in S ;

(viii) if $r \leq s \leq t$ in S , then $j_{t,r} = j_{t,s} \circ j_{s,r}$;

(ix) if $s \leq t$ and $u \leq v$ in S , then $j_{t,s}(a) \cdot j_{v,u}(b) = j_{tv,su}(a \cdot b)$ for all $a \in \mathcal{A}_s$ and $b \in \mathcal{A}_u$; in other words, the following diagram, where $\mu_{s,u}$ and $\mu_{t,v}$ denote the multiplication maps, commutes:

$$\begin{array}{ccc} \mathcal{A}_s \times \mathcal{A}_u & \xrightarrow{\mu_{s,u}} & \mathcal{A}_{su} \\ j_{t,s} \times j_{v,u} \downarrow & & \downarrow j_{tv,su} \\ \mathcal{A}_t \times \mathcal{A}_v & \xrightarrow{\mu_{t,v}} & \mathcal{A}_{tv} \end{array}$$

(x) if $s \leq t$ in S , then $j_{t,s}(a)^* = j_{t^*,s^*}(a^*)$ for all $a \in \mathcal{A}_s$, that is, the diagram

$$\begin{array}{ccc} \mathcal{A}_s & \xrightarrow{*} & \mathcal{A}_{s^*} \\ j_{t,s} \downarrow & & \downarrow j_{t^*,s^*} \\ \mathcal{A}_t & \xrightarrow{*} & \mathcal{A}_{t^*} \end{array}$$

commutes; if $\mathcal{A}_s \cdot \mathcal{A}_t$ spans a dense subspace of \mathcal{A}_{st} for all $s, t \in S$, we say that \mathcal{A} is *saturated*.

EXAMPLE 2.9. Let \mathcal{G} be an étale groupoid, and let $\mathcal{B} = \{\mathcal{B}_\gamma\}_{\gamma \in \mathcal{G}}$ be a Fell bundle over \mathcal{G} (as in Definition 2.6). Let $S(\mathcal{G})$ be the inverse semigroup of all bisections in \mathcal{G} . Recall that an open subset $U \subseteq \mathcal{G}$ is a *bisection* if the restrictions $d_U : U \rightarrow d(U)$ and $r_U : U \rightarrow r(U)$ of the source and range maps $d : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$ and $r : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$ are homeomorphisms. Let $S \subseteq S(\mathcal{G})$ be an inverse subsemigroup. Given $U \in S$, define \mathcal{A}_U to be the space $\mathcal{C}_0(\mathcal{B}_U)$ of continuous sections vanishing at infinity of the restriction \mathcal{B}_U of \mathcal{B} to $U \subseteq \mathcal{G}$. Then the collection $\mathcal{A} = \{\mathcal{A}_U\}_{U \in S}$ (disjoint union of the \mathcal{A}_U 's) is a Fell bundle over S with respect to the following structure:

(i) the multiplication $\mathcal{A}_U \times \mathcal{A}_V \rightarrow \mathcal{A}_{UV}$ is defined by

$$(\xi \cdot \eta)(\gamma) := \xi(r_U^{-1}(r(\gamma))) \cdot \eta(d_V^{-1}(d(\gamma)))$$

whenever $\xi \in \mathcal{C}_0(\mathcal{B}_U)$, $\eta \in \mathcal{C}_0(\mathcal{B}_V)$ and $\gamma \in UV$ (recall that UV is defined as the set of all products $\gamma_1\gamma_2$ with $\gamma_1 \in U$, $\gamma_2 \in V$ and $d(\gamma_1) = r(\gamma_2)$);

(ii) the involution $\mathcal{A}_U \rightarrow \mathcal{A}_{U^*}$ is defined by

$$\xi^*(\gamma) := \xi(\gamma^{-1})^*$$

whenever $\xi \in \mathcal{C}_0(\mathcal{B}_U)$ and $\gamma \in U^* := \{\gamma^{-1} : \gamma \in U\}$;

(iii) the inclusion maps $j_{V,U} : \mathcal{A}_U \hookrightarrow \mathcal{A}_V$ are defined in the canonical way: if $U \leq V$, that is, $U \subseteq V$, then we may extend a section $\xi \in \mathcal{C}_0(\mathcal{B}_U)$ by zero outside U and view it as a section $\tilde{\xi} \in \mathcal{C}_0(\mathcal{B}_V)$. Thus $j_{V,U}(\xi) := \tilde{\xi}$, where $\tilde{\xi}$ denotes the extension of ξ by zero.

The proof that \mathcal{A} is in fact a Fell bundle is not difficult and is left to the reader. Let us just remark that the multiplication above is well-defined. Moreover, since U, V are bisections, there is a unique way to write $\gamma \in UV$ as a product $\gamma = \gamma_1\gamma_2$ with $\gamma_1 \in U$ and $\gamma_2 \in V$. Indeed, we must have $\gamma_1 = r_U^{-1}(r(\gamma))$ and $\gamma_2 = d_V^{-1}(d(\gamma))$. Note that the multiplication we defined in \mathcal{A} uses the multiplication of \mathcal{B} . Thus $(\zeta \cdot \eta)(\gamma) = \zeta(\gamma_1) \cdot \eta(\gamma_2) \in \mathcal{B}_{\gamma_1} \cdot \mathcal{B}_{\gamma_2} \subseteq \mathcal{B}_{\gamma_1\gamma_2} = \mathcal{B}_\gamma$, so $\zeta \cdot \eta$ is in fact a section. Note that the associativity of the multiplication in \mathcal{A} now follows easily from the associativity of the products in \mathcal{G} and \mathcal{B} . Moreover, it also follows that the multiplication in \mathcal{A} is the usual convolution:

$$(\zeta \cdot \eta)(\gamma) = (\zeta * \eta)(\gamma) = \sum_{\gamma=\gamma_1\gamma_2} \zeta(\gamma_1) \cdot \eta(\gamma_2).$$

We want to prove that the Fell bundles described in Example 2.9 have same universal C^* -algebras. We consider two cases. First, if the groupoid \mathcal{G} is Hausdorff, we shall follow an idea appearing in Theorem 7.1 of [20] which uses a partition-of-unit argument. This idea does not seem to work in the non-Hausdorff case. However, under suitable separability conditions, we adapt an idea appearing in [5] together with a disintegration result of [17] to solve the non-Hausdorff case as well.

We need to work with bundles with incomplete fibers, and for this we shall need the following technical result:

LEMMA 2.10. *Let $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$ be a Fell bundle over an inverse semigroup S , and suppose that $\mathcal{A}^0 = \{\mathcal{A}_s^0\}_{s \in S}$ is a sub-bundle of \mathcal{A} satisfying the following properties:*

- (i) \mathcal{A}_s^0 is a dense subspace of \mathcal{A}_s for all $s \in S$;
- (ii) $\mathcal{A}_s^0 \cdot \mathcal{A}_t^0 \subseteq \mathcal{A}_{st}^0$ and $(\mathcal{A}_s^0)^* \subseteq \mathcal{A}_{s^*}^0$ for all $s, t \in S$;
- (iii) $\mathcal{A}_{ss^*}^0 \mathcal{A}_s \subseteq \mathcal{A}_s^0$ and $\mathcal{A}_s \mathcal{A}_{s^*}^0 \subseteq \mathcal{A}_s^0$ for all $s \in S$; and
- (iv) $j_{t,s}(\mathcal{A}_s^0) \subseteq \mathcal{A}_t^0$ whenever $s \leq t$.

Then $C^(\mathcal{A})$ is the enveloping C^* -algebra of the quotient $\mathcal{C}_c(\mathcal{A}^0)/\mathcal{I}_{\mathcal{A}}^0$, where $\mathcal{I}_{\mathcal{A}}^0$ is the ideal of $\mathcal{C}_c(\mathcal{A}^0)$ defined by*

$$\mathcal{I}_{\mathcal{A}}^0 := \text{span}\{a\delta_s - j_{t,s}(a)\delta_t : a \in \mathcal{A}_s^0, s \leq t\}.$$

Proof. Recall that $\mathcal{I}_{\mathcal{A}} = \text{span}\{a\delta_s - j_{t,s}(a)\delta_t : a \in \mathcal{A}_s, s \leq t\}$ is an ideal of $\mathcal{C}_c(\mathcal{A})$ and $C^*(\mathcal{A})$ is the enveloping C^* -algebra of the quotient $\mathcal{C}_c(\mathcal{A})/\mathcal{I}_{\mathcal{A}}$. By definition, a pre-representation of \mathcal{A} , once linearly extended to $\mathcal{C}_c(\mathcal{A}) = \text{span}\{a\delta_s : a \in \mathcal{A}_s\}$, is a representation if and only if it vanishes on the ideal $\mathcal{I}_{\mathcal{A}}$. Let π be a pre-representation of \mathcal{A}^0 on some C^* -algebra C , that is, π is a map $\mathcal{A}^0 \rightarrow C$ which respects the multiplication and involution of \mathcal{A}^0 (which are well-defined by (ii)) and is linear when restricted to the fibers \mathcal{A}_s^0 . Then π extends to a pre-representation $\tilde{\pi}$ of \mathcal{A} into C . In fact, (ii) implies that \mathcal{A}_e^0 is an ideal of \mathcal{A}_e for all $e \in E(S)$. It follows from Lemma 2.3 of [20] that the restriction $\pi_e : \mathcal{A}_e^0 \rightarrow C$ of π to \mathcal{A}_e^0 (which is a $*$ -homomorphism) is contractive, that is, $\|\pi_e(a)\| \leq \|a\|$ for all $a \in \mathcal{A}_e^0$. Using the $*$ -identity and the fact that π respects multiplication

and involution, this implies that all restrictions $\pi_s : \mathcal{A}_s^0 \rightarrow C$ are contractive and therefore extend to $\tilde{\pi}_s : \mathcal{A}_s \rightarrow C$. Of course, $\tilde{\pi} = \{\tilde{\pi}_s\}_{s \in S}$ is a pre-representation since π is. Now assume that π vanishes on the ideal $\mathcal{I}_{\mathcal{A}}^0$, that is, π is *coherent* in the sense that $\pi(a) = \pi(j_{t,s}(a))$ whenever $a \in \mathcal{A}_s^0$ and $s \leq t$. Then the extension $\tilde{\pi}$ is also coherent. Indeed, take a bounded approximate unit (e_i) for \mathcal{A}_{s^*s} contained in $\mathcal{A}_{s^*s}^0$. If $a \in \mathcal{A}_s$, then $ae_i \in \mathcal{A}_s^0$ by (iii), so that

$$\tilde{\pi}(a) = \tilde{\pi}\left(\lim_i ae_i\right) = \lim_i \pi(ae_i) = \lim_i \pi(j_{t,s}(ae_i)) = \lim_i \tilde{\pi}(j_{t,s}(ae_i)) = \tilde{\pi}(j_{t,s}(a)).$$

Conversely, if $\tilde{\pi}$ is coherent, so is its restriction π . We conclude that representations (that is, coherent pre-representations) of \mathcal{A}^0 correspond bijectively to representations of \mathcal{A} , whence the result follows. ■

As in Example 2.9, let $\mathcal{B} = \{\mathcal{B}_\gamma\}_{\gamma \in \mathcal{G}}$ be a Fell bundle over an étale groupoid \mathcal{G} , and let $\mathcal{A} = \{\mathcal{C}_0(\mathcal{B}_U)\}_{U \in S}$ be the associated Fell bundle over an inverse subsemigroup $S \subseteq S(\mathcal{G})$. Given $U \in S$, we define $\mathcal{A}_U^0 := \mathcal{C}_c(\mathcal{B}_U)$. Then the sub-bundle $\mathcal{A}^0 = \{\mathcal{A}_U^0\}_{U \in S}$ of \mathcal{A} satisfies the properties (i)–(iv) in statement of Lemma 2.10. The only non-trivial property to be checked is (iii). But if $\xi \in \mathcal{C}_0(\mathcal{B}_U)$, $\eta \in \mathcal{C}_c(\mathcal{B}_{U^*U})$ and $K = \text{supp}(\eta) \subseteq U^*U = d(U)$, then $(\xi \cdot \eta)(\gamma) = \xi(\gamma)\eta(d(\gamma))$ for all $\gamma \in U$, so that $\text{supp}(\xi \cdot \eta) \subseteq d_U^{-1}(K)$ is a compact subset of U , and hence $\xi \cdot \eta \in \mathcal{C}_c(\mathcal{B}_U)$. Analogously, $\mathcal{C}_c(\mathcal{B}_{UU^*}) \cdot \mathcal{C}_0(\mathcal{B}_U) \subseteq \mathcal{C}_c(\mathcal{B}_U)$. As a consequence of Lemma 2.10, $C^*(\mathcal{A})$ is the enveloping C^* -algebra of $\mathcal{C}_c(\mathcal{A}^0)/\mathcal{I}_{\mathcal{A}}^0$. Under some mild conditions, we shall prove in what follows that $C^*(\mathcal{A})$ is isomorphic to the full C^* -algebra $C^*(\mathcal{B})$ of the Fell bundle \mathcal{B} . First, let us recall the definition of $C^*(\mathcal{B})$.

DEFINITION 2.11. Given a Fell bundle $\mathcal{B} = \{\mathcal{B}_\gamma\}_{\gamma \in \mathcal{G}}$ over a locally compact étale groupoid \mathcal{G} , we write $\mathcal{C}_c(\mathcal{B})$ for the vector space of sections ξ of \mathcal{B} which can be written as a finite sum of the form $\xi = \sum_{i=1}^n \xi_i$, where each $\xi_i : U_i \rightarrow \mathcal{B}$ is a compactly supported, continuous local section of \mathcal{B} over some Hausdorff open subset $U_i \subseteq \mathcal{G}$, extended by zero outside U_i and viewed as a global section $\xi_i : \mathcal{G} \rightarrow \mathcal{B}$.

Alternatively, since the bisections in \mathcal{G} form a basis for its topology ([5], Proposition 3.5), we may restrict to local sections $\xi_i : U_i \rightarrow \mathcal{B}$ supported on bisections U_i in the above definition. Notice that bisections are open and Hausdorff by definition. Also note that any open Hausdorff subset $U \subseteq \mathcal{G}$ is locally compact with respect to the induced topology ([5], Proposition 3.7).

Let us warn the reader that, in general, if \mathcal{G} is not Hausdorff, sections in $\mathcal{C}_c(\mathcal{B})$ are not continuous with respect to the global topology. Of course, if \mathcal{G} is Hausdorff, $\mathcal{C}_c(\mathcal{B})$ coincides with the usual space of compactly supported, continuous (global) sections of \mathcal{B} . In any case, the vector space $\mathcal{C}_c(\mathcal{B})$ always has a canonical $*$ -algebra structure. The multiplication on $\mathcal{C}_c(\mathcal{B})$ is the convolution

product

$$(\xi * \eta)(\gamma) = \sum_{\gamma = \gamma_1 \gamma_2} \xi(\gamma_1) \eta(\gamma_2)$$

and the involution is defined by $\xi^*(\gamma) = \xi(\gamma^{-1})^*$ for all $\xi, \eta \in \mathcal{C}_c(\mathcal{B})$. As observed in Example 2.9, if ξ is supported in a bisection $U \subseteq \mathcal{G}$ and η is supported in a bisection $V \subseteq \mathcal{G}$, then $\xi * \eta$ is supported in the bisection UV and $(\xi * \eta)(\gamma) = \xi(\gamma_1) \eta(\gamma_2)$ whenever $\gamma \in UV$ is (uniquely) written in the form $\gamma = \gamma_1 \gamma_2$ with $\gamma_1 \in U$ and $\gamma_2 \in V$. In particular, this shows that the convolution product is well-defined on $\mathcal{C}_c(\mathcal{B})$.

By definition, $C^*(\mathcal{B})$ is the enveloping C^* -algebra of $\mathcal{C}_c(\mathcal{B})$. The same argument presented in Proposition 3.14 of [5] implies that $\|\pi(\xi)\| \leq \|\xi\|_\infty$ for any $*$ -representation π of $\mathcal{C}_c(\mathcal{B})$ whenever $\xi \in \mathcal{C}_c(\mathcal{B})$ is supported in some bisection of \mathcal{G} . Therefore the enveloping C^* -algebra of $\mathcal{C}_c(\mathcal{B})$ in fact exists.

DEFINITION 2.12 (Compare Proposition 5.4 of [5]). Let \mathcal{G} be an étale groupoid. We say that an inverse subsemigroup $S \subseteq S(\mathcal{G})$ is *wide* if the following properties hold:

- (i) S is a covering for \mathcal{G} , that is, $\mathcal{G} = \bigcup_{U \in S} U$, and
- (ii) given $U, V \in S$ and $\gamma \in \mathcal{G}$, there is $W \in S$ such that $\gamma \in W \subseteq U \cap V$.

THEOREM 2.13. Let $\mathcal{B} = \{\mathcal{B}_\gamma\}_{\gamma \in \mathcal{G}}$ be a Fell bundle over an étale groupoid, and let $\mathcal{A} = \{\mathcal{C}_0(\mathcal{B}_U)\}_{U \in S}$ be the associated Fell bundle over a wide inverse subsemigroup $S \subseteq S(\mathcal{G})$. Consider the sub-bundle $\mathcal{A}^0 = \{\mathcal{C}_c(\mathcal{B}_U)\}_{U \in S}$ of \mathcal{A} as above. If \mathcal{G} is either Hausdorff, or \mathcal{G} is second countable and the section algebras $\mathcal{C}_0(\mathcal{B}_U)$ are separable for all $U \in S$, then the canonical map $\Psi : \mathcal{C}_c(\mathcal{A}^0) \rightarrow \mathcal{C}_c(\mathcal{B})$ defined by

$$\Psi\left(\sum_{U \in F} \xi_U \delta_U\right) = \sum_{U \in F} \xi_U$$

whenever F is a finite subset of S and $\xi_U \in \mathcal{C}_c(\mathcal{B}_U)$ for all $U \in F$, induces an isomorphism of C^* -algebras $C^*(\mathcal{A}) \cong C^*(\mathcal{B})$.

Proof. By linearity, to show that Ψ is a $*$ -homomorphism, it suffices to check the equalities

$$\Psi((\xi \delta_U) \cdot (\eta \delta_V)) = \xi * \eta \quad \text{and} \quad \Psi((\xi \delta_U)^*) = \xi^*$$

for all $\xi \in \mathcal{C}_c(\mathcal{B}_U)$ and $\eta \in \mathcal{C}_c(\mathcal{B}_V)$, where $U, V \in S$. By definition of the $*$ -algebra structure of $\mathcal{C}_c(\mathcal{A}^0)$, we have $(\xi \delta_U) \cdot (\eta \delta_V) = (\xi \cdot \eta) \delta_{UV}$ and $(\xi \delta_U)^* = \xi^* \delta_{U^*}$. And we have already observed in Example 2.9 that $\xi \cdot \eta = \xi * \eta$. Moreover, $\mathcal{C}_c(\mathcal{B})$ is generated by sums of the form $\sum_{U \in F} \xi_U$ as above because S covers \mathcal{G} . Therefore,

Ψ is a surjective $*$ -homomorphism as stated. Note that Ψ vanishes on the ideal $\mathcal{I}_{\mathcal{A}}^0 \subseteq \mathcal{C}_c(\mathcal{A}^0)$ defined in Lemma 2.10 and hence induces a $*$ -homomorphism $\tilde{\Psi} : C^*(\mathcal{A}) \rightarrow C^*(\mathcal{B})$ by the same lemma. Since Ψ is surjective, so is $\tilde{\Psi}$. The most difficult part of the proof is to show that $\tilde{\Psi}$ is injective. Here we consider two cases:

Case 1. G is Hausdorff. In this case we are going to follow the same idea as in the proof of Theorem 7.1 in [20]. We first show that the kernel of Ψ is equal to

$$\mathcal{I} = \text{span}\{\zeta\delta_U - \zeta\delta_V : \zeta \in \mathcal{C}_c(\mathcal{B}) \text{ and } \text{supp}(\zeta) \subseteq U \cap V\}.$$

Of course, $\mathcal{I} \subseteq \ker(\Psi)$. For the opposite inclusion, take a finite sum of the form $\sum_{i=1}^n \zeta_i \delta_{U_i}$, where $U_i \in S$ and $\zeta_i \in \mathcal{C}_c(\mathcal{B}_{U_i})$, and suppose

$$\Psi\left(\sum_{i=1}^n \zeta_i \delta_{U_i}\right) = \sum_{i=1}^n \zeta_i = 0.$$

If $n = 1$, then $\zeta_1 = 0$ so that $\zeta_1 \delta_{U_1} = 0 \in \ker(\Psi)$. Thus we may assume $n > 1$. Consider the set \mathfrak{J} of all subsets of $\{1, \dots, n\}$ with at least two elements. Given $\mathfrak{s} \in \mathfrak{J}$, we define

$$V_{\mathfrak{s}} := \left(\bigcap_{i \in \mathfrak{s}} U_i\right) \setminus \left(\bigcup_{i \notin \mathfrak{s}} \text{supp}(\zeta_i)\right).$$

Then $\mathcal{V} = \{V_{\mathfrak{s}}\}_{\mathfrak{s} \in \mathfrak{J}}$ is an open cover of the compact subset $K = \bigcup_{i=1}^n \text{supp}(\zeta_i) \subseteq \mathcal{G}$.

