J. OPERATOR THEORY
67:1(2012), 289–298

AN ABSTRACT CHARACTERIZATION OF UNITAL OPERATOR SPACES

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Communicated by William Arveson

ABSTRACT. In this article, we characterize the "identity" of an operator space through an analogue of the abstract numerical radius. From this, we give a simple proof of the fact that quotients of unital operator spaces by complete M-ideals are unital. Moreover, we show that both $\mathrm{CB}(A)$ and $\mathrm{CB}(\mathcal{M}_*)$ are unital operator spaces, when A is a C^*-algebra and $\mathcal{M}$ is a von Neumann algebra. We also show that if X is a normed space with numerical index 1, then $\mathrm{CB}(\max \mathbb{X}; \min \mathbb{X})$ is a unital operator space. Using the idea in our characterization, we consider unital tensor products of unital operator spaces.

KEYWORDS: *Unital operator spaces.*

MSC (2000): 46L07.

1. INTRODUCTION AND NOTATIONS

Operator spaces are important objects in functional analysis and there is a ground breaking characterization of operator spaces by Effros and Ruan (see e.g. [18] or [12]). However, in applications, sometimes the starting point is not general subspaces of $\mathcal{L}(H)$ but unital subspaces of $\mathcal{L}(H)$ (i.e. subspaces that contain the identity; see e.g. [2]). Therefore, it seems important to have an abstract characterization of unital operator spaces.

The aim of this paper is to find the operator space analogue of the well-known notion of geometric unitaries in Banach spaces theory and use it to characterize unital operator spaces. Moreover, we will give some applications of this characterization. After we finished an earlier version of this work, we learned that Blecher and Neal gave a metric characterization of unital operator spaces in more or less the same period as we. Nevertheless, there seems to be no direct relation between these two works.

In Section 2, we will define a quantity $n_{\rm cb}(\mathbb{V};u)$ for an operator space $\mathbb{V}$ and a norm-one element $u \in \mathbb{V}$, which is an operator space analogue of the well-known notion of numerical radius $n(X,u)$ in Banach space theory (it is also a generalization of the k-numerical radius as defined in [4]). We will show that there exists a complete isometry (respectively, complete isomorphism) from $\mathbb{V}$ to some $\mathcal{L}(H)$ that sends u to id_H if and only if $n_{\rm cb}(\mathbb{V};u) = 1$ (respectively, $n_{\rm cb}(\mathbb{V};u) > 0$). We will also study some properties of $n_{\rm cb}(\mathbb{V};u)$.

Although the argument for our characterization theorem is easy, the idea behind it seems to open a door for the discovery and manipulation of unital operator spaces, and we will give such examples in Section 3. First of all, we use our characterization to give a simple proof for the fact that the quotient of a unital (respectively, quasi-unital) operator space by a complete M-ideal is also a unital (respectively, quasi-unital) operator space. Even though one can use the theory of the C^*-envelope to prove this fact, our proof is more elementary. Secondly, we use the idea of our characterization to construct unital tensor products of unital operator spaces. More precisely, one can always "unitize", in a functorial way, an operator space tensor product of two unital operator spaces. Thirdly, we use our characterization to show that $\mathrm{CB}(A)$ and $\mathrm{CB}(\mathcal{M}_*)$ are unital operator spaces with identity I_A and $I_{\mathcal{M}_*}$ respectively, when A is a C^*-algebra and $\mathcal{M}$ is a von Neumann algebra. Finally, our characterization can be used to show that if X is a normed space with numerical index 1, then I_X is the identity for the operator space $\mathrm{CB}(\max\mathbb{X};\min\mathbb{X})$.

NOTATION 1.1. Throughout this article, unless specified, H is a Hilbert space, X is a Banach space, and $\mathbb{U}$, $\mathbb{V}$ as well as $\mathbb{W}$ are matricially normed spaces (i.e. a matrix normed space that satisfies only condition (M2) in [12], p. 20), while V is the underlying normed space of $\mathbb{V}$. Moreover, we will denote by B_X and $\mathfrak{S}_1(X)$ the closed unit ball and the unit sphere of X.

