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HYPERCYCLICITY OF SHIFTS AS A ZERO-ONE LAW OF ORBITAL LIMIT POINTS

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ABSTRACT. On a separable, infinite dimensional Banach space X, a bounded linear operator $T : X \to X$ is said to be *hypercyclic* if there exists a vector x in X such that its orbit $\mathrm{Orb}(T, x) = \{x, Tx, T^2x, \dots\}$ is dense in X. However, for a unilateral weighted backward shift or a bilateral weighted shift T to be hypercyclic, we show that it suffices to merely require the operator to have an orbit $\mathrm{Orb}(T, x)$ with a non-zero limit point.

KEYWORDS: *Orbital limit points, hypercyclicity, shift operators.*

MSC (2000): 47A16, 46B45, 47B37.

INTRODUCTION

Let X be a separable, infinite dimensional Banach space. A bounded linear operator $T : X \to X$ is said to be *hypercyclic* if there exists a vector x in X such that the orbit $\mathrm{Orb}(T, x) = \{x, Tx, T^2x, \dots\}$ is dense in X. Such a vector x in X is called a *hypercyclic vector* for T.

Among the many examples of operators that admit hypercyclic vectors are the weighted shifts, which constitute a favorite testing ground for the literature of operator theory. Given the canonical base $\{e_n : n \geqslant 0\}$ for $\ell^p(\mathbb{Z}_+)$ with $p \geqslant 1$, we say that a bounded linear operator $T : \ell^p(\mathbb{Z}_+) \to \ell^p(\mathbb{Z}_+)$ is a *unilateral weighted backward shift* if there is a sequence of bounded positive weights $\{w_n\}_{n\geqslant 1}$ such that $Te_n = w_n e_{n-1}$, if $n \geqslant 1$ and $Te_0 = 0$. Similarly, for the canonical base $\{e_n : n \in \mathbb{Z}\}$ of $\ell^p(\mathbb{Z})$ with $p \geqslant 1$ a bounded linear operator $T : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ is a *bilateral weighted backward shift* if there is a bounded positive weight sequence $\{w_n\}_{n\in\mathbb{Z}}$ with $Te_n = w_n e_{n-1}$ for all $n \in \mathbb{Z}$. The weighted backward shifts have been well studied in the area of hypercyclicity, in fact relatively early on Salas [9] characterized the hypercyclic shifts by offering a necessary and sufficient condition in terms of their weight sequences. In the present paper we relate the concept of hypercyclicity to the geometry of an orbit, thus obtaining a new equivalent condition for the hypercyclicity of these shifts.

We would like to point out that our results are also a continuation of the work of Bourdon and Feldman [2], who proved that if an operator T has a somewhere dense orbit $\mathrm{Orb}(T, x)$, then the same orbit will be everywhere dense in X, and thus the operator T is hypercyclic. As it turns out, in the case of weighted backward shifts, either unilateral or bilateral, their remarkable insight can be carried even further. Indeed, in the present paper we show that for a shift to be hypercyclic it suffices to require the operator to have an orbit $\mathrm{Orb}(T, x)$ with a single non-zero limit point, thus relaxing Bourdon and Feldman's condition of having a dense orbit in some open subset of X. However, our condition does not guarantee that the original orbit $\mathrm{Orb}(T, x)$ is dense in X, but we can demonstrate how to construct a hypercyclic vector for T using the non-zero limit point of the orbit. Even more interestingly, the condition above can be relaxed to simply requiring that the orbit has infinitely many members in a ball whose closure avoids the zero vector.

To summarize this behavior of weighted backward shifts we emphasize that a shift T is not hypercyclic if and only if every set of the form $\mathrm{Orb}(T, x) \cup \{0\}$ is closed in X. Thus we uncover the existence of a zero-one law for the hypercyclicity of these shifts, which states that either no orbit has a non-zero limit point in X or some orbit has every vector in X as a limit point.

In [4] we show that this zero-one law for the hypercyclic behavior of shifts is also shared by other classes of operators, in particular the adjoints of the multiplication operators. However at this point it has been shown that this behavior does not generalize to all classes of operators, namely we provide in [4] an example of a linear fractional composition operator that is not hypercyclic and yet it has an orbit with a non-constant limit point.

In Section 1, we provide several conditions that characterize the hypercyclicity of a unilateral weighted backward shift. We then offer a technique for constructing a hypercyclic vector for a unilateral weighted backward shift T having e_0 as a limit point of one of its orbits $\mathrm{Orb}(T, x)$. In Section 2, we give a proof for the bilateral analogue of the above results which calls for different techniques.

Lastly, we would like to point out that very recently many authors have studied the geometry of orbits in relation to hypercyclicity; for instance Badea, Grivaux and Müller [1], Costakis and Manoussos [5] and Prăjitura [8].

1. THE UNILATERAL WEIGHTED BACKWARD SHIFT

Let $\{e_n : n \geqslant 0\}$ be the canonical base for $\ell^p(\mathbb{Z}_+)$ for $p \geqslant 1$, denoted by ℓ^p in the following. A vector x in ℓ^p is denoted by $x = (\widehat{x}(0), \widehat{x}(1), \dots) = \sum\limits_{i=0}^{\infty} \widehat{x}(i) e_i$, where $\sum\limits_{i=0}^{\infty} |\widehat{x}(i)|^p < \infty$. A bounded and linear operator $T : \ell^p \to \ell^p$ is said to be a *unilateral weighted backward shift* if there is a sequence of positive weights

$\{w_n\}_{n\geqslant 1}$ such that $Te_n = w_n e_{n-1}$, if $n \geqslant 1$ and $Te_0 = 0$. In fact, the weight sequence $\{w_n\}_{n\geqslant 1}$ is necessarily bounded because of the boundedness of T, and indeed $\|T\| = \sup\{w_n : n \geqslant 1\}$.

For a unilateral weighted backward shift T to be hypercyclic, Salas [9] provided a necessary and sufficient condition on the weights that $\sup_{n\geqslant 1}\prod_{j=1}^{n} w_j = \infty$. Another characterization was obtained by Chan and Sanders [3], who showed that T is hypercyclic if and only if T is weakly hypercyclic, which means that T has an orbit $\mathrm{Orb}(T,x)$ that is dense in the weak topology of ℓ^p. In the following, we show that the above equivalent conditions can be carried forward to other conditions in terms of the geometry of an orbit.

THEOREM 1.1. *Let $T : \ell^p \to \ell^p$ be a unilateral weighted backward shift. The following are equivalent:*

(i) *T is hypercyclic.*

(ii) *For any vector f in ℓ^p, there exists a vector $x = x(f)$ in ℓ^p whose orbit under T* $\mathrm{Orb}(T,x)$= $\{x, Tx, T^2x, \ldots\}$ *has f as a limit point.*

(iii) *The vector e_0 is a limit point of a certain orbit* $\mathrm{Orb}(T,x)$ *with x in ℓ^p.*

(iv) *There exists an x in ℓ^p whose orbit* $\mathrm{Orb}(T,x)$ *has a non-zero limit point.*

(v) *There exists an x in ℓ^p whose orbit* $\mathrm{Orb}(T,x)$ *has a non-zero weak limit point.*

(vi) *There exists a vector x in ℓ^p whose orbit* $\mathrm{Orb}(T,x)$ *has infinitely many members in an open ball whose closure avoids the origin; that is, there are a non-zero vector f in ℓ^p and a positive r with $r < \|f\|$ such that* $\mathrm{Orb}(T,x) \cap B(f,r)$ *is infinite.*

Before we provide a proof for the theorem, we have a few remarks to illustrate the result.

(1) For statement (iv), the limit point f cannot be chosen to be the zero vector. Take $w_j = \frac{1}{2}$ for all $j \geqslant 1$. Let $x = (x_1, x_2, \ldots)$ be a vector in ℓ^p with infinitely many non-zero entries x_i. Then $T^n x \to 0$ as $n \to \infty$, so the zero vector is a limit point of $\mathrm{Orb}(T,x)$, but T is clearly not hypercyclic.

(2) Regarding the equivalence of statements (i) and (iv), we remark that if an orbit $\mathrm{Orb}(T,x)$ has a non-zero limit point, the vector x that generates the orbit is not necessarily a hypercyclic vector. Take, for instance, the unilateral weighted backward shift $T : \ell^1 \to \ell^1$ whose weight sequence is given by $w_j = 2$ for all $j \geqslant 1$, and the vector $x = (x_1, x_2, \ldots)$ given by $x_{2^k} = 2^{-2^k}$ if $k \geqslant 1$ and $x_j = 0$ if $j \neq 2^k$. Clearly $x \in \ell^1$ and $\|T^{2^k}x - e_0\| = \sum_{j=k+1}^{\infty} 2^{2^k} 2^{-2^j}$, which goes to 0 as a limit, as $k \to \infty$. However, $\widehat{T^n x}(0) = 0$ if $n \neq 2^k$, and $\widehat{T^n x}(0) = 1$ if $n = 2^k$. Hence x is not a hypercyclic vector. In fact, for a similar reason x is not even a supercyclic vector because the scalar multiples of $\mathrm{Orb}(T,x)$ cannot approximate $e_0 + e_1$.

(3) Statement (vi) cannot be relaxed to weakly open sets. That is, a unilateral weighted backward shift T on ℓ^p may not be hypercyclic even if T has an orbit

$\mathrm{Orb}(T,x)$ with infinitely many members inside a weakly open set whose weak closure does not contain zero.