Let $\{\psi_{\mathfrak{s}}\}_{\mathfrak{s} \in \mathfrak{J}}$ be a partition of unit subordinate to the cover \mathcal{V} . Note that $\psi_{\mathfrak{s}} \zeta_i = 0$ whenever $i \notin \mathfrak{s}$, so that

$$\sum_{i \in \mathfrak{s}} \psi_{\mathfrak{s}} \zeta_i = \sum_{i=1}^n \psi_{\mathfrak{s}} \zeta_i = \psi_{\mathfrak{s}} \cdot 0 = 0.$$

Moreover, we have

$$\sum_{i=1}^n \zeta_i \delta_{U_i} = \sum_{i=1}^n \left(\sum_{\mathfrak{s} \in \mathfrak{J}} \psi_{\mathfrak{s}} \zeta_i \delta_{U_i}\right) = \sum_{\mathfrak{s} \in \mathfrak{J}} \sum_{i=1}^n \psi_{\mathfrak{s}} \zeta_i \delta_{U_i} = \sum_{\mathfrak{s} \in \mathfrak{J}} \left(\sum_{i \in \mathfrak{s}} \psi_{\mathfrak{s}} \zeta_i \delta_{U_i}\right).$$

Thus, it suffices to show that $\sum_{i \in \mathfrak{s}} \psi_{\mathfrak{s}} \zeta_i \delta_{U_i} \in \mathcal{I}$ for all $\mathfrak{s} \in \mathfrak{J}$. Now, given distinct elements $j, k \in \mathfrak{s}$, we have

$$\begin{aligned} \sum_{i \in \mathfrak{s}} \psi_{\mathfrak{s}} \zeta_i \delta_{U_i} &= \psi_{\mathfrak{s}} \zeta_j \delta_{U_j} + \psi_{\mathfrak{s}} \zeta_k \delta_{U_k} + \sum_{i \in \mathfrak{s} \setminus \{j, k\}} \psi_{\mathfrak{s}} \zeta_i \delta_{U_i} \\ &= \psi_{\mathfrak{s}} \zeta_j \delta_{U_j} + \psi_{\mathfrak{s}} \zeta_k \delta_{U_k} + \sum_{i \in \mathfrak{s} \setminus \{j, k\}} \psi_{\mathfrak{s}} \zeta_i \delta_{U_i} + \sum_{i \in \mathfrak{s} \setminus \{j, k\}} \psi_{\mathfrak{s}} \zeta_i \delta_{U_k} - \sum_{i \in \mathfrak{s} \setminus \{j, k\}} \psi_{\mathfrak{s}} \zeta_i \delta_{U_k} \\ &= \psi_{\mathfrak{s}} \zeta_j \delta_{U_j} + \sum_{i \in \mathfrak{s} \setminus \{j\}} \psi_{\mathfrak{s}} \zeta_i \delta_{U_k} + \sum_{i \in \mathfrak{s} \setminus \{j, k\}} (\psi_{\mathfrak{s}} \zeta_i \delta_{U_i} - \psi_{\mathfrak{s}} \zeta_i \delta_{U_k}) \\ &= \psi_{\mathfrak{s}} \zeta_j \delta_{U_j} - \psi_{\mathfrak{s}} \zeta_j \delta_{U_k} + \sum_{i \in \mathfrak{s} \setminus \{j, k\}} (\psi_{\mathfrak{s}} \zeta_i \delta_{U_i} - \psi_{\mathfrak{s}} \zeta_i \delta_{U_k}), \end{aligned}$$

where the last equality follows from $\psi_{\mathfrak{s}} \zeta_j + \sum_{i \in \mathfrak{s} \setminus \{j\}} \psi_{\mathfrak{s}} \zeta_i = \sum_{i \in \mathfrak{s}} \psi_{\mathfrak{s}} \zeta_i = 0$. Since $\text{supp}(\psi_{\mathfrak{s}} \zeta_i) \subseteq U_i$ for all $i, l \in \mathfrak{s}$, we get $\sum_{i \in \mathfrak{s}} \psi_{\mathfrak{s}} \zeta_i \delta_{U_i} \in \mathcal{I}$ as desired. Next, we show that $\mathcal{I} = \mathcal{I}_{\mathcal{A}}^0$. Of course, $\mathcal{I}_{\mathcal{A}}^0 \subseteq \ker(\Psi) = \mathcal{I}$. On the other hand, suppose $\zeta \in \mathcal{C}_c(\mathcal{B})$

and $\text{supp}(\xi) \subseteq U \cap V$ with $U, V \in S$. Since S is wide and $\text{supp}(\xi)$ is compact, there are $W_1, \dots, W_m \in S$ such that

$$\text{supp}(\xi) \subseteq \bigcup_{i=1}^m W_i \subseteq U \cap V.$$

Let $\{\phi_i\}_{i=1}^m$ be a partition of unit subordinate to the cover $\{W_1, \dots, W_m\}$ of $\text{supp}(\xi)$. Then

$$\begin{aligned} \xi\delta_U - \xi\delta_V &= \sum_{i=1}^m \phi_i \xi \delta_U - \sum_{i=1}^m \phi_i \xi \delta_V = \sum_{i=1}^m \phi_i \xi \delta_U - \sum_{i=1}^m \phi_i \xi \delta_{W_i} + \sum_{i=1}^m \phi_i \xi \delta_{W_i} - \sum_{i=1}^m \phi_i \xi \delta_V \\ &= \sum_{i=1}^m (\phi_i \xi \delta_U - \phi_i \xi \delta_{W_i}) + \sum_{i=1}^m (\phi_i \xi \delta_{W_i} - \phi_i \xi \delta_V) \end{aligned}$$

is an element of \mathcal{I}_A^0 because $\text{supp}(\phi_i \xi) \subseteq W_i \cap U \cap V$ for all $i = 1, \dots, m$. We conclude that $\ker(\Psi) = \mathcal{I}_A^0$. Hence Ψ induces an isomorphism $\mathcal{C}_c(\mathcal{A}^0)/\mathcal{I}_A^0 \cong \mathcal{C}_c(\mathcal{B})$ of $*$ -algebras and therefore also between their corresponding enveloping C^* -algebras $C^*(\mathcal{A}) \cong C^*(\mathcal{B})$.

Case 2. \mathcal{G} is second countable and $\mathcal{C}_0(\mathcal{B}_U)$ is separable for all $U \in S$. In this case, we shall adapt an idea appearing in Lemma 8.4 of [5] to our situation, supported by Proposition 3.3 of [17], which is a kind of disintegration result for linear functionals on Banach bundles. To show that $\tilde{\Psi} : C^*(\mathcal{A}) \rightarrow C^*(\mathcal{B})$ is injective, it is enough to prove that any representation π of $C^*(\mathcal{A})$ factors through a representation $\tilde{\pi}$ of $C^*(\mathcal{B})$ in the sense that $\tilde{\pi} \circ \tilde{\Psi} = \pi$. By Lemma 2.10, $C^*(\mathcal{A})$ is the enveloping C^* -algebra of $\mathcal{C}_c(\mathcal{A}^0)/\mathcal{I}_A^0$, so it is enough to work with representations of the latter. Now, every representation π of $\mathcal{C}_c(\mathcal{A}^0)/\mathcal{I}_A^0$ on a Hilbert space \mathcal{H} , once composed with the quotient homomorphism $\mathcal{C}_c(\mathcal{A}^0) \rightarrow \mathcal{C}_c(\mathcal{A}^0)/\mathcal{I}_A^0$ and hence viewed as a representation of $\mathcal{C}_c(\mathcal{A}^0)$, has the form

$$\pi\left(\sum_{U \in F} \xi_U \delta_U\right) = \sum_{U \in F} \pi_U(\xi_U)$$

where $\{\pi_U\}_{U \in S}$ is a representation of $\mathcal{A}^0 = \{\mathcal{C}_c(\mathcal{B}_U)\}_{U \in S}$ on \mathcal{H} . In fact, π_U is just the restriction of π to the copy of $\mathcal{A}_U^0 = \mathcal{C}_c(\mathcal{B}_U)$ inside $\mathcal{C}_c(\mathcal{A}^0)$. Given $\eta, \zeta \in \mathcal{H}$, we define the linear functional

$$\omega_U : \mathcal{C}_c(\mathcal{B}_U) \rightarrow \mathbb{C}, \quad \omega_U(\xi) := \langle \pi_U(\xi)\eta \mid \zeta \rangle \quad \text{for all } \xi \in \mathcal{C}_c(\mathcal{B}_U).$$

Since π_U is norm contractive, that is, $\|\pi_U(\xi)\| \leq \|\xi\|_\infty$ (see proof of Lemma 2.10), it follows that ω_U is continuous with respect to the inductive limit topology. In other words, ω_U is a *generalized Radon measure* in the sense of [17]. By Proposition 3.3 in [17], there are bounded linear functionals $\varepsilon_{U,\gamma} \in \mathcal{B}_\gamma^*$ with norm at most one for each $\gamma \in U$, such that $\gamma \mapsto \varepsilon_{U,\gamma}(\xi(\gamma))$ is a bounded measurable function on U for all $\xi \in \mathcal{C}_c(\mathcal{B}_U)$, and

$$\omega_U(\xi) = \int_U \varepsilon_{U,\gamma}(\xi(\gamma)) d|\omega_U|(\gamma),$$

where $|\omega_U|$ is the total variation measure associated to ω_U as in Lemma 3.1 of [17]:

$$(2.3) \quad |\omega_U|(\varphi) := \sup\{|\omega_U(\xi) : \|\xi\|_\infty \leq \varphi\} \quad \text{for all } \varphi \in \mathcal{C}_c^+(U).$$

As we have emphasize in the notation, all these objects depend *a priori* on the bisection $U \in S$. However, we are now going to show that they are compatible on intersections of bisections. First, given $U, V \in S$, we claim that

$$(2.4) \quad \omega_U(\xi) = \omega_V(\xi) \quad \text{whenever } \xi \in \mathcal{C}_c(\mathcal{B}) \text{ and } \text{supp}(\xi) \subseteq U \cap V.$$

In fact, since S is wide and $\text{supp}(\xi)$ is compact, we may find a cover $\{W_i\}_{i=1}^m$ of $\text{supp}(\xi)$ consisting of bisections $W_i \in S$ contained in $U \cap V$. Since $U \cap V$ is Hausdorff, there is a partition of unit $\{\psi_i\}_{i=1}^m$ subordinate to the cover $\{W_i\}_{i=1}^m$ of $\text{supp}(\xi)$. Since $\text{supp}(\psi_i \cdot \xi) \subseteq W_i \subseteq U \cap V$ and π is a representation, we have

$$\pi_U(\psi_i \cdot \xi) = \pi_{W_i}(\psi_i \cdot \xi) = \pi_V(\psi_i \cdot \xi) \quad \text{for all } i = 1, \dots, m.$$

Thus,

$$\pi_U(\xi) = \pi_U\left(\sum_{i=1}^m \psi_i \cdot \xi\right) = \sum_{i=1}^m \pi_U(\psi_i \cdot \xi) = \sum_{i=1}^m \pi_V(\psi_i \cdot \xi) = \pi_V\left(\sum_{i=1}^m \psi_i \cdot \xi\right) = \pi_V(\xi).$$

Therefore, $\omega_U(\xi) = \omega_V(\xi)$ as claimed. It follows directly from (2.3) that $|\omega_U|(\varphi) = |\omega_V|(\varphi)$ whenever $\text{supp}(\varphi) \subseteq U \cap V$ or, equivalently, $|\omega_U|(A) = |\omega_V|(A)$ whenever A is a Borel measurable subset of $U \cap V$. Since S countably covers \mathcal{G} , there is a positive Radon measure μ on \mathcal{G} whose restriction to U equals $|\omega_U|$ for all $U \in S$ (see Lemma A.1 of [16]). Moreover, following the construction of $\varepsilon_{U,\gamma}$ in Proposition 3.3 of [17] and using equation (2.4), it follows that $\varepsilon_{U,\gamma} = \varepsilon_{V,\gamma}$ whenever $\gamma \in U \cap V$. Therefore, we may define a linear functional

$$\omega : \mathcal{C}_c(\mathcal{B}) \rightarrow \mathbb{C}, \quad \omega(\xi) := \int_{\mathcal{G}} \varepsilon_\gamma(\xi(\gamma)) d\mu(\gamma),$$

where $\varepsilon_\gamma := \varepsilon_{U,\gamma}$ whenever $\gamma \in U$. By definition, the restriction of ω to $\mathcal{C}_c(\mathcal{B}_U)$ coincides with ω_U for all $U \in S$.

All this implies that the map

$$\tilde{\pi} : \mathcal{C}_c(\mathcal{B}) \rightarrow \mathbb{B}(\mathcal{H}), \quad \tilde{\pi}\left(\sum_{U \in \mathcal{F}} \xi_U\right) := \sum_{U \in \mathcal{F}} \pi_U(\xi_U) = \pi\left(\sum_{U \in \mathcal{F}} \xi_U \delta_U\right)$$

is well-defined. In fact, if $\sum_{U \in \mathcal{F}} \xi_U = 0$, then

$$\left\langle \sum_{U \in \mathcal{F}} \pi_U(\xi_U) \eta \mid \zeta \right\rangle = \sum_{U \in \mathcal{F}} \omega_U(\xi_U) \langle \eta \mid \zeta \rangle = \sum_{U \in \mathcal{F}} \omega(\xi_U) \langle \eta \mid \zeta \rangle = \omega\left(\sum_{U \in \mathcal{F}} \xi_U\right) \langle \eta \mid \zeta \rangle = 0.$$

Since $\eta, \zeta \in \mathcal{H}$ are arbitrary, we conclude that $\sum_{U \in \mathcal{F}} \pi_U(\xi_U) = 0$. Therefore $\tilde{\pi}$ is a well-defined map. Obviously it is linear, and since π is a representation of $\mathcal{C}_c(\mathcal{A}^0)$, it is easy to see that $\tilde{\pi}$ is a representation of $\mathcal{C}_c(\mathcal{B})$. Moreover, by construction we have $\tilde{\pi}(\Psi(x)) = \pi(x)$ for all $x \in \mathcal{C}_c(\mathcal{A}^0)$, and this concludes the proof. ■

2.3. TWISTED ÉTALE GROUPOIDS AND FELL LINE BUNDLES. If \mathcal{G} is a locally compact groupoid it is well known [2] that, at least in the Hausdorff case, there is a one-to-one correspondence between *twists* over \mathcal{G} (namely exact sequences

$$\mathbb{T} \times \mathcal{G}^{(0)} \xrightarrow{\iota} \Sigma \xrightarrow{\pi} \mathcal{G} ,$$

where Σ is a locally compact groupoid, ι is a homeomorphism onto its image, and π is a continuous open surjection) and Fell line bundles (namely Fell bundles with one-dimensional fibers) over \mathcal{G} . In particular, the so called full (respectively reduced) twisted groupoid C^* -algebra of (\mathcal{G}, Σ) turns out to be precisely the full (respectively reduced) cross-sectional C^* -algebra of the corresponding Fell line bundle.

Our techniques are specially well adapted to deal with Fell bundles and hence we have decided to emphasize these as opposed to twists. Of course, should one be interested in the underlying twist, it is readily available by considering unitary elements as explained in [2].

Now consider a Fell line bundle L over an étale locally compact groupoid \mathcal{G} . As a special case of Example 2.9, we may consider the Fell bundle $\mathcal{A} = \{\mathcal{C}_0(L_U)\}_{U \in S}$ associated to L , where S is an inverse subsemigroup of $S(\mathcal{G})$.

PROPOSITION 2.14. *Let notation be as above. If \mathcal{G} is Hausdorff or second countable, and if S is a wide inverse subsemigroup in $S(\mathcal{G})$, then there is a canonical isomorphism $C^*(L) \cong C^*(\mathcal{A})$.*

The proof is a direct consequence of Theorem 2.13.

3. SEMI-ABELIAN FELL BUNDLES AND TWISTED ÉTALE GROUPOIDS

This section contains the main result of this work. Given a semi-abelian Fell bundle over an inverse semigroup, we are going to construct a twisted étale groupoid in a canonical way. Later we are going to show that our construction preserves the associated $*$ -algebras. Our techniques are inspired by those of Kumjian and Renault in [13], [22].

3.1. THE CANONICAL ACTION AND THE GROUPOID OF GERMS.

DEFINITION 3.1. Let $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$ be a Fell bundle over an inverse semigroup S . We say that \mathcal{A} is *semi-abelian* if for each idempotent $e \in E = E(S)$, the fiber \mathcal{A}_e is a commutative C^* -algebra.

Let \mathcal{E} be the restriction of \mathcal{A} to the idempotent semilattice E of S . Note that \mathcal{E} is a Fell bundle over E . Moreover, \mathcal{A} is semi-abelian if and only if \mathcal{E} is an *abelian* Fell bundle in the sense that $ab = ba$ for all $a, b \in \mathcal{E}$ (see proof of Lemma 3.2 below). Of course, we could also say that an arbitrary Fell bundle $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$ is *abelian* if $ab = ba$ for all $a, b \in \mathcal{A}$, but we actually are not going to consider this more restrictive notion in this work. (Note that in this case the underlying

inverse semigroup S has to be commutative. Moreover, it is easy to see that \mathcal{A} is an abelian Fell bundle if and only if $C^*(\mathcal{A})$ is a commutative $*$ -algebra.) This terminology is compatible with the one appearing in Chapter X of [10], where the authors use the synonymous word "commutative" instead. However, we should mention that our terminology conflicts with the one introduced in [2], where they call a Fell bundle $\mathcal{B} = \{\mathcal{B}_\gamma\}_{\gamma \in \mathcal{G}}$ over a groupoid \mathcal{G} *abelian* if the fibers \mathcal{B}_x over the units $x \in \mathcal{G}^{(0)}$ are commutative $*$ -algebras.

LEMMA 3.2. *A Fell bundle \mathcal{A} is semi-abelian if and only if $C^*(\mathcal{E})$ is a commutative C^* -algebra.*

Proof. If $C^*(\mathcal{E})$ is commutative, then so is each \mathcal{A}_e since \mathcal{A}_e is an ideal of $C^*(\mathcal{E})$ by Corollary 4.6 of [6]. Conversely, assume that \mathcal{A}_e is commutative for all $e \in E$. To prove that $C^*(\mathcal{E})$ is commutative it suffices to show that $ab = ba$ for every $a \in \mathcal{A}_e$ and $b \in \mathcal{A}_f$, where $e, f \in E$. Let (e_i) be an approximate unit of \mathcal{A}_e . Since \mathcal{A}_e is an ideal of $C^*(\mathcal{E})$ we have $ab, e_i b, ba \in \mathcal{A}_e$. Using that \mathcal{A}_e is commutative, we get the result

$$ab = \lim_i e_i ab = \lim_i a e_i b = \lim_i e_i ba = ba. \quad \blacksquare$$

Let us assume from now on that \mathcal{A} is a semi-abelian saturated Fell bundle. Suppose that X is the spectrum of $C^*(\mathcal{E})$ so that (we may identify) $C^*(\mathcal{E}) = \mathcal{C}_0(X)$.

LEMMA 3.3. *Given $a \in \mathcal{A}$, we write $\text{dom}(a) := \{x \in X : (a^*a)(x) > 0\}$ and $\text{ran}(a) := \{x \in X : (aa^*)(x) > 0\} = \text{dom}(a^*)$. Then there is a unique homeomorphism $\theta_a : \text{dom}(a) \rightarrow \text{ran}(a)$ satisfying*

$$(a^*ba)(x) = (a^*a)(x)b(\theta_a(x)) \quad \text{for all } x \in \text{dom}(a) \text{ and } b \in \mathcal{C}_0(X).$$

If $a = u|a|$ is the polar decomposition of a in A'' , the enveloping von Neumann algebra of $A = C^(\mathcal{A})$, where we view each fiber of \mathcal{A} as a subspace of A , then*

$$(u^*bu)(x) = b(\theta_a(x)) \quad \text{for all } x \in \text{dom}(a) \text{ and } b \in \mathcal{C}_0(X).$$

Moreover, the following properties hold:

- (i) *If $a \in \mathcal{A}_e$, where $e \in E$, then $\theta_a = \text{id}_{\text{dom}(a)}$.*
- (ii) *If $a, b \in \mathcal{A}$, then $\theta_{ab} = \theta_a \circ \theta_b$ (composition of partial homeomorphisms).*
- (iii) *If $a \in \mathcal{A}$, then $\theta_{a^*} = \theta_a^{-1}$.*

Proof. Essentially this follows from Proposition 6 and Corollary 7 in [13]. In fact, it is enough to observe that each fiber $\mathcal{A}_s \subseteq A$ is contained in the normalizer of $B = C^*(\mathcal{E})$ in A . Recall that $a \in A$ normalizes B if $a^*Ba \subseteq B$ and $aBa^* \subseteq B$. \blacksquare

LEMMA 3.4. *Suppose that $s \in S$, $a_1, a_2 \in \mathcal{A}_s$ and $x \in \text{dom}(a_1) \cap \text{dom}(a_2)$. Then we have $\theta_{a_1}(x) = \theta_{a_2}(x)$.*

Proof. Since $a_1^*a_2 \in \mathcal{A}_{s^*s}$, Lemma 3.3 yields

$$\theta_{a_1}^{-1} \circ \theta_{a_2} = \theta_{a_1^*} \circ \theta_{a_2} = \theta_{a_1^*a_2} = \text{id}_{\text{dom}(a_1^*a_2)}.$$

So, it is enough to show that $x \in \text{dom}(a_1^*a_2)$, that is,

$$((a_1^*a_2)^*(a_1^*a_2))(x) = (a_2^*a_1a_1^*a_2)(x) > 0.$$

It is equivalent to show that $(a_2^*a_2)(x)(a_2^*a_1a_1^*a_2)(x) > 0$ because $x \in \text{dom}(a_2)$. Since the fibers over the idempotents commute with each other, we have the following and the result now follows:

$$(a_2^*a_2)(a_2^*a_1a_1^*a_2) = a_2^*a_1a_2^*a_2a_1^*a_2 = (a_2^*a_2)(a_1^*a_1)(a_2^*a_2). \quad \blacksquare$$

Let $e \in E$. Since \mathcal{A}_e is an ideal in $C^*(\mathcal{E})$, there is an open subset $\mathcal{U}_e \subseteq X$ such that $\mathcal{A}_e = \mathcal{C}_0(\mathcal{U}_e)$.

PROPOSITION 3.5. *Given $s \in S$, there is a homeomorphism $\theta_s : \mathcal{U}_{s^*s} \rightarrow \mathcal{U}_{ss^*}$ such that $\theta_s|_{\text{dom}(a)} = \theta_a$ for all $a \in \mathcal{A}_s$. Moreover, we have $\theta_s \circ \theta_t = \theta_{st}$ for all $s, t \in S$. In other words, θ is an action of S on X .*

Proof. Given $x \in \mathcal{U}_{s^*s}$, we define $\theta_s(x) := \theta_a(x)$, where a is any element in \mathcal{A}_s with $(a^*a)(x) > 0$. Note that such an element exists because \mathcal{A} is saturated. By Lemma 3.4, θ_s is a well-defined map $\mathcal{U}_{s^*s} \rightarrow \mathcal{U}_{ss^*}$ and, by definition, the restriction of θ_s to $\text{dom}(a)$ is equal to θ_a . Since each θ_a is a homeomorphism, we deduce that θ_s is continuous.