Note that given a matricially normed space V, the image $\mathbb{V}_0$, of $\mathbb{V}$ in $\mathbb{V}^{**}$ is an operator space. Moreover, it is easy to see that the canonical map from $\mathrm{CB}(\mathbb{V}_0,\mathbb{W})$ to $\mathrm{CB}(\mathbb{V},\mathbb{W})$ given by the canonical map $\kappa_V : \mathbb{V} \to \mathbb{V}_0$ is a complete isometry (see e.g. [16]).

We end this section by recalling the notion of "numerical radius" and "geometric unitary" for normed spaces (see e.g. [1] and [13]). For any $u \in \mathfrak{S}_1(X)$, we set $\mathcal{S}^{X;u} := \{f \in X^* : \|f\| = 1 = f(u)\}$, $\gamma^u(x) := \sup\{\|f(x)\| : f \in \mathcal{S}^{X;u}\}$ ($x \in X$), as well as $n(X;u) := \inf\{\gamma^u(x) : x \in \mathfrak{S}_1(X)\}$.

A norm one element u in a normed space X is called a *geometric unitary* (respectively, *strict geometric unitary*) if the numerical radius $n(X;u)$ is non-zero (respectively, equals 1).

REMARK 1.2. Let X be a normed space and $u \in \mathfrak{S}_1(X)$. Suppose that Y is another normed space and $\Psi : X \to Y$ is a contractive topological injection. If $\Psi(u) \in \mathfrak{S}_1(Y)$, then $n(Y;\Psi(u)) \leqslant \|\Psi^{-1}\| \cdot n(X;u)$.

2. COMPLETE GEOMETRIC UNITARIES

2.1. DEFINITION AND MAIN RESULTS. Let $\mathbb{V}$ and $\mathbb{W}$ be matrically normed spaces, $v \in \mathfrak{S}_1(\mathbb{V})$ and $w \in \mathfrak{S}_1(\mathbb{W})$. For any $n \in \mathbb{N}$, we put $\mathrm{Mor}_w^v(\mathbb{V};\mathbb{W}) := \{\varphi \in \mathrm{CB}(\mathbb{V},\mathbb{W}) : \|\varphi\|_{\mathrm{cb}} \leqslant 1; \varphi(v) = w\}$, $\mathcal{S}_n^{\mathbb{V};v} := \mathrm{Mor}_{I_n}^v(\mathbb{V};M_n)$,

$$\gamma_k^v(x) := \sup\{\|\varphi_k(x)\| : \varphi \in \mathcal{S}_n^{\mathbb{V};v}; n \in \mathbb{N}\} \quad (k \in \mathbb{N}; x \in M_k(\mathbb{V}))$$

(where $\varphi_k : M_k(V) \to M_{nk}$ is given by $\varphi_k([x_{ij}]) = [\varphi(x_{ij})]$), as well as

$$n_{\mathrm{cb}}(\mathbb{V};v) := \inf\{\gamma_k^v(x) : x \in \mathfrak{S}_1(M_k(\mathbb{V})); k \in \mathbb{N}\}.$$

Moreover, we denote by $A^v(\mathbb{V})$ the unital C^*-algebra $\bigoplus\limits_{n=1}^{\infty} C(\mathcal{S}_n^{\mathbb{V},v} M_n)$ and define $j_V^v : \mathbb{V} \to A^v(\mathbb{V})$ to be the map given by evaluations.

In the case when $\mathbb{V}$ is an operator algebra and u being the identity of $\mathbb{V}$, the quantity $\gamma_k^u(x)$ was defined in [4], p. 192, and is called the *k-th numerical radius* of $x \in \mathbb{V}$.

DEFINITION 2.1. $u \in \mathfrak{S}_1(\mathbb{V})$ is called a *complete strict geometric unitary* (respectively, *complete geometric unitary*) if $n_{\mathrm{cb}}(\mathbb{V};u) = 1$ (respectively, $n_{\mathrm{cb}}(\mathbb{V};u) > 0$). In this case, we say that $\mathbb{V}$ is a *unital operator space with identity u* (respectively, *a quasi-unital operator space with quasi-identity u*).

REMARK 2.2. (i) $\mathcal{S}_n^{\mathbb{V};u}$ is a compact set under the point-norm topology, and $\mathcal{S}_1^{\mathbb{V};u} = \mathcal{S}^{V;u}$.