For example, let $g = \sum_{j=1}^{\infty} \frac{1}{2^j} e_{3\cdot 4^{j+1}}$. Clearly $\|g\|^2 = \sum_{j=1}^{\infty} \left(\frac{1}{2^j}\right)^2 = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$. Consider the weakly open set $U = \left\{f \in \ell^2 : |\langle f-g, g\rangle| < \frac{1}{10}\right\}$. Then if $f \in U$, we have that $||\langle f,g\rangle| - \|g\|^2| < \frac{1}{10}$, and thus $\frac{7}{30} < |\langle f,g\rangle| < \frac{13}{30}$. Let $V = \{f \in \ell^2 : |\langle f,g\rangle| < \frac{1}{30}\}$. Then if $f \in V$, $f \notin U$, and hence $U \cap V = \emptyset$. But $0 \in V$, so U is a weakly open set whose weak closure avoids the origin.

We now proceed to define a bounded positive weight sequence for T as follows. For any positive integer j in an interval of the form $[1 + 2\cdot 4^k, 4^{k+1}]$, where $k \geqslant 1$, we define

$$w_{1+2\cdot 4^k} = \cdots = w_{3\cdot 4^k} = \left[\frac{1}{k\cdot 2^{k-1}}\right]^{1/4^k}, \quad w_{1+3\cdot 4^k} = \cdots = w_{4^{k+1}} = [k\cdot 2^{k-1}]^{1/4^k}.$$

For those positive integers j outside the intervals of the form $[1+2\cdot 4^k, 4^{k+1}]$, for $k \geqslant 1$, we simply take $w_j = 1$.

Note that $\sup\{x^{1/4^x} : x \geqslant 1\} < \infty$, and so $\{w_j\}_{j\geqslant 1}$ is a bounded sequence. Hence the unilateral weighted backward shift T with the weight sequence $\{w_j\}_{j\geqslant 1}$ is a bounded linear operator. Furthermore, from Salas' criterion for hypercyclicity for unilateral backward shifts (see [9]) we immediately see that T is not hypercyclic since $w_1 \cdots w_j \leqslant 1$ for all integers $j \geqslant 1$.

Now, let $x = \sum_{j=1}^{\infty} \frac{1}{3j} e_{4^{j+1}}$. Clearly, $\|x\|^2 = \sum_{j=1}^{\infty} (\frac{1}{3j})^2 = \frac{1}{9}\sum_{j=1}^{\infty}\frac{1}{j^2} < \infty$.

Furthermore, for any integer $k \geqslant 2$,

$$T^{4^k} x = \frac{1}{3(k-1)} e_0 + \frac{2^{k-1}}{3} e_{3\cdot 4^k} + \sum_{j=k+1}^{\infty} \frac{1}{3j} w_{4^{j+1}-4^k+1} \cdots w_{4^{j+1}} e_{4^{j+1}-4^k}.$$

Since $4^{j+1} - 4^k > 3\cdot 4^{j+1}$ whenever $j \geqslant k+1$, the above summation is obviously orthogonal to g. Hence, $|\langle T^{4^k} x, g\rangle - \|g\|^2| = |\langle \frac{2^{k-1}}{3} e_{3\cdot 4^k}, g\rangle - \frac{1}{3}| = |\frac{1}{3} - \frac{1}{3}| = 0$ for all $k \geqslant 2$. So, $T^{4^k} x \in U$ for all $k \geqslant 2$, however T is not hypercyclic.

We are now ready to prove the theorem.

Proof. It is clear that (i) implies (vi), (ii) implies (iii), (iii) implies (iv), which in turn implies (v).

To show (i) implies (ii), suppose that T is hypercyclic. Then by definition, there exists $x \in \ell^p$ such that $\mathrm{Orb}(T,x)$ is dense in ℓ^p. Let $f \in \ell^p$. If $f \notin \mathrm{Orb}(T,x)$, then clearly f is a limit point of $\mathrm{Orb}(T,x)$. On the other hand, if $f \in \mathrm{Orb}(T,x)$ and f is not a limit point of the orbit, then there is a neighborhood U of f that contains no point of $\mathrm{Orb}(T,x)$ other than f. But then the points of $U \setminus \{f\}$ are not in the closure of $\mathrm{Orb}(T,x)$, which gives a contradiction.

To show (iv) implies (i), we suppose that there exist a vector x and a nonzero vector $f = (f_0, f_1, f_2, f_3, \ldots)$ in ℓ^p such that f is a limit point of the orbit $\mathrm{Orb}(T,x)$. Since $f_j \neq 0$ for some $j \geqslant 0$, we assume without loss of generality that

$f_0 \neq 0$. Hence there exists an increasing sequence $\{n_k\}_{k\geqslant 1} \subset \mathbb{N}$ and an $N > 0$ such that

$$\|T^{n_k}x - f\| < \frac{1}{2^k} < \frac{|f_0|}{2}, \quad \text{for all } k \geqslant N.$$

Let $x = (x_0, x_1, x_2, \ldots) \in \ell^p$. Then

$$T^{n_k}x = T^{n_k}(x_0, x_1, x_2, \ldots) = (w_1 \cdots w_{n_k} x_{n_k}, \ldots).$$

Hence $\|T^{n_k}x - f\| \geqslant |w_1 \cdots w_{n_k} x_{n_k} - f_0|$. So there exists a sequence $\{n_k\}_{k\geqslant 1}$ such that $|w_1 \cdots w_{n_k} x_{n_k} - f_0| < \frac{|f_0|}{2}$, for all $k \geqslant N$.

Thus $\frac{|f_0|}{2} < |w_1 \cdots w_{n_k} x_{n_k}|$ and so $\frac{|f_0|}{2(w_1 \cdots w_{n_k})} < |x_{n_k}|$ for all $k \geqslant N$. Hence we get that

$$\frac{|f_0|^p}{2^p (w_1 \cdots w_{n_k})^p} < |x_{n_k}|^p, \quad \text{for all } k \geqslant N.$$

Now, since $x \in \ell^p$ we have

$$\frac{|f_0|^p}{2^p} \sum_{k=N}^{\infty} \frac{1}{(w_1 \cdots w_{n_k})^p} \leqslant \sum_{k=N}^{\infty} |x_{n_k}|^p \leqslant \|x\|^p < \infty.$$

It follows that $\frac{1}{(w_1 \cdots w_{n_k})^p} \to 0$, i.e. there exists an increasing sequence $\{n_k\}$ such that $w_1 \cdots w_{n_k} \to \infty$ as $k \to \infty$.

Thus by Salas' criterion for hypercyclicity of unilateral backward shifts that $\sup\limits_{n\geqslant 1} \prod\limits_{j=1}^{n} w_j = \infty$ (see [9]), we have that T is hypercyclic.

To show (v) implies (iv), we suppose there exists a vector x in ℓ^p such that the $\mathrm{Orb}(T, x)$ has $f \in \ell^p$ as a non-zero weak limit point. Since $f \neq 0$, let $k \geqslant 0$ such that $f_k \neq 0$.

Considering the weakly open sets that contain f, we get that for all $j \geqslant 1$ there exists an $n_j \geqslant 1$ such that for k as above, $|\langle T^{n_j}x - f, e_k\rangle| < \frac{1}{j}$.

That is $|w_{k+1} \cdots w_{k+n_j} x_{k+n_j} - f_k| < \frac{1}{j}$, for all $j \geqslant 1$.

Next, we inductively pick a subsequence $\{n_{j_k}\}$ of $\{n_j\}$ as follows:

(1) Let $j_1 = 1$.

(2) Once we have chosen j_m we pick $j_{m+1} > j_m$ such that $k + n_{j_m} < n_{j_{m+1}}$ and $\sum\limits_{i=j_{m+1}}^{\infty} |x_{k+n_i}|^p \leqslant \frac{1}{j_m \cdot \|T\|^{p \cdot n_{j_m}}}$. This can be done since $x \in \ell^p$, so $\sum\limits_{i=1}^{\infty} |x_{k+n_i}|^p \leqslant \|x\|^p < \infty$ and $\frac{1}{j_m \cdot \|T\|^{p \cdot n_{j_m}}}$ has been fixed in the previous m-th step.

Now, without loss of generality we can assume, by taking a subsequence if necessary, that $\{n_j\}$ satisfies $k + n_j < n_{j+1}$ and $\sum\limits_{i=j+1}^{\infty} |x_{k+n_i}|^p \leqslant \frac{1}{j \cdot \|T\|^{p \cdot n_j}}$.

Let $y = \sum\limits_{i=1}^{\infty} x_{k+n_i} \cdot e_{k+n_i}$. Clearly since x is in ℓ^p, so is y, as $\|y\| \leqslant \|x\| < \infty$.

Then $T^{n_m}y = \sum_{i=1}^{\infty} x_{k+n_i} \cdot T^{n_m}e_{k+n_i}$. But $k + n_i < n_{i+1}$ for all $i \geqslant 1$, and so $k + n_i < n_m$ for all $i < m$. Thus since T is a unilateral backward shift we conclude that $T^{n_m}y = \sum_{i=m}^{\infty} x_{k+n_i} \cdot T^{n_m}e_{k+n_i}$.