It remains to prove that $\theta_s \circ \theta_t = \theta_{st}$. Now, since $\theta_a \circ \theta_b = \theta_{ab}$ for all $a \in \mathcal{A}_s$ and $b \in \mathcal{A}_t$, it is enough to show that the domains of $\theta_s \circ \theta_t$ and θ_{st} coincide. If $x \in \text{dom}(\theta_s \circ \theta_t)$, that is, if $x \in \mathcal{U}_{t^*t} = \text{dom}(\theta_t)$ and $\theta_t(x) \in \mathcal{U}_{s^*s} = \text{dom}(\theta_s)$, then there is $a \in \mathcal{A}_s$ and $b \in \mathcal{A}_t$ with $x \in \text{dom}(b)$ and $\theta_b(x) \in \text{dom}(a)$, that is,

$$x \in \text{dom}(\theta_a \circ \theta_b) = \text{dom}(\theta_{ab}) = \text{dom}(ab) \subseteq \mathcal{U}_{(st)^*(st)} = \text{dom}(\theta_{st}).$$

Conversely, if $x \in \text{dom}(\theta_{st})$, there is $c \in \mathcal{A}_{st}$ such that $x \in \text{dom}(c)$, that is, $(c^*c)(x) > 0$. We claim that there is $a \in \mathcal{A}_s$ and $b \in \mathcal{A}_t$ with $x \in \text{dom}(ab)$. In fact, suppose this is not the case, so that $(ab)^*(ab)(x) = 0$ for all $a \in \mathcal{A}_s$ and $b \in \mathcal{A}_t$. Polarization in a and b yields

$$(a_1b_1)^*(a_2b_2)(x) = 0$$

for all $a_1, a_2 \in \mathcal{A}_s$ and $b_1, b_2 \in \mathcal{A}_t$. This implies that $(c_1^*c_2)(x) = 0$ for all $c_1, c_2 \in \overline{\text{span}} \mathcal{A}_s \mathcal{A}_t = \mathcal{A}_{st}$, which is a contradiction. This proves our claim. Therefore, there is $a \in \mathcal{A}_s$ and $b \in \mathcal{A}_t$ with

$$x \in \text{dom}(ab) = \text{dom}(\theta_{ab}) = \text{dom}(\theta_a \circ \theta_b) \subseteq \text{dom}(\theta_s \circ \theta_t). \quad \blacksquare$$

We are ready to define the groupoid $\mathcal{G} = \mathcal{G}(\mathcal{A})$ associated to the semi-abelian Fell bundle \mathcal{A} . Indeed, we define the groupoid \mathcal{G} to be the groupoid of germs of the action θ of S on X constructed above:

$$\mathcal{G} := \{[s, x] : s \in S, x \in \text{dom}(\theta_s) = \mathcal{U}_{s^*s}\},$$

where, by definition, $[s, x] = [t, y]$ if and only if $x = y$ and there is $e \in E$ with $x \in \mathcal{U}_e$ and $se = te$. The source and range maps are defined by $d([s, x]) = x$ and

$r([s, x]) = \theta_s(x)$, respectively. Multiplication and inversion are given by

$$[s, x] \cdot [t, y] := [st, y] \quad \text{whenever } \theta_t(y) = x, \quad \text{and} \quad [s, x]^{-1} := [s^*, \theta_s(x)].$$

The topology on \mathcal{G} is, by definition, generated by the basic open sets

$$(3.1) \quad \mathcal{O}(s, U) := \{[s, x] : x \in U\}$$

where $s \in S$ and $U \subseteq \mathcal{U}_{s^*s}$ is an open subset. In particular, $\mathcal{O}_s := \mathcal{O}(s, \mathcal{U}_{s^*s})$ is an open subset of \mathcal{G} . Moreover, the restriction of d defines a homeomorphism $d_s : \mathcal{O}_s \rightarrow \mathcal{U}_{s^*s}$.

See Section 4 of [5] for more details on the construction of groupoids of germs, but please notice that our notion of germs differs from the one described in Section 3 of [22].

With this structure, \mathcal{G} is an étale groupoid and the unit space of \mathcal{G} may be identified with X through the following map, where $e_x \in E(S)$ is any idempotent with $x \in \mathcal{U}_{e_x}$:

$$X \ni x \mapsto [e_x, x] \in \mathcal{G}^{(0)}.$$

3.2. THE CONSTRUCTION OF THE FELL LINE BUNDLE. Let A and B be C^* -algebras, and let ${}_A\mathcal{X}_B$ be an imprimitivity Hilbert A - B -bimodule. If I is a closed ideal of A , then $\mathcal{F}(I) = I \cdot \mathcal{X}$ is a closed submodule of \mathcal{X} , and $\text{Ind}_{\mathcal{X}}(I) = \overline{\text{span}}\langle \mathcal{F}(I) | \mathcal{F}(I) \rangle_B$ is a closed ideal of B . Moreover, the maps $I \mapsto \mathcal{F}(I)$ and $I \mapsto \text{Ind}_{\mathcal{X}}(I)$ define bijective correspondences between closed ideals of A , closed submodules of \mathcal{X} and closed ideals of B . Given a closed ideal J in B , the corresponding submodule is $\mathcal{X} \cdot J$, and the corresponding ideal in A is $\overline{\text{span}}_A\langle \mathcal{X} \cdot J | \mathcal{X} \cdot J \rangle$. This fact is known as the Rieffel correspondence.

If I is a closed ideal of A , then using an approximate unit for I , it is easy to see that

$$(3.2) \quad \mathcal{F}(I) = I \cdot \mathcal{X} = \{\xi \in \mathcal{X} : {}_A\langle \xi | \xi \rangle \in I\}.$$

And similarly if J is a closed ideal of B , then

$$(3.3) \quad \mathcal{X} \cdot J = \{\xi \in \mathcal{X} : \langle \xi | \xi \rangle_B \in J\}.$$

Given a saturated Fell bundle $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$ over an inverse semigroup S , note that each fiber \mathcal{A}_s is an imprimitivity Hilbert \mathcal{A}_{ss^*} - \mathcal{A}_{s^*s} -bimodule in the canonical way. For instance, the inner products are defined by

$$\langle a | b \rangle_{\mathcal{A}_{s^*s}} := a^*b, \quad \mathcal{A}_{ss^*}\langle a | b \rangle := ab^* \quad \text{for all } a, b \in \mathcal{A}_s.$$

Now assume that $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$ is a fixed saturated, semi-abelian Fell bundle. Let X be the spectrum of the commutative C^* -algebra $C^*(\mathcal{E}_{\mathcal{A}})$. Recall that \mathcal{U}_e denotes the open subset of X corresponding to the spectrum of the ideal \mathcal{A}_e in $C^*(\mathcal{E}_{\mathcal{A}}) \cong \mathcal{C}_0(X)$.

DEFINITION 3.6. Given $s \in S$ and $x \in X$, we define $\mathcal{A}_{(s,x)}$ to be the closed submodule of \mathcal{A}_s corresponding to the ideal $\{b \in \mathcal{A}_{s^*s} : b(x) = 0\}$ in \mathcal{A}_{s^*s} under the Rieffel correspondence.

LEMMA 3.7. *With the notations above, we have the following properties for all $s, t \in S$, $a \in \mathcal{A}_s$ and $x, y \in X$:*

- (i) $\mathcal{A}_{(s,x)} = \mathcal{A}_{(ss^*,\theta_s(x))} \cdot \mathcal{A}_s = \mathcal{A}_s \cdot \mathcal{A}_{(s^*s,x)}$ whenever $x \in \mathcal{U}_{s^*s}$.
- Moreover, if $x \notin \mathcal{U}_{s^*s}$, then $\mathcal{A}_{(s,x)} = \mathcal{A}_s$.
- (ii) $a \in \mathcal{A}_{(s,x)} \Leftrightarrow (a^*a)(x) = 0 \Leftrightarrow (aa^*)(\theta_s(x)) = 0$.
- (iii) $\mathcal{A}_s \cdot \mathcal{A}_{(t,y)} = \mathcal{A}_{(st,y)}$ and, if $x \in \mathcal{U}_{tt^*}$, then $\mathcal{A}_{(s,x)} \mathcal{A}_t = \mathcal{A}_{(st,\theta_{t^*}(x))}$.
- (iv) $\mathcal{A}_{(s,x)} \cdot \mathcal{A}_{(t,y)} = \mathcal{A}_{(st,y)}$ whenever $x \in \mathcal{U}_{s^*s}$, $y \in \mathcal{U}_{t^*t}$ and $\theta_t(y) = x$.
- (v) $\mathcal{A}_{(s,x)}^* = \mathcal{A}_{(s^*,\theta_s(x))}$ whenever $x \in \mathcal{U}_{s^*s}$.

Proof. Note that $\mathcal{A}_{(s^*s,x)} = \{b \in \mathcal{A}_{s^*s} : b(x) = 0\}$. Thus, by definition, $\mathcal{A}_{(s,x)} = \mathcal{A}_s \cdot \mathcal{A}_{(s^*s,x)}$. Now, by equation (3.3), we have

$$\mathcal{A}_{(s,x)} = \{a \in \mathcal{A}_s : (a^*a)(x) = 0\}.$$

We have $(aa^*)(\theta_s(x)) = (a^*a)(x)$ for all $a \in \mathcal{A}_s$ and $x \in \mathcal{U}_{s^*s}$. Thus, if $x \in \mathcal{U}_{s^*s}$, then

$$\mathcal{A}_{(s,x)} = \{a \in \mathcal{A}_s : (aa^*)(\theta_s(x)) = 0\}.$$

Again, by equation (3.2) this is equal to $\mathcal{A}_{(ss^*,\theta_s(x))} \cdot \mathcal{A}_s$ because

$$\mathcal{A}_{(ss^*,\theta_s(x))} = \{b \in \mathcal{A}_{ss^*} : b(\theta_s(x)) = 0\}.$$

If $x \notin \mathcal{U}_{s^*s}$, then $\mathcal{A}_{(s^*s,x)} = \mathcal{A}_{s^*s}$ because, by definition, \mathcal{U}_{s^*s} is the spectrum of \mathcal{A}_{s^*s} . This proves (i) and (ii). To prove (iii), we use (i) to conclude that

$$\mathcal{A}_s \cdot \mathcal{A}_{(t,y)} = \mathcal{A}_s \cdot \mathcal{A}_t \cdot \mathcal{A}_{(t^*t,y)} = \mathcal{A}_{st} \cdot \mathcal{A}_{(t^*t,y)} = \mathcal{A}_{st} \cdot \mathcal{A}_{t^*s^*st} \cdot \mathcal{A}_{(t^*t,y)}$$

and

$$\mathcal{A}_{(st,y)} = \mathcal{A}_{st} \cdot \mathcal{A}_{(t^*s^*st,y)}.$$

Since $t^*s^*st \leq t^*t$, we have $\mathcal{A}_{t^*s^*st} \subseteq \mathcal{A}_{t^*t}$ and hence also $\mathcal{A}_{(t^*s^*st,y)} \subseteq \mathcal{A}_{(t^*t,y)}$. Thus, $\mathcal{A}_{(st,y)} \subseteq \mathcal{A}_s \cdot \mathcal{A}_{(t,y)}$. On the other hand, both $\mathcal{A}_{t^*s^*st}$ and $\mathcal{A}_{(t^*t,y)}$ are ideals in $\mathcal{C}_0(X)$, so that

$$\mathcal{A}_{t^*s^*st} \cdot \mathcal{A}_{(t^*t,y)} = \mathcal{A}_{t^*s^*st} \cap \mathcal{A}_{(t^*t,y)} = \{b \in \mathcal{A}_{t^*s^*st} : b(y) = 0\} = \mathcal{A}_{(t^*s^*st,y)}.$$

This concludes the proof of the first assertion in (iii). To prove the second assertion in (iii) we use a similar argument. First, if $x \notin \mathcal{U}_{s^*s}$, then $\theta_{t^*}(x) \notin \theta_{t^*}(\mathcal{U}_{s^*s} \cap \mathcal{U}_{tt^*}) = \mathcal{U}_{t^*s^*st}$ (see Proposition 4.5 of [5]), so that $\mathcal{A}_{(s,x)} \mathcal{A}_t = \mathcal{A}_s \mathcal{A}_t = \mathcal{A}_{st} = \mathcal{A}_{(st,\theta_{t^*}(x))}$ by (i). Now assume that $x \in \mathcal{U}_{s^*s}$. Using (i) again, we get

$$\mathcal{A}_{(s,x)} \cdot \mathcal{A}_t = \mathcal{A}_{(ss^*,\theta_s(x))} \cdot \mathcal{A}_{st} = \mathcal{A}_{(ss^*,\theta_s(x))} \cdot \mathcal{A}_{stt^*s^*} \cdot \mathcal{A}_{st}$$

and, on the other hand,

$$\mathcal{A}_{(st,\theta_{t^*}(x))} = \mathcal{A}_{(stt^*s^*,\theta_{st}(\theta_{t^*}(x)))} \cdot \mathcal{A}_{st} = \mathcal{A}_{(stt^*s^*,\theta_s(x))} \cdot \mathcal{A}_{st}.$$

As above, it is easy to see that $\mathcal{A}_{(stt^*s^*, \theta_s(x))} = \mathcal{A}_{(ss^*, \theta_s(x))} \cdot \mathcal{A}_{stt^*s^*}$. To prove (iv), we use (i), (iii) and the hypothesis $\theta_t(y) = x$, to get

$$\begin{aligned} \mathcal{A}_{(s,x)} \cdot \mathcal{A}_{(t,y)} &\stackrel{(i)}{=} \mathcal{A}_{(ss^*, \theta_s(x))} \cdot \mathcal{A}_{st} \cdot \mathcal{A}_{(t^*t,y)} \\ &\stackrel{\theta_t(y)=x}{=} \mathcal{A}_{(ss^*, \theta_{st}(y))} \cdot \mathcal{A}_{stt^*s^*} \cdot \mathcal{A}_{st} \cdot \mathcal{A}_{t^*s^*st} \cdot \mathcal{A}_{(t^*t,y)} \\ &\stackrel{(iii)}{=} \mathcal{A}_{(stt^*s^*, \theta_{st}(y))} \cdot \mathcal{A}_{st} \cdot \mathcal{A}_{(t^*s^*st,y)} \stackrel{(i)}{=} \mathcal{A}_{(st,y)}. \end{aligned}$$

Finally, to prove (v), we use (i) and conclude that

$$\mathcal{A}_{(s^*, \theta_s(x))} = \mathcal{A}_{(s^*s, \theta_{s^*}(\theta_s(x)))} \cdot \mathcal{A}_{s^*} = \mathcal{A}_{(s^*s,x)} \cdot \mathcal{A}_{s^*} = (\mathcal{A}_s \cdot \mathcal{A}_{(s^*s,x)})^* = \mathcal{A}_{(s,x)}^*. \quad \blacksquare$$

Having fixed our semi-abelian Fell bundle $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$ above, we henceforth let \mathcal{G} be the associated groupoid of germs constructed in Section 3.1.

DEFINITION 3.8. Let $s \in S$, and $a, b \in \mathcal{A}_s$. Given $x \in X$, we shall say that

$$a \stackrel{x}{=} b,$$

if $((a - b)^*(a - b))(x) = 0$. Hence $a \stackrel{x}{=} b$ if and only if $a - b \in \mathcal{A}_{(s,x)}$.

In the following we present some elementary properties of the relation defined above.

LEMMA 3.9. Let $s \in S$, let $a, b, c \in \mathcal{A}_s$, and let $x \in X$.

(i) If $(a^*a)(x) = 0$ (in particular if $x \notin \mathcal{U}_{s^*s}$), then $a \stackrel{x}{=} 0_s$ (where “ 0_s ” stands for the zero element of \mathcal{A}_s).

(ii) If $a \stackrel{x}{=} b$, then for every $t \in S$, and $c \in \mathcal{A}_t$, one has $ca \stackrel{x}{=} cb$.

(iii) If $a \stackrel{x}{=} b$, then for every $e \in E(S)$, and every $c \in \mathcal{A}_e$, one has $ac \stackrel{x}{=} bc$.

(iv) If $a \stackrel{x}{=} b$, then for every $t \in S$ such that $x \in \mathcal{U}_{tt^*}$, and every $c \in \mathcal{A}_t$, one has $ac \stackrel{\theta_t^{-1}(x)}{=} bc$.

(v) If $a \stackrel{x}{=} b$ and $b \stackrel{x}{=} c$, then $a \stackrel{x}{=} c$.

(vi) If $a \stackrel{x}{=} b$, then $a^* \stackrel{\theta_s(x)}{=} b^*$.

Proof. Item (i) is obvious. To prove (ii), suppose $a \stackrel{x}{=} b$. Then $a - b \in \mathcal{A}_{(s,x)}$, so that $ca - cb = c(a - b) \in \mathcal{A}_t \mathcal{A}_{(s,x)} = \mathcal{A}_{(ts,x)}$ by Lemma 3.7(iii). Thus $ca \stackrel{x}{=} cb$.

To prove (iii), we compute

$$((ac - bc)^*(ac - bc))(x) = \overline{c(x)} ((a - b)^*(a - b))(x) c(x) = 0.$$

It is also possible to prove (iii) by showing that $\mathcal{A}_{(s,x)} \cdot \mathcal{A}_e = \mathcal{A}_{(se,x)}$. Note that this follows from Lemma 3.7(iii) if $x \in \mathcal{U}_e$. Item (iv) follows from Lemma 3.7(iii). Finally, if $a - b \in \mathcal{A}_{(s,x)}$ and $b - c \in \mathcal{A}_{(s,x)}$, then $a - c = (a - b) + (b - c) \in \mathcal{A}_{(s,x)}$. This proves (v). Finally, (vi) follows from the relation $(cc^*)(\theta_s(x)) = (c^*c)(x)$ applied to $c = a - b \in \mathcal{A}_s$. \blacksquare

DEFINITION 3.10. Consider the set \mathcal{F} of triples (a, s, x) such that $a \in \mathcal{A}_s$, and $x \in \mathcal{U}_{s^*s}$. If one is given $(a, s, x), (a', s', x') \in \mathcal{F}$, we will say that

$$(a, s, x) \sim (a', s', x')$$

if there exists some $e \in E(S)$, and some $b \in \mathcal{A}_e$, with

- (i) $x = x'$,
- (ii) $b(x) \neq 0$,
- (iii) $se = s'e$,
- (iv) $ab \stackrel{x}{=} a'b$.

REMARK 3.11. (i) Under the conditions of the above definition, the coordinate "s" in (a, s, x) could just as well be dropped, since one usually assumes that the fibers are pairwise disjoint, and hence there is only one s such that $a \in \mathcal{A}_s$. Nevertheless we believe it is convenient to mention s explicitly.

(ii) Since b identifies with a function on X , which is supported on \mathcal{U}_e , the fact that $b(x) \neq 0$ implies $x \in \mathcal{U}_e$.

(iii) Observe that $ab \in \mathcal{A}_s\mathcal{A}_e \subseteq \mathcal{A}_{se}$, and $a'b \in \mathcal{A}_{s'}\mathcal{A}_e \subseteq \mathcal{A}_{s'e}$. Since $se = s'e$, we see that both ab and $a'b$ lie in the same fiber of \mathcal{A} , and hence (iv) is meaningful.

PROPOSITION 3.12. *The relation " \sim " defined in Definition 3.10 is an equivalence relation.*

Proof. Given $(a, s, x) \in \mathcal{F}$, one has $(a, s, x) \sim (a, s, x)$ by taking $e = s^*s$, and any $b \in \mathcal{A}_e$, with $b(x) \neq 0$. That our relation is symmetric is obvious.

With respect to transitivity suppose that $(a, s, x) \sim (a', s', x') \sim (a'', s'', x'')$. Take e and f in $E(S)$, $b \in \mathcal{A}_e$, and $c \in \mathcal{A}_f$, satisfying the conditions of Definition 3.10 with respect to each one of the two occurrences of " \sim " above, respectively. Noticing that $x = x' = x''$, let $g = ef$, and $d = bc$. Clearly

$$d \in \mathcal{A}_e\mathcal{A}_f \subseteq \mathcal{A}_{ef} = \mathcal{A}_g.$$

Moreover $d(x) = b(x)c(x) \neq 0$, and $sg = sef = s'ef = s'fe = s''fe = s''g$. By Lemma 3.9(iii), we have

$$ad = abc \stackrel{x}{=} a'bc = a'cb \stackrel{x}{=} a''cb = a''d,$$

so the conclusion follows from Lemma (3.9)(v). ■

Recall that $\theta : S \rightarrow \mathcal{I}(X)$, $s \mapsto \theta_s$ is an action of S on $X = \mathcal{G}^{(0)}$. Here $\mathcal{I}(X)$ denotes the inverse semigroup of all partial bijections of X . Moreover, it induces an action $\tilde{\theta} : S \rightarrow \mathcal{I}(\mathcal{C}_0(X))$ on the commutative C^* -algebra $\mathcal{C}_0(X)$ in the canonical way: $\tilde{\theta}_s : \mathcal{C}_0(\mathcal{U}_{s^*s}) \rightarrow \mathcal{C}_0(\mathcal{U}_{ss^*})$ is defined by $\tilde{\theta}_s(f) := f \circ \theta_s^{-1}$ for all $s \in S$ and $f \in \mathcal{C}_0(\mathcal{U}_{s^*s})$. We are tacitly identifying $\mathcal{C}_0(X)$ with $C^*(\mathcal{E})$, where $\mathcal{E} = \mathcal{A}|_E$. Under this identification, $\mathcal{A}_e \subseteq C^*(\mathcal{E})$ corresponds to the ideal $\mathcal{C}_0(\mathcal{U}_{s^*s}) \subseteq \mathcal{C}_0(X)$. Thus, we may view $\tilde{\theta}$ as an action of S on $C^*(\mathcal{E})$.

LEMMA 3.13. *If $s \in S$, $a \in \mathcal{A}_s$ and $d \in \mathcal{A}_{s^*s}$, then $ad = \tilde{\theta}_s(d)a$. Moreover,*

$$ab^*c = cb^*a \quad \text{for all } a, b, c \in \mathcal{A}_s.$$

Proof. We first prove that $ab^*b = bb^*a$ for all $b \in \mathcal{A}_s$. We have

$$\begin{aligned} (ab^*b - bb^*a)(ab^*b - bb^*a)^* &= (ab^*b - bb^*a)(b^*ba^* - a^*bb^*) \\ &= ab^*bb^*ba^* - ab^*(ba^*)(bb^*) - b(b^*a)(b^*b)a^* + bb^*aa^*bb^* \\ &= ab^*bb^*ba^* - ab^*bb^*ba^* - bb^*bb^*aa^* + bb^*aa^*bb^* = 0. \end{aligned}$$

Therefore, $ab^*b = bb^*a$ for all $b \in \mathcal{A}_s$. By polarization, $ab^*c = cb^*a$ for all $b, c \in \mathcal{A}_s$. Note that $\tilde{\theta}_s(b^*c) = cb^*$ because $(b^*c)(x) = (cb^*)(\theta_s(x))$ for all $x \in \mathcal{U}_{s^*s}$. We conclude that $ab^*c = \tilde{\theta}_s(b^*c)a$ for all $b, c \in \mathcal{A}_s$. Since $\mathcal{A}_s^*\mathcal{A}_s$ spans a dense subspace of \mathcal{A}_{s^*s} the assertion follows. ■

Now, we can show that the equivalence relation defined in Definition 3.10 also has a left hand side version in the following sense.