(ii) Suppose that $\mathbb{V}$ is a matrix normed subspace of a matricially normed space $\mathbb{W}$ and $\gamma_k^{u,W}$ is defined by $\mathcal{S}_n^{\mathbb{W};u}$ in a similar way as γ_k^u. It is easy to see that $\gamma_k^u = \gamma_k^{u,W}$ for all $k \in \mathbb{N}$ and so $n_{\mathrm{cb}}(\mathbb{W},u) \leqslant n_{\mathrm{cb}}(\mathbb{V},u)$.

Suppose that X is a vector space and σ_k is a seminorm on $M_k(X)$ ($k \in \mathbb{N}$) satisfying conditions (M1) and (M2) in [12], p. 20. Let $N := \{v \in V : \sigma_1^u(v) = 0\}$. Then σ_k induces a semi-norm $\widetilde{\sigma}_k$ on $M_k(X/N)$ which, by 2.3.6 of [12], provides an operator space structure on X/N. This gives the following lemma.

LEMMA 2.3. *γ_k^u induces an operator space structure on $\mathbb{V}_u := \mathbb{V}/N_u$ (where $N_u := \{v \in \mathbb{V} : \gamma_1^u(v) = 0\}$).*

Whenever we talk about the operator space $\mathbb{V}_u$, we consider the operator space structure as given in the above lemma (even when $N_u = (0)$). Moreover, we denote by Q_u the canonical complete contraction from $\mathbb{V}$ to $\mathbb{V}_u$.

LEMMA 2.4. *Let $\mathbb{V}$ be a matricially normed space and $u \in \mathfrak{S}_1(\mathbb{V})$.*

(i) $n_{\mathrm{cb}}(\mathbb{V};u) > 0$ *if and only if Q_u is a complete isomorphism. In this case, we have* $n_{\mathrm{cb}}(\mathbb{V};u) = \|Q_u^{-1}\|_{\mathrm{cb}}^{-1}$.

(ii) $n_{\mathrm{cb}}(\mathbb{V}_u, Q_u(u)) = 1$.

Proof. (i) The first statement is more or less obvious. If $n_{\mathrm{cb}}(\mathbb{V};u) > 0$, then we have the following which gives the second statement:

$$n_{\mathrm{cb}}(\mathbb{V};u)^{-1} = \inf\{\mu \in \mathbb{R}_+ : \|(Q_u^{-1})_n(y)\| \leqslant \mu\widetilde{\gamma}_n^u(y); n \in \mathbb{N}; y \in M_n(\mathbb{V}_u)\}.$$

(ii) It is not hard to see that Q_u induces a bijection from $\mathcal{S}_n^{\mathbb{V}_u;Q_u(u)}$ to $\mathcal{S}_n^{\mathbb{V};u}$. Consequently, $N_{Q_u(u)} = (0)$ and $\widetilde{\gamma}_k^u = \gamma_k^{Q_u(u)}$ on $M_k(V_u) = M_k(V_{Q_u(u)})$, which implies that $n_{\mathrm{cb}}(\mathbb{V}_u;Q_u(u)) = 1$. ∎

PROPOSITION 2.5. *Let $\mathbb{V}$ and $\mathbb{W}$ be matricially normed spaces. Suppose that $\Psi : \mathbb{V} \to \mathbb{W}$ is a complete contraction and $u \in \mathfrak{S}_1(\mathbb{V})$ such that $\Psi(u) \in \mathfrak{S}_1(\mathbb{W})$.*

(i) *There exists a complete contraction $\Psi_u : \mathbb{V}_u \to \mathbb{W}_{\Psi(u)}$ with $Q_{\Psi(u)} \circ \Psi = \Psi_u \circ Q_u$.*

(ii) *If Ψ is a complete topological injection, then $n_{\mathrm{cb}}(\mathbb{W};\Psi(u)) \leqslant \|\Psi^{-1}\|_{\mathrm{cb}} \cdot n_{\mathrm{cb}}(\mathbb{V};u)$.*

Proof. (i) Ψ induces a map $\widetilde{\Psi}_n : \mathcal{S}_n^{\mathbb{W};\Psi(u)} \to \mathcal{S}_n^{\mathbb{V};u}$ and we have $\gamma_n^{\Psi(u)} \circ \Psi \leqslant \gamma_n^u$ ($n \in \mathbb{N}$). This produces the required map Ψ_u.