Furthermore, since the vectors $T^{n_m}e_{k+n_i}$ and $T^{n_m}e_{k+n_j}$ have disjoint support for $i \neq j$, that is $\widehat{T^{n_m}e_{k+n_i}}(s) = 0$ whenever $\widehat{T^{n_m}e_{k+n_j}}(s) \neq 0$, we have that

$$\begin{aligned}
\|T^{n_m}y - f_k e_k\| &\leqslant \|(w_{k+1}\cdots w_{k+n_m}x_{k+n_m} - f_k)\cdot e_k\| + \Big\|\sum_{i=m+1}^{\infty} x_{k+n_i}\cdot T^{n_m}e_{k+n_i}\Big\| \\
&\leqslant |w_{k+1}\cdots w_{k+n_m}x_{k+n_m} - f_k| + \Big[\sum_{i=m+1}^{\infty} |x_{k+n_i}|^p \cdot \|T^{n_m}e_{k+n_i}\|^p\Big]^{1/p} \\
&\leqslant \frac{1}{m^p} + \Big[\sum_{i=m+1}^{\infty} |x_{k+n_i}|^p \cdot \|T\|^{p\cdot n_m}\Big]^{1/p} \leqslant \frac{1}{m^p} + \frac{1}{\sqrt[p]{m}} \to 0 \quad \text{as } m \to \infty.
\end{aligned}$$

Thus $T^{n_m}y \to f_k e_k$ in norm as $m \to \infty$, where $f_k e_k \neq 0$ in ℓ^p. So Orb(T, y) has a non-zero limit point.

To show (vi) implies (i), we suppose there exist non-zero vectors x and f in ℓ^p, and a positive number r with $0 < r < \|f\|$ such that $\mathrm{Orb}(T,x) \cap B(f,r)$ is infinite.

For $p > 1$, we have that ℓ^p is reflexive, so the convex ball $\overline{\mathrm{Ball}(\ell^p)} := \overline{\mathrm{Ball}(\ell^p)}^{\mathrm{wk}} = \overline{\mathrm{Ball}(\ell^p)}^{\|\cdot\|}$ is weakly compact.

Now, since ℓ^p is a Banach space, by the Eberlein–Smulian Theorem, $\overline{\mathrm{Ball}(\ell^p)}$ is weak limit point compact, so every infinite set has a weak limit point.

Since $\mathrm{Orb}(T,x) \cap B(f,r)$ is infinite and included in $\overline{B(f,r)}$ which is weak limit point compact, we conclude that the $\mathrm{Orb}(T,x)$ has a non-zero weak limit point in ℓ^p as $0 \notin \overline{B(f,r)}$. Thus since (v) implies (i) we have that T is hypercyclic.

For the remaining case that $p = 1$, we will show that there exists a vector $y \in \ell^1$ such that $\mathrm{Orb}(T,y)$ has a non-zero limit point. Thus, since (iv) implies (i), it follows that T is hypercyclic.

Claim 1. *Without loss of generality we can assume that f has at most finitely many non-zero entries.*

Proof of Claim 1. Suppose f has infinitely many non-zero entries.

Since the set $\{h \in \ell^1 : h \text{ has finitely many non-zero entries}\}$ is dense in ℓ^1, we have that there exists $h \in \ell^1$ with finitely many non-zero entries such that $\|f - h\| < \frac{\|f\|-r}{2}$.

Thus for $g \in \ell^1$ with $\|g - f\| < r$, we have $\|g - h\| \leqslant \|g - f\| + \|f - h\| < r + \frac{\|f\|-r}{2} = \frac{\|f\|+r}{2} < \|h\|$ (since $\|f\| + \frac{\|f\|-r}{2} < \|h\|$).

Therefore $B(f,r) \subset B(h, \frac{\|f\|+r}{2})$, and hence for $r' = \frac{\|f\|+r}{2}$ we have that the intersection $\mathrm{Orb}(T,x) \cap B(h,r')$ is infinite with $0 < r' < \|h\|$. ∎

Now, by our Claim 1 assume that there exists an $N > 0$ such that $f_k = 0$ for all $k \geqslant N$. By assumption there exist a vector $x \in \ell^1$ and a strictly increasing sequence $\{n_j\} \subset \mathbb{N}$ such that $T^{n_j}x \in B(f, r)$ for all j, where $0 < r < \|f\|$.

Let $E = \{i \geqslant 0 : 0 \leqslant i \leqslant N, f_i \neq 0\}$.

Claim 2. *For all $j \geqslant 0$ there exists $i \in E$ such that $|(\widehat{T^{n_j}x - f})(i)| < \frac{r \cdot |f_i|}{\|f\|}$.*

Proof of Claim 2. Suppose that there exists $j_0 \geqslant 0$ such that for all $i \in E$, $|(\widehat{T^{n_{j_0}}x - f})(i)| \geqslant \frac{r \cdot |f_i|}{\|f\|}$. Then $\|T^{n_{j_0}}x - f\| \geqslant \sum_{i \in E} |(\widehat{T^{n_{j_0}}x - f})(i)| \geqslant \sum_{i \in E} \frac{r \cdot |f_i|}{\|f\|} = \frac{r}{\|f\|} \cdot \sum_{i \in E} |f_i| = r$, which gives a contradiction with $T^{n_{j_0}}x \in B(f, r)$. ∎

Now, since $E \subset \{1, 2, \dots, N\}$ is a finite set we get by our Claim 2 that there exists an $i \in E$ such that for infinitely many j we have $|(\widehat{T^{n_j}x - f})(i)| < \frac{r \cdot |f_i|}{\|f\|}$.

Without loss of generality we can assume, by taking a subsequence if necessary, that there exists $i \in E$ such that for all $j \geqslant 0$, $|(\widehat{T^{n_j}x - f})(i)| < \frac{r \cdot |f_i|}{\|f\|}$.

For notational simplicity assume further that $i = 0$.

By the reverse triangle inequality, $|f_0| - |\widehat{T^{n_j}x}(0)| < \frac{r \cdot |f_0|}{\|f\|}$, and hence for $\alpha := |f_0| \cdot \frac{\|f\| - r}{\|f\|} > 0$ we get that $|\widehat{T^{n_j}x}(0)| > |f_0| - \frac{r \cdot |f_0|}{\|f\|} = |f_0| \cdot \frac{\|f\| - r}{\|f\|} = \alpha > 0$. That is

$$|\widehat{T^{n_j}x}(0)| > \alpha > 0 \quad \text{for all } j \geqslant 0. \tag{1.1}$$

Next, we pick a subsequence $\{n_{j_k}\}$ of $\{n_j\}$ as follows:

(1) Set $j_1 = 1$.

(2) Inductively choose j_{k+1} so that $j_{k+1} > j_k$ and $\sum_{i=j_{k+1}}^{\infty} |x_{n_i}| \leqslant \frac{1}{\|T\|^{n_{j_k}} \cdot (j_k + 1)}$.

Without loss of generality we assume that

$$\sum_{i=k+1}^{\infty} |x_{n_i}| \leqslant \frac{1}{\|T\|^{n_k} \cdot (k+1)}. \tag{1.2}$$

Let $y = \sum_{j=1}^{\infty} \frac{\alpha}{|\widehat{T^{n_j}x}(0)|} |x_{n_j}| e_{n_j}$. Clearly, by (1.1) we have that $\|y\| \leqslant \|x\| < \infty$, and thus $y \in \ell^1$.

Now since T is a backward shift and $n_j < n_{j+1}$ for all $j \geqslant 1$ we have that $T^{n_m}y - \alpha e_0 = \left(\frac{\alpha}{|\widehat{T^{n_m}x}(0)|} |x_{n_m}| \cdot T^{n_m} e_{n_m} - \alpha e_0 \right) + \sum_{j=m+1}^{\infty} \frac{\alpha}{|\widehat{T^{n_j}x}(0)|} |x_{n_j}| \cdot T^{n_m} e_{n_j}$, where $|x_{n_m}| \cdot T^{n_m} e_{n_m} = |x_{n_m}| \cdot w_1 \cdot w_2 \cdots w_{n_m} \cdot e_0 = |\widehat{T^{n_m}x}(0)| e_0$.

So $\|T^{n_m}y - \alpha e_0\| \leqslant 0 + \sum_{j=m+1}^{\infty} \frac{\alpha}{|\widehat{T^{n_j}x}(0)|} |x_{n_j}| \|T^{n_m} e_{n_j}\|$.

Hence (1.1), (1.2) and continuity of T give that $\|T^{n_m}y - \alpha e_0\| \leqslant \sum_{j=m+1}^{\infty} |x_{n_j}| \|T\|^{n_m} \leqslant \|T\|^{n_m} \cdot \frac{1}{\|T\|^{n_m} \cdot (m+1)} = \frac{1}{m+1}$ for all $m \geqslant 1$.

Thus $\|T^{n_m}y - \alpha e_0\| \to 0$ as $m \to \infty$, and hence $\mathrm{Orb}(T, y)$ has the non-zero limit point αe_0. So T is hypercyclic. ∎

As an easy consequence of the equivalence of statements (i) and (iv) in the above theorem, we have the following result.

COROLLARY 1.2. *A unilateral weighted backward shift $T : \ell^p \to \ell^p$ is not hypercyclic if and only if for every x in ℓ^p the set* $\mathrm{Orb}(T, x) \cup \{0\}$ *is closed.*

As we have pointed out in remark (2) after the statement of Thorem 1.1, an orbit $\mathrm{Orb}(T, x)$ may have a non-zero limit point without having the vector x that generates the orbit be hypercyclic for T. Nonetheless, we can demonstrate how to construct a hypercyclic vector for T using the non-zero limit point of the orbit, which is assumed to be e_0 in the following.

Let $1 \leqslant p < \infty$. Let $T : \ell^p \to \ell^p$ be a unilateral weighted backward shift with weight sequence $\{w_j\}_{j\geqslant 1}$ and let $c = \|T\| < \infty$.

Suppose that there exists a vector x in ℓ^p such that the orbit $\mathrm{Orb}(T, x)$ has e_0 as a limit point. We want to construct a hypercyclic vector y for T.

Let $D = \{(a_0, a_1, a_2, \ldots) \in \ell^p : a_i \in \mathbb{Q}$ and $a_i \neq 0$ for all but finitely many $i \in \mathbb{Z}_+\}$. Clearly D is a dense and countable set, so we can enumerate $D = \{d_1, d_2, d_3, \ldots\}$.