LEMMA 3.14. *Given $(a, s, x), (a', s', x') \in \mathcal{F}$, we have $(a, s, x) \sim (a', s', x')$ if and only if there is $f \in E(S)$ and $c \in \mathcal{A}_f$ satisfying:*

- (i) $x = x'$,
- (ii) $c(y) \neq 0$, where $y = \theta_s(x) = \theta_{s'}(x)$,
- (iii) $fs = fs'$, and
- (iv) $ca \stackrel{x}{=} ca'$.

Proof. Suppose that $(a, s, x) \sim (a', s', x')$, so that $x = x'$ and there is $e \in E(S)$ and $b \in \mathcal{A}_e$ with $b(x) \neq 0$, $se = s'e$ and $ab \stackrel{x}{=} a'b$. Replacing e by $es^*ss'^*s'$ and using Lemma 3.9(iii) to multiply the equation $ab \stackrel{x}{=} a'b$ on the right by a function in $d \in \mathcal{A}_{(s^*s)(s'^*s')}$ with $d(x) \neq 0$, we may assume that $e \leq s^*s$ and $e \leq s'^*s'$. Thus, we may assume $b \in \mathcal{A}_e \subseteq \mathcal{A}_{s^*s} \cap \mathcal{A}_{s'^*s'}$. By Lemma 3.13, we have $ab = \tilde{\theta}_s(b)a$ and $a'b = \tilde{\theta}_{s'}(b)a$. Note that $\tilde{\theta}_s(b) = \tilde{\theta}_{se}(b) = \tilde{\theta}_{s'e}(b) = \tilde{\theta}_{s'}(b)$. Define c to be this common value. We have $c = \tilde{\theta}_s(b) \in \tilde{\theta}_s(\mathcal{A}_e \cap \mathcal{A}_{s^*s}) = \mathcal{A}_{ses^*}$ by Proposition 4.5 of [5], and similarly $c = \tilde{\theta}_{s'}(b) \in \mathcal{A}_{s'es'^*}$. Defining f to be gg' , where $g := ses^*$ and $g' := s'es'^*$, we therefore have $c \in \mathcal{A}_f$. Moreover, $gs = ses^*s = se = s'e = s'es'^*s' = g's'$, so that $fs = fs'$. We conclude that $c \in \mathcal{A}_f$ and $ca \stackrel{x}{=} ca'$. Finally, note that $c(y) = \tilde{\theta}_s(b)(y) = b(\theta_s^{-1}(y)) = b(x) \neq 0$. Thus we have shown (i)–(iv) provided $(a, s, x) \sim (a', s', x')$. Conversely, if (i)–(iv) hold, then we may use a similar argument to show that $(a, s, x) \sim (a', s', x')$. In fact, as before, we may assume $f \leq ss^*$ and $f \leq s's'^*$. Then we use Lemma 3.13 again to conclude that $ca = a\tilde{\theta}_s^{-1}(c)$ and $ca' = a\tilde{\theta}_{s'}^{-1}(c)$. Defining $b := \tilde{\theta}_s^{-1}(c) = \tilde{\theta}_{s'}^{-1}(c)$ and $e = (s^*fs)(s'^*fs)$, and proceeding as before, we conclude that $b \in \mathcal{A}_e$, $b(x) \neq 0$, and $ab \stackrel{x}{=} a'b$. Therefore, $(a, s, x) \sim (a', s', x')$, as desired. ■

DEFINITION 3.15. The equivalence class of each (a, s, x) in \mathcal{F} will be denoted by $[a, s, x]$. The quotient of \mathcal{F} by this equivalence relation will be denoted by L .

In what follows we shall give L the structure of a Fell line bundle over the groupoid \mathcal{G} constructed in Section 3.1.

PROPOSITION 3.16. *Given $[a, s, x], [a', s', x']$ in L , we have $[a, s, x] = [a', s', x']$ if and only if $[s, x] = [s', x']$ in \mathcal{G} , and $ac \stackrel{x}{\sim} a'c$ for all $e \in E(S)$ with $se = s'e$ and for all $c \in \mathcal{A}_e$. In particular, the following correspondence is a well-defined surjection from L to \mathcal{G} :*

$$\pi : [a, s, x] \mapsto [s, x].$$

Proof. If $[a, s, x] = [a', s', x']$, then, by definition, we have $x = x'$ and there is $f \in E(S)$ and $d \in \mathcal{A}_f$ with $sf = s'f$, $d(x) \neq 0$ and $ad \stackrel{x}{\sim} a'd$. In particular, this implies $[s, x] = [s', x']$. Now, take any $e \in E(S)$ with $se = s'e$ and any element $c \in \mathcal{A}_e$. By Lemma 3.9(iii), we have $adc \stackrel{x}{\sim} a'dc$. Since $dc = cd$ and $d(x) \neq 0$, this implies $ac \stackrel{x}{\sim} a'c$. Conversely, assume that $[s, x] = [s', x']$ and $ac \stackrel{x}{\sim} a'c$ for all $e \in E(S)$ with $se = s'e$ and $c \in \mathcal{A}_e$. The equality $[s, x] = [s', x']$ means that $x = x'$ and there is $f \in E(S)$ with $x \in \mathcal{U}_f$ and $sf = s'f$. Since \mathcal{U}_f is the spectrum of the commutative C^* -algebra \mathcal{A}_f , there is $d \in \mathcal{A}_e$ with $d(x) \neq 0$. And by hypothesis, $ad \stackrel{x}{\sim} a'd$, so that $[a, s, x] = [a', s', x']$. ■

From now on, for each $\gamma \in \mathcal{G}$ we let L_γ denote $\pi^{-1}(\gamma)$, and call it the *fiber of L over γ* . The next result provides a linear structure on each fiber of L .

PROPOSITION 3.17. *Given $\gamma \in \mathcal{G}$ and elements $[a, s, x], [b, t, x] \in L$ which are in the same fiber L_γ , we define*

$$[a, s, x] + [b, t, x] := [ac + bc, se, x],$$

where $e \in E(S)$ is any idempotent satisfying $se = te$ and $x \in \mathcal{U}_e$, and $c \in \mathcal{A}_e$ is any function with $c(x) = 1$. Then this is a well-defined addition operation on L_γ . Moreover, if $s = t$ (so that a, b belong to the same fiber \mathcal{A}_s), then

$$[a, s, x] + [b, s, x] = [a + b, s, x].$$

Given $\lambda \in \mathbb{C}$, we define

$$\lambda \cdot [a, s, x] := [\lambda a, s, x].$$

Then, this is a well-defined scalar product on L_γ . With this structure, L_γ is a complex vector space. Moreover, the assignment

$$[a, s, x] \mapsto \|[a, s, x]\| := \sqrt{(a^*a)(x)}$$

is a well-defined norm on the fiber L_γ . Hence L_γ is a normed vector space.

Proof. First, note that if $[a, s, x]$ and $[b, t, x]$ are in the same fiber L_γ , then $[s, x] = \pi([a, s, x]) = \pi([b, t, x]) = [t, x]$, so that there is an idempotent $e \in E(S)$

with $se = te$ and $x \in \mathcal{U}_e$. To show that the definition of the sum above does not depend on the choices of $e \in E(S)$ and $c \in \mathcal{A}_e$, take another idempotent $e' \in E(S)$ with $se' = te'$ and $x \in \mathcal{U}_{e'}$, and another function $c' \in \mathcal{A}_{e'}$ with $c'(x) = 1$. We have to show that

$$(3.4) \quad [ac + bc, se, x] = [ac' + bc', se', x].$$

Define $f := ee' \in E(S)$, and take any function $d \in \mathcal{A}_f$ with $d(x) \neq 0$. Note that $x \in \mathcal{U}_f$ and $r := (se)f = sf = (se')f = (te')f = tf = (te)f$. Since $c(x) = c'(x) = 1$, it is easy to check that $\zeta c \stackrel{x}{=} \zeta c'$ for all $\zeta \in \mathcal{A}_r$. In particular, $(ad + bd)c \stackrel{x}{=} (ad + bd)c'$, and hence (using that $cd = dc$ and $c'd = dc'$)

$$(ac + bc)d = (ad + bd)c \stackrel{x}{=} (ad + bd)c' = (ac' + bc')d.$$

This verifies equation (3.4). Next, we show that the sum on L does not depend on representatives: suppose that $[a', s', x] = [a, s, x]$ and $[b', t', x] = [b, t, x]$. All these elements belong to the same fiber L_γ . Hence, there is $e \in E(S)$ with $x \in \mathcal{U}_e$ and $r := se = s'e = t'e = te$. The equality $[a', s', x] = [a, s, x]$ yields $f \in E(S)$ and $c \in \mathcal{A}_f$ with $c(x) \neq 0$, $s'f = sf$ and $a'c \stackrel{x}{=} ac$. And the equality $[b', t', x] = [b, t, x]$ yields $g \in E(S)$ and $d \in \mathcal{A}_g$ with $d(x) \neq 0$, $t'g = tg$ and $b'd \stackrel{x}{=} bd$. Rescaling c and d , we may assume that $c(x) = d(x) = 1$. Moreover, replacing the idempotents e, f, g by the product efg , and using Lemma 3.9(iii) to replace the functions c and d by a function of the form $hcd \in \mathcal{A}_{efg}$, where h is any function in \mathcal{A}_e with $h(x) = 1$, we may further assume that $e = f = g$ and $c = d$. Thus, all the elements $ac, bc, a'c, b'c$ belong to the same fiber \mathcal{A}_r , and we have $ac \stackrel{x}{=} a'c$ and $bc \stackrel{x}{=} b'c$, that is $ac - a'c \in \mathcal{A}_{(r,x)}$ and $bc - b'c \in \mathcal{A}_{(r,x)}$. Therefore,

$$(ac + bc) - (a'c + b'c) = (ac - a'c) + (bc - b'c) \in \mathcal{A}_{(r,x)},$$

that is, $ac + bc \stackrel{x}{=} a'c + b'c$. This shows that the sum on L_γ is well-defined. If a, b belong to the same fiber \mathcal{A}_s and if $c \in \mathcal{A}_e$ is such that $c(x) = 1$, where $e \in E(S)$ and $x \in \mathcal{U}_e$, then $[ac + bc, se, x] = [(a + b)c, se, x] = [a + b, s, x]$, so that

$$[a, s, x] + [b, t, x] = [a + b, s, x].$$

It is easy to see that the scalar product is also well-defined and that L_γ is a complex vector space with this structure. To see that the map $[a, s, x] \mapsto (a^*a)(x)^{1/2}$ is well-defined, assume that $[a, s, x] = [b, t, x]$. First suppose that $s = t$, so that a, b are in the same fiber $\mathcal{A}_s = \mathcal{A}_t$. In this case, the equality $[a, s, x] = [b, t, x]$ means that $a \stackrel{x}{=} b$. Multiplying this equation on the left by a^* and using Lemma 3.9(ii) we get $a^*a \stackrel{x}{=} a^*b$, that is, $(a^*a)(x) = (a^*b)(x)$. Similarly, $(b^*b)(x) = (b^*a)(x)$. Since $(a^*b)(x) = \overline{(b^*a)(x)}$, we get $(a^*a)(x) = (b^*b)(x)$. In the general case, if a and b are in different fibers \mathcal{A}_s and \mathcal{A}_t , the equality $[a, s, x] = [b, t, x]$ yields $e \in E(S)$ and $c \in \mathcal{A}_e$ with $c(x) \neq 0$, $se = te$ and $ac \stackrel{x}{=} bc$. Now, ac and bc are in the same fiber $\mathcal{A}_{se} = \mathcal{A}_{te}$. The previous argument implies $(ac)^*(ac)(x) = (bc)^*(bc)(x)$. But $(ac)^*(ac) = (c^*a^*ac)(x) = (c^*c)(x)(a^*a)(x)$ and similarly $(bc)^*(bc)(x) =$

$(c^*c)(x)(b^*b)(x)$. Since $(c^*c)(x) > 0$, we conclude that $(a^*a)(x) = (b^*b)(x)$. Therefore $[a, s, x] \mapsto (a^*a)(x)^{1/2}$ is a well-defined map, which is easily seen to be a norm on L_γ . ■

Next, we prove that L has one-dimensional fibers:

PROPOSITION 3.18. *Given $\gamma = [s, x] \in \mathcal{G}$, take $a \in \mathcal{A}_s$ with $(a^*a)(x) > 0$. Then, for any element $[b, t, y]$ in the fiber L_γ , there is a unique $\lambda \in \mathbb{C}$ such that $[b, t, y] = \lambda \cdot [a, s, x]$. Hence, the singleton formed by the element $[a, s, x]$ is a basis for L_γ and we have $L_\gamma \cong \mathbb{C}$ as complex vector spaces. Moreover, the map*

$$\lambda \mapsto \frac{\lambda \cdot [a, s, x]}{\sqrt{(a^*a)(x)}}$$

defines an isomorphism $\mathbb{C} \xrightarrow{\sim} L_\gamma$ of normed vector spaces. In particular, each fiber L_γ of L is a Banach space.

Proof. Since $[b, t, y] \in L_\gamma$, we have $[t, y] = \pi([b, t, y]) = \gamma = [s, x]$. Thus, $x = y$ and there is $e \in E(S)$ such that $x \in \mathcal{U}_e$ and $te = se$. If $c \in \mathcal{A}_e = \mathcal{C}_0(\mathcal{U}_e)$ is any function with $c(x) = 1$, then we have $[b, t, x] = [bc, te, x]$ and $[a, s, x] = [ac, se, x]$. In fact, if $d \in \mathcal{A}_e$ is any function with $d(x) \neq 0$, then it is easy to see that $bcd \stackrel{x}{=} bd$ and $acd \stackrel{x}{=} ad$. Hence, replacing b by bc , and a by ac , we may assume that both a, b are in the same fiber, say \mathcal{A}_s . Thus, we want to show that there is a unique $\lambda \in \mathbb{C}$ satisfying $[b, s, x] = \lambda \cdot [a, s, x] = [\lambda a, s, x]$. Since both a, b belong to the same fiber \mathcal{A}_s , this is the same as to show that $b \stackrel{x}{=} \lambda a$. Multiplying this equation on the left by a^* (and using Lemma 3.9(ii)), we see that if λ exists, it has to be $\frac{(a^*b)(x)}{(a^*a)(x)}$. Now, to see that this λ works, we compute

$$(3.5) \quad (b - \lambda a)^*(b - \lambda a)(x) = (b^*b)(x) - \lambda(b^*a)(x) - \bar{\lambda}(a^*b)(x) + |\lambda|^2(a^*a)(x).$$

Note that

$$\lambda(b^*a)(x) = \frac{(a^*b)(x)}{(a^*a)(x)}(b^*a)(x) = \frac{(a^*bb^*a)(x)}{(a^*a)(x)} = \frac{(a^*a)(x)(bb^*)(\theta_s(x))}{(a^*a)(x)} = (b^*b)(x).$$

Similarly, one proves that $\bar{\lambda}(a^*b)(x) = |\lambda|^2(a^*a)(x) = (b^*b)(x)$, so that equation (3.5) equals zero. Therefore, $b \stackrel{x}{=} \lambda a$ for $\lambda = \frac{(a^*b)(x)}{(a^*a)(x)}$. Finally, it is easy to see that the map $\mathbb{C} \ni \lambda \mapsto \frac{\lambda \cdot [a, s, x]}{\sqrt{(a^*a)(x)}} \in L_\gamma$ is an isometric isomorphism of complex vector spaces. Its inverse is the map $[b, s, x] \mapsto \frac{(a^*b)(x)}{\sqrt{(a^*a)(x)}}$. The element $[a, s, x]$ is therefore a basis vector for L_γ and $L_\gamma \cong \mathbb{C}$ as complex normed vector spaces. ■

In order to define a topology on L we shall use Proposition 2.4. Given $a \in \mathcal{A}_s$, we define the local section \hat{a} of L by the formula

$$(3.6) \quad \hat{a}([s, x]) := [a, s, x] \quad \text{for all } [s, x] \in \mathcal{O}_s.$$

Thus, by definition, the domain of \hat{a} is the open subset $\text{dom}(\hat{a}) := \mathcal{O}_s \subseteq \mathcal{G}$. Recall that $\mathcal{O}_s = \mathcal{O}(s, \mathcal{U}_{s^*s}) = \{[s, x] : x \in \mathcal{U}_{s^*s}\}$.

PROPOSITION 3.19. *There is a unique topology on L making it a continuous Banach bundle and making all the local sections \widehat{a} with $a \in \mathcal{A}$ continuous for this topology. Moreover, with this topology, L is a complex line bundle, that is, a locally trivial one-dimensional complex vector bundle.*

Proof. We are going to prove (i) and (ii) in Proposition 2.4 in order to find the required topology. Property (i) is obvious since every element of L has the form $[a, s, x]$ for some $a \in \mathcal{A}_s$ and $[s, x] \in \mathcal{O}_s$. To prove (ii), suppose we have finitely many elements $s_i \in S$, $a_i \in \mathcal{A}_{s_i}$ and $\lambda_i \in \mathbb{C}$ for $i = 1, \dots, n$. We want to show that the set

$$\mathcal{V} = \left\{ \gamma \in \bigcap_{i=1}^n \text{dom}(\widehat{a}_i) : \left\| \sum_{i=1}^n \lambda_i \widehat{a}_i(\gamma) \right\| < \alpha \right\}$$

is open in \mathcal{G} for all $\alpha > 0$. Given γ_0 in \mathcal{V} , it belongs to $\text{dom}(\widehat{a}_i) = \mathcal{O}_{s_i}$ for all $i = 1, \dots, n$, so it has equivalent representations of the form $\gamma_0 = [s_i, x_0]$ for all $i = 1, \dots, n$. Thus, there is $e \in E(S)$ such that $x_0 \in \mathcal{U}_e$ and $t := s_1 e = \dots = s_n e$. Replacing e by the product $(s_1^* s_1) \cdots (s_n^* s_n) e$, we may assume $e \leq s_i^* s_i$ and hence $\mathcal{U}_e \subseteq \mathcal{U}_{s_i^* s_i}$ for all i . Take a function $b \in \mathcal{C}_0(\mathcal{U}_e) \cong \mathcal{A}_e$ which is identically 1 on a neighborhood $U_0 \subseteq \mathcal{U}_e$ of x_0 . It is easy to see that $b(x) a_i c \stackrel{x}{=} a_i b c$ for all $x \in \mathcal{U}_e$ and all $c \in \mathcal{A}_e$. In particular, $a_i c \stackrel{x}{=} a_i b c$ and hence $[a_i, s_i, x] = [a_i b, s_i e, x]$ for all $x \in U_0$. Hence, $\mathcal{O}(t, U_0) = \mathcal{O}(s_i, U_0) \subseteq \bigcap_i \mathcal{O}_{s_i} = \bigcap_i \text{dom}(\widehat{a}_i)$, and if $\gamma = [t, x] = [s_i, x] \in \mathcal{O}(t, U_0)$, we have

$$\lambda_i \widehat{a}_i(\gamma) = [\lambda_i a_i, s_i, x] = [\lambda_i a_i b, s_i e, x] = \widehat{\lambda_i a_i b}([s_i e, x]) = \widehat{\lambda_i a_i b}(\gamma).$$

Note that $\lambda_i a_i b \in \mathcal{A}_{s_i e} = \mathcal{A}_t$ for all i . Defining $a := \sum_{i=1}^n \lambda_i a_i b \in \mathcal{A}_t$, we conclude that

$$\sum_{i=1}^n \lambda_i \widehat{a}_i(\gamma) = \widehat{a}(\gamma) \quad \text{for all } \gamma \in \mathcal{O}(t, U_0).$$

Note that $\|\widehat{a}(\gamma)\| = \sqrt{(a^* a)(x)}$ for all $\gamma = [t, x] \in \mathcal{O}(t, U_0)$. Since $\sqrt{(a^* a)(x_0)} = \|\widehat{a}(\gamma_0)\| < \alpha$ and $a^* a$ is continuous, there is a neighborhood U of x_0 contained in U_0 such that $\|\widehat{a}(\gamma)\| < \alpha$ for all g in the open subset $\mathcal{U} := \mathcal{O}(t, U)$ of \mathcal{G} . All this implies that $\gamma_0 \in \mathcal{U} \subseteq \mathcal{V}$ and therefore \mathcal{V} is an open subset of \mathcal{G} . By Proposition 2.4, there is a unique topology on L turning it into an upper-semicontinuous Banach bundle and making the local section $\widehat{a} : \mathcal{O}_s \rightarrow L$ continuous for all $a \in \mathcal{A}_s$, $s \in S$. As we have already noted, $\|\widehat{a}(\gamma)\| = \sqrt{(a^* a)(d(\gamma))}$ for all $\gamma \in \text{dom}(\widehat{a})$. Since the map $\gamma \mapsto \sqrt{(a^* a)(d(\gamma))}$ is continuous from \mathcal{G} to \mathbb{R}^+ , we conclude that the norm on L is continuous (again by Proposition 2.4). Therefore, L is in fact a continuous Banach bundle, as desired. It remains to show that L is locally trivial. However, the local sections \widehat{a} are continuous and do not vanish on $\mathcal{O}(s, \text{dom}(a)) \subseteq \mathcal{O}_s$. Since $(\mathcal{O}_s$ and hence also) $\mathcal{O}(s, \text{dom}(a))$ is a locally compact Hausdorff space, we may apply the usual methods to conclude that L

is trivializable over $\mathcal{O}(s, \text{dom}(a))$. A local trivialization is provided by the map $(\lambda, \gamma) \mapsto \lambda \cdot \hat{a}(\gamma)$ from $\mathbb{C} \times \mathcal{O}(s, \text{dom}(a))$ to L . See also Remark II.13.9 of [10]. ■

Given $v, w \in L$, with $(\pi(v), \pi(w)) \in \mathcal{G}^{(2)}$, we shall define the product vw as follows. Write $v = [a, s, x]$ and $w = [b, t, y]$, so our assumption translates into

$$([s, x], [t, y]) \in \mathcal{G}^{(2)},$$

and hence $x = \theta_t(y)$.

PROPOSITION 3.20. *With notation as above, putting*

$$vw = [ab, st, y],$$

we get a well-defined operation on L , that is, the right-hand side does not depend on the choice of representatives for v and w .

