J. OPERATOR THEORY
67:1(2012), 237–256

WEYL SEQUENCES AND THE ESSENTIAL SPECTRUM OF SOME JACOBI OPERATORS

MARCIN MOSZYŃSKI

Communicated by William Arveson

ABSTRACT. This work is devoted to finding Weyl sequences for some Jacobi operators and, consequently, describing some of the essential spectra subsets. In particular we study those problems for perturbations of the Jacobi operators with the k-th weight given by k^α or by $k^\alpha + b_k$, where $\alpha \in (0;1)$ and $\{b_k\}$ is a 2-periodic sequence.

KEYWORDS: *Jacobi matrix (operator), spectrum, essential spectrum, approximative spectrum, Weyl sequence, perturbations.*

MSC (2000): 47B36, 47B25, 47A10, 81Q10.

1. INTRODUCTION

The approximative spectrum method and its particular variant, Weyl sequences method, are elementary and, we can say, classical tools of spectral analysis. They can help in describing the spectrum and, respectively, the essential spectrum of operators. The price paid for the formal simplicity of those methods is that one can give no direct answer on some more delicate spectral questions, e.g., on absolute or singular continuity of the spectrum. Also "input information" on the operator sufficient to construct an approximative eigenvector is not so often accessible in practice. However, there are various examples of operators for which Weyl sequences can be constructed and the spectral information obtained in this way seems not to be easy to gain with the use of some "more advanced" methods. For Jacobi operators case we can mention for instance the results of [10], [11], [15].

This work is devoted to the problem of finding Weyl sequences for some special classes of Jacobi operators and, consequently, describing some subsets of their essential spectra.

We proceed here with the following idea. The construction of a Weyl sequence can be relatively simple, when we know something about the asymptotics of a generalized eigenvector of the operator. Thus, as the first step, we consider such an operator J_0 and $\lambda \in \mathbb{R}$ for which such asymptotics is known. Then we construct a possibly large family of various Weyl sequences for J_0 and λ. In the second step we try to perturb J_0 by a Jacobi operator P — we formulate some sufficient conditions for P guaranteeing that one of Weyl sequences just constructed for J_0 is also a Weyl sequence for the perturbation J of J_0 by P. Such method allows us to find Weyl sequences for quite "complicated" J, for which no asymptotic information on generalized eigenvectors is usually known. As we shall see, this method can often occur much more fruitful than the classical Weyl compact perturbation method for the essential spectrum.

This general idea is realized here for a class of Jacobi operators described in Theorem 5.8, the main result of this paper. It seems to work, in particular, in many cases where the various currently known spectral methods (e.g., subordination theory methods) cannot be used.

A special attention is devoted here to two particular cases. The first — when the weights w_{0k} of the unperturbed operator J_0 are given by $w_{0k} = k^\alpha$, and the second — more general, when $w_{0k} = k^\alpha + b_k$, where $\alpha \in (0;1)$ and $\{b_k\}$ is a 2-periodic sequence (see Corollaries 5.10 and 5.11). Note that some examples of the first case were studied in [10], and a construction of Weyl sequence similar to our construction was shown there.

The paper is organized as follows.

In Section 1 we give some more detailed description of the idea presented above, and we make an estimate for the Weyl sequences construction for J_0 (from the first step). We also find "the optimal construction" from the point of view of the chosen estimate (see Lemma 2.3 and Remarks 2.4).

The main results of Section 2 are Proposition 3.1 and Theorem 3.2. The first concerns asymptotics of generalized eigenvectors for such J_0 and λ, for which the transfer matrix sequence is in the so-called class H. The second result describes a certain family of Weyl sequences for the above J_0 and λ, this way closing the first step.

In Section 3 we check the H class assumptions for the two examples of J_0 with the weights mentioned earlier.

Finally, Section 4 deals with the second step and it contains the main spectral results of the paper, the general, as well as some more particular ones.

Let us explain now some notation and terminology used in this paper. Sequences are denoted here in two ways: $\{x_n\}_{n \geqslant n_0}$ or $\{x(n)\}_{n \geqslant n_0}$. This second, „functional", way is used mostly for sequences which we treat as elements of some vector space, e.g. of $l(\mathbb{N})$ — the vector space of all complex sequences on $\mathbb{N}$ ($0 \notin \mathbb{N}$ here) or $l^2(\mathbb{N})$ — our fundamental Hilbert space of square-summable complex sequences on $\mathbb{N}$. When the starting subscript n_0 equals 1 and the choice of the letter used to denote the variable (index) is clear, then we often simplify the notation and write $\{x_n\}$ or $\{x(n)\}$. The symbol Δ denotes the discrete derivative of sequences, $(\Delta x)(n) = x(n+1) - x(n)$ for $x = \{x(n)\}_{n \geqslant n_0}$ and $n \geqslant n_0$. By $\rightarrow$

we denote the convergence of sequences in the usual sense, and by $\xrightarrow{w}$ we denote the weak convergence of a sequence in a Hilbert space.

For a given real weight sequence $\{w_k\}$ and a real diagonal sequence $\{q_k\}$ we consider the Jacobi matrix

$$\begin{pmatrix} q_1 & w_1 & & & \\ w_1 & q_2 & w_2 & & \\ & w_2 & q_3 & w_3 & \\ & & w_3 & q_4 & \ddots \\ & & & \ddots & \ddots \end{pmatrix},$$

which we identify with the *formal Jacobi operator* $\mathcal{J}$ acting in $l(\mathbb{N})$. So, for any complex sequence $x = \{x(k)\}$

$$(\mathcal{J}x)(k) := w_{k-1}x(k-1) + q_k x(k) + w_k x(k+1), \quad k \in \mathbb{N}, \tag{1.1}$$

with the convention that $w_k = x(k) = 0$, if $k < 1$. As usual, the Jacobi operator J is the appropriate maximal operator acting in $l^2(\mathbb{N})$, i.e., the restriction of the formal operator $\mathcal{J}$ to the maximal domain $D(J) := \{x \in l^2(\mathbb{N}) : \mathcal{J}x \in l^2(\mathbb{N})\}$.

As it has just been announced, we shall usually denote by J_0 a Jacobi operator treated as „unperturbed" operator, and by J, a perturbation of J_0. The weight and diagonal sequences for J_0 will be denoted by $\{w_{0k}\}$ and $\{q_{0k}\}$, respectively, and the appropriate formal Jacobi operator, by $\mathcal{J}_0$.

We use the symbol $\asymp$ to express the possibility of the global mutual estimate between two functions, i.e., for some set Y and $f, g : Y \to \mathbb{C}$ we write $f \asymp g$ if and only if there exist two constants $0 < c < C < +\infty$ such that for any $y \in Y$

$$c < \left|\frac{f(y)}{g(y)}\right| < C.$$

We shall use also "$f(y) \asymp g(y), y \ldots$" to express the above condition for the functions restricted to the set of y satisfying the condition "...", or simply "$f(y) \asymp g(y)$", when the choice of Y is clear (e.g., sometimes we shall write $a_k \asymp b_k$, omitting "$k \in \mathbb{N}$").

2. INITIAL IDEAS AND ESTIMATES FOR J_0

We present here a general idea how to find Weyl sequences for some Jacobi operators, and we show some preliminary estimates, which could help in practical use of this idea.

Let us start with recalling the main definitions and abstract spectral results. Let A be a linear operator acting in Banach space X with the domain $D(A)$, let $\lambda \in \mathbb{C}$, and let $\{f_n\}_{n \geqslant n_0}$ be a sequence of non-zero vectors from $D(A)$. We call $\{f_n\}_{n \geqslant n_0}$ *approximative eigenvector* for A and λ if and only if $\|(A-\lambda)f_n\| / \|f_n\| \to 0$. By a *Weyl sequence* for A and λ we call an approximative eigenvector $\{f_n\}_{n \geqslant n_0}$, such that $f_n/\|f_n\| \xrightarrow{w} 0$. We shall use the following classical results (for (ii) see, e.g., Chapter IX of [2]).