Since Orb(T,x) has e_0 as a limit point, there exists a sequence of positive integers $n_k \nearrow \infty$ such that $\|T^{n_k}x - e_0\| < \frac{1}{2^k} < \frac{1}{2}$ for all $k \geqslant 1$. By our proof in Theorem 1.1 we get that $\frac{1}{w_1 \cdots w_{n_k}} \to 0$ as $k \to \infty$.

In the next steps we will construct a sequence of positive integers $\{k_j\}_{j\geqslant 1}$ whose terms we will then use to define the desired hypercyclic vector y for T.

For this purpose we will first construct the sequence $\{k_j\}_{j\geqslant 1}$ subject to the restriction that $y \in \ell^p$, that is $\sum_{j=0}^{\infty} |\widehat{y}(j)|^p < \infty$. Furthermore, we require that there exists a sequence $\{m_l\}_{l\geqslant 1}$ such that for each $l \geqslant 1$ and each $\varepsilon > 0$ there exists an $l_0 \geqslant 1$ having $\|T^{m_{l_0}}y - d_l\| < \varepsilon$. Thus each d_l in D can be approximated arbitrarily close by an element in the Orb(T, y).

We note that for the second condition it suffices to require that there exists a sequence $\{m_l\}_{l\geqslant 1}$ such that $\|T^{m_l}y - d_l\| < \frac{1}{2^l}$ for all $l \geqslant 1$. For if $l \geqslant 1$ and $\varepsilon > 0$, there exists l_0 large enough such that $\frac{1}{2^{l_0}} < \frac{\varepsilon}{2}$ and $\|d_l - d_{l_0}\| < \frac{\varepsilon}{2}$, by the density of the set D. Then $\|T^{m_{l_0}}y - d_l\| \leqslant \|T^{m_{l_0}}y - d_{l_0}\| + \|d_l - d_{l_0}\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon$.

Step 1. In this step we will choose $k_1 \in \mathbb{N}$ and define the first block of entries of y. For this we require the following two conditions:

(a) Since $d_1 \in D$ we can write $d_1 = \sum_{j=0}^{N_1} \alpha_j(1)e_j$ for some $N_1 \geqslant 0$.

We choose k_1 by requiring first that $n_{k_1} > N_1$, and thus having the entire first block defined below fit inside $y \in \ell^p$. The first block of y will have $(N_1 + 1)$

entries and will end at position $n_{k_1} \in \{n_k\}_{k\geqslant 1}$. We further note that in between the blocks of y defined in each step, we set the vector y to have only zero entries.

Position	$n_{k_1} - N_1$	$n_{k_1} - (N_1 - 1)$	$\dots$	n_{k_1}
Entry	$\frac{\alpha_0(1)}{w_1 \cdot w_2 \cdots w_{n_{k_1}-N_1}}$	$\frac{\alpha_1(1)}{w_2 \cdot w_3 \cdots w_{n_{k_1}-(N_1-1)}}$	$\dots$	$\frac{\alpha_{N_1}(1)}{w_{N_1+1} \cdot w_{N_1+2} \cdots w_{n_{k_1}}}$

(b) Dealing with the requirement that $y \in \ell^p$, we need

$$\frac{|\alpha_0(1)|}{w_1 \cdot w_2 \cdots w_{n_{k_1}-N_1}} < \frac{1}{2}, \qquad (1)$$

$$\frac{|\alpha_1(1)|}{w_2 \cdot w_3 \cdots w_{n_{k_1}-(N_1-1)}} < \frac{1}{2^2}, \qquad (2)$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{|\alpha_{N_1}(1)|}{w_{N_1+1} \cdot w_{N_1+2} \cdots w_{n_{k_1}}} < \frac{1}{2^{N_1+1}}, \qquad (N_1+1).$$

Similar conditions for $k_2, k_3, \dots$ will give us that $\|y\|^p = \sum_{j=1}^{\infty} \left(\frac{1}{2^j}\right)^p \leqslant \sum_{j=1}^{\infty} \frac{1}{2^j} = 1$, so $y \in \ell^p$.

We achieve the above finite number of inequalities by using the fact that the product $\frac{1}{w_1 \cdots w_{n_k}} \to 0$ as $k \to \infty$. Namely if k_1 is large enough, $\frac{1}{w_1 \cdots w_{n_{k_1}}}$ can be made as small as needed.

Now we note that

$$\frac{|\alpha_0(1)|}{w_1 \cdot w_2 \cdots w_{n_{k_1}-N_1}} = \frac{|\alpha_0(1)| w_{n_{k_1}-(N_1-1)} \cdots w_{n_{k_1}}}{w_1 \cdot w_2 \cdots w_{n_{k_1}-N_1} \cdot w_{n_{k_1}-(N_1-1)} \cdots w_{n_{k_1}}} \leqslant \frac{|\alpha_0(1)| \cdot c^{N_1}}{w_1 \cdot w_2 \cdots w_{n_{k_1}}}.$$

So condition (1) can be satisfied by choosing the term k_1 large enough such that $\frac{1}{w_1 \cdot w_2 \cdots w_{n_{k_1}}} < \frac{1}{2|\alpha_0(1)| \cdot c^{N_1}}$. Similarly, all conditions (1) trough $(N_1 + 1)$ can be satisfied by choosing k_1 large enough such that $\frac{1}{w_1 \cdot w_2 \cdots w_{n_{k_1}}} < P_1$, where $P_1 = \min\left\{\frac{1}{2^{j+1}|\alpha_j(1)| \cdot c^{N_1}} : j = 0, \dots, N_1\right\}$.

We note that $P_1 = \frac{1}{2^{N_1+1} M_1 \cdot c^{N_1}}$, where $M_1 = \max\{|\alpha_j(1)| : j = 0, 1, \dots, N_1\}$.

We have now chosen $k_1 \in \mathbb{N}$.

Step 2. This step is for choosing $k_2 \in \mathbb{N}$ and defining the second block of entries of y. Write $d_2 = \sum_{j=0}^{N_2} \alpha_j(2) e_j$ for some $N_2 \geqslant 0$. We require the following of k_2:

(a) To avoid the overlapping of entries in the second block of $y \in \ell^p$ with entries in the first block, we need that $n_{k_2} - n_{k_1} > N_2$. Also we would need that $n_{k_2} > N_2$, but this follows from the previous condition. Clearly we require that $k_2 > k_1$. The second block will have $(N_2 + 1)$ entries, ending at position $n_{k_2} \in \{n_k\}_{k\geqslant 1}$.

Position	$n_{k_2} - N_2$	$n_{k_2} - (N_2 - 1)$	$\dots$	n_{k_2}
Entry	$\frac{\alpha_0(2)}{w_1 \cdot w_2 \cdots w_{n_{k_2}-N_2}}$	$\frac{\alpha_1(2)}{w_2 \cdot w_3 \cdots w_{n_{k_2}-(N_2-1)}}$	$\dots$	$\frac{\alpha_{N_2}(2)}{w_{N_2+1} \cdot w_{N_2+2} \cdots w_{n_{k_2}}}$

(b) For the requirement that $y \in \ell^p$ we need to choose k_2 large enough such that $\frac{1}{w_1 \cdot w_2 \cdots w_{n_{k_2}}} < P_2$, where $P_2 = \min\left\{\frac{1}{2^{j+1+(N_1+1)}|\alpha_j(2)| \cdot c^{N_2}} : j = 0, 1, \dots, N_2\right\}$.

We note that $P_2 = \frac{1}{2^{N_1+N_2+2} M_2 \cdot c^{N_2}}$, where $M_2 = \max\left\{|\alpha_j(2)| : j = 0, \dots N_2, \right\}$.

(c) For the choice of k_2, we need an extra condition since we now want to also verify the condition that $\|T^{m_1} y - d_1\| < \frac{1}{2}$.

Let $m_1 = n_{k_1} - N_1$. The next conditions will clearly give us this last requirement.

Shifting the vector y by $m_1 = n_{k_1} - N_1$ will produce the following changes to the coefficients of the second block:

Position	$n_{k_2} - N_2 - m_1$	$\dots$	$n_{k_2} - m_1$
Entry	$\frac{\alpha_0(2) \cdot w_{n_{k_2}-N_2-(m_1-1)} \cdots w_{n_{k_2}-N_2}}{w_1 \cdot w_2 \cdots w_{n_{k_2}-N_2}}$	$\dots$	$\frac{\alpha_{N_2}(2) \cdot w_{n_{k_2}-(m_1-1)} \cdots w_{n_{k_2}}}{w_{N_2+1} \cdot w_{N_2+2} \cdots w_{n_{k_2}}}$

But each entry above is in absolute value bounded above by $\frac{|\alpha_j(2)| \cdot c^{N_2} \cdot c^{m_1}}{w_1 \cdots w_{n_{k_2}}}$, which can be made as small as needed.

So we now choose k_2 large enough such that $\frac{1}{w_1 \cdot w_2 \cdots w_{n_{k_2}}} < Q_2$, where $Q_2 = \min\left\{\frac{1}{2^2(N_2+1)|\alpha_j(2)| \cdot c^{N_2+m_1}} : j = 0, 1, \dots, N_2\right\} = \frac{1}{2^2(N_2+1) \cdot M_2 \cdot c^{N_2+m_1}}$.

We have now chosen $k_2 \in \mathbb{N}$.