Proof. Suppose that v has another representation as $v = [a', s', x']$, in which case $x = x'$, and there exists $e \in E(S)$ and $c \in \mathcal{A}_e$, such that

$$c(x) \neq 0, \quad se = s'e \quad \text{and} \quad ac \stackrel{x}{=} a'c.$$

To show that

$$(3.7) \quad [ab, st, y] = [a'b, s't, y],$$

we will check that the conditions of Definition 3.10 are satisfied for the idempotent $f = t^*et$, and the element $d = b^*cb \in \mathcal{A}_f$, under the special case in which $(b^*b)(y) \neq 0$. With respect to Definition 3.10(ii), Lemma 3.3 yields

$$d(y) = (b^*cb)(y) = (b^*b)(y) c(\theta_t(y)) = (b^*b)(y) c(x) \neq 0.$$

Checking Definition 3.10(iii) we have $stf = stt^*et = sett^*t = s'ett^*t = s'tt^*et = s'tf$. As for Definition 3.10(iv), recall that the fibers over idempotent elements are commutative, so

$$abd = abb^*cb = acbb^*b \stackrel{y}{=} a'cbb^*b = a'bb^*cb = a'bd,$$

where the crucial middle step is a consequence of Lemma 3.9(iv), given that $bb^*b \in \mathcal{A}_t$, and $\theta_t^{-1}(x) = y$. The verification of equation (3.7) is thus complete when $(b^*b)(y) \neq 0$.

Suppose now that $(b^*b)(y) = 0$. Then it follows from Lemma 3.9(i) that $b \stackrel{y}{=} 0_t$, and hence from Lemma 3.9(ii) we have

$$ab \stackrel{y}{=} 0_{st} \quad \text{and} \quad a'b \stackrel{y}{=} 0_{s't}.$$

It follows that

$$[ab, st, y] = [0, st, y] \quad \text{and} \quad [a'b, s't, y] = [0, s't, y].$$

and it suffices to show that $[0, st, y] = [0, s't, y]$. Let $f = t^*et$, as above, and notice that since $x \in \mathcal{U}_e \cap \mathcal{U}_{tt^*}$, one has

$$y = \theta_{t^*}(x) \in \theta_{t^*}(\mathcal{U}_e \cap \mathcal{U}_{tt^*}) = \mathcal{U}_{t^*et} = \mathcal{U}_f.$$

Choosing any $c \in \mathcal{A}_f$, such that $c(x) \neq 0$, one verifies the conditions in Definition 3.10, hence proving equation (3.7).

Next suppose that one is given another representation of w as $w = [b', t', y']$, in which case $y = y'$, and there exists $e \in E(S)$ and $c \in \mathcal{A}_e$, such that $c(x) \neq 0$, $te = t'e$ and $bc \stackrel{y}{=} b'c$. It follows that

$$ste = st'e \quad \text{and} \quad abc \stackrel{y}{=} ab'c,$$

the last relation being a consequence of Lemma 3.9(ii). Therefore

$$[ab, st, y] = [ab', st', y]. \quad \blacksquare$$

PROPOSITION 3.21. *The following assignment is a well-defined operation on L :*

$$[a, s, x] \mapsto [a, s, x]^* := [a^*, s^*, \theta_s(x)].$$

Proof. Suppose $[a, s, x] = [b, t, y]$ in L , that is, $x = y$ and there is $e \in E(S)$ and $c \in \mathcal{A}_e$ such that $se = te$, $c(x) \neq 0$ and $ac \stackrel{x}{=} bc$. Then $\theta_s(x) = \theta_t(y)$, $es^* = et^*$ and $c^*a^* \stackrel{\theta_s(x)}{=} c^*b^*$ by Lemma 3.9(vi). And by Lemma 3.14 this implies that $[a^*, s^*, \theta_s(x)] = [b^*, t^*, \theta_t(y)]$. \blacksquare

THEOREM 3.22. *With the multiplication defined in Proposition 3.20 and the involution defined in Proposition 3.21, L is a Fell line bundle, that is, a one-dimensional, locally trivial (continuous) Fell bundle over the étale groupoid \mathcal{G} .*

Proof. We already know that L is a one-dimensional, locally trivial continuous Banach bundle with the unique topology making the local sections \hat{a} continuous for all $a \in \mathcal{A}$. Since the algebraic operations on L are essentially inherited from \mathcal{A} , it is easy to see that all the algebraic properties required in Definition 2.6 are indeed satisfied. Let us check axioms (iv), (viii) and (ix) in Definition 2.6. Given $[a, s, x], [b, t, y] \in L$ with $\theta_t(y) = x$, Lemma 3.3 yields

$$\begin{aligned} \|[a, s, x] \cdot [b, t, y]\|^2 &= \|[ab, st, y]\|^2 = (b^*a^*ab)(y) = (b^*b)(y)(a^*a)(\theta_t(y)) \\ &= (b^*b)(y)(a^*a)(x) = \|[a, s, x]\|^2 \|[b, t, y]\|^2. \end{aligned}$$

Thus $\|[a, s, x] \cdot [b, t, y]\| = \|[a, s, x]\| \cdot \|[b, t, y]\|$. This, of course, proves (iv). To prove (viii), we compute

$$\|[a, s, x]^* \cdot [a, s, x]\| = \|[a^*a, s^*s, x]\| = \sqrt{((a^*a)^*(a^*a))(x)} = (a^*a)(x) = \|[a, s, x]\|^2.$$

To check (ix), it is enough to observe that

$$[a, s, x]^* \cdot [a, s, x] = [a^*a, s^*s, x] = [(a^*a)^{1/2}, s^*s, x]^* \cdot [(a^*a)^{1/2}, s^*s, x]$$

which is an element of the C^* -algebra $L_{[s^*s, x]}$ of the form w^*w , with $w \in L_{[s^*s, x]}$, and therefore positive. The relation $(a^*a)(x) = (aa^*)(\theta_s(x))$ for all $a \in \mathcal{A}_s$ and $x \in \mathcal{U}_{s^*s}$ implies that the involution is isometric: $\|[a, s, x]^*\| = \|[a, s, x]\|$. Finally, we show that the multiplication and the involution on L are continuous. By

Proposition 2.7 it is enough to prove the relations

$$(3.8) \quad \widehat{a}(\gamma) \cdot \widehat{b}(\gamma') = \widehat{ab}(\gamma\gamma') \quad \text{and} \quad \widehat{a}(\gamma)^* = \widehat{a^*}(\gamma^{-1})$$

for all $a, b \in \mathcal{A}$, $\gamma \in \text{dom}(\widehat{a})$ and $\gamma' \in \text{dom}(\widehat{b})$ with $r(\gamma') = d(\gamma)$. Suppose $a \in \mathcal{A}_s$ and $b \in \mathcal{A}_t$. By definition, we have

$$\text{dom}(\widehat{a}) = \mathcal{O}_s = \{[s, x] : x \in \mathcal{U}_{s^*s}\} \quad \text{and} \quad \text{dom}(\widehat{b}) = \mathcal{O}_t = \{[t, y] : y \in \mathcal{U}_{t^*t}\}.$$

Suppose $\gamma = [s, x]$ and $\gamma' = [t, y]$ with $r(\gamma') = \theta_t(y) = x = d(\gamma)$. Then

$$\widehat{a}(\gamma) \cdot \widehat{b}(\gamma') = [a, s, x] \cdot [b, t, y] = [ab, st, y] = \widehat{ab}([st, y]) = \widehat{ab}(\gamma\gamma'),$$

and

$$\widehat{a}(\gamma)^* = [a, s, x]^* = [a^*, s^*, \theta_s(x)] = \widehat{a^*}([s^*, \theta_s(x)]) = \widehat{a^*}(\gamma^{-1}). \quad \blacksquare$$

DEFINITION 3.23. The Fell line bundle $L = L(\mathcal{A})$ over $\mathcal{G} = \mathcal{G}(\mathcal{A})$ constructed above from the Fell bundle $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$ over S , will be called the *Fell line bundle associated to \mathcal{A}* .

As already observed in Section 2.3, Fell line bundles over \mathcal{G} correspond bijectively to twists over \mathcal{G} . The twisted groupoid $(\mathcal{G}(\mathcal{A}), \Sigma(\mathcal{A}))$ corresponding to $L(\mathcal{A})$ will be called the *twisted groupoid associated to \mathcal{A}* . Thus $\mathcal{G}(\mathcal{A})$ is the étale groupoid constructed in Section 3.1 and the extension groupoid $\Sigma(\mathcal{A})$ is the space of unitary elements of the Fell line bundle $L(\mathcal{A})$ constructed above. The algebraic and topological structure on $\Sigma(\mathcal{A})$ is canonically induced from $L(\mathcal{A})$. For instance, the multiplication on $\Sigma(\mathcal{A})$ is just the multiplication of $L(\mathcal{A})$ restricted to $\Sigma(\mathcal{A})$, and the inversion on $\Sigma(\mathcal{A})$ is the restricted involution from $L(\mathcal{A})$. The projection $\Sigma(\mathcal{A}) \rightarrow \mathcal{G}(\mathcal{A})$ is the restriction of the bundle projection $L(\mathcal{A}) \rightarrow \mathcal{G}(\mathcal{A})$. Finally, the inclusion $\mathbb{T} \times X \hookrightarrow \Sigma(\mathcal{A})$ is defined by $(z, x) \mapsto z \cdot 1_x$ for all $z \in \mathbb{T}$ and $x \in X = \mathcal{G}(\mathcal{A})^{(0)}$, where 1_x denotes the unit element of the C^* -algebra $L(\mathcal{A})_x \cong \mathbb{C}$ (the fiber over x).

It is also possible to construct the twist groupoid $\Sigma = \Sigma(\mathcal{A})$ directly from \mathcal{A} following the ideas appearing in [22]. For convenience, we outline here the main steps into this procedure. We define Σ as the set

$$\Sigma := \{[a, s, x] : a \in \mathcal{A}_s, x \in \text{dom}(a)\},$$

of equivalence classes $[a, s, x]$, where, by definition, $[a, s, x] = [a', s', x']$ if and only if $x = x'$, and there is $b, b' \in \mathcal{E}$ such that $b(y), b'(y) > 0$ and $ab = a'b'$. The surjection $\pi : \Sigma \rightarrow \mathcal{G}$ is defined by $\pi([a, s, x]) := [s, x]$. It is easy to see that π is well-defined.

The groupoid structure on Σ is defined in the same fashion as for \mathcal{G} : the source and range maps are $d([a, s, x]) = x$ and $r([a, s, x]) = \theta_s(x)$ and the operations are

$$\begin{aligned} [a, s, x] \cdot [a', s', x'] &= [aa', ss', x'] \quad \text{whenever } \theta_{s'}(x') = x \quad \text{and} \\ [a, s, x]^{-1} &= [a^*, s^*, \theta_s(x)]. \end{aligned}$$

With this structure, it is not difficult to see that Σ is in fact a groupoid. Before we define the appropriate topology on Σ , let us define the inclusion map $\iota : \mathbb{T} \times X \rightarrow \Sigma$: Given $(z, x) \in \mathbb{T} \times X$, define $\iota(z, x) := \llbracket b, e, x \rrbracket \in \Sigma$, where $e \in E(S)$ and b is any element of \mathcal{A}_e with $b(x) \neq 0$ and $\frac{b(x)}{|b(x)|} = z$. Then ι is a well-defined injective morphism of groupoids.

To specify a topology on Σ it is enough to define a system of open neighborhoods of a point $\llbracket a, s, x \rrbracket \in \Sigma$. This is given by the sets

$$(3.9) \quad \mathcal{O}(a, U, V) := \{ \llbracket za, s, y \rrbracket : y \in U \text{ and } z \in V \},$$

where U is an open subset of $\text{dom}(a)$ containing x and $V \subseteq \mathbb{T}$ is an open subset containing 1. With this topology it is not difficult to see that Σ is a topological groupoid, that $\pi : \Sigma \rightarrow \mathcal{G}$ is an open continuous map and that $\iota : \mathbb{T} \times X \hookrightarrow \Sigma$ is a homeomorphism onto its image:

$$I = \{ \llbracket b, e, x \rrbracket : x \in X, e \in E, b \in \mathcal{A}_e \text{ and } b(x) \neq 0 \}.$$

Moreover,

$$\mathbb{T} \times X \xrightarrow{\iota} \Sigma \xrightarrow{\pi} \mathcal{G}$$

is an exact sequence of topological groupoids and hence (\mathcal{G}, Σ) is a twisted groupoid.

Note that Σ has a canonical action of \mathbb{T} :

$$z \cdot \llbracket a, s, x \rrbracket := \llbracket za, s, x \rrbracket \quad \text{for all } z \in \mathbb{T} \text{ and } \llbracket a, s, x \rrbracket \in \Sigma.$$

It is easy to see that this is a well-defined free action of \mathbb{T} on Σ and the surjection $\pi : \Sigma \rightarrow \mathcal{G}$ induces an isomorphism from the orbit space Σ/\mathbb{T} onto \mathcal{G} . In other words, Σ is a principal \mathbb{T} -bundle over \mathcal{G} . It is also possible to exhibit local trivializations for the \mathbb{T} -bundle Σ . In fact, given $a \in \mathcal{A}_s$, the map

$$\psi : \mathbb{T} \times \text{dom}(a) \rightarrow \Sigma|_U, \quad \psi(z, x) := \llbracket za, s, x \rrbracket = z \cdot \llbracket a, s, x \rrbracket$$

defines a homeomorphism, where $\Sigma|_U := \pi^{-1}(U)$ is the restriction of Σ to the open subset $U = \mathcal{O}(s, \text{dom}(a)) \subseteq \mathcal{G}$. Note that ψ is \mathbb{T} -equivariant in the sense that $\psi(z, \gamma) = z \cdot \psi(1, \gamma)$. Recall that U is a bisection, so we have a canonical homeomorphism $U \cong \text{dom}(a)$ (which is given by the restriction of d to U). Through this homeomorphism we may also obtain homeomorphisms $\Sigma_U \cong \mathbb{T} \times U$. These homeomorphisms are compatible with the projections onto U , so they are in fact isomorphisms of bundles over U .

PROPOSITION 3.24. *Let (\mathcal{G}, Σ) be a twisted étale groupoid and let $L = (\mathbb{C} \times \Sigma)/\mathbb{T}$ be the associated Fell line bundle. If S is a wide inverse subsemigroup of $S(\mathcal{G})$ and if $\mathcal{A} = \{C_0(L_U)\}_{U \in S}$ is the Fell bundle over S defined from L as in Example 2.9, then the twisted groupoid $(\mathcal{G}(\mathcal{A}), \Sigma(\mathcal{A}))$ associated to \mathcal{A} is isomorphic to (\mathcal{G}, Σ) .*

Proof. By definition, the inclusion $\iota : \mathbb{T} \times X \hookrightarrow \Sigma$ is a homeomorphism onto its image $\iota(\mathbb{T} \times X) = \pi^{-1}(X)$, where π is the surjection $\Sigma \rightarrow \mathcal{G}$. Thus the

restriction $\Sigma|_X = \pi^{-1}(X)$ is trivializable. It follows that L is trivializable over X , and hence also over any open subset $U \subseteq X$. Thus $L_U \cong \mathbb{C} \times U$, so that

$$\mathcal{A}_U = \mathcal{C}_0(L_U) \cong \mathcal{C}_0(U) \subseteq \mathcal{C}_0(X) \quad \text{for every } U \in E(S).$$

This gives a faithful representation of $\mathcal{E}_{\mathcal{A}}$ into $\mathcal{C}_0(X)$. Since S covers \mathcal{G} , the idempotent semilattice $E(S)$ covers $\mathcal{G}^{(0)} = X$. Indeed, given $x \in X$, there is $U \in S$ with $x \in U$ and hence $x = d(x) \in d(U) = U^*U \in E(S)$. Thus the family $\{\mathcal{C}_0(U) : U \in E(S)\}$ spans a dense subspace of $\mathcal{C}_0(X)$. It follows from Proposition 4.3 of [6] that the representation $\mathcal{E}_{\mathcal{A}} \rightarrow \mathcal{C}_0(X)$ integrates to an isomorphism $C^*(\mathcal{E}_{\mathcal{A}}) \cong \mathcal{C}_0(X)$. So, the spectrum of $C^*(\mathcal{E}_{\mathcal{A}})$ may be identified with X , and through this identification, the spectrum $\widehat{\mathcal{A}}_U$ of \mathcal{A}_U is U for all $U \in E(S)$. Let us now describe the action θ associated to \mathcal{A} . Given $U \in S$, θ_U is a homeomorphism from $\widehat{\mathcal{A}}_{U^*U} \cong U^*U = d(U)$ onto $\widehat{\mathcal{A}}_{UU^*} \cong UU^* = r(U)$ which satisfies

$$(a^*ba)(x) = (a^*a)(x)b(\theta_U(x))$$

for all $a \in \mathcal{A}_U = \mathcal{C}_0(L_U)$, $b \in \mathcal{C}_0(X)$ and $x \in d(U)$. It is enough to consider $b \in \mathcal{C}_0(r(U))$ in order to characterize θ_U . Now, if $\gamma \in U$ and $x = d(\gamma)$, note that $x = \gamma^{-1}\gamma = \gamma^{-1}r(\gamma)\gamma$. By definition of the multiplication on \mathcal{A} (see Example 2.9), we have

$$(a^*a)(x) = a^*(\gamma^{-1})a(\gamma) = \overline{a(\gamma)}a(\gamma) = |a(\gamma)|^2$$

and

$$(a^*ba)(x) = a^*(\gamma^{-1})b(r(\gamma))a(\gamma) = |a(\gamma)|^2b(r(\gamma)).$$

As a consequence, $\theta_U(x) = \theta_U(d(\gamma)) = \theta_U(r(\gamma))$. In other words, θ_U is the homeomorphism $\tilde{\theta}_U : d(U) \rightarrow r(U)$ given by $\tilde{\theta}_U(x) = r_U(d_U^{-1}(x))$. The maps $\tilde{\theta}_U$ always give an action of S on X . And by Proposition 5.4 of [5], the map $\phi : \mathcal{G}(\mathcal{A}) \rightarrow \mathcal{G}$ defined by $\phi([U, y]) = d_U^{-1}(y)$ for all $[U, y] \in \mathcal{G}(\mathcal{A})$ is an isomorphism of étale groupoids provided S is wide, which is our case here. Next, we are going to find an isomorphism $\Sigma(\mathcal{A}) \cong \Sigma$. Recall that Σ may be identified with the set of unitary elements in the Fell line bundle L through the map $\sigma \mapsto [1, \sigma]$. In this way, we define $\psi : \Sigma(\mathcal{A}) \rightarrow \Sigma$ by

$$\psi(\llbracket a, U, y \rrbracket) := \frac{a(\gamma)}{|a(\gamma)|}, \quad \text{where } \gamma = d_U^{-1}(y) \in \mathcal{G},$$

whenever $a \in \mathcal{A}_U = \mathcal{C}_0(L_U)$, $y \in \text{dom}(a) \subseteq d(U)$ and $x = \theta_U(y) = r_U(d_U^{-1}(y))$. Notice that $|a(\gamma)|^2 = |a(\gamma)^*a(\gamma)| = |a^*(\gamma^{-1})a(\gamma)| = (a^*a)(y) > 0$. To show that ψ is well-defined, assume $\llbracket a, U, y \rrbracket = \llbracket a', U', y \rrbracket$ in $\Sigma(\mathcal{A})$, so there are $b, b' \in \mathcal{E}_{\mathcal{A}}$ such that $ab = a'b'$ and $b(y), b'(y) > 0$. Suppose $a \in \mathcal{A}_U$, $a' \in \mathcal{A}_{U'}$, $b \in \mathcal{A}_V$ and $b' \in \mathcal{A}_{V'}$, where $U, U', V, V' \in S$. Since $b, b' \in \mathcal{E}_{\mathcal{A}}$, we have $V, V' \in E(S)$, so that $V, V' \subseteq X$. If $\gamma = d_U^{-1}(y)$, then $\gamma = \gamma d(\gamma) = \gamma y \in UV \cap U'V'$. Thus $(ab)(\gamma) = a(\gamma)b(y)$ and $(a'b')(\gamma) = a'(\gamma)b'(y)$. Since $b(y)$ and $b'(y)$ are positive numbers, we get

$$\frac{a(\gamma)}{|a(\gamma)|} = \frac{a(\gamma)b(y)}{|a(\gamma)b(y)|} = \frac{(ab)(\gamma)}{|(ab)(\gamma)|} = \frac{(a'b')(\gamma)}{|(a'b')(\gamma)|} = \frac{a'(\gamma)b'(y)}{|a'(\gamma)b'(y)|} = \frac{a'(\gamma)}{|a'(\gamma)|}.$$

Let us check that ψ is a groupoid homomorphism. Take $[[a, U, y]], [[b, V, z]] \in \Sigma(\mathcal{A})$ with $a \in \mathcal{A}_U$, $b \in \mathcal{A}_V$, and let $\gamma_1 \in U$ and $\gamma_2 \in V$ such that $d(\gamma_1) = y$ and $d(\gamma_2) = z$. Then $\gamma = \gamma_1\gamma_2 \in UV$ and $d(\gamma) = d(\gamma_2) = z$, so that $(ab)(\gamma) = a(\gamma_1)b(\gamma_2)$. Hence,

$$\begin{aligned} \psi([[a, U, y]][[b, V, z]]) &= \psi([[ab, UV, z]]) = \frac{(ab)(\gamma)}{|(ab)(\gamma)|} \\ &= \frac{a(\gamma_1)}{|a(\gamma_1)|} \frac{b(\gamma_2)}{|b(\gamma_2)|} = \psi([[a, U, y]])\psi([[b, V, z]]). \end{aligned}$$

This shows that ψ respects multiplication. Here we have used that the multiplication in the Fell line bundle L restricts to the multiplication in the groupoid $\Sigma \subseteq L$. Similarly, since the involution in L restricts to the inverse in $\Sigma \subseteq L$, we get that ψ preserves inversion:

$$\psi([[a, U, y]]^{-1}) = \psi([[a^*, U^{-1}, \theta_U(y)]]) = \frac{a^*(\gamma_1^{-1})}{|a^*(\gamma_1^{-1})|} = \frac{a(\gamma_1)^*}{|a(\gamma_1)|} = \psi([[a, U, y]])^{-1}.$$

The pair (ψ, ϕ) of groupoid homomorphisms $\psi : \Sigma(\mathcal{A}) \rightarrow \Sigma$ and $\phi : \mathcal{G}(\mathcal{A}) \rightarrow \mathcal{G}$ we have defined is a morphism of extensions, that is, the following diagram commutes:

$$\begin{array}{ccccc} \mathbb{T} \times X & \xrightarrow{\iota_{\mathcal{A}}} & \Sigma(\mathcal{A}) & \xrightarrow{\pi_{\mathcal{A}}} & \mathcal{G}(\mathcal{A}) \\ \text{id} \downarrow & & \psi \downarrow & & \phi \downarrow \\ \mathbb{T} \times X & \xrightarrow{\iota} & \Sigma & \xrightarrow{\pi} & \mathcal{G} \end{array}$$