PROPOSITION 2.1. *Let X be a Hilbert space and A a self-adjoint operator in X. Then:*

(i) *$\lambda \in \sigma(A)$ if and only if there exists an approximative eigenvector for A and λ;*
(ii) *$\lambda \in \sigma_{\mathrm{ess}}(A)$ if and only if there exists a Weyl sequence for A and λ.*

Our general idea is the following. We consider an "unperturbed" Jacobi operator J_0 and its *generalized eigenvector* u for some $\lambda \in \mathbb{R}$, i.e., $u = \{u(k)\}_{k\geqslant 1} \in l(\mathbb{N})$ satisfying

$$((\mathcal{J}_0 - \lambda)u)(k) = 0, \quad k \geqslant 2. \tag{2.1}$$

The assumption that J_0 is „unperturbed" means also that it is „not very complicated", so that we can find some asymptotic information on the sequence u. Having the above, we make two steps.

Step 1. We construct a wide family of Weyl sequences for J_0 and λ by modifying the generalized eigenvector u.

Step 2. We find some conditions on a perturbing Jacobi operator P guaranteeing that there exists such an element of the family constructed in Step 1, which is a Weyl sequence also for the Jacobi operator J formally equal to „$J_0 + P$". Roughly speaking, such an operator P has to be „small" on one of Weyl sequences from the above family.

To make Step 1, we shall search first for an approximative eigenvector $\{f_n\}_{n\geqslant n_0}$ of the form $f_n = \varphi_n \cdot u$, where $\{\varphi_n\}_{n\geqslant n_0}$ is an appropriately chosen „modifying function sequence" with terms in $l(\mathbb{N})$. Let us omit for a moment the dependence on the subscript n, and consider only one modifying function $\varphi = \{\varphi(k)\}_{k\geqslant 1} \in l(\mathbb{N})$. It can be easily calculated that

$$(\mathcal{J}_0 - \lambda)(\varphi \cdot u) = \varphi \cdot [(\mathcal{J}_0 - \lambda)u] + M_\varphi u, \tag{2.2}$$

where $M_\varphi : l(\mathbb{N}) \to l(\mathbb{N})$ is the linear operator given by the matrix

$$\begin{pmatrix} 0 & \alpha(1) & & & \\ -\alpha(1) & 0 & \alpha(2) & & \\ & -\alpha(2) & 0 & \alpha(3) & \\ & & -\alpha(3) & 0 & \ddots \\ & & & \ddots & \ddots \end{pmatrix},$$

with $\alpha = \{\alpha(k)\}$ defined by

$$\alpha(k) := (\Delta\varphi)(k) w_{0k}, \quad k \in \mathbb{N}.$$

We shall call α *the weight sequence* of M_φ. Observe, that α is simply the product of the weight sequence of J_0 and $\Delta\varphi$, so, in particular, it is not dependent on u, λ

and on the diagonal sequence of J_0. The formula (2.2) is very convenient for our goal, since by (2.1) we have

$$((\mathcal{J}_0 - \lambda)(\varphi \cdot u))(k) = (M_\varphi u)(k) \tag{2.3}$$

for any $k \geqslant 2$.

For φ such that $0 < \|\varphi \cdot u\| < +\infty$ we denote

$$\rho(\varphi) := \frac{\|(\mathcal{J}_0 - \lambda)(\varphi \cdot u)\|^2}{\|\varphi \cdot u\|^2}, \quad \widetilde{\rho}(\varphi) := \frac{\|M_\varphi u\|^2}{\|\varphi \cdot u\|^2}.$$

By (2.3)

$$\rho(\varphi) - \widetilde{\rho}(\varphi) = \frac{|\varphi(1)\tau + w_{01}(\Delta\varphi)(1)u(2)|^2 - |w_{01}(\Delta\varphi)(1)u(2)|^2}{\|\varphi \cdot u\|^2}, \tag{2.4}$$

where $\tau = ((\mathcal{J}_0 - \lambda)u)(1)$. By definition, to get an approximative eigenvector $\{f_n\}_{n \geqslant n_0}$ of the form $f_n = \varphi_n \cdot u$ it is enough to find $\{\varphi_n\}_{n \geqslant n_0}$ satisfying

$$\rho(\varphi_n) \to 0 \tag{2.5}$$

(note, that $f_n \in D(J_0)$ automatically, since both f_n and $(\mathcal{J}_0 - \lambda)f_n$ are in $l^2(\mathbb{N})$ in such case). Using (2.4) we see, that under the extra assumption

$$\varphi_n(1) = \varphi_n(2) = 0 \quad \text{for } n \text{ large enough} \tag{2.6}$$

one can replace (2.5) by

$$\widetilde{\rho}(\varphi_n) \to 0 \tag{2.7}$$

(note that (2.5) and (2.7) are equivalent under some weaker assumptions, which could be easily formulated using (2.4)).

Let us formulate now a lemma containing our basic estimate for $\widetilde{\rho}$. For $d \in \mathbb{N}$ denote by F_d the function defined as follows: $F_d : \mathbb{R}^{d+1} \setminus \{0\} \to \mathbb{R}$,

$$F_d(t) := \frac{t_0^2 + \sum\limits_{k=1}^{d} (t_k - t_{k-1})^2 + t_d^2}{\sum\limits_{k=0}^{d} t_k^2}, \quad t = (t_0, \ldots, t_d).$$

LEMMA 2.2. *Let $u, \varphi \in l(\mathbb{N})$ with φ being real.*

(i) *If a non-negative scalar sequence $\{\eta_k\}$ and a constant $0 \leqslant C < +\infty$ are such, that*

$$|w_{0k}|(|u(k)| + |u(k+1)|) \leqslant C\eta_k, \quad k \in \mathbb{N}, \tag{2.8}$$

then

$$\|M_\varphi u\|^2 \leqslant 2C^2 \sum_{k=1}^{+\infty} |(\Delta\varphi)(k)|^2 \eta_k^2.$$

(ii) *If a non-negative scalar sequence $\{\xi_k\}_{k \geqslant l_0}$ and a constant $0 < c < +\infty$ are such, that*

$$|u(k)| \geqslant c\xi_k, \quad k \geqslant l_0, \tag{2.9}$$

then

$$\|\varphi \cdot u\|^2 \geqslant c^2 \sum_{k \geqslant l_0} |\varphi(k)|^2 \xi_k^2.$$

(iii) *Suppose that the assumptions of the parts* (i) *and* (ii) *hold. If moreover* $\{\eta_k\}$ *is increasing,* $\{\xi_k\}_{k \geqslant l_0}$ *is decreasing,* $\varphi \neq 0$*, and* $\operatorname{supp}\varphi \subset I$*, where* $I = [p; p+d] \cap \mathbb{N}$ *for some* $p, d \in \mathbb{N}$ *such that* $p \geqslant \max\{2; l_0\}$*, then*

$$\widetilde{\rho}(\varphi) \leqslant \frac{2C^2}{c^2} \Big(\frac{\eta_{p+d}}{\xi_{p+d}}\Big)^2 F_d(\varphi(p), \ldots, \varphi(p+d)). \tag{2.10}$$

Proof. To prove (i) we use $|a+b|^2 \leqslant 2(|a|^2 + |b|^2)$, and we obtain

$$\|M_\varphi u\|^2 = |(\Delta\varphi)(1) w_{01} u(2)|^2 + \sum_{k=2}^{+\infty} |-(\Delta\varphi)(k-1) w_{0(k-1)} u(k+1) + (\Delta\varphi)(k) w_{0k} u(k+1)|^2$$

$$\leqslant 2\sum_{k=1}^{+\infty} |(\Delta\varphi)(k) w_{0k} u(k)|^2 + 2\sum_{k=1}^{+\infty} |(\Delta\varphi)(k) w_{0k} u(k+1)|^2 \leqslant 2C^2 \sum_{k=1}^{+\infty} |(\Delta\varphi)(k)|^2 \eta_k^2.$$

The part (ii) is obvious. To obtain the part (iii), using (i) and (ii) we estimate

$$\widetilde{\rho}(\varphi) \leqslant \frac{2C^2}{c^2} \frac{\sum_{k=p-1}^{p+d} |(\Delta\varphi)(k)|^2 \eta_k^2}{\sum_{k=p}^{p+d} |\varphi(k)|^2 \xi_k^2}.$$

Now, by the monotonicity of both estimating sequences, we get the assertion. ∎

Certainly the estimates leading to the result (iii) above for some choices of J_0, u, φ can be far from optimal. Nevertheless, for many interesting cases they are good enough, and moreover, the final result, the RHS of (2.10), has a rather simple form. In particular, the function F_d depends only on the number d being the length of the interval I supporting the modifying function φ. If we want to base our search on this result, we should only optimize the choice of φ for any given d (i.e., find the inf of the values of F_d).