Step 3. We will choose the general term $k_j \in \mathbb{N}$ for $j \geqslant 3$ satisfying:

(a) $k_j > k_{j-1}$ and $n_{k_j} - n_{k_{j-1}} > N_j$;

(b) $\frac{1}{w_1 \cdot w_2 \cdots w_{n_{k_j}}} < \frac{1}{M_j \cdot c^{N_j} \cdot 2^{j + \sum_{t=1}^{j} N_t}}$;

(c) $\frac{1}{w_1 \cdot w_2 \cdots w_{n_{k_j}}} < \frac{1}{M_j \cdot c^{N_j + m_l} \cdot (N_j+1) \cdot 2^{j-1+l}}$, for all $l = 1, 2, \dots, j-1$, where $m_l = n_{k_l} - N_l$ and $m_1 < m_2 < m_3 < \cdots$.

Note that the entries of y in the j-th block are: $\frac{\alpha_0(j)}{w_1 \cdot w_2 \cdots w_{n_{k_j}-N_j}}$, $\frac{\alpha_1(j)}{w_2 \cdot w_3 \cdots w_{n_{k_j}-(N_j-1)}}$, $\dots$, $\frac{\alpha_{N_j}(j)}{w_{N_j+1} \cdot w_{N_j+2} \cdots w_{n_{k_j}}}$.

Thus in Steps 1–3 we have chosen the sequence $\{k_l\}_{l \geqslant 1}$ such that $y \in \ell^p$ and $\|T^{m_l} y - d_l\| < \frac{1}{2^l}$ for all $l \geqslant 1$ where $d_l \in D$. Thus for all d_l in the dense set D, we have that $d_l \in \overline{\mathrm{Orb}(T, y)}$, so y is a hypercyclic vector for T.

2. THE BILATERAL WEIGHTED BACKWARDS SHIFT

After examining how hypercyclicity relates to having an orbit with a non-zero limit point for the unilateral weighted backward shifts in Section 1, we turn to the study of bilateral weighted shifts.

Let $\{e_n : n \in \mathbb{Z}\}$ be the canonical basis for $\ell^p(\mathbb{Z})$ for $p \geqslant 1$. Then, a bounded and linear operator $T : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ is said to be a *bilateral weighted backward shift* if there is a sequence of bounded positive weights $\{w_n : n \in \mathbb{Z}\}$ such that $Te_n = w_n e_{n-1}$ for all $n \in \mathbb{Z}$.

Analogous to the unilateral weighted shift we have the following result.

THEOREM 2.1. *Suppose that $T : \ell^p(\mathbb{Z}) \to \ell^p(\mathbb{Z})$ is a bilateral weighted backward shift. The following are equivalent:*

(i) *T is hypercyclic.*

(ii) *There exists an x in $\ell^p(\mathbb{Z})$ whose orbit* $\mathrm{Orb}(T, x)$ *has a non-zero limit point.*

(iii) *There exists a vector x in $\ell^p(\mathbb{Z})$ whose orbit* $\mathrm{Orb}(T, x)$ *has infinitely many members in an open ball $B(h, r)$, where $0 < r < \|h\|$.*

Before we give a proof of the theorem, we remark that statement (ii) cannot be relaxed to having an orbit with a non-zero weak limit point in $\ell^p(\mathbb{Z})$, as is the case in Theorem 1.1. Chan and Sanders [3] showed the existence of a bilateral weighted backward shift that is weakly hypercyclic, and thus has a non-zero weak limit point, but the shift is not norm hypercyclic.

For a bilateral weighted shift to be hypercyclic, Salas [9] provided a necessary and sufficient condition in terms of the weights: for every $\varepsilon > 0$ and every $q \in \mathbb{N}$ there exists n arbitrarily large such that for every $j \in \mathbb{Z}$ with $|j| \leqslant q$ we have $\prod_{s=0}^{n-1} w_{s+j} > \frac{1}{\varepsilon}$ and $\prod_{s=1}^{n} w_{j-s} < \varepsilon$. However, as it turns out, this condition is not as helpful in proving the above theorem as its counterpart for the unilateral case that we have studied in Section 1. For that reason, we now offer a constructive argument to prove Theorem 2.1.

Proof. It is clear that (i) implies (iii).

To show that statement (ii) implies (i), we suppose without loss of generality that $x = (\ldots, \widehat{x}(-1), \widehat{x}(0), \widehat{x}(1), \ldots)$ is a vector in $\ell^p(\mathbb{Z})$ such that e_0 is the non-zero limit point of the orbit Orb(T,x). We set $p = 2$ for notational simplicity.

Hence there is an increasing sequence of positive integers $\{n_i\}$ such that

$$\|T^{n_i} x - e_0\| < \frac{1}{2^i} \quad \text{for all } i \geqslant 1.$$

Thus $|w_1 \cdot w_2 \cdots w_{n_i} \widehat{x}(n_i) - 1| < \frac{1}{2^i} < \frac{1}{2}$ for all $i \geqslant 1$. This then implies that if $i \geqslant 1$,

$$\frac{1}{2w_1 \cdots w_{n_i}} < |\widehat{x}(n_i)|. \tag{2.1}$$

Since $\sum_{i=1}^{\infty} |\widehat{x}(n_i)|^2 \leqslant \|x\|^2 < \infty$, the above inequality gives that

$$\sum_{i=1}^{\infty} \Big[\frac{1}{w_1 \cdots w_{n_i}}\Big]^2 < \infty,$$

and thus

$$\frac{1}{w_1 \cdots w_{n_i}} \to 0 \quad \text{as } i \to \infty. \tag{2.2}$$

By considering the terms of the vector $T^{n_j}x$ with negative indices we have that

$$\sum_{i=1}^{j-1} |w_{-(n_j-n_i-1)} \cdots w_0 \cdots w_{n_i} \cdot \widehat{x}(n_i)|^2 < \frac{1}{2^{2j}}.$$

Now, focusing on the i-th term of the above summation and using (2.1), we see that if $1 \leqslant i \leqslant j-1$, then

$$w_{-(n_j-n_i-1)} \cdots w_0 < \frac{1}{2^{j-1}}. \tag{2.3}$$

Claim. Let $y = (\dots, 0, \widehat{y}(0), \dots, \widehat{y}(k), 0, \dots)$ *and* $a = (\dots, 0, \widehat{a}(-l), \dots, \widehat{a}(0), \dots, \widehat{a}(l), 0, \dots)$ *in* $\ell^2(\mathbb{Z})$ *with* k *and* $l \geqslant 1$. *For all positive* ε, *there exist an integer* $m > k + l$ *and a vector* z *of the form* $z = (\dots, 0, \widehat{z}(m-l), \dots, \widehat{z}(m), \dots, \widehat{z}(m+l), 0, \dots)$ *such that:*

(i) $T^m z = a$;
(ii) $\|z\| < \varepsilon$;
(iii) $\|T^s z\| < \varepsilon$, *if* $1 \leqslant s \leqslant k$;
(iv) $\|T^m y\| < \varepsilon$.

Proof of Claim. Note that $\|T\| = \sup\{|w_i| : i \in \mathbb{Z}\} > 1$.

Choose an integer i such that $n_i > k + l$, where k and l are positive integers given in the vectors y and a in the statement of the Claim. For this fixed n_i and for any $j > i$ we denote $m = n_j - n_i + k$.

We first observe by (2.3) that

$$\begin{aligned}
\|T^m y\|^2 &= \sum_{r=0}^{k} |w_{-(m-r-1)} \cdots w_{-(n_j-n_i-1)} \cdots w_0 \cdots w_r \cdot \widehat{y}(r)|^2 \\
&\leqslant \|T\|^{2k} \cdot \|y\|^2 \cdot \sum_{r=0}^{k} [w_{-(m-r-1)} \cdots w_{-(n_j-n_i-1)} \cdots w_0]^2 \\
&\leqslant \|T\|^{2k} \cdot \|y\|^2 \cdot \|T\|^{2k} \sum_{r=0}^{k} [w_{-(n_j-n_i-1)} \cdots w_0]^2 \\
&\leqslant \|T\|^{4k} \cdot \|y\|^2 \cdot (k+1) \cdot \frac{1}{2^{2(j-1)}}.
\end{aligned}$$

Next we need to find z such that $T^m z = a$, which is equivalent to having $w_{r+1} \cdots w_{m+r} \cdot \widehat{z}(m+r) = \widehat{a}(r)$ whenever $-l \leqslant r \leqslant l$. Then,

$$\|z\|^2 = \sum_{r=-l}^{l} |\widehat{z}(m+r)|^2 = \sum_{r=-l}^{l} \Big| \frac{\widehat{a}(r)}{w_{r+1} \cdots w_{m+r}} \Big|^2 \leqslant \|a\|^2 \sum_{r=-l}^{l} \Big[\frac{1}{w_{r+1} \cdots w_{m+r}} \Big]^2.$$

Note that if $-l \leqslant r \leqslant l$, then $m+r \leqslant m+l = n_j - n_i + k + l = n_j - (n_i - k - l) < n_j$ since $n_i > k + l$.

Thus, $\|z\|^2 \leqslant \|a\|^2 \sum_{r=-l}^{l} \big[\frac{w_{m+r+1} \cdots w_{n_j}}{w_{r+1} \cdots w_{n_j}}\big]^2$.

We now let $c_\ell = \max\big\{1, \frac{1}{w_0}, \frac{1}{w_{-1}w_0}, \ldots, \frac{1}{w_{-l+1} \cdots w_0}\big\}$ and observe that the numerator $w_{m+r+1} \cdots w_{n_j}$ is a product of $n_j - (m+r) = n_j - [n_j - n_i + k + r] = n_i - k - r$ factors.