In fact, recall that $\iota_{\mathcal{A}} : \mathbb{T} \times X \rightarrow \Sigma(\mathcal{A})$ is defined by $\iota_{\mathcal{A}}(z, x) = [[b, V, x]]$, where $V \in E(S)$ and $b \in \mathcal{A}_V$ is such that $b(x) \neq 0$ and $\frac{b(x)}{|b(x)|} = z$. And the surjection $\pi_{\mathcal{A}} : \Sigma(\mathcal{A}) \rightarrow \mathcal{G}(\mathcal{A})$ is defined by $\pi_{\mathcal{A}}([[a, U, y]]) = [U, y]$ whenever $a \in \mathcal{A}_U$. By definition, $\psi(\iota_{\mathcal{A}}(z, x)) = \psi([[b, V, x]]) = \frac{b(x)}{|b(x)|} = z$. Here we view $z \in \mathbb{T}$ as the element $z[1, x] = [z, x]$ of the fiber L_x . On the other hand, $\iota(z, x) = z \cdot x \in \Sigma$ is identified with the element $[1, z \cdot x] = [z, x] \in L_x$. This says that the left hand side of the diagram above commutes. To see that the right hand side also commutes, take $[[a, U, y]] \in \Sigma$. If $a \in \mathcal{A}_U$ and $\gamma = d_U^{-1}(y) \in U$, then $\phi(\pi_{\mathcal{A}}([[a, U, y]])) = \phi([U, y]) = d_U^{-1}(y) = \gamma$. On the other hand, $\psi([[a, U, y]]) = \frac{a(\gamma)}{|a(\gamma)|} \in \Sigma \subseteq L$ is also sent to γ via $\pi : \Sigma \rightarrow \mathcal{G}$ because $\frac{a(\gamma)}{|a(\gamma)|}$ belongs to the fiber L_γ and the restriction of the bundle projection $L \rightarrow \mathcal{G}$ to Σ equals π . Therefore the diagram commutes, as desired. Since ϕ is bijective, the commutativity of the diagram forces ψ to be bijective as well. For example, to prove the injectivity of ψ , assume that $\sigma_1, \sigma_2 \in \Sigma(\mathcal{A})$ and $\psi(\sigma_1) = \psi(\sigma_2)$. Applying π and using the commutativity of the right hand side, we get $\phi(\pi_{\mathcal{A}}(\sigma_1)) = \phi(\pi_{\mathcal{A}}(\sigma_2))$. The injectivity of ϕ yields $\pi_{\mathcal{A}}(\sigma_1) = \pi_{\mathcal{A}}(\sigma_2)$, so there is a unique $z \in \mathbb{T}$ with $\sigma_2 = z \cdot \sigma_1$. By the commutativity of the left hand side ψ must be \mathbb{T} -equivariant, so that $\psi(\sigma_1) = \psi(\sigma_2) = z \cdot \psi(\sigma_1)$. Since

the \mathbb{T} -action is free, we conclude that $z = 1$ and hence $\sigma_1 = \sigma_2$. Analogously one proves that ψ is surjective. It remains to check that ψ is a homeomorphism. Since this is a local issue, we may restrict to local trivializations. As we have seen above each $a \in \mathcal{A}_U$ yields a local trivialization $\mathbb{T} \times \text{dom}(a) \cong \Sigma_U$ through the map $(z, x) \mapsto \llbracket za, U, x \rrbracket$, where $U = \mathcal{O}(U, \text{dom}(a)) \subseteq \mathcal{G}(\mathcal{A})$. On the other hand, since $|a(d_U^{-1}(x))|^2 = (a^*a)(x) > 0$, a is a non-vanishing continuous section on $\mathcal{V} = \phi(U) = \{d_U^{-1}(x) : x \in \text{dom}(a)\}$. This yields a local trivialization $\mathbb{C} \times \mathcal{V} \cong L_{\mathcal{V}}$ through the map $(\lambda, \gamma) \mapsto \lambda a(\gamma)$, which induces a local trivialization $\mathbb{T} \times \mathcal{V} \cong \Sigma_{\mathcal{V}}$ through the map $(z, \gamma) \mapsto z \frac{a(\gamma)}{|a(\gamma)|}$. Once composed with these trivializations, $\psi : \Sigma_U \rightarrow \Sigma_{\mathcal{V}}$ gives the map $(z, x) \mapsto (z, d_U^{-1}(x))$ from $\mathbb{T} \times \text{dom}(a)$ to $\mathbb{T} \times \mathcal{V}$, which is a homeomorphism because d_U is. Therefore, ψ is a homeomorphism. ■

REMARK 3.25. Let $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$ be a saturated semi-abelian Fell bundle over an inverse semigroup S , and let \mathcal{G} be the étale groupoid of germs associated to \mathcal{A} as in Section 3.1. Then the inverse subsemigroup $T = \{\mathcal{O}_s : s \in S\} \subseteq S(\mathcal{G})$ is wide. In fact, the first property in Definition 2.12 is obvious because every element $\gamma \in \mathcal{G}$ has the form $\gamma = [s, x]$ with $x \in \mathcal{U}_{s^*s}$, so that $\gamma \in \mathcal{O}_s$ for some $s \in S$. To prove the second property, take $s, t \in S$ and suppose that $\gamma \in \mathcal{O}_s \cap \mathcal{O}_t$. Then $\gamma = [s, x] = [t, x]$ for some $x \in \mathcal{U}_{s^*s} \cap \mathcal{U}_{t^*t}$. Thus, there is $e \in E(S)$ such that $x \in \mathcal{U}_e$ and $se = te$. Define $r := se = te \in S$ and note that $r^*r = s^*se = t^*te$, so that $\mathcal{U}_{r^*r} = \mathcal{U}_{s^*s} \cap \mathcal{U}_{t^*t} \cap \mathcal{U}_e$. In particular $x \in \mathcal{U}_{r^*r}$. Since $re = se = te$, it follows that $[r, x] = [s, x] = [t, x] = \gamma$ belongs to \mathcal{O}_r . And if $[r, y], y \in \mathcal{U}_{r^*r}$, is an arbitrary element of \mathcal{O}_r , it belongs to $\mathcal{O}_s \cap \mathcal{O}_t$ because $[r, y] = [s, y] = [t, y]$. In fact, $y \in \mathcal{U}_e$ and $re = se = te$.

3.3. CHARACTERIZATION OF SEMI-ABELIAN FELL BUNDLES. Let \mathcal{A} be a semi-abelian, saturated Fell bundle over S , let \mathcal{G} be the étale groupoid of germs constructed in Section 3.1 with unit space $\mathcal{G}^{(0)} = X$, the spectrum of the commutative C^* -algebra $C^*(\mathcal{E}_{\mathcal{A}})$. Consider the Fell line bundle L over \mathcal{G} associated to \mathcal{A} as in Section 3.2.

Given $s \in S$, we define $L_s = L|_{\mathcal{O}_s}$ to be the restriction of L to the open subset $\mathcal{O}_s = \mathcal{O}(s, \mathcal{U}_{s^*s}) \subseteq \mathcal{G}$. We shall write $\mathcal{C}_s = \mathcal{C}_0(L_s)$ for the space of continuous sections of L_s vanishing at infinity. Recall that each $\mathcal{O}_s = \{[s, x] : x \in \mathcal{U}_{s^*s}\}$ is a bisection of \mathcal{G} and we have (see Proposition 7.4 of [5])

$$\mathcal{O}_s \cdot \mathcal{O}_t = \mathcal{O}_{st} \quad \text{and} \quad \mathcal{O}_s^{-1} = \mathcal{O}_{s^*} \quad \text{for all } s, t \in S.$$

This says that the map $s \mapsto \mathcal{O}_s$ is a homomorphism from S to the inverse semigroup $S(\mathcal{G})$ of all bisections in \mathcal{G} . The restrictions of d and r to \mathcal{O}_s will be denoted by d_s and r_s . Since \mathcal{O}_s is a bisection, $d_s : \mathcal{O}_s \rightarrow \mathcal{U}_{s^*s}$ and $r_s : \mathcal{O}_s \rightarrow \mathcal{U}_{ss^*}$ are homeomorphisms. Moreover, from the definitions of d and r , it follows that $r_s \circ d_s^{-1} = \theta_s$.

PROPOSITION 3.26. *With notations as above, the family of Banach spaces $\mathcal{C} = \{\mathcal{C}_s\}_{s \in S}$ is a Fell bundle over S with respect to the following algebraic operations:*

(i) the multiplication $\mathcal{C}_s \times \mathcal{C}_t \rightarrow \mathcal{C}_{st}$ is defined by

$$(\xi \cdot \eta)(\gamma) := \xi(r_s^{-1}(r(\gamma)))\eta(d_t^{-1}(d(\gamma))) \quad \text{for all } \gamma \in \mathcal{O}_{st}, \xi \in \mathcal{C}_s, \eta \in \mathcal{C}_t;$$

(ii) and the involution $\mathcal{C}_s \rightarrow \mathcal{C}_{s^*}$ is defined by

$$\xi^*(\gamma) = \overline{\xi(\gamma^{-1})} \quad \text{for all } \gamma \in \mathcal{O}_{s^*} \text{ and } \xi \in \mathcal{C}_s.$$

The inclusion maps are defined in the canonical way: if $s \leq t$ in S , then $\mathcal{O}_s \leq \mathcal{O}_t$ in $S(\mathcal{G})$, that is, $\mathcal{O}_s \subseteq \mathcal{O}_t$. Thus each section ξ of L_s may be viewed as a section of L_t extending it by zero outside \mathcal{O}_s . Hence we define the inclusion map $j_{t,s} : \mathcal{C}_s \rightarrow \mathcal{C}_t$ by $j_{t,s}(\xi) = \tilde{\xi}$ for all $\xi \in \mathcal{C}_s$, where $\tilde{\xi}$ denotes the extension of ξ by zero.

Proof. The proof consists of straightforward calculations and is left to the reader. We just remark that the multiplication is well-defined. In fact, if $\gamma \in \mathcal{O}_{st} = \mathcal{O}_s \cdot \mathcal{O}_t$, there is a unique way to write $\gamma = \gamma_1 \cdot \gamma_2$ with $\gamma_1 \in \mathcal{O}_s$ and $\gamma_2 \in \mathcal{O}_t$ because \mathcal{O}_s and \mathcal{O}_t are bisections. Moreover, this unique way is given by $\gamma_1 = r_s^{-1}(r(\gamma))$ and $\gamma_2 = d_t^{-1}(d(\gamma))$. Note that the multiplication we defined on \mathcal{C} uses the multiplication of L as a Fell line bundle. Thus $(\xi \cdot \eta)(\gamma) = \xi(\gamma_1)\eta(\gamma_2) \in L_{\gamma_1}L_{\gamma_2} \subseteq L_{\gamma_1\gamma_2} = L_\gamma$, so that $\xi \cdot \eta$ is a section of L . Since all maps involved in the multiplication are continuous, $\xi \cdot \eta$ is a continuous section and it vanishes at infinity because ξ and η do. ■

THEOREM 3.27. *Let $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$ be a semi-abelian, saturated Fell bundle and let $\mathcal{C} = \{\mathcal{C}_s\}_{s \in S} = \{\mathcal{C}_0(L_s)\}_{s \in S}$ be the (semi-abelian, saturated) Fell bundle constructed above. Given $a \in \mathcal{A}_s$, we define the function $\hat{a} : \mathcal{O}_s \rightarrow L$ by*

$$(3.10) \quad \hat{a}([s, x]) := [a, s, x] \quad \text{for all } [s, x] \in \mathcal{O}_s.$$

Then \hat{a} is a section of L_s and belongs to $\mathcal{C}_0(L_s)$. Moreover, the map $a \mapsto \hat{a}$ from \mathcal{A} to \mathcal{C} , which shall henceforth be called the Gelfand map, is an isomorphism of Fell bundles $\mathcal{A} \cong \mathcal{C}$. In particular, we have isomorphisms of imprimitivity Hilbert bimodules

$$\mathcal{A}_{ss^*} \mathcal{A}_s \mathcal{A}_{s^*s} \cong \mathcal{C}_0(\mathcal{U}_{ss^*}) \mathcal{C}_0(L_s) \mathcal{C}_0(\mathcal{U}_{s^*s}) \quad \text{for all } s \in S.$$

Proof. It is clear that \hat{a} is a section of L_s since the bundle projection $p : L \rightarrow \mathcal{G}$ is given by $p([a, s, x]) = [s, x]$. Moreover, by definition of the topology on L , all the sections \hat{a} are continuous. Note that

$$(3.11) \quad \|\hat{a}([s, x])\|^2 = \|[a, s, x]\|^2 = (a^*a)(x).$$

Since $a^*a \in \mathcal{C}_0(\mathcal{U}_{s^*s})$, this implies that \hat{a} vanishes at infinity, that is, $\hat{a} \in \mathcal{C}_0(L_s)$ and

$$\|\hat{a}\| = \sup_{x \in \mathcal{U}_{s^*s}} \sqrt{(a^*a)(x)} = \|a^*a\|^{1/2} = \|a\|.$$

Thus the map $\mathcal{A}_s \ni a \mapsto \hat{a} \in \mathcal{C}_0(L_s)$ is isometric. It is obviously linear by definition of the linear structure on the fibers of L (see Proposition 3.17). To prove its surjectivity, first take a section $\xi \in \mathcal{C}_c(L_s)$ with support contained in $\mathcal{O}(s, \text{dom}(a))$, where a is a fixed element of \mathcal{A}_s . Since \hat{a} does not vanish on $\mathcal{O}(s, \text{dom}(a))$ by equation (3.11), the line bundle L is trivializable over $\mathcal{O}(s, \text{dom}(a))$

$\cong \text{dom}(a)$. More precisely, the restriction $L|_{\mathcal{O}(s, \text{dom}(a))}$ is isomorphic to $\mathbb{C} \times \mathcal{O}(s, \text{dom}(a)) \cong \mathbb{C} \times \text{dom}(a)$ through the map $\mathbb{C} \times \text{dom}(a) \ni (\lambda, x) \mapsto \lambda \widehat{a}([s, x]) \in L$. Therefore, there is a continuous function $h : \text{dom}(a) \rightarrow \mathbb{C}$ such that

$$(3.12) \quad \zeta([s, x]) = h(x) \widehat{a}([s, x]) \quad \text{for all } x \in \text{dom}(a).$$

Since ζ is supported in $\mathcal{O}(s, \text{dom}(a))$, h belongs to $\mathcal{C}_c(\text{dom}(a)) \subseteq \mathcal{C}_c(\mathcal{U}_{s^*s})$ and hence may be viewed as an element of $\mathcal{A}_{s^*s} \cong \mathcal{C}_0(\mathcal{U}_{s^*s})$. Moreover, in this way equation (3.12) holds for all $x \in \mathcal{U}_{s^*s}$ and from the (easily verified) relation $ah \stackrel{x}{=} h(x)a$, we obtain

$$\widehat{ah}([s, x]) = [ah, s, x] = h(x)[a, s, x] = h(x) \widehat{a}([s, x]) = \zeta([s, x]).$$

Thus $\widehat{ah} = \zeta$ and therefore the image of the Gelfand map $\mathcal{A}_s \rightarrow \mathcal{C}_0(L_s)$ contains all the functions with compact support contained in $\mathcal{O}(s, \text{dom}(a))$. Since the open subsets $\mathcal{O}(s, \text{dom}(a))$ with $a \in \mathcal{A}_s$ cover \mathcal{O}_s , a partition-of-unit argument shows that any function in $\mathcal{C}_c(L_s)$ is in the image of the Gelfand map and therefore it is surjective.

We have already seen in the proof of Theorem 3.22 (see (3.8)) that the Gelfand map preserves the Fell bundle multiplications and involutions, that is,

$$\widehat{a \cdot b} = \widehat{a} \cdot \widehat{b} \quad \text{and} \quad \widehat{a^*} = \widehat{a}^* \quad \text{for all } s, t \in S, a \in \mathcal{A}_s, b \in \mathcal{A}_t.$$

Finally, we show that the Gelfand map preserves the inclusion maps $\mathcal{A}_s \hookrightarrow \mathcal{A}_t$ and $\mathcal{C}_0(L_s) \hookrightarrow \mathcal{C}_0(L_t)$ whenever $s \leq t$. For this all we have to check is the following: if $a \in \mathcal{A}_s$ and we consider it as an element of \mathcal{A}_t (so we are in fact identifying $\mathcal{A}_s \subseteq \mathcal{A}_t$), then the function \widehat{a} vanishes outside \mathcal{O}_s . But as we have already observed above, equation (3.11) implies that \widehat{a} is supported in $\mathcal{O}(s, \text{dom}(a)) \subseteq \mathcal{O}_s$. ■

The Fell bundle $\mathcal{C} = \{\mathcal{C}_0(L_s)\}_{s \in S}$ over S constructed in Proposition 3.26 may be also considered as a Fell bundle over the inverse semigroup

$$T = \{\mathcal{O}_s : s \in S\} \subseteq S(\mathcal{G}).$$

The structure is basically the same. To avoid confusion, let us write $\mathcal{B} = \{\mathcal{B}_t\}_{t \in T}$ for the Fell bundle \mathcal{C} considered over T . Thus, if $t = \mathcal{O}_s$, then the fiber \mathcal{B}_t is by definition $\mathcal{C}_0(L_s)$, and the algebraic operations and inclusion maps are defined as in Proposition 3.26. Note that \mathcal{C} is the pullback of \mathcal{B} along the map $\mathcal{O} : S \rightarrow T$, $s \mapsto \mathcal{O}_s$. The map \mathcal{O} is a surjective homomorphism of inverse semigroups, but it is not injective in general. The most trivial example is when the Fell bundle $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$ is the *zero* Fell bundle, that is, $\mathcal{A}_s = \{0\}$ for all $s \in S$. In this case, the associated twisted groupoid (\mathcal{G}, Σ) is the *empty* groupoid, that is, $\mathcal{G} = \Sigma = \emptyset$. Thus $S(\mathcal{G})$ (and hence also T) is the inverse semigroup with just one element, the zero element (empty set): $S(\mathcal{G}) = T = \{0\}$. Of course, the map $\mathcal{O} : S \rightarrow \{0\}$ is not injective since S might be an arbitrary inverse semigroup.

If \mathcal{O} is injective, then \mathcal{B} and \mathcal{C} are isomorphic Fell bundles. Thus, by Theorem 3.27, \mathcal{B} is also isomorphic to the original Fell bundle \mathcal{A} in this case. So, it is interesting to give conditions on \mathcal{A} that imply the injectivity of $\mathcal{O} : S \rightarrow S(\mathcal{G})$.

Before we go into this problem, let us prove that the C^* -algebras of \mathcal{A} and \mathcal{B} are always isomorphic:

PROPOSITION 3.28. *Let notation be as above. Then the Gelfand map from \mathcal{A} to \mathcal{B} induces an isomorphism $C^*(\mathcal{A}) \cong C^*(\mathcal{B})$ which restricts to an isomorphism $C^*(\mathcal{E}_\mathcal{A}) \cong C^*(\mathcal{E}_\mathcal{B}) \cong C_0(X)$, where $\mathcal{E}_\mathcal{A} = \mathcal{A}|_{E(S)}$ and $\mathcal{E}_\mathcal{B} = \mathcal{B}|_{E(T)}$.*

Proof. Since the Gelfand map from \mathcal{A} to $\mathcal{C} = \{\mathcal{C}_0(L_s)\}_{s \in S}$ is an isomorphism of Fell bundles, it is enough to show that the canonical morphism from $\mathcal{C} = \{\mathcal{C}_0(L_s)\}_{s \in S}$ to $\mathcal{B} = \{\mathcal{C}_0(L_t)\}_{t \in T}$ (consisting of the homomorphism $\mathcal{O} : S \rightarrow T = \{\mathcal{O}_s : s \in S\}$ and the identity maps $\mathcal{C}_0(L_s) \rightarrow \mathcal{C}_0(L_t)$ between the fibers whenever $t = \mathcal{O}_s$) induces an isomorphism $C^*(\mathcal{C}) \cong C^*(\mathcal{B})$ which restricts to an isomorphism $C^*(\mathcal{E}_\mathcal{C}) \cong C^*(\mathcal{E}_\mathcal{B})$. The canonical morphism $\mathcal{C} \rightarrow \mathcal{B}$ induces the map $\text{Rep}(\mathcal{B}) \rightarrow \text{Rep}(\mathcal{C})$ which assigns to a representation (see Definition 3.1 of [6]) $\pi = \{\pi_t\}_{t \in T}$ of \mathcal{B} , the representation $\tilde{\pi} = \{\tilde{\pi}_s\}_{s \in S}$ with $\tilde{\pi}_s = \pi_{\mathcal{O}_s}$ for all $s \in S$. The induced $*$ -homomorphism $C^*(\mathcal{C}) \rightarrow C^*(\mathcal{B})$ is automatically surjective, and to prove its injectivity, we show that every representation $\rho = \{\rho_s\}_{s \in S}$ of \mathcal{C} is equal to $\tilde{\pi}$ for some (necessarily unique) representation $\pi \in \text{Rep}(\mathcal{B})$. In fact, all we have to show is that, for all $r, s \in S$,

$$(3.13) \quad \pi_r(f) = \pi_s(f) \quad \text{whenever } \mathcal{O}_r = \mathcal{O}_s \text{ and } f \in \mathcal{C}_0(L_r) = \mathcal{C}_0(L_s).$$

The equality $\mathcal{O}_r = \mathcal{O}_s$ implies that $\mathcal{U}_{r^*r} = \mathcal{U}_{s^*s}$, $\mathcal{U}_{rr^*} = \mathcal{U}_{ss^*}$ and $\theta_r = \theta_s$. Moreover, given $x \in \mathcal{U}_{r^*r} = \mathcal{U}_{s^*s}$, we have $[r, x] = [s, x]$ because $\mathcal{O}_r = \mathcal{O}_s$. Hence, there is $e \in E(S)$ with $x \in \mathcal{U}_e$ and $re = se$. Multiplying e by $(r^*r)(s^*s)$, we may assume that $e \leq (r^*r)(s^*s)$, that is, $e \leq r^*r$ and $e \leq s^*s$. Defining $E_{r,s} = \{e \in E(S) : re = se, e \leq (r^*r)(s^*s)\}$, we conclude that

$$\mathcal{U}_{r^*r} = \mathcal{U}_{s^*s} = \bigcup_{e \in E_{r,s}} \mathcal{U}_e.$$

As a consequence, we get

$$(3.14) \quad C_0(\mathcal{U}_{r^*r}) = C_0(\mathcal{U}_{s^*s}) = \overline{\text{span}}_{e \in E_{r,s}} C_0(\mathcal{U}_e).$$

Now, given $e \in E_{r,s}$, $f \in \mathcal{C}_0(L_r) = \mathcal{C}_0(L_s)$ and $h \in \mathcal{C}_0(\mathcal{U}_e)$, we have

$$\rho_r(f)\rho_e(h) = \rho_{re}(fh) = \rho_{se}(fh) = \rho_s(f)\rho_e(h).$$

Since $e \leq r^*r$ and ρ is a representation, we have $\rho_e(h) = \rho_{r^*r}(h)$, so that

$$\rho_r(f)\rho_e(h) = \rho_r(f)\rho_{r^*r}(h) = \rho_r(fh).$$

Analogously, $\rho_s(f)\rho_e(h) = \rho_s(fh)$. We conclude that $\rho_r(fh) = \rho_s(fh)$ for all $h \in \mathcal{C}_0(\mathcal{U}_e)$ with $e \in E_{r,s}$, and by equation (3.14) this also holds for every h in $\mathcal{C}_0(\mathcal{U}_{r^*r}) = \mathcal{C}_0(\mathcal{U}_{s^*s})$. This is enough to prove (3.13) because by Cohen's Factorization Theorem, every element of $\mathcal{C}_0(L_r) = \mathcal{C}_0(L_s)$ is a product of the form fh with $f \in \mathcal{C}_0(L_r) = \mathcal{C}_0(L_s)$ and $h \in \mathcal{C}_0(\mathcal{U}_{r^*r}) = \mathcal{C}_0(\mathcal{U}_{s^*s})$. Therefore we get an isomorphism $C^*(\mathcal{C}) \cong C^*(\mathcal{B})$. Its restriction to the idempotent parts gives an injective $*$ -homomorphism $C^*(\mathcal{E}_\mathcal{C}) \rightarrow C^*(\mathcal{E}_\mathcal{B})$ (which is the identity map on the fibers). To

see that it is surjective, suppose $s \in S$ and $f = \mathcal{O}_s$ is idempotent in T . Although s is not necessarily idempotent in S , we must have $f = \mathcal{O}_s = \mathcal{O}_s^* \mathcal{O}_s = \mathcal{O}_{s^*s}$. Since the morphism $\mathcal{E}_C \rightarrow \mathcal{E}_B$ maps $\mathcal{C}_{s^*s} = \mathcal{C}_0(L_{s^*s})$ onto $\mathcal{B}_f = \mathcal{C}_0(L_s) = \mathcal{C}_0(L_{s^*s})$ (it is just the identity map), the surjectivity of the induced map $C^*(\mathcal{E}_C) \rightarrow C^*(\mathcal{E}_B)$ follows, and therefore $C^*(\mathcal{E}_C) \cong C^*(\mathcal{E}_B) \cong \mathcal{C}_0(X)$, as desired. ■

Let us now return to the injectivity problem of the map $\mathcal{O} : S \rightarrow S(\mathcal{G})$.