LEMMA 2.3. *The function* F_d *reaches its infimum on the unit sphere* $S_d = \Big\{t \in \mathbb{R}^{d+1} : \sum_{k=0}^{d} t_k^2 = 1\Big\}$ *and this infimum equals* $2 - 2\cos(\pi/(d+2))$.

Proof. F_d is a continuous homogeneous function of degree 0, so the infimum of its values is reached on the unit sphere S_d. For $t \in S_d$ we have

$$F_d(t) = t_0^2 + \sum_{k=1}^{d} (t_k^2 - 2t_{k-1}t_k + t_{k-1}^2) + t_d^2 = 2 - 2\sum_{k=1}^{d} t_{k-1}t_k$$

$$= 2 - \Big(\sum_{k=1}^{d} t_{k-1}t_k + \sum_{k=0}^{d-1} t_{k+1}t_k\Big) = 2 - \langle H_{d+1}t, t\rangle,$$

where $\langle\cdot\,,\,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^{d+1}$ and H_{d+1} is the discrete free Shrödinger operator in $\mathbb{R}^{d+1}$, i.e., it is the operator given by the $(d+1)\times(d+1)$ matrix

$$\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & 1 & 0 \end{pmatrix}.$$

Hence

$$\inf_{t\in S_d} F_d(t) = 2 - \sup_{t\in S_d}\langle H_{d+1}t,t\rangle = 2-\mu(H_{d+1}),$$

where $\mu(H_{d+1})$ is the maximal eigenvalue of H_{d+1}, which equals to $2\cos(\pi/(d+2))$ (see e.g. [3], [4]). ∎

REMARKS 2.4. (i) The exact value $2-2\cos(\pi/(d+2))$ of the minimum for F_d is not very important for us. In fact, the information that

$$\min_{t\in S_d} F_d(t) \asymp \frac{1}{d^2}, \quad d\in\mathbb{N} \tag{2.11}$$

can be more convenient.

(ii) It is not necessary to choose the optimal $t = (t_0\dots,t_d)$ (i.e., t with the minimal value of $F_d(t)$) for the entries $(\varphi(p),\dots,\varphi(p+d))$ (from the part (iii) of Lemma 2.2). A commonly used choice is the following "triangle choice":

$$t_0 = t_k - t_{k-1} = t_{d-k} - t_{d-k+1} = t_d = \nu, \quad k = 1,\dots,\left[\frac{d}{2}\right],$$

where ν is an arbitrary positive number. Let us denote the above $t\in\mathbb{R}^{d+1}$ by t_{trig_d} (e.g., with ν such that $t_{\mathrm{trig}_d}\in S_d$). One can easily check that for our purposes this choice is not worse than the optimal one, since we also have $F_d(t_{\mathrm{trig}_d}) \asymp 1/d^2$, $d\in\mathbb{N}$. Such "triangle choice" for Weyl sequence construction can be found for instance in [10], [11].

(iii) In the part (iii) of Lemma 2.2 it is assumed that $\{\eta_k\}$ is increasing and $\{\xi_k\}_{k\geqslant l_0}$ is decreasing. Observe, that assuming other kinds of monotonicity for these two sequences we shall also obtain some estimates similar to (2.10), with slightly changed part '(η_{p+d}/ξ_{p+d})'.

Summing up the above two lemmas and Remark 2.4(i) we immediately obtain the following general result concerning the existence of approximative eigenvector $\{f_n\}_{n\geqslant n_0}$ (for the unperturbed Jacobi operator J_0) determined by a modifying function sequence $\{\varphi_n\}_{n\geqslant n_0}$ with $\operatorname{supp}\varphi_n\subset I_n$, where

$$I_n = [p_n; p_n+d_n]\cap\mathbb{N} \tag{2.12}$$

for some $p_n, d_n\in\mathbb{N}$.

PROPOSITION 2.5. *Let u be a generalized eigenvector for J_0 and $\lambda \in \mathbb{R}$. Assume that a non-negative increasing scalar sequence $\{\eta_k\}$, a positive decreasing scalar sequence $\{\xi_k\}_{k \geqslant l_0}$ and sequences $\{p_n\}_{n \geqslant n_0}$, $\{d_n\}_{n \geqslant n_0}$ of natural numbers with $p_n \geqslant \max\{2; l_0\}$ for large n, are such that* (2.8), (2.9) *hold with some positive constants c, C and*

$$\frac{\eta_{p_n+d_n}}{d_n \xi_{p_n+d_n}} \to 0. \tag{2.13}$$

Then there exists an approximative eigenvector $\{f_n\}_{n \geqslant n_0}$ for J_0 and λ such that $\|f_n\| = 1$ and $\operatorname{supp} f_n \subset I_n$ for $n \geqslant n_0$, where I_n is given by (2.12)*. If, moreover, $p_n \to +\infty$, then the above $\{f_n\}_{n \geqslant n_0}$ is a Weyl sequence for J_0 and λ.*

Proof. We only need to prove the last sentence of the assertion, i.e., that $f_n \xrightarrow{\mathrm{w}} 0$. And this follows from the well-known criterion of week convergence in Hilbert space (see, e.g., [14]). ∎

REMARKS 2.6. (i) By Remark 2.4(ii), constructing the modifying function sequence $\{\varphi_n\}_{n \geqslant n_0}$ we can choose the entries from $t_{\mathrm{trig}_{d_n}}$ for the values of φ_n on I_n instead of making the „optimal" choice.

(ii) We do not need the assumption on self-adjointness of J_0, until we do not claim any spectral consequences of the existence of the approximative eigenvector. When we assume also that J_0 is self-adjoint, then by Proposition 2.1 we get $\lambda \in \sigma(J_0)$ (and $\lambda \in \sigma_{\mathrm{ess}}(J_0)$ if $p_n \to +\infty$).

3. PARTICULAR RESULTS FOR H CLASS

In this section we show a particular case of Proposition 2.5 being very convenient for some applications. The extra assumption we shall make is that the sequence of transfer matrices for J_0 and λ belongs to the H class (see, e.g., [12], [13]). Recall that the transfer matrix $B_k(\lambda)$ for J_0 and λ is given for $k \geqslant 2$ by

$$B_k(\lambda) := \begin{pmatrix} 0 & 1 \\ -\frac{w_{0(k-1)}}{w_{0k}} & \frac{\lambda - q_{0k}}{w_{0k}} \end{pmatrix}, \tag{3.1}$$

if the weights satisfy

$$w_{0k} \neq 0, \quad k \in \mathbb{N}. \tag{3.2}$$

Recall also that a sequence $\{C_k\}_{k \geqslant k_0}$ of complex 2×2 matrices is in the H class if and only if there exists $0 < M < +\infty$ such that

$$\Big\| \prod_{j=k_0}^{k} C_j \Big\|^2 \leqslant M \prod_{j=k_0}^{k} |\det C_j|, \quad k \geqslant k_0 \tag{3.3}$$

(in this paper $\prod\limits_{j=m}^{n} A_j$ denotes $A_n \cdots A_m$ if $n > m$; if $n = m$ it equals A_m, and if $n < m$ it equals I). This class is a convenient tool to study the absolutely continuous part of some Jacobi operators. For some "typical situations", if $\lambda \in \mathbb{R}$ and $\{B_k(\lambda)\}_{k\geqslant 2} \in H$, then $\lambda \in \sigma_{\mathrm{ac}}(J_0)$. More details concerning H class can be found in [12], [13], in particular, several criteria containing sufficient conditions for belonging to H class.