(ii) By Remark 2.2(ii) and the argument for part (i),

$$\begin{aligned} n_{\mathrm{cb}}(\mathbb{W};\Psi(u)) &\leqslant n_{\mathrm{cb}}(\Psi(\mathbb{V});\Psi(u)) \\ &= \sup\left\{\lambda \in \mathbb{R}_+ : \lambda\|\Psi_n(x)\| \leqslant \gamma_n^{\Psi(u)}(\Psi_n(x)); n \in \mathbb{N}; x \in M_n(\mathbb{V})\right\} \\ &\leqslant \sup\{\lambda \in \mathbb{R}_+ : \lambda\|\Psi^{-1}\|_{\mathrm{cb}}^{-1}\|x\| \leqslant \gamma_n^u(x); n \in \mathbb{N}; x \in M_n(\mathbb{V})\} \\ &= \|\Psi^{-1}\|_{\mathrm{cb}} \cdot n_{\mathrm{cb}}(\mathbb{V};u). \quad ∎ \end{aligned}$$

Consequently, for any complete contraction $\Psi : \mathbb{V} \to \mathcal{L}(H)$ with $\Psi(u) = \mathrm{id}_H$ (where $u \in \mathfrak{S}_1(\mathbb{V})$), there exists a complete contraction $\Psi_u : \mathbb{V}_u \to \mathcal{L}(H)$ with $\Psi = \Psi_u \circ Q_u$.

The following is our first theorem which tells us that one can use strict complete geometric unitary to describe the "identity" of an operator space. Let us first recall the following well-known fact.

LEMMA 2.6. *For any $T \in \mathcal{L}(H^n)$, one has $\|T\| = \sup\{\|(P \otimes I_n)T(P \otimes I_n)\| : P \in \mathcal{L}(H)$ is a finite rank projection$\}$.*

THEOREM 2.7. *Let $\mathbb{V}$ be a matricially normed space and $u \in \mathfrak{S}_1(\mathbb{V})$. Then u is a complete geometric unitary if and only if there exists a Hilbert space H and a completely contractive complete topological injection $\Theta : \mathbb{V} \to \mathcal{L}(H)$ such that $\Theta(u) = \mathrm{id}_H$ and $n_{\mathrm{cb}}(\mathbb{V};u) = \|\Theta^{-1}\|_{\mathrm{cb}}^{-1}$.*

Proof. Suppose that $n_{\mathrm{cb}}(\mathbb{V};u) = 1$. If $\pi : A^u(\mathbb{V}) \to \mathcal{L}(H)$ is a unital $*$-representation and $\Theta := \pi \circ j_V^u$, then $\Theta(u) = \mathrm{id}_H$, and Θ is a complete isometry (because $n_{\mathrm{cb}}(\mathbb{V},u) = 1$). Conversely, suppose there exists such a Θ which is a complete isometry. For any rank n projection P on H, the map $x \mapsto P\Theta(x)P$ is an element of $\mathcal{S}_n^{\mathbb{V};u}$. Therefore by Lemma 2.6, $\|\cdot\|_k = \gamma_k^u$ ($k \in \mathbb{N}$), and $n_{\mathrm{cb}}(\mathbb{V},u) = 1$ (see Lemma 2.4(i)). The general case follows from the case of $n_{\mathrm{cb}}(\mathbb{V};u) = 1$ as well as Lemma 2.4(i) and Proposition 2.5(ii). ∎

In the case of $n_{\mathrm{cb}}(\mathbb{V};u) = 1$, one can also prove the necessity of the above theorem by taking $H = \bigoplus_{n=1}^{\infty} \bigoplus_{\varphi \in \mathcal{S}_n^{\mathbb{V};u}} \ell^2(n)$ and adapting the argument of 2.3.5 in [12].