DEFINITION 3.29. Let $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$ be a Fell bundle over an inverse semigroup S . Let $C^*(\mathcal{A})$ be the (full) cross-sectional C^* -algebra of \mathcal{A} , and let $\pi_u : \mathcal{A} \rightarrow C^*(\mathcal{A})$ be the universal representation of \mathcal{A} . We say that \mathcal{A} is *faithful* if the map $s \mapsto \pi_u(\mathcal{A}_s)$ is injective, that is, $\pi_u(\mathcal{A}_s) = \pi_u(\mathcal{A}_t)$ if and only if $s = t$. We say that \mathcal{A} is *semi-faithful* if the restriction $\mathcal{E}_{\mathcal{A}} = \mathcal{A}|_E$ of \mathcal{A} to the semilattice of idempotents $E = E(S)$ is faithful.

Recall from [4] that an inverse semigroup S with zero is said to be *continuous* if $s \equiv t$ implies $s = t$, where \equiv is the following equivalence relation:

$$s \equiv t \Leftrightarrow s^*s = t^*t \text{ and for any nonzero idempotent } f \leq s^*s,$$

there is a nonzero idempotent $e \leq f$ with $se = te$.

PROPOSITION 3.30. Let $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$ be a saturated, semi-abelian Fell bundle over an inverse semigroup S with zero element 0 such that $\mathcal{A}_0 = \{0\}$, and let L be the associated Fell line bundle. If S is continuous and \mathcal{A} is semi-faithful, then the map $s \mapsto \mathcal{O}_s$ from S to the inverse semigroup $T = \{\mathcal{O}_s : s \in S\} \subseteq S(\mathcal{G})$ is injective. Hence $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$ and $\mathcal{B} = \{\mathcal{C}_0(L_t)\}_{t \in T}$ are isomorphic Fell bundles.

Proof. Recall that $\mathcal{O}_s = \{[s, x] : x \in \mathcal{U}_{s^*s}\}$ for all $s \in S$. If $s, t \in S$ are such that $\mathcal{O}_s = \mathcal{O}_t$, then $\mathcal{U}_{s^*s} = \mathcal{U}_{t^*t}$, $\mathcal{U}_{ss^*} = \mathcal{U}_{tt^*}$ and $\theta_s = \theta_t$. In particular $\mathcal{A}_{s^*s} \cong \mathcal{C}_0(\mathcal{U}_{s^*s}) = \mathcal{C}_0(\mathcal{U}_{t^*t}) \cong \mathcal{A}_{t^*t}$ in $\mathcal{C}_0(X) \cong C^*(\mathcal{E})$. Since \mathcal{A} is semi-faithful, we get $s^*s = t^*t$. Moreover, from the semi-faithfulness we have $\mathcal{A}_e = \{0\}$ if and only if $e = 0$. Now, take any nonzero idempotent $f \leq s^*s$. Then $\emptyset \neq \mathcal{U}_f \subseteq \mathcal{U}_{s^*s} = \mathcal{U}_{t^*t}$. Thus, if $y \in \mathcal{U}_f$, then $[s, x] = [t, x]$, and hence there is $e \in E(S)$ with $se = te$ and $x \in \mathcal{U}_e$. In particular, $\mathcal{U}_e \neq \emptyset$ so that $\mathcal{A}_e \neq \{0\}$ and hence e is a nonzero idempotent. The product $g = ef$ is a nonzero idempotent because $x \in \mathcal{U}_g = \mathcal{U}_e \cap \mathcal{U}_f$, and we have $g \leq f$ and $sg = tg$. Hence $s \equiv t$ and the continuity of S implies $s = t$. ■

REMARK 3.31. The hypothesis $\mathcal{A}_0 = \{0\}$ is not necessary for the injectivity of $s \mapsto \mathcal{O}_s$. As a simple example, consider $S = \{0, 1\}$ (which is a semilattice and therefore a continuous inverse semigroup). Define $\mathcal{A}_0 = \mathbb{C}$ and $\mathcal{A}_1 = \mathbb{C} \times \mathbb{C}$. With the inclusion $\mathbb{C} \hookrightarrow \mathbb{C} \times \{0\} \subseteq \mathbb{C} \times \mathbb{C}$ and the canonical algebraic operations (inherited from $\mathbb{C} \times \mathbb{C}$), \mathcal{A} is a semi-abelian, saturated Fell bundle. Note that $C^*(\mathcal{A}) = C^*(\mathcal{E}) \cong \mathbb{C} \times \mathbb{C} \cong \mathcal{C}(\{x_0, x_1\})$, where x_0, x_1 are two distinct points. With these identifications, we may say that $\mathcal{U}_0 = \{x_0\}$ and $\mathcal{U}_1 = \{x_0, x_1\} = X$. Note that $\mathcal{G} \cong X$ and $\Sigma \cong \mathbb{T} \times X$ in this case (this happens whenever S is a

semilattice). Moreover, we may identify $\mathcal{O}_0 \cong \mathcal{U}_0$ and $\mathcal{O}_1 \cong \mathcal{U}_1$. Thus the map $s \mapsto \mathcal{O}_s$ is injective.

EXAMPLE 3.32. Consider a Fell bundle \mathcal{A} over a discrete group G with unit 1. Rigorously, G is not continuous as an inverse semigroup because it has no zero element. However, this is the only problem. One may add a zero element 0 to G turning it into a continuous inverse semigroup $S = G \cup \{0\}$ (not a group anymore, of course), and extend \mathcal{A} to a Fell bundle $\tilde{\mathcal{A}}$ over S simply defining $\tilde{\mathcal{A}}_0 = \{0\}$. Note that \mathcal{A} is saturated and semi-abelian if and only if $\tilde{\mathcal{A}}$ is. Moreover, if \mathcal{A} is saturated, then the following assertions are equivalent:

- (i) \mathcal{A} is the zero Fell bundle;
- (ii) $\mathcal{A}_1 = \{0\}$;
- (iii) there is $g \in G$ with $\mathcal{A}_g = \{0\}$.

In fact, if $g \in G$ is such that $\mathcal{A}_g = \{0\}$, then $\mathcal{A}_1 = \mathcal{A}_g^* \mathcal{A}_g = \{0\}$. And if $\mathcal{A}_1 = \{0\}$, then $\mathcal{A}_g^* \mathcal{A}_g = \mathcal{A}_1 = \{0\}$, so that $\mathcal{A}_g = \{0\}$ for all $g \in G$. For groups, the zero Fell bundle is the only case where the injectivity of $s \mapsto \mathcal{O}_s$ fails (unless $G = \{1\}$ is the trivial group). In fact, if \mathcal{A} is a nonzero, semi-abelian, saturated Fell bundle over G , then all the fibers \mathcal{A}_g are nonzero by the equivalences above. It follows that $\tilde{\mathcal{A}}$ is faithful. In particular, it is semi-faithful, so we may apply Proposition 3.30 to conclude that $s \mapsto \mathcal{O}_s$ is injective from S to $S(\mathcal{G})$. In particular, its restriction from G to $S(\mathcal{G})$ is also injective. One may also prove this directly in the group case: the associated groupoid \mathcal{G} is the transformation groupoid $\mathcal{G} \cong G \rtimes_{\theta} X$, which as a topological space is just $G \times X$ and the groupoid operations are $d(s, x) = x$, $r(s, x) = \theta_s(x)$, $(s, x) \cdot (t, y) = (st, y)$ whenever $\theta_t(y) = x$, and $(s, x)^{-1} = (s^{-1}, \theta_s(x))$. Bisections of this groupoid are sets of the form $\{s\} \times U$ where $U \subseteq X$ is some open subset. Moreover, the bisection \mathcal{O}_s is just $\{s\} \times X$ (here we are assuming $\mathcal{A}_s \neq \{0\}$ for all $s \in G$; otherwise we have $\mathcal{O}_s = \emptyset$ for all $s \in G$). Hence the map $s \mapsto \mathcal{O}_s$ is clearly injective.

4. ISOMORPHISM OF REDUCED ALGEBRAS

4.1. THE REGULAR REPRESENTATION. In this section we will study reduced C^* -algebras of Fell bundles over inverse semigroups and their relationship to the reduced C^* -algebra of the associated Fell line bundle.

We begin by briefly recalling some facts about the reduced C^* -algebra of Fell line bundles over (non necessarily Hausdorff) étale groupoids, referring the reader to [21] for more details, such as the construction of the reduced C^* -algebra in the non-étale case, although Renault only treats the Hausdorff case.

Throughout this section we suppose we are given an étale groupoid \mathcal{G} and a Fell line bundle L over \mathcal{G} .

For the time being we will also fix $x \in \mathcal{G}^{(0)}$. Observe that L_x is a one-dimensional C^* -algebra which is therefore isomorphic to \mathbb{C} , so we will henceforth tacitly identify L_x with \mathbb{C} .

Denoting by $\mathcal{G}_x = d^{-1}(x)$, let \mathcal{H}_x be the collection of all square-summable sections of L over \mathcal{G}_x , that is all functions $\xi : \mathcal{G}_x \rightarrow \dot{\bigcup}_{\gamma \in \mathcal{G}_x} L_\gamma$, such that $\xi(\gamma) \in L_\gamma$, for all $\gamma \in \mathcal{G}_x$, and such that

$$(4.1) \quad \sum_{\gamma \in \mathcal{G}_x} \xi(\gamma)^* \xi(\gamma) < \infty.$$

In regards to this sum notice that $\xi(\gamma)^* \xi(\gamma) \in L_{\gamma^* \gamma} = L_x$, which we are identifying with \mathbb{C} , as mentioned above.

It is well known that \mathcal{H}_x becomes a Hilbert space with inner product

$$(4.2) \quad \langle \xi | \eta \rangle = \sum_{\gamma \in \mathcal{G}_x} \xi(\gamma)^* \eta(\gamma) \quad \text{for all } \xi, \eta \in \mathcal{H}_x.$$

Recall from Definition 2.11 that $\mathcal{C}_c(L)$ denotes the space of all sections of the form $\sum_{i=1}^n f_i$ where each f_i is a compactly supported, continuous local section $f_i : U_i \rightarrow L$ over some open Hausdorff subset $U_i \subseteq \mathcal{G}$ (which can be taken to be a bisection of \mathcal{G}), extended by zero outside U_i and viewed as a global section $f_i : \mathcal{G} \rightarrow L$.

PROPOSITION 4.1. *For every $f \in \mathcal{C}_c(L)$ there exists a bounded linear operator $\pi_x(f)$ on \mathcal{H}_x such that*

$$\pi_x(f)\xi|_\gamma = \sum_{\gamma_1 \gamma_2 = \gamma} f(\gamma_1)\xi(\gamma_2) \quad \text{for all } \xi \in \mathcal{H}_x, \gamma \in \mathcal{G}_x.$$

Proof. Before we begin observe that any given summand “ $f(\gamma_1)\xi(\gamma_2)$ ” above lies in the same fiber of L , namely $L_{\gamma_1} L_{\gamma_2} = L_{\gamma_1 \gamma_2} = L_\gamma$.

Let us now argue that the sum in the statement does indeed converge by showing that only finitely many summands are nonzero. We begin by treating the case in which f is supported on a given bisection U of \mathcal{G} .

Given $\gamma \in \mathcal{G}_x$ suppose first that $r(\gamma) \notin r(U)$. Therefore there is no γ_1 in U such that $r(\gamma) = r(\gamma_1)$, and hence the above sum admits no nonzero summand, and $\pi_x(f)\xi|_\gamma = 0$.

On the other hand, if $r(\gamma) \in r(U)$, then $r(\gamma) = r(\gamma_1)$, for some $\gamma_1 \in U$, which is unique, given that U is a bisection. Setting $\gamma_2 = \gamma_1^{-1}\gamma$, we see that (γ_1, γ_2) is the unique pair satisfying $\gamma_1 \gamma_2 = \gamma$, with $\gamma_1 \in U$, and hence

$$(4.3) \quad \pi_x(f)\xi|_\gamma = f(\gamma_1)\xi(\gamma_2).$$

This shows that the sum in the statement converges as it has at most one nonzero summand.

Still supposing that f is supported on the bisection U , let us prove that $\pi_x(f)$ is well-defined and bounded. In order to do this we claim that the correspondence $\gamma \mapsto \gamma_2$, defined as above for $\gamma \in \mathcal{G}_x \cap r(U)$, is injective. In fact,

suppose that γ and γ' lie in $\mathcal{G}_x \cap r(U)$ and that both lead up to the same γ_2 . This means that there are γ_1 and γ'_1 in U such that

$$\gamma = \gamma_1\gamma_2 \quad \text{and} \quad \gamma' = \gamma'_1\gamma_2.$$

Therefore $d(\gamma_1) = r(\gamma_2) = d(\gamma'_1)$, which implies that $\gamma_1 = \gamma'_1$, again because U is a bisection. This obviously gives $\gamma = \gamma'$, concluding the proof of our claim.

Viewing γ_2 as a function of γ , as above, we then have for all $\xi \in \mathcal{H}_x$,

$$\begin{aligned} \sum_{\gamma \in \mathcal{G}_x} \|(\pi_x(f)\xi)(\gamma)\|^2 &= \sum_{\gamma \in \mathcal{G}_x \cap r(U)} \|(\pi_x(f)\xi)(\gamma)\|^2 \\ &= \sum_{\gamma \in \mathcal{G}_x \cap r(U)} \|f(\gamma\gamma_2^{-1})\xi(\gamma_2)\|^2 \leq \|f\|_\infty^2 \sum_{\gamma \in \mathcal{G}_x \cap r(U)} \|\xi(\gamma_2)\|^2 \leq \|f\|_\infty^2 \|\xi\|^2, \end{aligned}$$

where the last inequality holds because γ_2 is an injective function of γ . This shows that $\|\pi_x(f)\xi\| \leq \|f\|_\infty \|\xi\|$, and hence that $\pi_x(f)$ is well-defined and bounded with $\|\pi_x(f)\| \leq \|f\|_\infty$.

In order to treat the general case, let $f \in C_c(L)$. Then we may write f as a finite sum $f = \sum_{i=1}^n f_i$, where each f_i is supported in some bisection, in which case it is clear that $\pi_x(f) = \sum_{i=1}^n \pi_x(f_i)$, and we see that $\pi_x(f)$ is indeed bounded. ■

It is now easy to see that the correspondence $f \mapsto \pi_x(f)$ is a $*$ -representation of $C_c(L)$, and hence extends continuously to a representation, by abuse of language also denoted π_x , of $C^*(L)$ on \mathcal{H}_x .

For each $\gamma \in \mathcal{G}_x$ choose a unit vector v_γ in L_γ and set

$$\delta_\gamma = \begin{cases} v_\gamma & \text{if } \gamma' = \gamma, \\ 0 & \text{otherwise.} \end{cases}$$

As we shall see, the random choice of v_γ above will have little, if any effect in what follows. It is then easy to see that $\{\delta_\gamma\}_{\gamma \in \mathcal{G}_x}$ is an orthonormal basis of \mathcal{H}_x .

Among the elements of \mathcal{G}_x one obviously finds x itself, so δ_x is one of our basis elements.

PROPOSITION 4.2. *For every $x \in \mathcal{G}^{(0)}$ one has δ_x is a cyclic vector for π_x .*

Proof. Let $\gamma' \in \mathcal{G}_x$, and let U be a bisection of \mathcal{G} containing γ' . Choose $f \in C_c(L)$ supported on U and such that $f(\gamma') \neq 0$. We claim that $\pi_x(f)\delta_x$ is a nonzero multiple of $\delta'_{\gamma'}$. In order to see this, suppose that $\gamma \in \mathcal{G}$ is such that

$$\pi_x(f)\delta_x|_\gamma \neq 0.$$

By definition there exists at least one pair (γ_1, γ_2) such that $\gamma_1\gamma_2 = \gamma$, and $f(\gamma_1)\delta_x(\gamma_2) \neq 0$. This obviously implies that $\gamma_1 \in U$, and $\gamma_2 = x$. In particular this says that $\gamma_2 \in \mathcal{G}^{(0)}$ and hence $\gamma_1 = \gamma$, so we deduce that $\gamma \in U$. In addition

$$d(\gamma) = d(\gamma_1) = r(\gamma_2) = x.$$

It follows that the source of both γ and γ' coincide with x , and that both γ and γ' lie in U . Since U is a bisection we conclude that $\gamma = \gamma'$, thus showing that $\pi_x(f)\delta_x|_\gamma$ vanishes whenever $\gamma \neq \gamma'$.

In order to compute the value of $\pi_x(f)\delta_x|_{\gamma'}$ one observes that $\gamma' = \gamma'x$, so the unique pair (γ_1, γ_2) with $\gamma_1\gamma_2 = \gamma'$ and $\gamma_1 \in U$, according to equation (4.3), is $(\gamma_1, \gamma_2) = (\gamma', x)$. We then have

$$\pi_x(f)\delta_x|_{\gamma'} = f(\gamma')\delta_x(x).$$

Since this is nonzero we conclude that $\pi_x(f)\delta_x$ is indeed a nonzero multiple of $\delta_{\gamma'}$. This shows that any $\delta_{\gamma'}$ lies in the cyclic space spanned by δ_x , and hence that δ_x is a cyclic vector for π_x , thus concluding the proof. ■

PROPOSITION 4.3. *Given $x \in \mathcal{G}^{(0)}$ let ϕ_x be the state associated to the representation π_x and the cyclic vector δ_x , namely*

$$\phi_x(f) = \langle \pi_x(f)\delta_x | \delta_x \rangle$$

for all $f \in \mathcal{C}_c(L)$. Then $\phi_x(f) = f(x)$.

Proof. It is clearly enough to prove the statement under the assumption that f is supported on a bisection U of \mathcal{G} . We begin by claiming that

$$(4.4) \quad \pi_x(f)\delta_x|_x = f(x)\delta_x(x).$$

Suppose first that $r(x) \in r(U)$. So we have by equation (4.3) that $\pi_x(f)\delta_x|_x = f(\gamma_1)\delta_x(\gamma_2)$, where (γ_1, γ_2) is the unique pair of elements in \mathcal{G} such that $\gamma_1\gamma_2 = x$, and $\gamma_1 \in U$.

In order to find γ_1 and γ_2 , recall that $r(x) \in r(U)$, so there exists $\gamma \in U$ such that $r(\gamma) = r(x) = x$. Then $\gamma\gamma^{-1} = x$, and we see that $(\gamma_1, \gamma_2) = (\gamma, \gamma^{-1})$. Using brackets to indicate boolean value we have

$$\pi_x(f)\delta_x|_x = f(\gamma)\delta_x(\gamma^{-1}) = [x=\gamma^{-1}]f(\gamma)\delta_x(\gamma^{-1}) = [x=\gamma]f(x)\delta_x(x).$$

Notice that the last term above equals $f(x)\delta_x(x)$. While this is obvious when $f(x) = 0$, notice that if $f(x) \neq 0$ we must have $x \in U$, and then the unique element $\gamma \in U$ with $r(\gamma) = x$ is x itself, so $x = \gamma$, or equivalently $[x=\gamma] = 1$.

This proves equation (4.4) under the assumption that $r(x) \in r(U)$, so suppose now that $r(x) \notin r(U)$. In this case we have already seen that $\pi_x(f)\delta_x|_x$ vanishes, so it is enough to check that the right hand side of equation (4.4) also vanishes. But this is immediate since otherwise $x \in U$, whence $r(x) \in r(U)$.

Having finished the proof of equation (4.4) we have

$$\phi_x(f) = \langle \pi_x(f)\delta_x | \delta_x \rangle = (\pi_x(f)\delta_x|_x)\overline{\delta_x(x)} \stackrel{(4.4)}{=} f(x)\delta_x(x)\overline{\delta_x(x)} = f(x),$$

because $\delta_x(x)$ is a unit vector by construction. ■

4.2. THE ISOMORPHISM. Let $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$ be a semi-abelian Fell bundle over the inverse semigroup S and let \mathcal{E} be the restriction of \mathcal{A} to the idempotent semilattice $E(S)$. Therefore $C^*(\mathcal{E})$ is an abelian C^* -algebra whose spectrum will be denoted by X , so $C^*(\mathcal{E})$ is isomorphic to $\mathcal{C}_0(X)$.