Let us begin here with a result showing that the assumption $\{B_k(\lambda)\}_{k\geqslant 2} \in H$ has important asymptotic consequences for generalized eigenvectors.

PROPOSITION 3.1. *If* (3.2) *holds,* $\lambda \in \mathbb{C}$, *and* $\{B_k(\lambda)\}_{k\geqslant 2} \in H$, *then for any non-zero generalized eigenvector* u *for* J_0 *and* λ

$$|u(k)|^2 + |u(k+1)|^2 \asymp w_{0k}^{-1}, \quad k \in \mathbb{N}. \tag{3.4}$$

If, moreover, $\lambda \in \mathbb{R}$ *and*

$$\det \begin{pmatrix} u(1) & \overline{u(1)} \\ u(2) & \overline{u(2)} \end{pmatrix} \neq 0,$$

then there exists a constant $c > 0$ *such that*

$$|u(k)| \geqslant c|w_{0k}|^{-1/2}, \quad k \in \mathbb{N}. \tag{3.5}$$

Note, that in general (3.5) does not follow from (3.4), which gives some estimates from below only for $|u(k)|$, $|u(k+1)|$ "being together". Typical initial conditions satisfying the extra condition above are $u(1) = 1$, $u(2) = i$.

Proof. Denote

$$\Phi_k := \prod_{j=2}^{k} B_j(\lambda).$$

We have

$$\begin{pmatrix} u(k) \\ u(k+1) \end{pmatrix} = \Phi_k \begin{pmatrix} u(1) \\ u(2) \end{pmatrix}, \quad k \geqslant 2 \tag{3.6}$$

and hence, by the definition of H class, for any $k \geqslant 2$

$$|u(k)|^2 + |u(k+1)|^2 \leqslant C_1 \|\Phi_k\|^2 \leqslant C_2 \prod_{j=2}^{k} |\det B_j(\lambda)| = C_2 \prod_{j=2}^{k} \left| \frac{w_{0(j-1)}}{w_{0j}} \right| = C_3 |w_{0k}|^{-1},$$

with some positive constants C_1, C_2, C_3. To get an estimate from below, let us observe first that by the explicit formula for the inverse matrix the following general result holds for 2×2 matrices:

$$\|A^{-1}\| \asymp |\det A|^{-1} \|A\|, \quad A \in \mathrm{Inv}(2), \tag{3.7}$$

where $\mathrm{Inv}(2)$ denotes the set of invertible 2×2 complex matrices. Now, by (3.6), by "$\|Ax\| \geqslant \|x\| / \|A^{-1}\|$" and by (3.7) we have

$$|u(k)|^2 + |u(k+1)|^2 = \left\| \Phi_k \begin{pmatrix} u(1) \\ u(2) \end{pmatrix} \right\|^2 \geqslant \frac{C_4}{\|\Phi_k^{-1}\|^2} \geqslant C_5 \left(\frac{|\det \Phi_k|}{\|\Phi_k\|} \right)^2 \geqslant \cdots$$

and continuing with the use of the definition of H class again

$$\cdots \geqslant C_7 |\det \Phi_k| = C_8 |w_{0k}|^{-1}$$

for any $k \geqslant 2$, with some positive constants $C_4, \dots, C_8$.

To prove the second part of the lemma, observe that since $\lambda \in \mathbb{R}$ and the weights and the diagonals are real, the sequence $\overline{u}$ is also a generalized eigenvector for J_0 and λ. So, we have

$$\begin{pmatrix} u(k) & \overline{u(k)} \\ u(k+1) & \overline{u(k+1)} \end{pmatrix} = B_k(\lambda) \begin{pmatrix} u(k-1) & \overline{u(k-1)} \\ u(k) & \overline{u(k)} \end{pmatrix}.$$

Thus, denoting $z_k = u(k+1)\overline{u(k)}$ and computing the determinants of both sides above, we get $w_{0k}\mathrm{Im}z_k = w_{0(k-1)}\mathrm{Im}z_{(k-1)}$ for any $k \geqslant 2$, which means that $w_{0k}\mathrm{Im}z_k$ does not depend on k. Moreover, by the assumption on the initial conditions, the constant value C' of $w_{0k}\mathrm{Im}z_k$ is non-zero. We have

$$|u(k+1)||u(k)| = |z_k| \geqslant |\mathrm{Im}z_k| = \frac{|C'|}{|w_{0k}|}$$

(in particular $u(k+1)$, $u(k)$ are non-zero), and by the first part of the lemma

$$|u(k+1)| \leqslant \frac{C''}{\sqrt{|w_{0k}|}}$$

for any $k \in \mathbb{N}$, with a positive constant C''. Hence

$$|u(k)| \geqslant \frac{|C'|}{|w_{0k}|\,|u(k+1)|} \geqslant \frac{|C'|\sqrt{|w_{0k}|}}{C''|w_{0k}|} = \frac{|C'|}{C''}|w_{0k}|^{-1/2}, \quad k \in \mathbb{N}. \quad ∎$$

We are ready now to formulate the main result of our Step 1.

THEOREM 3.2. *Suppose that $W : \mathbb{N} \to \mathbb{R}_+$ is an increasing function satisfying*

$$w_{0k} \asymp W(k), \quad k \in \mathbb{N},$$

and that (3.2) *holds. Assume also that $\lambda \in \mathbb{R}$ and $\{B_k(\lambda)\}_{k \geqslant 2} \in H$.*

Then for any pair of sequences $\{p_n\}_{n \geqslant n_0}$, $\{d_n\}_{n \geqslant n_0}$ of natural numbers satisfying $p_n \to +\infty$ and

$$\frac{W(p_n + d_n)}{d_n} \to 0, \tag{3.8}$$

there exists a Weyl sequence $\{f_n\}_{n \geqslant n_0}$ for J_0 and λ such that $\|f_n\| = 1$ and $\mathrm{supp} f_n \subset I_n$ *for $n \geqslant n_0$, where I_n is given by* (2.12).

Proof. Let us consider the generalized eigenvector u for J_0 and λ with the initial conditions $u(1) = 1$, $u(2) = i$. By Proposition 3.1 (note that (3.2) holds by $w_{0k} \asymp W(k)$) we have

$$|w_{0k}|(|u(k)| + |u(k+1)|) \leqslant C\sqrt{|w_{0k}|} \leqslant C'\sqrt{W(k)}$$

and

$$|u(k)| \geqslant \frac{c}{\sqrt{|w_{0k}|}} \geqslant \frac{c'}{\sqrt{W(k)}}$$

for any $k \in \mathbb{N}$ with some positive constants c, C, c', C'. Now we can use Proposition 2.5, taking monotonic $\{\eta_k\}$, $\{\xi_k\}_{k\geqslant l_0}$ with $l_0 = 1$ and

$$\eta_k = \sqrt{W(k)}, \quad \xi_k = \frac{1}{\sqrt{W(k)}},$$

and we obtain our assertion. ∎

REMARK 3.3. Suppose that J_0 is self-adjoint and that the assumptions of the above theorem hold. Then for "typical situations" $\lambda \in \sigma_{\mathrm{ac}}(J_0)$, as it has already been mentioned. In particular $\lambda \in \sigma_{\mathrm{ess}}(J_0)$, and thus the spectral consequence of Theorem 3.2 (see also Remarks 2.6(ii)) is not very interesting in fact. Yet, the main goal of this theorem is *not* this kind of spectral information, but exactly the information about the existence of many Weyl sequences satisfying some special "support" conditions.

4. TWO EXAMPLES

We show here two non-trivial examples of classes of Jacobi operators for which Theorem 3.2 can be used.