COROLLARY 2.8. *Let $\mathbb{V}$ be a matricially normed space and $u \in \mathfrak{S}_1(\mathbb{V})$.*

(i) *If u is a complete strict geometric unitary, then $\mathbb{V}$ is an operator space.*

(ii) $n_{\mathrm{cb}}(\mathbb{V};u) = n_{\mathrm{cb}}(\mathbb{V}^{**};u)$ *(where V^{**} is the bidual of V).*

2.2. RELATIONSHIPS BETWEEN $n_{\mathrm{cb}}(\mathbb{V},u)$ AND $n(V,u)$. In this subsection, we will compare $n_{\mathrm{cb}}(\mathbb{V},u)$ and $n(V,u)$.

LEMMA 2.9. *Let $\mathbb{V}$ be a matricially normed space. If $n_{\mathrm{cb}}(\mathbb{V};u) > 0$ (respectively, $n_{\mathrm{cb}}(\mathbb{V};u) = 1$), then $n(V;u) > 0$ (respectively, $n(V;u) \geqslant \frac{1}{2}$).*

Proof. If Ψ is the completely contractive complete topological injection (respectively, complete isometry) given by Theorem 2.7, then Remark 1.2(ii) and Theorem 3 of [9] show that $n(V;u) \geqslant \frac{n(\mathcal{L}(H);1)}{\|\Psi^{-1}\|} \geqslant \frac{1}{2\|\Psi^{-1}\|}$. ∎

Next, we consider the case of minimal quantization.

PROPOSITION 2.10. *Let X be a normed space and $u \in \mathfrak{S}_1(X)$.*

(i) $n(X;u) \leqslant n_{\mathrm{cb}}(\min \mathbb{X};u)$.

(ii) $n(X;u) > 0$ *if and only if* $n_{\mathrm{cb}}(\min \mathbb{X};u) > 0$.

Proof. (i) Since $\mathcal{S}_1^{\min \mathbb{X};u} = \mathcal{S}^{X;u}$, it suffices to prove that for any $k \in \mathbb{N}$ and $x = (x_{ij}) \in \mathfrak{S}_1(M_k(\min \mathbb{X}))$, we have

$$n(X;u) \leqslant \sup\{\|f_k(x)\| : f \in \mathcal{S}^{X;u}\}. \tag{2.1}$$

Suppose that Ω is a compact Hausdorff space such that $\min \mathbb{X} \subseteq C(\Omega)$ as operator subspace. There exist $\omega \in \Omega$ and $(c_i), (d_i) \in \mathfrak{S}_1(\ell^2(n))$ with $1 = \|x\| = \|x(\omega)\| = \Big|\sum_{ij} \bar{c}_i x_{ij}(\omega) d_j\Big|$, which implies that $\Big\|\sum_{ij} \bar{c}_i x_{ij} d_j\Big\| \geqslant 1$. On the other hand, it is easy to see that $\Big\|\sum_{ij} \bar{c}_i x_{ij} d_j\Big\| \leqslant \|x\| = 1$. Now, for any $f \in \mathcal{S}^{X;u}$,

$$\|f_k(x)\| \geqslant \left|\left\langle (f(x_{ij})) \begin{pmatrix} d_1 \\ \vdots \\ d_k \end{pmatrix}, \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} \right\rangle\right| = \left| f\Big(\sum_{ij} \bar{c}_i x_{ij} d_j\Big)\right|.$$

Hence, $\gamma^u\Big(\sum_{ij} \bar{c}_i x_{ij} d_j\Big) \leqslant \sup\{\|f_k(x)\| : f \in \mathcal{S}^{X;u}\}$ and Equality (2.1) is verified.

(ii) This part follows from part (i) as well as Lemma 2.9. ∎

Consequently, if $\mathbb{V}$ is an operator space and $u \in \mathfrak{S}_1(\mathbb{V})$ such that $n_{\mathrm{cb}}(\mathbb{V};u) > 0$, then we have $n_{\mathrm{cb}}(\min \mathbb{V};u) > 0$ (by Lemma 2.9 and Proposition 2.10). This fact is not easy to obtain directly from the definition.