For each $e \in E(S)$ we will identify \mathcal{A}_e as a closed two-sided ideal in $\mathcal{C}_0(X)$, and therefore there exists an open set $\mathcal{U}_e \subseteq X$, such $\mathcal{A}_e = \mathcal{C}_0(\mathcal{U}_e)$. By Proposition 3.5, for each $s \in S$, there exists a homeomorphism

$$\theta_s : \mathcal{U}_{s^*s} \rightarrow \mathcal{U}_{ss^*},$$

such that for every $f \in \mathcal{C}_0(\mathcal{U}_{ss^*})$, and every $a_s \in \mathcal{A}_s$,

$$(a_s^* f a_s)x = (a_s^* a_s)(x) f(\theta_s(x)) \quad \text{for all } f \in \mathcal{C}_0(\mathcal{U}_{ss^*}), a_s \in \mathcal{A}_s, x \in \mathcal{U}_{s^*s}.$$

Moreover, $s \mapsto \theta_s$ gives an action of S on X . Let \mathcal{G} be the groupoid of germs for this action as in Section 3.1 and let L be the Fell line bundle over \mathcal{G} associated to \mathcal{A} as constructed in Section 3.2.

It is our goal in this section to prove that the reduced C^* -algebra of \mathcal{A} is isomorphic to the reduced C^* -algebra of L .

LEMMA 4.4. *A necessary and sufficient condition for a given element $[s, x]$ in \mathcal{G} to lie in $\mathcal{G}^{(0)}$ is that there exists $e \in E(S)$ such that $e \leq s$, and $x \in \mathcal{U}_e$.*

Proof. Given e as in the statement notice that $se = e = ee$, and hence in view of the equivalence relation leading up to the notion of germs, we have

$$[s, x] = [e, x] \in \mathcal{G}^{(0)}.$$

Conversely, suppose that $[s, x] \in \mathcal{G}^{(0)}$. Then

$$[s, x] = [s, x]^{-1}[s, x] = [s^*, \theta_s(x)][s, x] = [s^*s, x].$$

Therefore, there exists $f \in E(S)$ such that $x \in \mathcal{U}_f$, and $sf = s^*sf$. Setting $e = s^*sf$, we get

$$\mathcal{U}_e = \mathcal{U}_{s^*s} \cap \mathcal{U}_f \ni x, \quad \text{and} \quad se = ss^*sf = sf = s^*sf = e,$$

so that $e \leq s$, and the proof is complete. ■

Given a pure state ϕ on $C^*(\mathcal{E}) = \mathcal{C}_0(X)$, we wish to identify the state $\tilde{\phi}$ on $C^*(\mathcal{A})$ described in Proposition 7.4 of [6]. Since the pure states on $\mathcal{C}_0(X)$ are precisely the point evaluations, there must exist some $x_0 \in X$, such that

$$\phi(f) = f(x_0) \quad \text{for all } f \in \mathcal{C}_0(X).$$

Recall from equation (7.3) of [6] that, given $e \in E(S)$, we let ϕ_e be the state on $\mathcal{A}_e = \mathcal{C}_0(\mathcal{U}_e)$ given by restriction of ϕ . Evidently

$$(4.5) \quad \phi_e \neq 0 \Leftrightarrow x_0 \in \mathcal{U}_e.$$

Using Proposition 5.5 of [6], the above can be used as a characterization of when ϕ is supported on \mathcal{A}_e .

If $s \in S$ and $e \in E(S)$ are such that $e \leq s$ and $\phi_e \neq 0$, then ϕ_e has a canonical extension $\tilde{\phi}_e^s$ to \mathcal{A}_s given by Proposition 6.1 of [6]. In order to describe it, choose $h \in \mathcal{A}_e$ such that $h(x_0) = 1$. Then it is clear that

$$\phi_e(f) = \phi_e(h)\phi_e(f) = \phi_e(hf),$$

so, by Proposition 6.3 of [6] we have

$$\tilde{\phi}_e^s(a) = \phi_e(ha) = (ha)(x_0) \quad \text{for all } a \in \mathcal{A}_s.$$

In order to compute the expression $(ha)(x_0)$, we use the isomorphism $\mathcal{A}_s \cong \mathcal{C}_0(L_s)$, that is, the Gelfand map constructed in Theorem 3.27, which we have so far also used implicitly in identifying $\mathcal{A}_e = \mathcal{C}_0(L_e) = \mathcal{C}_0(\mathcal{U}_e)$. Thus, we are going to identify $a \in \mathcal{A}_s$ with $\hat{a} \in \mathcal{C}_0(L_s)$. Under our identification we have

$$(ha)(x_0) = \hat{h}\hat{a}([e, x_0]) = \hat{h}\hat{a}([e, x_0]) = \hat{h}([e, x_0])\hat{a}([s, x_0]) = \dots$$

because the only way of writing $[e, x_0]$ as a product of elements in \mathcal{O}_e and \mathcal{O}_s is

$$[e, x_0] = [e, x_0][s, x_0].$$

Continuing with our computation of $(ha)(x_0)$ above, we have

$$\dots = 1 \cdot \hat{a}([s, x_0]) = \hat{a}([e, x_0]),$$

where the last equality is simply due to the fact that $[s, x_0] = [e, x_0]$. Returning with the identification between X and $\mathcal{G}^{(0)}$, we may then write

$$\tilde{\phi}_e^s(a) = a(x_0).$$

Suppose, on the other hand that s is such that there is no e in $\text{supp}(\phi)$ with $e \leq s$. By the characterization of $\text{supp}(\phi)$ given in equation (4.5), we deduce that there is no e in $E(S)$ such that $x_0 \in \mathcal{U}_e$ and $e \leq s$. Choosing any idempotent f such that $x \in \mathcal{U}_f$ we then claim that $[f, x_0] \notin \mathcal{O}_s$. In order to prove this suppose the contrary and hence $[f, x_0] = [s, x]$, for some $x \in \mathcal{U}_{s^*s}$. This would imply $x = x_0$ and the existence of $e \in E(S)$, with $x_0 \in \mathcal{U}_e$ and $fe = se$. Setting $e' = fe$ we would then have $x_0 \in \mathcal{U}_{e'}$ and $e' \leq s$, a situation which has been explicitly ruled out by our hypothesis. Therefore $[f, x_0] \notin \mathcal{O}_s$ and hence any $f \in \mathcal{C}_c(L_s)$ vanishes on $[f, x_0]$. In particular, for any $a_s \in \mathcal{A}_s$, we have

$$\hat{a}_s([f, x_0]) = 0 = \tilde{\phi}(a_s).$$

We have therefore proved the following:

PROPOSITION 4.5. *Let $x_0 \in X$ and let ϕ be the pure state on $\mathcal{C}_0(X)$ given by evaluation on x_0 . Then the canonical extension $\tilde{\phi}$ of ϕ to $C^*(\mathcal{A})$ given by Proposition 7.4 of [6] is such that*

$$\tilde{\phi}(a_s) = \hat{a}_s(x_0) \quad \text{for all } s \in S \text{ and } a_s \in \mathcal{A}_s.$$

Consider the canonical inclusion from $\mathcal{C}_c(L_s)$ into $\mathcal{C}_c(L) \subseteq C^*(L)$. An argument similar to that given in Proposition 3.14 of [5] shows that this inclusion is continuous for the sup-norm on $\mathcal{C}_c(L_s)$, and hence it extends to $\mathcal{C}_0(L_s) \rightarrow C^*(L)$. Moreover, these maps form a representation of the Fell bundle $\{\mathcal{C}_0(L_s)\}_{s \in S}$ into $C^*(L)$. Using the Gelfand isomorphisms $\mathcal{A}_s \cong \mathcal{C}_0(L_s)$ (see Theorem 3.27), we get a representation of \mathcal{A} into $C^*(L)$, which therefore integrates to a (surjective) $*$ -homomorphism

$$\Psi : C^*(\mathcal{A}) \rightarrow C^*(L).$$

In fact, this is essentially the same $*$ -homomorphism appearing in Theorem 2.13 for the case $\mathcal{B} = L$ (which is therefore an isomorphism if \mathcal{G} is Hausdorff or second countable).

THEOREM 4.6. *The homomorphism $\Psi : C^*(\mathcal{A}) \rightarrow C^*(L)$ above factors through the corresponding reduced C^* -algebras providing an isomorphism*

$$\Psi_r : C_r^*(\mathcal{A}) \rightarrow C_r^*(L).$$

Proof. For $x_0 \in X$ denote by π_{x_0} the representation of $C_r^*(L)$ given by Proposition 4.1. On the other hand, let $\tilde{\phi}$ be the state on $C^*(\mathcal{A})$ mentioned in Proposition 4.5 in terms of x_0 , and let ρ_{x_0} be the GNS representation of $C^*(\mathcal{A})$ associated to $\tilde{\phi}$.

We claim that $\pi_{x_0} \circ \Psi$ is a representation equivalent to ρ_{x_0} . Since δ_{x_0} is a cyclic vector for π_{x_0} by Proposition 4.2, we see that it is also cyclic for $\pi_{x_0} \circ \Psi$ simply because Ψ is onto. The associated vector state is given, on any $a_s \in \mathcal{A}_s$, by

$$\langle \pi_{x_0}(\Psi(a_s))\delta_{x_0} | \delta_{x_0} \rangle \stackrel{(4.3)}{=} \Psi(a_s)(x_0) \stackrel{(4.5)}{=} \tilde{\phi}(a_s).$$

The claim therefore follows from the uniqueness of the GNS representation. With respect to the respective reduced norms $\|\cdot\|_r$, we then have, for every $a \in C^*(\mathcal{A})$,

$$\|\Psi(a)\|_r = \sup_{x_0 \in X} \|\pi_{x_0}(\Psi(a))\| = \sup_{x_0 \in X} \|\rho_{x_0}(a)\| = \|a\|_r,$$

from where the result readily follows. ■

Let (\mathcal{G}, Σ) be the twisted groupoid associated to \mathcal{A} as in Section 2.3. Since the reduced C^* -algebra of (\mathcal{G}, Σ) is, by definition, the reduced C^* -algebra of L , the above result also gives a canonical isomorphism

$$C_r^*(\mathcal{A}) \cong C_r^*(\mathcal{G}, \Sigma).$$

5. APPLICATION: CARTAN SUBALGEBRAS

In this section we will apply the results so far developed to prove part of Renault’s Theorem [22] on the characterization of Cartan subalgebras of C^* -algebras.

Based on previous work by Vershik, Feldman, Moore, and Kumjian [24], [8], [9], [13], Renault gave the following:

DEFINITION 5.1 ([22], Definition 5.1). A closed $*$ -subalgebra B of a separable C^* -algebra A is a *Cartan subalgebra* if:

- (i) B contains an approximate unit of A ;
- (ii) B is maximal abelian in A ;
- (iii) B is regular in the sense that the normalizer of B in A , namely

$$N(B) = \{a \in A : aBa^* \subseteq B, a^*Ba \subseteq B\},$$

generates A ; and

- (iv) there exists a faithful conditional expectation from A to B .

Renault has proved ([22], Theorem 5.6) that, whenever B is a Cartan subalgebra of A , there exists a twisted, essentially principal, étale, Hausdorff groupoid (\mathcal{G}, Σ) , such that A is isomorphic to $C_r^*(\mathcal{G}, \Sigma)$ via an isomorphism which carries B onto $C_0(\mathcal{G}^{(0)})$.

In [6] the second named author has recently introduced a generalization of the notion of Cartan subalgebras to include situations in which the *maximal abelian algebra B is no longer abelian*. To describe this result we need to recall that, given a closed $*$ -subalgebra B of a C^* -algebra A , a *virtual commutant of B in A* is a B -bimodule map

$$\phi : J \rightarrow A,$$

where J is an ideal in B . Virtual commutants are akin to elements in the relative commutant $B' \cap A$, since given any $a \in B' \cap A$, the following map is a virtual commutant with domain $J = B$:

$$\phi_a : b \in B \mapsto ab \in A.$$

DEFINITION 5.2 ([6], Definition 12.1). A closed $*$ -subalgebra B of a separable C^* -algebra A is said to be a *generalized Cartan subalgebra* if it satisfies (i), (iii) and (iv) of Definition 5.1 and, instead of (ii), it is required that:

- (ii)' every virtual commutant of B in A has its range contained in B .

In Theorem 14.5 of [6] it is proved that if B is a generalized Cartan subalgebra of A , there exists a Fell bundle $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$, over a countable inverse semigroup S , such that A is isomorphic to $C_r^*(\mathcal{A})$ via an isomorphism which carries B onto $C_r^*(\mathcal{E})$, where \mathcal{E} is the restriction of \mathcal{A} to the idempotent semilattice of S . We observe that by Proposition 4.3 of [6] there is no difference between $C_r^*(\mathcal{E})$ and $C^*(\mathcal{E})$, but we use the former to standardize our notation.

In the remainder of this section we will show how Theorem 14.5 of [6] combines with the results of this paper to prove most of the conclusions of Renault's Theorem. We therefore fix, throughout, a separable C^* -algebra A and a Cartan subalgebra B , according to Definition 5.1.

Given that B is abelian it is easy to prove that property of Definition 5.1(ii) implies Definition 5.2(ii)' (see Proposition 9.8 of [6]), so B is also a generalized Cartan subalgebra. By Theorem 14.5 of [6] we therefore deduce that there exists a countable inverse semigroup S and a Fell bundle $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$, such that $A \simeq C_r^*(\mathcal{A})$, and $B \simeq C_r^*(\mathcal{E})$, as above.

If $e \in E(S)$ then, by Corollary 8.9 of [6], \mathcal{A}_e identifies with an ideal of B , and hence \mathcal{A}_e is commutative. In other words, \mathcal{A} is a semi-abelian Fell bundle. Moreover by the construction of the Fell bundle in Theorem 14.5 of [6], it is also saturated (see Proposition 13.3 of [6]).

Let \mathcal{G} be the étale groupoid associated to \mathcal{A} as in Section 3.1, and let L be the Fell line bundle over \mathcal{G} associated to \mathcal{A} as constructed in Section 3.2. As observed in Section 2.3 this is tantamount to saying that we have a twisted groupoid (\mathcal{G}, Σ) . We conclude from Theorem 4.6 that

$$A \simeq C_r^*(\mathcal{A}) \simeq C_r^*(L) = C_r^*(\mathcal{G}, \Sigma),$$

where the isomorphisms involved restrict to give

$$B \simeq C^*(\mathcal{E}) \simeq C_0(\mathcal{G}^{(0)}).$$

In order to get the whole of Renault's result we still need to show that \mathcal{G} is Hausdorff and essentially principal but, unfortunately, we cannot offer an argument entirely based on our results which leads to the conclusion that \mathcal{G} is Hausdorff.

Being unable to fully prove Renault's result we conclude this section with some remarks on the Hausdorff question.

Of course, should we know that \mathcal{G} is Hausdorff, it would quickly follow that \mathcal{G} is essentially principal by Proposition 4.2 of [22]. However we remark that there are non-Hausdorff, essentially principal, étale groupoids for which $C_0(\mathcal{G}^{(0)})$ is *not* maximal abelian in $C_r^*(\mathcal{G})$ ([7], Proposition 2.4), and hence Proposition 4.2 of [22] is not valid for non-Hausdorff groupoids.

Returning to the question of whether the groupoid \mathcal{G} associated to \mathcal{A} is Hausdorff, one might wonder if it is possible to answer it affirmatively via some quick argument based on the postulated existence of the conditional expectation. (In Proposition 5.4 of [22] Renault does achieve this in a non-trivial way.) An indication that lured us in this direction is the fact that Hausdorff étale groupoids do possess a very standard conditional expectation obtained by restricting functions to the unit space ([22], Proposition 4.3), a process which is not available in the non-Hausdorff case.

However we have found an example which proves that the existence of the conditional expectation, by itself (without the assumption that the groupoid be essentially principal), is not enough to guarantee that the groupoid is Hausdorff.

PROPOSITION 5.3. *There exists a non-Hausdorff, étale groupoid \mathcal{G} , such that $C_0(\mathcal{G}^{(0)})$ is the image of a faithful conditional expectation defined on $C_r^*(\mathcal{G})$.*

Proof. Partly as an illustration of our methods, we will first construct a semi-abelian Fell bundle over an inverse semigroup, which will then give rise by Section 3.1 to the groupoid we need.

Consider the commutative semigroup $S = \{e, 1, \sigma\}$ endowed with the multiplication operation

\cdot	e	1	σ
e	e	e	e
1	e	1	σ
σ	e	σ	1

Notice that e is a zero-element and 1 is a unit for S . It is easy to see that S is an inverse semigroup and that all of its elements are self-adjoint.

In presenting our Fell bundle $\mathcal{A} = \{\mathcal{A}_s\}_{s \in S}$, each fiber \mathcal{A}_s will be taken to be a subset of the cartesian product $\mathcal{C}([-1, 1]) \times S$, as follows:

- (i) $\mathcal{A}_e = \mathcal{C}_0[-1, 0) \times \{e\}$,
- (ii) $\mathcal{A}_1 = \mathcal{C}[-1, 1] \times \{1\}$,
- (iii) $\mathcal{A}_\sigma = \mathcal{C}[-1, 1] \times \{\sigma\}$.

The Banach space structure of each \mathcal{A}_s is that of its first coordinate, while the multiplication and involution on \mathcal{A} are defined coordinatewise. The inclusions $j_{1,e}$ and $j_{\sigma,e}$ are given by

$$j_{1,e}(f, e) = (f, 1) \quad \text{and} \quad j_{\sigma,e}(f, e) = (f, \sigma) \quad \text{for all } f \in \mathcal{C}_0[-1, 0).$$

Clearly $C_r^*(\mathcal{E})$ identifies with $\mathcal{A}_1 = \mathcal{C}[-1, 1]$, so that $\mathcal{G}^{(0)} = [-1, 1]$, and one may check that the groupoid of germs \mathcal{G} consists of the following distinct elements:

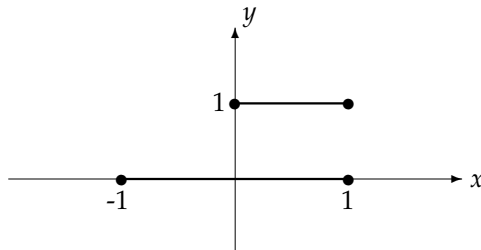
$$[e, x] \quad \text{for } x \in [-1, 0); \quad [1, x] \quad \text{for } x \in [0, 1]; \quad [\sigma, x] \quad \text{for } x \in [0, 1].$$

Incidentally, notice that $[e, x] = [1, x] = [\sigma, x]$ for all $x \in [-1, 0)$. The topology of \mathcal{G} is such that the open bisections

$$\mathcal{O}_1 = \{[1, x] : x \in [-1, 1]\} = \mathcal{G}^{(0)} \quad \text{and} \quad \mathcal{O}_\sigma = \{[\sigma, x] : x \in [-1, 1]\}$$

are each canonically homeomorphic to $[-1, 1]$, but \mathcal{G} is not Hausdorff since it is impossible to separate the germs $[1, 0]$ and $[\sigma, 0]$ from one another.

The reduced C^* -algebra of \mathcal{G} , which is isomorphic to the reduced C^* -algebra of \mathcal{A} , may be described as the algebra of all continuous complex-valued functions on the topological subspace X of \mathbb{R}^2 given by $X = ([-1, 1] \times \{0\}) \cup ([0, 1] \times \{1\})$.



The natural identification of fibers within $\mathcal{C}(X)$, say $\pi_s : A_s \rightarrow \mathcal{C}(X)$ for $s \in S$, may be given as follows: for each $f \in \mathcal{C}_0[-1, 0)$, one puts

$$\pi_e(f, e) |_{(x,y)} = \begin{cases} f(x) & \text{if } x < 0, \text{ and } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $f \in \mathcal{C}_0[-1, 1]$,

$$\pi_\sigma(f, \sigma) |_{(x,y)} = \begin{cases} f(x) & \text{if } y = 0, \\ -f(x) & \text{if } x \geq 0, \text{ and } y = 1, \end{cases}$$

while

$$\pi_1(f, 1) |_{(x,y)} = f(x) \quad \text{for all } (x, y) \in X.$$

The subalgebra $\mathcal{C}_0(\mathcal{G}^{(0)})$, or equivalently $C_r^*(\mathcal{E}) (= \pi_1(A_1))$, may be described as the subalgebra of $\mathcal{C}(X)$ formed by the functions which do not depend on the second variable y .

It therefore remains to show that there does indeed exist a conditional expectation as required. But this may be simply given, for every $g \in \mathcal{C}(X)$, by

$$E(g) |_{(x,y)} = \begin{cases} g(x, 0) & \text{if } x < 0, \text{ and } y = 0, \\ p(x)g(x, 0) + (1 - p(x))g(x, 1) & \text{if } x \geq 0, \end{cases}$$

where $p : [0, 1] \rightarrow [0, 1]$ is any continuous function such that $p(0) = 1$. As long as the set of points x where $p(x) \in (0, 1)$ is dense in $[0, 1]$, one can prove that E is a faithful conditional expectation as desired. ■

6. CONCLUDING REMARKS

In this work we have studied a natural connection between Fell bundles over étale groupoids and inverse semigroups. As we have seen in Section 2.2, Fell bundles over étale groupoids give rise to Fell bundles over inverse semigroups, but the latter seems to be slightly more general objects. This idea essentially appears in the unpublished work [23] by Nándor Sieben, and we would like to thank him for providing access to his work. In [6] the second named author gives a first application to Fell bundles over inverse semigroups proving that they provide examples of noncommutative Cartan subalgebras. Of course, this gives a special reason to study those objects and we are strongly inspired by this work and some of his references, especially the articles [13], [22] by Kumjian and Renault.

Although arbitrary Fell bundles over inverse semigroups are more general than over étale groupoids, our main result shows that in the semi-abelian case both can be "disintegrated" and are essentially equivalent to twisted étale groupoids or, equivalently, Fell line bundles (over étale groupoids). For groupoids, this result is proved in Theorem 5.6 of [2], but it can be essentially also

obtained from our main result by first "integrating" a given semi-abelian Fell bundle (in [2] those Fell bundles are called just "abelian") over an étale groupoid as in Section 2.2, obtaining in this way a semi-abelian Fell bundle over an inverse semigroup and then "disintegrating" (that is, applying our main result to) it as in Section 3 yielding the desired twisted étale groupoid.

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