For a sequence $a = \{a_k\}_{k\geqslant k_0}$ of elements of a normed space (e.g., for a scalar or matrix sequence) we write $a \in D^1$ (or a is *a D^1 sequence*) if and only if $\Delta a \in l^1$ (we use the same symbols D^1, l^1 for all spaces and any k_0). We denote by discr A the discriminant of the characteristic polynomial of 2×2 matrix A, i.e., $\operatorname{discr} A = (\operatorname{tr} A)^2 - 4\det A$.

EXAMPLE 4.1. Consider J_0 with weights and diagonals given for $k \in \mathbb{N}$ by

$$w_{0k} = k^\alpha + b, \quad q_{0k} = a, \tag{4.1}$$

where $\alpha \in (0;1)$ and a, b are arbitrary real constants, such that $k^\alpha + b \neq 0$ for all $k \in \mathbb{N}$. Let $\lambda \in \mathbb{R}$. We have

$$B_k(\lambda) := \begin{pmatrix} 0 & 1 \\ -\frac{(1-\frac{1}{k})^\alpha + \frac{b}{k^\alpha}}{1+\frac{b}{k^\alpha}} & \frac{\frac{\lambda-a}{k^\alpha}}{1+\frac{b}{k^\alpha}} \end{pmatrix},$$

so using properties of D^1 class (see, e.g., some generalizations for more general classes in Appendix 5.1 of [13] or in Section 2.1 of [6]) we can easily see that $\{B_k(\lambda)\}_{k\geqslant 2} \in D^1$. Moreover we have

$$B_k(\lambda) \to B_\infty := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

with $\operatorname{discr} B_\infty = -4 < 0$, which gives $\{B_k(\lambda)\}_{k\geqslant 2} \in H$ by Criterion 1 of [12] (see also Criterion 5.8 (1) of [13]). Let us choose W:

$$W(k) = k^\alpha.$$

Now condition (3.8) has the form

$$\frac{(p_n + d_n)^\alpha}{d_n} \longrightarrow 0, \tag{4.2}$$

but by Lemma 4.2 below, for $p_n \longrightarrow +\infty$ this condition is equivalent to

$$\frac{p_n^\alpha}{d_n} \longrightarrow 0. \tag{4.3}$$

Hence, in this case, by Theorem 3.2, for any $\lambda \in \mathbb{R}$ and any sequences $\{p_n\}_{n\geqslant n_0}$, $\{d_n\}_{n\geqslant n_0}$ of natural numbers such that $p_n \longrightarrow +\infty$ and (4.3) holds, there exists a Weyl sequence $\{f_n\}_{n\geqslant n_0}$ for J_0 and λ such that $\|f_n\| = 1$ and $\operatorname{supp} f_n \subset [p_n; p_n+d_n]$.

Above we have used the following simple result.

LEMMA 4.2. *Suppose that $\{p_n\}_{n\geqslant n_0}$ and $\{d_n\}_{n\geqslant n_0}$ are sequences of natural numbers, and that $p_n \longrightarrow +\infty$. Then for $\alpha \in (0;1)$ conditions* (4.2) *and* (4.3) *are equivalent. Moreover $d_n \longrightarrow +\infty$, if those conditions hold.*

Proof. The first assertion immediately follows from the estimates

$$\frac{p_n^\alpha}{d_n} \leqslant \frac{(p_n + d_n)^\alpha}{d_n} \leqslant \frac{p_n^\alpha + d_n^\alpha}{d_n} = \frac{p_n^\alpha}{d_n} + \Big(\frac{p_n^\alpha}{d_n}\frac{1}{p_n^\alpha}\Big)^{1-\alpha},$$

and the second from $d_n = (p_n^\alpha/d_n)^{-1} p_n^\alpha$. ∎

EXAMPLE 4.3. As the second example consider J_0 with weights and diagonals more complicated, given by formula more general than the one in the previous case:

$$w_{0k} = k^\alpha + b_k, \quad q_{0k} = a_k \tag{4.4}$$

for $k \in \mathbb{N}$, where $\alpha \in (0;1)$ and $\{a_k\}_{k\geqslant 1}$, $\{b_k\}_{k\geqslant 1}$ are arbitrary real 2-periodic sequences, such that $k^\alpha + b_k \neq 0$ for all $k \in \mathbb{N}$. Let $\lambda \in \mathbb{R}$. In this case $\{B_k(\lambda)\}_{k\geqslant 2}$ need not to be a D^1-sequence, but we shall study $\{B_{2k+1}(\lambda)B_{2k}(\lambda)\}_{k\geqslant 1}$. Using properties of D^1 class (in particular, it is convenient to use Lemma 2.1 b) of [5]) we can compute

$$\begin{aligned} B_{2k+1}(\lambda)B_{2k}(\lambda) &:= \begin{pmatrix} 0 & 1 \\ -\frac{(2k)^\alpha+b_2}{(2k+1)^\alpha+b_1} & \frac{\lambda-a_1}{(2k+1)^\alpha+b_1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\frac{(2k-1)^\alpha+b_1}{(2k)^\alpha+b_2} & \frac{\lambda-a_2}{(2k)^\alpha+b_2} \end{pmatrix} \\ &= \begin{pmatrix} C_k^{(1)} & C_k^{(2)} \\ C_k^{(3)} & C_k^{(4)} \end{pmatrix}, \end{aligned}$$

where

$$C_k^{(1)} = -1 + \frac{1}{k^\alpha}\Big(\frac{b_2 - b_1}{2^\alpha}\Big)\nu_k^{(1)} + r_k^{(1)},$$
$$C_k^{(2)} = \frac{1}{k^\alpha}\Big(\frac{\lambda - a_2}{2^\alpha}\Big)\nu_k^{(2)} + r_k^{(2)},$$
$$C_k^{(3)} = \frac{1}{k^\alpha}\Big(\frac{a_1 - \lambda}{2^\alpha}\Big)\nu_k^{(3)} + r_k^{(3)},$$
$$C_k^{(4)} = -1 + \frac{1}{k^\alpha}\Big(\frac{b_1 - b_2}{2^\alpha}\Big)\nu_k^{(4)} + r_k^{(4)},$$

with $\{\nu_k^{(j)}\}_{k\geqslant 1} \in D^1$, $\nu_k^{(j)} \to 1$ and $\{r_k^{(j)}\}_{k\geqslant 1} \in l^1$ for $j = 1,2,3,4$. Hence

$$B_{2k+1}(\lambda)B_{2k}(\lambda) = -I + \frac{1}{k^\alpha}V_k(\lambda) + R_k(\lambda),$$

where $\{V_k(\lambda)\}_{k\geqslant 1} \in D^1$, $\{R_k(\lambda)\}_{k\geqslant 1} \in l^1$ and $V_k(\lambda) \to V_\infty(\lambda)$, with

$$V_\infty(\lambda) = \begin{pmatrix} \frac{b_2-b_1}{2^\alpha} & \frac{\lambda-a_2}{2^\alpha} \\ \frac{a_1-\lambda}{2^\alpha} & \frac{b_1-b_2}{2^\alpha} \end{pmatrix}.$$

Denote

$$G = \{\lambda \in \mathbb{R} : (\lambda - a_2)(\lambda - a_1) \leqslant (b_2 - b_1)^2\} = [g_-; g_+],$$
$$\breve{G} = \{\lambda \in \mathbb{R} : (\lambda - a_2)(\lambda - a_1) < (b_2 - b_1)^2\} = (g_-; g_+),$$

where

$$g_\pm = \frac{\pm\sqrt{(a_2 - a_1)^2 + 4(b_2 - b_1)^2} + a_1 + a_2}{2}. \tag{4.5}$$

We have discr $V_\infty(\lambda) < 0$ if and only if $\lambda \in \mathbb{R} \setminus G$. Therefore, by Criterion 2 of [12] (see also Criterion 5.8(2) of [13]), when $\lambda \in \mathbb{R}\setminus G$, then $\{B_{2k+1}(\lambda)B_{2k}(\lambda)\}_{k\geqslant 1} \in H$, which gives also $\{B_k(\lambda)\}_{k\geqslant 2} \in H$ (see Proposition 1.1 b) of [12]). Now choosing $W(k) = k^\alpha$ as in the previous example, we get similarly as before: for any $\lambda \in \mathbb{R}\setminus G$ and any sequences $\{p_n\}_{n\geqslant n_0}$, $\{d_n\}_{n\geqslant n_0}$ of natural numbers such that $p_n \to +\infty$ and (4.3) holds, there exists a Weyl sequence $\{f_n\}_{n\geqslant n_0}$ for J_0 and λ such that $\|f_n\| = 1$ and $\operatorname{supp} f_n \subset [p_n; p_n + d_n]$.