For a C^*-algebra A and $u \in \mathfrak{S}_1(A)$, it is easy to see, by using 4.1 of [5] and the above discussions, that u is a unitary if and only if $n_{\mathrm{cb}}(A;u) = 1$, which is also equivalent to $n_{\mathrm{cb}}(A;u) > 0$. However, for a general operator spaces $\mathbb{V}$ it is possible to have elements $u, v \in \mathfrak{S}_1(\mathbb{V})$ such that $n_{\mathrm{cb}}(\mathbb{V}, u) > 0$ and $n_{\mathrm{cb}}(\mathbb{V}, u) > 0$ but $n_{\mathrm{cb}}(\mathbb{V}; u) \neq n_{\mathrm{cb}}(\mathbb{V}; v)$. In order to give such an example, we need the following lemma which is probably known but we include a simple argument here (as we do not find it explicitly stated in the literature).

LEMMA 2.11. *Suppose that X and Y are two normed spaces and $u \in \mathfrak{S}_1(X)$. Then $n(E;(u,0)) = n(X;u)$ where $E = X \oplus^1 Y$.*

Proof. Note that $\mathcal{S}^{E;(u,0)} = \mathcal{S}^{X;u} \times B_{Y^*}$, which implies $\gamma^{(u,0)}(x,y) \leqslant \gamma^u(x) + \|y\|$. For any $\varepsilon > 0$, there are $f \in \mathcal{S}^{X;u}$ and $g \in B_{Y^*}$ with $\gamma^u(x) < |f(x)| + \varepsilon$ and $g(y) = \|y\|$. If $f(x) = |f(x)|\mathrm{e}^{\mathrm{i}\theta}$, then $|f(x) + (\mathrm{e}^{\mathrm{i}\theta} g)(y)| \geqslant \gamma^u(x) + \|y\| - \varepsilon$. Consequently, $\gamma^{(u,0)}(x,y) = \gamma^u(x) + \|y\|$, and $n(E;(u,0)) \leqslant n(X;u)$. On the other hand, for any $n \in \mathbb{N}$, there exists $(x_n, y_n) \in E$ with $\|x_n\| + \|y_n\| = 1$ and $\gamma^u(x_n) + \|y_n\| < n(E;(u,0)) + \frac{1}{n}$. If there are infinitely many n with $x_n = 0$, then there is a subsequence with $1 = \|y_{n_k}\| < n(E;(u,0)) + \frac{1}{n_k}$, which implies that $n(E;(u,0)) = 1$ and $n(X;u) = 1$ as well. Otherwise, we may assume that all x_n are non-zero and take $z_n = \frac{x_n}{1 - \|y_n\|} \in \mathfrak{S}_1(X)$. Since $(\gamma^u(x_n) + \|y_n\|)(1 - \|y_n\|) \geqslant \gamma^u(x_n)$ (because $\gamma^u(x_n) \leqslant \|x_n\| = 1 - \|y_n\|$), we have $\gamma^u(z_n) \leqslant \gamma^u(x_n) + \|y_n\| < n(E;(u,0)) + \frac{1}{n}$. This completes the proof. ∎

EXAMPLE 2.12. If E is a finite dimensional Banach space with $n(E;u) = \frac{1}{e}$ (see e.g. 3.5 of [10]), then for any quantization $\mathbb{E}$ of E, we have $0 < n_{\mathrm{cb}}(\mathbb{E};u) < 1$ (by Propositions 2.10(ii) and the fact that $\frac{1}{e} < \frac{1}{2}$ together with Lemma 2.9). Moreover, if $F = \mathbb{C} \oplus^1 E$, then $n(F;(0,u)) = \frac{1}{e}$ and $n(F;(1,0)) = 1$ (by Lemma 2.11) and so, by the above, $0 < n_{\mathrm{cb}}(\min \mathbb{F};(0,u)) < 1$ but $n_{\mathrm{cb}}(\min \mathbb{F};(1,0)) = 1$ (by Proposition 2.10(i)).

3. APPLICATIONS

The idea of characterizing unital operator spaces through the quantity $n_{\mathrm{cb}}(\mathbb{V};v)$ can be used to give an abstract characterization for non-unital operator systems in a similar fashion as [19]. However, we leave the details to the readers (see also [17] for a even more general situation). In the following, we will give another four interesting applications.