Hence, $\|z\|^2$ is bounded above by

$$\|a\|^2 \cdot \|T\|^{2(n_i-k+l)} \cdot \sum_{r=-l}^{0} \Big[\frac{1}{w_{r+1} \cdots w_{n_j}} \Big]^2 + \|a\|^2 \cdot \|T\|^{2(n_i-k-1)} \cdot \sum_{r=1}^{l} \Big[\frac{1}{w_{r+1} \cdots w_{n_j}} \Big]^2$$

$$\leqslant \|a\|^2 \cdot \|T\|^{2(n_i-k+l)} \cdot c_\ell^2 \sum_{r=-l}^{0} \Big[\frac{1}{w_1 \cdots w_{n_j}} \Big]^2 + \|a\|^2 \cdot \|T\|^{2(n_i-k-1)} \cdot \sum_{r=1}^{l} \Big[\frac{w_1 \cdots w_r}{w_1 \cdots w_{n_j}} \Big]^2$$

$$\leqslant \Big[\frac{1}{w_1 \cdots w_{n_j}} \Big]^2 \{ \|a\|^2 \cdot \|T\|^{2(n_i-k+l)} \cdot c_\ell^2 \cdot (l+1) + \|a\|^2 \cdot \|T\|^{2(n_i-k-1)} \cdot \|T\|^{2l} \cdot l \}.$$

Note that if $1 \leqslant s \leqslant k$, then $\|T^s z\| \leqslant \|T\|^k \|z\|$.

It is now evident that for any given $\varepsilon > 0$, we can use (2.2) to choose an integer $j > i$ so that (ii), (iii) and (iv) in the Claim are satisfied. ∎

To finish the proof, it remains to show that our Claim implies that T is hypercyclic. For this, let $D = \{d_1, d_2, \ldots\}$ be a dense subset of $\ell^2(\mathbb{Z})$, where each d_i is of the form $d_i = (\ldots, 0, \widehat{d}(-l_i), \ldots, \widehat{d}(0), \ldots, \widehat{d}(l_i), 0, \ldots)$.

First, take $y_1 = 0$ with $k_1 = 1$, $a_1 = d_1$ and $\varepsilon = \frac{1}{4^1}$ as in the statement of the Claim. Then our Claim gives us that there exists an $m_1 \in \mathbb{N}$ and a vector z_1 with $\widehat{z_1}(i) = 0$ whenever $i \geqslant m_1 + l_1$ such that:

(1) $T^{m_1} z_1 = d_1$;
(2) $\|z_1\| < \frac{1}{4^1}$;
(3) $\|Tz_1\| < \frac{1}{4^1}$;
(4) $\|T^{m_1} y_1\| = 0$.

Inductively, we take $k_j = m_{j-1} + l_{j-1}$ and $y_j = z_1 + z_2 + \cdots + z_{j-1}$. Clearly, $\{k_j\}_{j \geqslant 1}$ is an increasing sequence, as $k_{j+1} = m_j + l_j > k_j + 2l_j$ for all $j \geqslant 1$.

Thus we note that $\widehat{y_j}(i) = 0$ whenever $i \geqslant k_j$.

Let $\varepsilon = \frac{1}{4^j}$ in the statement of the Claim. Then our Claim implies that there exist $m_j \in \mathbb{N}$ and a vector z_j with $\widehat{z_j}(i) = 0$ whenever $i \geqslant m_j + l_j$ so that:

(1) $T^{m_j} z_j = d_j$;

(2) $\|z_j\| < \frac{1}{4^j}$;
(3) $\|T^s z_j\| < \frac{1}{4^j}$, if $1 \leqslant s \leqslant k_j$;
(4) $\|T^{m_j} y_j\| < \frac{1}{4^j}$.

Define the vector b by setting $b = \sum_{i=1}^{\infty} z_i$. Since the sum is absolutely convergent, we have that $\|b\| = \left\| \sum_{i=1}^{\infty} z_i \right\| \leqslant \sum_{i=1}^{\infty} \|z_i\| < \sum_{i=1}^{\infty} \frac{1}{4^i} < \infty$, so $b \in \ell^2(\mathbb{Z})$.

Also note that $b = y_j + z_j + \sum_{i=j+1}^{\infty} z_i$, so

$$\begin{aligned}\|T^{m_j} b - d_j\| &\leqslant \|T^{m_j} y_j\| + \|T^{m_j} z_j - d_j\| + \left\| T^{m_j} \Big(\sum_{i=j+1}^{\infty} z_i \Big) \right\| \\ &< \frac{1}{4^j} + 0 + \sum_{i=j+1}^{\infty} \|T^{m_j} z_i\| < \frac{1}{4^j} + \sum_{i=j+1}^{\infty} \frac{1}{4^i} = \sum_{i=j}^{\infty} \frac{1}{4^i} \quad \to 0 \text{ as } j \to \infty.\end{aligned}$$

Thus $\|T^{m_j} b - d_j\| \to 0$ as $j \to \infty$, so by density of the set $D = \{d_1, d_2, \ldots\}$ we have that b is a hypercyclic vector for T.

To show that (iii) implies (ii), we suppose that there exists a vector x in $\ell^p(\mathbb{Z})$ whose orbit Orb(T,x) has infinitely many members in an open ball $B(h, r)$, where $0 < r < \|h\|$. For notational simplicity we set $p = 2$.

Let $s > 0$ such that $\frac{1-s}{1+s} > \frac{r}{\|h\|}$. Since $h \in \ell^2(\mathbb{Z})$, there exists $N \geqslant 1$ such that $\frac{\|h\|^2}{(1+s)^2} < \sum_{i=-N}^{N} |\widehat{h}(i)|^2$. Let $\{n_j\}_{j \geqslant 1}$ be an increasing sequence of integers such that $T^{n_j} x \in B(h, r)$ and set $f_j = T^{n_j} x$.

We now show that for each f_j there exists an integer i with $i \leqslant N$ such that $0 < s \cdot |\widehat{h}(i)| < |\widehat{f_j}(i)|$ whenever $|i| \leqslant N$.

To do that, we suppose on the contrary that there exists an f_j so that $|\widehat{f_j}(i)| \leqslant s \cdot |\widehat{h}(i)|$ whenever $|i| \leqslant N$. This would imply that, by our choice of N,

$$(1-s)^2 \frac{\|h\|^2}{(1+s)^2} < (1-s)^2 \sum_{i=-N}^{N} |\widehat{h}(i)|^2 \leqslant \sum_{i=-N,\, \widehat{h}(i) \neq 0}^{N} |\widehat{h}(i) - \widehat{f_j}(i)|^2 \leqslant \|h - f_j\|^2 < r^2,$$

which would contradict our choice of s.

Since there are finitely many integers i with $|i| \leqslant N$, but infinitely many $j \geqslant 1$, there exists an integer i_0 with $|i_0| \leqslant N$ such that $0 < s \cdot |\widehat{h}(i_0)| < |\widehat{f_j}(i_0)|$, for infinitely many j.

We now assume $i_0 = 0$ for notational simplicity. In addition, by taking a subsequence of $\{n_j\}$ if necessary, we further assume that

$$0 < s \cdot |\widehat{h}(0)| < |\widehat{f_j}(0)|, \quad \text{whenever } j \geqslant 1. \tag{2.4}$$

Recall that $f_j = T^{n_j}x$, and thus

$$\widehat{f}_j(0) = w_1 \cdots w_{n_j}\widehat{x}(n_j), \tag{2.5}$$

from which it follows that

$$\infty > \|x\|^2 \geqslant \sum_{j=-\infty}^{\infty} |\widehat{x}(n_j)|^2 = \sum_{j=-\infty}^{\infty} \frac{|\widehat{f}_j(0)|^2}{(w_1 \cdots w_{n_j})^2} \geqslant s^2|\widehat{h}(0)|^2 \sum_{j=-\infty}^{\infty} \Big(\frac{1}{w_1 \cdots w_{n_j}}\Big)^2.$$

Hence

$$\frac{1}{w_1 \cdots w_{n_j}} \to 0 \quad \text{as } j \to \infty. \tag{2.6}$$

This means that $\|T\| = \sup\{w_j : j \in \mathbb{Z}\} > 1$.

We now switch our attention to the weights with negative indices. First, we note that $T^{n_j}x \in B(h, r)$ for all j, and hence, there exists $M > 0$ so that $\|T^{n_j}x\| \leqslant M$ for all j.

Secondly we note that if $1 \leqslant i < j$, then

$$T^{n_j}(\widehat{x}(n_i)e_{n_i}) = w_{-(n_j-n_i-1)} \cdots w_0 \cdots w_{n_i}\widehat{x}(n_i)e_{n_i-n_j}.$$

Thus,

$$\begin{aligned}
M^2 &\geqslant \|T^{n_j}x\|^2 \geqslant \Big\|\sum_{i=1}^{j-1} T^{n_j}(\widehat{x}(n_i)e_{n_i})\Big\|^2 = \sum_{i=1}^{j-1}(w_{-(n_j-n_i-1)} \cdots w_0 \cdots w_{n_i})^2|\widehat{x}(n_i)|^2 \\
&= \sum_{i=1}^{j-1}(w_{-(n_j-n_i-1)} \cdots w_0)^2|\widehat{f}_i(0)|^2, \quad \text{by (2.3)} \\
&> s^2|\widehat{h}(0)|^2\sum_{i=1}^{j-1}(w_{-(n_j-n_i-1)} \cdots w_0)^2, \quad \text{by (2.2)}.
\end{aligned}$$

In other words, if $1 \leqslant i < j$, then

$$\sum_{i=1}^{j-1}(w_{-(n_j-n_i-1)} \cdots w_0)^2 < \frac{M^2}{s^2|\widehat{h}(0)|^2}. \tag{2.7}$$

Claim. For every $\varepsilon > 0$ and every integer $k \geqslant 1$, there exist positive integers j and m with $m < j$ so that:

(i) $k < m$;

(ii) $2n_m < n_j$;

(iii) $w_{-(n_j-n_m-1)} \cdots w_0 < \frac{\varepsilon}{\sqrt{n_k}\|T\|^{n_k}}$;

(iv) $\frac{1}{w_1 \cdots w_{n_j}} < \frac{\varepsilon}{\|T\|^{2n_m}}$.