REMARKS 4.4. (i) In both examples J_0 is self-adjoint (the Carleman condition is satisfied), and both are the "typical situations" mentioned in the Remark 3.3. In particular, using methods of subordination theory developed e.g. in [5], [12], it can be easily proved that $\sigma_{\mathrm{ac}}(J_0) = \mathbb{R}$ in the first example, and $\sigma_{\mathrm{ac}}(J_0) = \mathbb{R}\setminus\breve{G}$ in the second one.

(ii) Taking $\alpha = 1$ instead of $0 < \alpha < 1$, in both examples we unfortunately cannot obtain any result on Weyl sequences, by our methods. It is related to the fact that $(p_n + d_n)^\alpha/d_n = (p_n + d_n)/d_n \geqslant 1$ for such case.

5. WEYL SEQUENCES AND THE ESSENTIAL SPECTRUM FOR PERTURBATIONS OF J_0

In this section we shall try to find some Weyl sequences for perturbations J of J_0, using the family of Weyl sequences found for J_0 in the previous sections. In other words, we make here Step 2 described in Section 1.

Let J be the Jacobi operator with a weight sequence $\{w_k\}$ and a diagonal sequence $\{q_k\}$. We treat J as a perturbation of J_0. More precisely, we assume

$$w_k = w_{0k} + v_k, \quad q_k = q_{0k} + \delta_k, \quad k \in \mathbb{N}, \tag{5.1}$$

where $\{v_k\}_{k\geqslant 1}$, $\{\delta_k\}_{k\geqslant 1}$ are real sequences. Denoting by P the Jacobi operator given by weight sequence $\{v_k\}$ and diagonal sequence $\{\delta_k\}$ we get in particular

$$Jf = J_0 f + Pf$$

for any $f \in l_{\mathrm{fin}}(\mathbb{N})$, where $l_{\mathrm{fin}}(\mathbb{N})$ denotes the space of "finite sequences", i.e., of such sequences from $l(\mathbb{N})$ which have only finite number of non-zero entries. Our further considerations are based on the following obvious observation.

LEMMA 5.1. *Suppose that $\{f_n\}_{n\geqslant n_0}$ is a Weyl sequence for J_0 and λ, consisting of vectors f_n from $l_{\mathrm{fin}}(\mathbb{N})$. If*

$$\frac{\|Pf_n\|}{\|f_n\|} \to 0, \tag{5.2}$$

then $\{f_n\}_{n\geqslant n_0}$ is also a Weyl sequence for J and λ.

In practice, the condition (5.2) will be obtained here, by the use of the estimate from the lemma below.

LEMMA 5.2. *Suppose that $f \in l^2(\mathbb{N})$ is such that* $\operatorname{supp} f \subset I$, *where $I = [p; p + d] \cap \mathbb{N}$ with $p, d \in \mathbb{N}$, $p \geqslant 2$. Then*

$$\|Pf\| \leqslant \Big(2 \sup_{j\in\widetilde{I}} |v_j| + \sup_{j\in\widetilde{I}} |\delta_j|\Big) \|f\|,$$

where $\widetilde{I} = [p-1; p+d+1] \cap \mathbb{N}$.

Proof. Denote by P_1, P_2, P_3 the operators in $l^2(\mathbb{N})$ with the domain $l_{\mathrm{fin}}(\mathbb{N})$ determined by the sub-diagonal, the diagonal and the super-diagonal of the Jacobi matrix for P, respectively. Hence, for $f \in l_{\mathrm{fin}}(\mathbb{N})$ we have $Pf = P_1 f + P_2 f + P_3 f$, and in particular, if $\operatorname{supp} f \subset I$, then

$$\|Pf\| \leqslant \|P_1 f\| + \|P_2 f\| + \|P_3 f\| \leqslant 2 \sup_{j\in\widetilde{I}} |v_j| \|f\| + \sup_{j\in I} |\delta_j| \|f\|,$$

which gives the final estimate. ∎

In this section, to simplify the formulation of the main theorem, we shall consider some stronger assumptions on the function (sequence) W from Theorem 3.2. Namely

$$(\mathbf{Z}) \begin{cases} \text{(i) } W : \mathbb{N} \to \mathbb{R}_+, \quad W(n) \nearrow +\infty; \\ \text{(ii) } \exists_{C\in\mathbb{R}} \forall_{k\in\mathbb{N}}\, W(2k) \leqslant CW(k); \\ \text{(iii) } \frac{W(k)}{k} \to 0. \end{cases}$$

Note that such W satisfies also

$$\exists_{C\in\mathbb{R}} \forall_{p,d\in\mathbb{N}}\, W(p+d) \leqslant C(W(p)+W(d)), \tag{5.3}$$

because by (i) and (ii)

$$W(p+d) \leqslant W(2\max(p,d)) \leqslant CW(\max(p,d)) = C\max(W(p),W(d)) \leqslant C(W(p)+W(d)).$$

In particular, from (5.3) and (**Z**) (i) we easily get

$$\exists_{0<c<C<+\infty} \forall_{p\in\mathbb{N},p>1}\, c \leqslant \frac{W(p-1)}{W(p)} \leqslant C. \tag{5.4}$$

Positive increasing sequences given for large k by the formula $k^\alpha(\ln k)^\beta$ with $0<\alpha<1$, $\beta\in\mathbb{R}$ are examples of W satisfying (**Z**). In particular, (**Z**) holds for the function $W(k)=k^\alpha$, which was used in our examples from the previous section. One of benefits from (**Z**) is the possibility of a simple generalization of Lemma 4.2 for such W.

LEMMA 5.3. *Assume* (**Z**). *Suppose that* $\{p_n\}_{n\geqslant n_0}$ *and* $\{d_n\}_{n\geqslant n_0}$ *are sequences of natural numbers, and that* $p_n \to +\infty$. *Then conditions* (3.8) *and*

$$\frac{W(p_n)}{d_n} \to 0, \tag{5.5}$$

are equivalent. Moreover $d_n \to +\infty$, *if those conditions hold.*

Proof. "(3.8) $\Rightarrow$ (5.5)" is obvious by the monotonicity of W. Assume (5.5). By (5.3) for some $C\in\mathbb{R}$ we have

$$\frac{W(p_n+d_n)}{d_n} \leqslant C\Big(\frac{W(p_n)}{d_n}+\frac{W(d_n)}{d_n}\Big)$$

and $d_n = (W(p_n)/d_n)^{-1}W(p_n) \to +\infty$, thus by (iii) of (**Z**) and by (5.5) we get (3.8). ∎

We introduce now a convenient class 0_W of scalar sequences.

DEFINITION 5.4. The scalar sequence $\{x_k\}_{k\geqslant k_0}$ is in the 0_W class if and only if it possesses a subsequence convergent to 0, and

$$(\Delta x)_k = o(W(k)^{-1}), \quad k\to+\infty. \tag{5.6}$$

In the particular cases $W(k)=k^\alpha$ we shall also write 0_α instead of 0_W.

EXAMPLE 5.5. Let $f : [0;+\infty) \to \mathbb{R}$ be a C^1 periodic function such that $\inf f([0;+\infty)) \leqslant 0 \leqslant \sup f([0;+\infty))$, let $0 < \alpha < 1$ and $0 < \gamma < 1-\alpha$. Taking $x_k = f(k^\gamma)$, we get $\{x_k\} \in 0_\alpha$. To prove this fact let us formulate the following lemma.

LEMMA 5.6. *If $a_k \to +\infty$ and $(\triangle a)_k \to 0$, then for any $T > 0$ the sequence $\{a_k \pmod{T}\}$ is dense in $[0;T]$.*

We omit here a proof of this lemma, being rather elementary. Using now this result for $a_k = k^\gamma$, by the periodicity and the continuity of f, we get that $\{x_k\}$ possesses a subsequence convergent to any limit between $\inf f([0;+\infty))$ and $\sup f([0;+\infty)$ (to get this, it has been enough to assume $0 < \gamma < 1$ for γ). Moreover f' is bounded, hence for any k

$$|(\Delta x)_k| = |f((k+1)^\gamma) - f(k^\gamma)| \leqslant C|(k+1)^\gamma - k^\gamma| \leqslant C\gamma \frac{1}{k^{1-\gamma}},$$

with a constant C, which gives (5.6).

The role of the assumptions (**Z**) and of the class 0_W, and their relation to the results of the previous section becomes more clear in the following lemma.

LEMMA 5.7. *Assume* (**Z**). *If $\{x_k\}_{k\geqslant 1} \in 0_W$, then there exist sequences of natural numbers $\{p_n\}_{n\geqslant 1}$ and $\{d_n\}_{n\geqslant 1}$ such that $p_n \to +\infty$,* (3.8) *holds and*

$$\sup_{k\in\widetilde{I}_n} |x_k| \to 0,$$

where $\widetilde{I}_n = [p_n - 1; p_n + d_n + 1] \cap \mathbb{N}$.

Proof. For $p, d \in \mathbb{N}$, $p > 1$ denote $\widetilde{I} := [p-1; p+d+1] \cap \mathbb{N}$ and

$$y(p,d) := \sup_{k\in\widetilde{I}} |x_k|.$$

We have $(\Delta x)_k = \varepsilon_k / W(k)$, where $\{\varepsilon_k\}$ is such that $\varepsilon_k \to 0$. Define

$$\omega_k := \frac{1}{k} + \sup_{j\geqslant k} |\varepsilon_j|.$$

Thus $\omega_k > 0$, and we can estimate $|(\Delta x)_k| \leqslant \omega_k / W(k)$. For $k \in \widetilde{I}$ we have

$$|x_k| \leqslant |x_{p-1}| + |x_{p-1} - x_k| \leqslant |x_p| + |(\Delta x)_{p-1}| + \sum_{j=p-1}^{k-1} |(\Delta x)_j| \leqslant |x_p| + 2\sum_{j=p-1}^{p+d} |(\Delta x)_j|.$$

Hence, since W is increasing and $\{\omega_k\}$, decreasing, we have

$$y(p,d) \leqslant 2\Big(|x_p| + \frac{\omega_{p-1}(d+2)}{W(p-1)}\Big). \tag{5.7}$$

Let us choose now $\{p_n\}_{n\geqslant 1}$ such that $p_n \geqslant 2$, $p_n \to +\infty$ and $x_{p_n} \to 0$. The proof is completed by showing that there exists $\{d_n\}_{n\geqslant 1}$ such that (3.8) and $y(p_n, d_n) \to 0$

hold. Observe that $\omega_k \to 0$. Thus, by (5.7), (5.4) and Lemma 5.3, it is sufficient to construct $\{d_n\}_{n\geqslant 1}$ satisfying

$$\frac{W(p_n)}{d_n} \to 0 \quad \text{and} \quad \frac{d_n}{W(p_n)}\omega_{p_n-1} \to 0.$$

Define

$$d_n := \Big[\frac{W(p_n)}{\sqrt{\beta_n}}\Big],$$

where $\beta_n := \omega_{p_n-1}$, and $[\,\cdot\,]$ denotes the integer part. We have $\beta_n \to 0$ and $W(p_n)/\sqrt{\beta_n} \to +\infty$ (since $W(k) \to +\infty$, $\omega_k \to 0$, $p_n \to +\infty$), which gives $d_n \to +\infty$. Moreover, for large n

$$\frac{W(p_n)}{d_n} \leqslant 2\frac{W(p_n)}{\frac{W(p_n)}{\sqrt{\beta_n}}} = 2\sqrt{\beta_n}, \quad \frac{d_n}{W(p_n)}\omega_{p_n-1} \leqslant \frac{\frac{W(p_n)}{\sqrt{\beta_n}}}{W(p_n)}\beta_n = \sqrt{\beta_n},$$

which proves the desired conditions on $\{d_n\}_{n\geqslant 1}$. ∎

We are ready now to formulate the main result of the paper.

THEOREM 5.8. *Suppose that J, with weights and diagonals given by* (5.1), *is self-adjoint, and $\lambda \in \mathbb{R}$. Assume that $w_{0k} \asymp W(k)$, where* ($\mathbf{Z}$) *and* (3.2) *hold, and that the sequence $\{B_k(\lambda)\}_{k\geqslant 2}$ of transfer matrices for J_0 and λ is in the H class. Assume also that the sequences $\{v_k\}_{k\geqslant 1}$, $\{\delta_k\}_{k\geqslant 1}$ have the form*

$$v_k = z_k^{(1)} + t_k^{(1)}x_k, \quad \delta_k = z_k^{(2)} + t_k^{(2)}x_k,$$

with some real sequences $\{x_k\}_{k\geqslant 1}$, $\{t_k^{(j)}\}_{k\geqslant 1}$, $\{z_k^{(j)}\}_{k\geqslant 1}$ satisfying: $\{x_k\}_{k\geqslant 1} \in 0_W$, $\{t_k^{(j)}\}_{k\geqslant 1} \in l^\infty$ and $z_k^{(j)} \to 0$ for $j = 1,2$. Then $\lambda \in \sigma_{\mathrm{ess}}(J)$.

Proof. Let us first choose some sequences $\{p_n\}_{n\geqslant 1}$ and $\{d_n\}_{n\geqslant 1}$ of natural numbers for $\{x_k\}_{k\geqslant 1}$, as in Lemma 5.7. By Theorem 3.2, there exists a Weyl sequence $\{f_n\}_{n\geqslant n_0}$ for J_0 and λ such that $\|f_n\| = 1$ and $\operatorname{supp} f_n \subset I_n$ for $n \geqslant n_0$, where $I_n = [p_n; p_n + d_n] \cap \mathbb{N}$. Hence, by Lemma 5.2 and by the assumption that $\{t_k^{(j)}\}_{k\geqslant 1} \in l^\infty$, there exists a constant C such that

$$\begin{aligned}\frac{\|Pf_n\|}{\|f_n\|} &\leqslant 2\sup_{j\in\widetilde{I}_n}|z_j^{(1)}| + \sup_{j\in\widetilde{I}_n}|z_j^{(2)}| + C\sup_{j\in\widetilde{I}_n}|x_j| \\ &\leqslant 2\sup_{j\geqslant p_n-1}|z_j^{(1)}| + \sup_{j\geqslant p_n-1}|z_j^{(2)}| + C\sup_{j\in\widetilde{I}_n}|x_j|\end{aligned}$$

for n large enough. Using now $z_k^{(j)} \to 0$, $p_n \to +\infty$ and the assertion of Lemma 5.7 we obtain $\|Pf_n\|/\|f_n\| \to 0$. Therefore our assertion follows from Lemma 5.1 and Proposition 2.1(ii). ∎

REMARK 5.9. The terms $\{z_k^{(j)}\}$ tending to zero are rather of less importance, since compact perturbations do not change the essential spectrum. More important are the bounded terms $\{t_k^{(j)}\}$, which can be chosen independently for the weights and for the diagonals of J. Unfortunately, such independence is rather impossible for "the main perturbation term" $\{x_k\}$.

Special cases of Theorem 5.8 can be obtained, for instance, by taking the examples of J_0 considered in Section 3. In those examples $W(k) = k^\alpha$ with $\alpha \in (0;1)$, and the weights w_{0k} for J_0 are estimated by a constant multiplied by k^α. So, as it is easy to see, the weights w_k for J from our theorem are estimated by $\mathrm{const} \cdot (k^\alpha + k^{1-\alpha})$. In particular, the Carleman condition holds for J, which guarantees its self-adjointness. Hence, the following two results are the direct consequences of Theorem 5.8 and Examples 4.1 and 4.3, respectively.