Proof of Claim. Let $\varepsilon > 0$ and $k \geqslant 1$ be given. Determine a positive integer $t > k + 1$ so that $\frac{M^2}{s^2|\widehat{h}(0)|^2(t-k)} < \big[\frac{\varepsilon}{\sqrt{n_k}\|T\|^{n_k}}\big]^2$.

For that choice of t, we can determine $j > t$ such that $n_j > 2n_t$ and $\frac{1}{w_1 \cdots w_{n_j}} < \frac{\varepsilon}{\|T\|^{2n_t}}$, by (2.6).

Since $k+1 < t < j$, we have, by (2.7),

$$\sum_{i=k+1}^{t} (w_{-(n_j-n_i-1)} \cdots w_0)^2 < \frac{M^2}{s^2|\widehat{h}(0)|^2}.$$

The summation on the left-hand side has $t-k$ positive terms, so there must exist an integer m with $k+1 \leqslant m \leqslant t$ such that

$$(w_{-(n_j-n_m-1)} \cdots w_0)^2 < \frac{M^2}{s^2|\widehat{h}(0)|^2(t-k)} < \Big[\frac{\varepsilon}{\sqrt{n_k}\|T\|^{n_k}}\Big]^2,$$

and hence (iii) is satisfied.

Since $k+1 \leqslant m \leqslant t < j$, we see that (i), (ii) and (iv) are satisfied too. ∎

To finish the proof, we use our Claim to construct a vector z whose orbit $\mathrm{Orb}(T,z)$ has e_0 as a limit point.

Take $k=1$ and $\varepsilon = \frac{1}{2}$ in our Claim. We then have positive integers j and m with $j > m$ so that (i) through (iv) are satisfied. For notational simplicity, we can assume that $m=2$ and $j=3$, because we can certainly achieve that by taking a subsequence of $\{n_i\}$ having the first three terms n_k, n_m and n_j. Hence, $2n_2 < n_3, w_{-(n_3-n_2-1)} \cdots w_0 < \frac{1}{2\sqrt{n_1}\|T\|^{n_1}}$, and $\frac{1}{w_1 \cdots w_{n_3}} < \frac{1}{2\|T\|^{2n_2}}$.

Inductively, for every odd integer $k \geqslant 1$, we take $\varepsilon = 2^{-(k+1)/2}$ in the Claim, and by choosing a subsequence of $\{n_i\}$, we get three consecutive integers $k < m < j$ such that (i) through (iv) are satisfied. In this way we can assume that the original sequence $\{n_i\}$ satisfies that, for any $q \geqslant 1$,:

(1) $n_{2q-1} < n_{2q}$;

(2) $2n_{2q} < n_{2q+1}$;

(3) $w_{-(n_{2q+1}-n_{2q}-1)} \cdots w_0 < \frac{1}{2^q\sqrt{n_{2q-1}}\|T\|^{n_{2q-1}}}$;

(4) $\frac{1}{w_1 \cdots w_{n_{2q+1}}} < \frac{1}{2^q\|T\|^{2n_{2q}}}$.

Let $m_i = n_{2i+1} - n_{2i} + n_{2i-1}$ for $i \geqslant 1$ and set $z = \sum_{i=1}^{\infty} \frac{1}{w_1 \cdots w_{m_i}} e_{m_i}$. Then,

$$\begin{aligned}
\|z\|^2 &= \sum_{i=1}^{\infty}\Big[\frac{1}{w_1 \cdots w_{m_i}}\Big]^2 = \sum_{i=1}^{\infty}\Big[\frac{w_{m_i+1} \cdots w_{n_{2i+1}}}{w_1 \cdots w_{n_{2i+1}}}\Big]^2 \\
&\leqslant \sum_{i=1}^{\infty}\Big[\frac{\|T\|^{n_{2i}-n_{2i-1}}}{w_1 \cdots w_{n_{2i+1}}}\Big]^2 \leqslant \sum_{i=1}^{\infty}\Big[\frac{\|T\|^{n_{2i}}}{2^i\|T\|^{2n_{2i}}}\Big]^2 \quad \text{(by (4))} \\
&< \sum_{i=1}^{\infty}\frac{1}{4^i\|T\|^{2n_{2i}}} < \sum_{i=1}^{\infty}\frac{1}{4^i} < \infty,
\end{aligned}$$

so clearly z is in $\ell^2(\mathbb{Z})$. Now,

$$T^{m_k}z = \sum_{i=1}^{k-1} w_{-(m_k-m_i-1)} \cdots w_0 e_{m_i-m_k} + e_0 + \sum_{i=k+1}^{\infty} \frac{1}{w_1 \cdots w_{m_i-m_k}} e_{m_i-m_k},$$

and so

$$\|T^{m_k}z - e_0\|^2 = \sum_{i=1}^{k-1}(w_{-(m_k-m_i-1)}\cdots w_0)^2 + \sum_{i=k+1}^{\infty}\Big(\frac{1}{w_1\cdots w_{m_i-m_k}}\Big)^2.$$

To estimate the first summation, we first note that its subindex $m_k - m_i - 1 = n_{2k+1} - n_{2k} + n_{2k-1} - m_i - 1 = (n_{2k+1} - n_{2k} - 1) + (n_{2k-1} - m_i)$.

Since the summation index runs between 1 and $k-1$, we have $m_i \leqslant m_{k-1} = n_{2k-1} - n_{2k-2} + n_{2k-3}$, and so $n_{2k-1} - m_i \geqslant 0$. Hence, using (3) and the fact that $\{n_i\}$ is increasing and so $n_{2k-1} \geqslant 2k-1 > k-1$, we have that

$$\begin{aligned}\sum_{i=1}^{k-1}(w_{-(m_k-m_i-1)}\cdots w_0)^2 &\leqslant \sum_{i=1}^{k-1}\|T\|^{2(n_{2k-1}-m_i)}(w_{-(n_{2k+1}-n_{2k}-1)}\cdots w_0)^2\\ &< \sum_{i=1}^{k-1}\frac{1}{4^k n_{2k-1}} < \frac{1}{4^i}.\end{aligned}$$

To estimate the second summand, we proceed as follows,

$$\sum_{i=k+1}^{\infty}\Big(\frac{1}{w_1\cdots w_{m_i-m_k}}\Big)^2 = \sum_{i=k+1}^{\infty}\Big(\frac{w_{m_i-m_k+1}\cdots w_{n_{2i+1}}}{w_1\cdots w_{n_{2i+1}}}\Big)^2.$$

Note that the number of weights in the numerator of the previous expression is given by $n_{2i+1} - m_i + m_k = n_{2i+1} - (n_{2i+1} - n_{2i} + n_{2i-1}) + m_k < n_{2i} + m_k < 2n_{2i}$, because $i \geqslant k+1$ and $m_k < n_{2k+1}$ by its definition.

Hence,

$$\begin{aligned}\sum_{i=k+1}^{\infty}\Big(\frac{1}{w_1\cdots w_{m_i-m_k}}\Big)^2 &< \sum_{i=k+1}^{\infty}\Big(\frac{\|T\|^{2n_{2i}}}{w_1\cdots w_{n_{2i+1}}}\Big)^2\\ &< \sum_{i=k+1}^{\infty}\Big(\frac{\|T\|^{2n_{2i}}}{2^i\|T\|^{2n_{2i}}}\Big)^2 \quad \text{(by (4))}\\ &= \sum_{i=k+1}^{\infty}\frac{1}{4^i} < \frac{1}{3\cdot 4^k}.\end{aligned}$$

By combining both estimates, we see that $\|T^{m_k}z - e_0\|^2 < \frac{1}{4^k} + \frac{1}{3\cdot 4^k}$, which goes to 0 as $k \to \infty$. Hence, e_0 is a limit point of the orbit $\mathrm{Orb}(T,z)$. ∎

A much simpler argument for showing (ii) implies (i) can be made if we require the bilateral weighted shift having e_0 as a limit point of $\mathrm{Orb}(T,x)$ to be invertible.

Since e_0 is a limit point, there exists a sequence of integers $n_k \nearrow \infty$ such that $\|T^{n_k}x - e_0\| < \frac{1}{2^k} < \frac{1}{2}$ for all $k \geqslant 1$. Thus we have shown previously that $|w_1\cdot w_2\cdots w_{n_k}\widehat{x}(n_k) - 1| < \frac{1}{2}$ for all $k \geqslant 1$, and consequently that $\prod_{j=1}^{n_k} w_j \to \infty$.

It also follows that $|w_{-(n_k-n_1-1)}\cdots w_{-1}\cdot w_0\cdot w_1\cdots w_{n_1}\widehat{x}(n_1)| < \frac{1}{2^k}$ for all $k\geqslant 2$, and thus since $\frac{1}{w_1\cdots w_{n_1}|\widehat{x}(n_1)|} < 2$ we observe that $w_{-(n_k-n_1-1)}\cdots w_{-1}\cdot w_0 < \frac{1}{2^k}\cdot 2 = \frac{1}{2^{k-1}}$ for all $k\geqslant 2$.

Furthermore, $w_{-n_k}\cdots w_{-1}\cdot w_0 < \frac{w_{-n_k}\cdots w_{-n_k+n_1}}{2^{k-1}}$ for all $k\geqslant 2$.

Hence $0 < w_{-n_k}\cdots w_{-1}\cdot w_0 < \frac{\|T^{n_1+1}\|}{2^{k-1}}$ for all $k\geqslant 2$. Letting $k\to\infty$ we get that $w_{-n_k}\cdots w_{-1}\cdot w_0\to 0$, and therefore $\prod_{j=1}^{n_k} w_{-j}\to 0$.

Thus there is a sequence $n_k\nearrow\infty$ so that $\prod_{j=1}^{n_k} w_j\to\infty$ and $\prod_{j=1}^{n_k} w_{-j}\to 0$, so by Feldman's criterion for invertible bilateral shifts [6] we have that T is hypercyclic.

From this we can now deduce the following statement, using the fact that if T is hypercyclic then so is T^{-1} (see Kitai [7]).

COROLLARY 2.2. *If $T:\ell^p(\mathbb{Z})\to\ell^p(\mathbb{Z})$ is an invertible bilateral weighted shift having an orbit with a non-zero limit point, then T^{-1} also has an orbit with a non-zero limit point.*

As we have pointed out before, Salas [9] proved that a bilateral weighted shift T is hypercyclic if and only if for every $\varepsilon > 0$ and every $q\in\mathbb{N}$ there exists n arbitrarily large such that for every $j\in\mathbb{Z}$ with $|j|\leqslant q$ we have $\prod_{s=0}^{n-1} w_{s+j} > \frac{1}{\varepsilon}$ and $\prod_{s=1}^{n} w_{j-s} < \varepsilon$. We would like to point out that Theorem 2.1 offers a new equivalent condition to check whether or not an operator T is hypercyclic. To illustrate the applicability of our condition we offer an example of an operator T for which it is not easy to check hypercyclicity using Salas' condition, but quite easy using ours.

Let T be a bilateral weighted backward shift on $\ell^2(\mathbb{Z})$ with weight sequence $\{w_j\}_{-\infty}^{\infty}$ defined as follows. Let $k_1 = 1$ and $n_1 = 1$. Recursively we take $k_j = n_{j-1}+j$ and $n_j = n_{j-1}+k_j+1$ for all $j\geqslant 2$. Let $w_0 = 1$. We define the weights with positive indices in consecutive blocks, setting the j-th block to have k_j entries of 2's followed by the entry $\frac{1}{2^{k_j}}$.

Similarly, we define the weights with negative indices in consecutive blocks having the following entries:

block 1: one weight with value $\frac{1}{2^{1+l_1}}$, followed by l_1 entries of 2's;

block 2: $\frac{1}{2^{3+l_3}}$, then l_3 entries of 2's; $\frac{1}{2^{2+l_2}}$, then l_2 entries of 2's;

block 3: $\frac{1}{2^{6+l_6}}$, then l_6 entries of 2's; $\frac{1}{2^{5+l_5}}$, then l_5 entries of 2's; $\frac{1}{2^{4+l_4}}$, then l_4 entries of 2's, where

$$l_1 = n_2-n_1-2,\quad l_2 = n_3-n_2-l_1-3,\quad l_3 = n_3-n_1-(l_1+l_2)-4,$$
$$l_4 = n_4-n_3-(l_1+l_2+l_3)-5,\quad l_5 = n_4-n_2-(l_1+l_2+l_3+l_4)-6,$$
$$l_6 = n_4-n_1-(l_1+l_2+l_3+l_4+l_5)-7,\ \text{etc.}$$

Finally for all $j \geqslant 1$ we let $p_j = -(l_1 + \cdots + l_j + 1 + \cdots + j)$.

We list below the values of the terms of the sequences n_j, k_j and l_j, that we have used to construct the next two tables, where the n_j positions have been marked by $*$:

$$\begin{aligned} &k_1 = 1, \quad n_1 = 1, \\ &k_2 = 3, \quad n_2 = 5, \quad l_1 = 2, \\ &k_3 = 8, \quad n_3 = 14, \quad l_2 = 4, \quad l_3 = 3, \\ &k_4 = 18, \quad n_4 = 33, \quad l_4 = 5, \quad l_5 = 8, \quad l_6 = 3. \end{aligned}$$

We define the vector $x \in \ell^2(\mathbb{Z})$ by setting $\widehat{x}(n_i) = \frac{1}{2^{k_i}}$ for $i \geqslant 1$ and 0 otherwise.

Looking at the effect of applying T^{n_j} to the coordinates of the vector x, we first note that the product of the positive weights in each block is 1. Thus, since $\widehat{x}(n_j) = \frac{1}{2^{k_j}}$ we have that $\widehat{T^{n_j}x}(0) = 1$, for all $j \geqslant 1$. Furthermore the non-zero coordinate of the vector x in the $(j+1)$-block is shifted by T^{n_j} to the value $\frac{1}{2^{k_{j+1}}} \cdot 2^{n_j} = \frac{1}{2^{n_j+j+1}} \cdot 2^{n_j} = \frac{1}{2^{j+1}}$. The other non-zero entries with positive indices of $T^{n_j}x$ are given by the value $\frac{1}{k_{j+s}-n_j}$, where each $k_{j+s} - n_j > 2^{j+1}$ and the sequence $\{k_{j+s} - n_j\}_{s>1}$ is strictly increasing to infinity as $s \to \infty$.

Considering the entries with negative indices of $T^{n_j}x$, we find that shifting by T^{n_j} moves the coordinate $\widehat{x}(n_j)$ in the 0-th position, while $\widehat{x}(n_s)$ for $1 \leqslant s < j$ is moved in the $p-1$ position for some p.

Furthermore the non-zero terms with negative indices of the vector $T^{n_j}x$ are:

$$2^{-[1+2+\cdots+(j+S_j)]}, 2^{-[1+2+\cdots+((j-1)+S_j)]}, \ldots, 2^{-[1+2+\cdots+(1+S_j)]},$$

where $S_j = \sum_{i=1}^{j-1} i$. Thus as $j \to \infty$ we have that the entries above go to zero.

Finally to avoid any overlapping of the entries with negative indices while shifting by T^{n_j} (that is we want the non-zero entries of T^{n_j} to be to the left of the non-zero entries of $T^{n_{j-1}}$), we verify that $k_j > n_{j-1} + 2$.

Therefore by the observations listed above, we have that for the sequence $\{n_j\}_{j \geqslant 1}$, the vector x in $\ell^2(\mathbb{Z})$ and the bilateral weighted shift T with weights $\{w_j\}_{j \in \mathbb{Z}}$, $T^{n_j}x \to e_0$ as $j \to \infty$. Hence by our theorem, T is hypercyclic.

Example of a bilateral weighted backward shift having an orbit $\mathrm{Orb}(T, x)$ with e_0 as a limit point

(i) The positive indices

Position	0	k_1	2	$\leftarrow k_2 = 3 \rightarrow$			6	$\leftarrow k_3 = 8 \rightarrow$					15	$\leftarrow k_4 = 18 \rightarrow$						34	$\dots$
Weights	1	2*	$\frac{1}{2}$	2	2	2*	$\frac{1}{2^3}$	2	2	...	2	2*	$\frac{1}{2^8}$	2	2	2	...	2	2*	$\frac{1}{2^{18}}$	$\dots$
$\widehat{x}(j)$	0	$\frac{1}{2}$	0	0	0	$\frac{1}{2^3}$	0	0	0	...	0	$\frac{1}{2^8}$	0	0	0	0	...	0	$\frac{1}{2^{18}}$	0	$\dots$
$\widehat{T^{n_1}x}(j)$	1	0	0	0	$\frac{1}{2^2}$	0	0	0	...	0	$\frac{1}{2^7}$	0	0	0	0	...	0	$\frac{1}{2^{17}}$	0	0	$\dots$
$\widehat{T^{n_2}x}(j)$	1	0	0	0	0	0	0	...	0	$\frac{1}{2^3}$	0	0	0	0	...	0	$\frac{1}{2^{13}}$	0	0	0	$\dots$
$\widehat{T^{n_3}x}(j)$	1	0	0	0	0	0	0	0	0	...	0	0	0	0	$\frac{1}{2^4}$	0	...	0	0	0	$\dots$
$\widehat{T^{n_4}x}(j)$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$\dots$

(ii) The negative indices

Position	...		p_6	$\leftarrow l_6 \rightarrow$			p_5	$\leftarrow l_5 \rightarrow$			p_4	$\leftarrow l_4 \rightarrow$			p_3	$\leftarrow l_3 \rightarrow$			p_2	$\leftarrow l_2 \rightarrow$			p_1	$\leftarrow l_1 \rightarrow$		0
Weights	...	2	$\frac{1}{2^9}$	2	2	2	$\frac{1}{2^{13}}$	2	...	2	$\frac{1}{2^9}$	2	...	2	$\frac{1}{2^6}$	2	2	2	$\frac{1}{2^6}$	2	...	2	$\frac{1}{2^3}$	2	2	1
$\widehat{x}(j)$																										0
$\widehat{T^{n_1}x}(j)$																										1
$\widehat{T^{n_2}x}(j)$																						$\frac{1}{2}$	0	0	0	1
$\widehat{T^{n_3}x}(j)$														$\frac{1}{2^6}$	0	0	0	$\frac{1}{2^3}$	0	0	...	0	0	0	0	1
$\widehat{T^{n_4}x}(j)$	...	$\frac{1}{2^{21}}$	0	0	0	$\frac{1}{2^{15}}$	0	0	...	$\frac{1}{2^{10}}$	0	0	...	0	0	0	0	0	0	0	...	0	0	0	0	1

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