COROLLARY 5.10. *Assume that J has weights and diagonals given by*

$$w_k = k^\alpha + b + t_k^{(1)} x_k + z_k^{(1)}, \quad q_k = a + t_k^{(2)} x_k + z_k^{(2)}, \tag{5.8}$$

where $\alpha \in (0;1)$, $a, b \in \mathbb{R}$, $k^\alpha + b \neq 0$ for all $k \in \mathbb{N}$, and $\{x_k\}_{k\geqslant 1}$, $\{t_k^{(j)}\}_{k\geqslant 1}$, $\{z_k^{(j)}\}_{k\geqslant 1}$ are real sequences satisfying $\{x_k\}_{k\geqslant 1} \in 0_\alpha$, $\{t_k^{(j)}\}_{k\geqslant 1} \in l^\infty$ and $z_k^{(j)} \to 0$ for $j = 1, 2$. Then $\sigma_{\mathrm{ess}}(J) = \mathbb{R}$.

COROLLARY 5.11. *Assume that J has weights and diagonals given by*

$$w_k = k^\alpha + b_k + t_k^{(1)} x_k + z_k^{(1)}, \quad q_k = a_k + t_k^{(2)} x_k + z_k^{(2)}, \tag{5.9}$$

where $\alpha \in (0;1)$, $\{a_k\}_{k\geqslant 1}$, $\{b_k\}_{k\geqslant 1}$ are arbitrary real 2-periodic sequences, such that $k^\alpha + b_k \neq 0$ for all $k \in \mathbb{N}$ and $\{x_k\}$, $\{t_k^{(j)}\}$, $\{z_k^{(j)}\}$ are as in Corollary 5.10. Then $\mathbb{R} \setminus \breve{G} \subset \sigma_{\mathrm{ess}}(J)$, where $\breve{G} = (g_-; g_+)$ (see (4.5)).

We have formulated Corollary 5.10 mainly to show the case with the full information on the essential spectrum — formally it was not necessary, since it is a special case of Corollary 5.11 ($\breve{G} = \emptyset$ when $a_1 = a_2$, $b_1 = b_2$). Note that spectral properties of several classes of Jacobi operators satisfying the above or similar assumptions have been presented in many papers, where some stronger spectral results were obtained (see, e.g., [1], [5], [6], [8], [9], [10], [12]). The typical results were a full or a partial description of the absolutely continuous spectrum, obtained by the use of the subordination theory. However, it seems that the corollaries presented above work also for some cases where the currently known subordination theory tools cannot be used.

EXAMPLE 5.12. Consider J with weights and diagonals given by

$$w_k = k^\alpha + b_k + c_k f(k^\gamma), \quad q_k = a_k, \tag{5.10}$$

where f is as in Example 5.5, $\alpha \in (0;1)$, $0 < \gamma < 1 - \alpha$, and $\{a_k\}_{k\geqslant 1}$, $\{b_k\}_{k\geqslant 1}$, $\{c_k\}_{k\geqslant 1}$ are arbitrary real 2-periodic sequences, such that $k^\alpha + b_k \neq 0$ for all $k \in \mathbb{N}$. Then, by Corollary 5.11, $\mathbb{R} \setminus (g_-; g_+) \subset \sigma_{\mathrm{ess}}(J)$, where $g_\pm$ are given by (4.5). Janas, Naboko and Stolz in [10] consider a special case of this example. They additionally assume that f is a C^2 function with $\inf f([0;+\infty)) = 0$, $\sup f([0;+\infty)) = 1$, that $a_k = b_k = 0$ for any k, $0 < c_1, c_2,\ c_1 \neq c_2$, and that $\gamma < 1/2(1-\alpha)$. For such a case we have $g_- = g_+$, i.e., $\sigma_{\mathrm{ess}}(J) = \mathbb{R}$. This result is obtained in [10] by the use of Weyl sequence, which is constructed essentially in the same way as here. But the authors prove much more for their case. Using some subordination theory tools based on weighted Stolz class (see [6]) they obtain also the absolute continuity of J in $\mathbb{R} \setminus [-|c_1 - c_2|; |c_1 - c_2|]$. Moreover, using some spectral averaging argumentation, they also prove that $\sigma_{\mathrm{ac}}(J) \cap (-|c_1 - c_2|; |c_1 - c_2|) = \emptyset$. Note that the subordination theory methods, mentioned above, do not work when $1/2(1-\alpha) \leqslant \gamma < 1-\alpha$. However, the Weyl sequences method presented here works also in such a case. But even for $\gamma < 1/2(1-\alpha)$ it seems to be not clear how to prove that $\sigma_{\mathrm{ess}}(J) = \mathbb{R}$, without using Weyl sequences...Note also that some detailed spectral properties of some other special cases, related to the general case considered in this example are studied in [7].

Acknowledgements. The author wishes to thank Professor Jerzy Browkin for his suggestion to employ Lemma 5.6, which essentially simplified considerations for the Example 5.5.

Research supported by the MSHE Grant N N 201 4265 33.

REFERENCES

[1] J. DOMBROWSKI, J. JANAS, M. MOSZYŃSKI, S. PEDERSEN, Spectral gaps resulting from periodic perturbations of a class of Jacobi operators, *Constr. Approx.* **20**(2004), 585–601.

[2] D.E. EDMUNDS, W.D. EVANS, *Spectral Theory and Differential Operators*, Oxford Math. Monographs, Clarendon Press, Oxford 1987.

[3] J.F. ELLIOTT, The characteristic roots of certain real symmetric matrices, Master's Thesis, Univ. of Tennessee, Tennessee 1953.

[4] R.T. GREGORY, D. KARNEY, *A Collection of Matrices for Testing Computational Algorithm*, Wiley Interscience, New York-London-Sydney 1969.

[5] J. JANAS, M. MOSZYŃSKI, Spectral properties of Jacobi matrices by asymptotic analysis, *J. Approx. Theory* **120**(2003), 309–336.

[6] J. JANAS, M. MOSZYŃSKI, New discrete Levinson type asymptotics of solutions of linear systems, *J. Difference Equations Appl.* **12**(2006), 133–163.

[7] J. JANAS, M. MOSZYŃSKI, Spectral analysis of unbounded Jacobi operators with oscillating entries, submitted.

[8] J. JANAS, S. NABOKO, Jacobi matrices with power-like weights — grouping in blocks approach, *J. Funct. Anal.* **166**(1999), 218–243.

[9] J. JANAS, S. NABOKO, Spectral analysis of selfadjoint Jacobi matrices with periodically modulated entries, *J. Funct. Anal.* **191**(2002), 318–342.

[10] J. JANAS, S. NABOKO, G. STOLZ, Decay bounds on eigenfunctions and the singular spectrum of unbounded Jacobi matrices, *Internat. Math. Res. Notices* **2009**(2009), 736–764.

[11] Y. LAST, B. SIMON, The essential spectrum of Schrödinger, Jacobi, and CMV operators, *J. Anal. Math.* **98**(2006), 183–220.

[12] M. MOSZYŃSKI, Spectral properties of some Jacobi matrices with double weights, *J. Math. Anal. Appl.* **280**(2003), 400–412.

[13] M. MOSZYŃSKI, Slowly oscillating perturbations of periodic Jacobi operators in $l^2(N)$, *Studia Math.* **192**(2009), 259–279.

[14] M. REED, B. SIMON, *Methods of Modern Mathematical Physics*, Vol. I, Academic Press, New York-London 1972.

[15] G. STOLZ, Spectral theory for slowly oscillating potentials. I. Jacobi matrices, *Manuscripta Math.* **84**(1994), 245–260.

MARCIN MOSZYŃSKI, WYDZIAŁ MATEMATYKI INFORMATYKI I MECHANIKI UNIWERSYTET WARSZAWSKI UL. BANACHA 2, 02-097 WARSZAWA, POLAND
E-mail address: mmoszyns@mimuw.edu.pl

Received March 30, 2009.