3.1. QUOTIENT OF QUASI-UNITAL SPACES.

In this subsection, we will show that the quotient of a quasi-unital (respectively, unital) operator space by a proper complete M-ideal is also quasi-unital (respectively, unital). In the case of unital operator space, one can also obtain the same result by considering the C^*-envelope (see the closing remark of [3]), but our argument is more elementary.

LEMMA 3.1. *Let X and Y be two normed spaces, and $E = X \oplus^\infty Y$. If $(u, v) \in \mathfrak{S}_1(E)$ with $n(E; (u, v)) > 0$, then $\|u\| = 1 = \|v\|$.*

Proof. Suppose that $\|u\| = 1$ but $\|v\| < 1$. For any $(f, g) \in \mathcal{S}^{E;(u,v)}$, we have $\|f\| + \|g\| = 1 = f(u) + g(v)$. Hence $\|g\| = 0$ and so, $\mathcal{S}^{E;(u,v)} = \mathcal{S}^{X;u} \times \{0\}$. Thus, we have a contradiction that $(f, 0)(0, y) = 0$ for any $y \in \mathfrak{S}_1(Y)$ and $(f, 0) \in \mathcal{S}^{E;(u,v)}$. ∎

THEOREM 3.2. *Let $\mathbb{V}$ be an operator space and $\mathbb{W} \subsetneq \mathbb{V}$ be a complete M-ideal. If $v \in \mathfrak{S}_1(\mathbb{V})$ with $n_{\rm cb}(\mathbb{V}; v) > 0$, then $\|Q(v)\| = 1$ and $n_{\rm cb}(\mathbb{V}; v) \leqslant n_{\rm cb}(\mathbb{V}/\mathbb{W}; Q(v))$ (where $Q : \mathbb{V} \to \mathbb{V}/\mathbb{W}$ is the canonical quotient map).*

Proof. By Corollary 2.8(ii), we assume that $\mathbb{W}$ is a complete M-summand of $\mathbb{V}$ (since $\mathbb{W}^{\perp\perp}$ is a complete M-summand of $\mathbb{V}^{**}$). Let $P : \mathbb{V} \to \mathbb{V}$ be the complete M-projection such that $P(\mathbb{V}) = \mathbb{W}$ and let $\mathbb{U} = (I - P)(\mathbb{V})$. Then $\mathbb{V} \cong \mathbb{U} \oplus^\infty \mathbb{W}$ as operator spaces and $Q(x) \mapsto (I - P)(x)$ is a complete isometry from $\mathbb{V}/\mathbb{W}$ to $\mathbb{U}$. Suppose that $v = (u, w)$ with $u \in \mathbb{U}$ and $w \in \mathbb{W}$. Then by Lemmas 2.9 and 3.1, we see that $\|u\| = 1 = \|w\|$. Pick any $f \in \mathfrak{S}_1(\mathbb{U}^*)$ with $f(u) = 1$ and define $\Psi : \mathbb{U} \to \mathbb{V}$ by $\Psi(x) = (x, f(x)w)$. It is not hard to check that Ψ is a complete isometry (as $\|f_n(x)w\| \leqslant \|x\|$ for any $x \in M_n(\mathbb{U})$) such that $\Psi(u) = (u, w)$. Now, $n_{\rm cb}(\mathbb{V}; (u, w)) \leqslant n_{\rm cb}(\mathbb{U}; u)$ (by Proposition 2.5(ii)). ∎

Consequently, the quotient of a unital operator system by a complete M-ideal is also a unital operator system.

3.2. UNITAL TENSOR PRODUCTS.

Let us start with the following easy lemma which follows from Proposition 2.5(i) and Theorem 2.7.

LEMMA 3.3. *Let $\mathbb{U}$ be an operator space, $u \in \mathfrak{S}_1(\mathbb{U})$ and $\mathbb{U}^u := j^u_U(\mathbb{U}) \subseteq A^u(\mathbb{U})$.*

(i) *If $\mathbb{V}$ is a unital operator space with identity e and $\varphi \in \mathrm{Mor}^u_e(\mathbb{U}; \mathbb{V})$, there is $\psi \in \mathrm{Mor}^u_e(\mathbb{U}^u, \mathbb{V})$ such that $\varphi = \psi \circ j^u_U$.*

(ii) *u is a complete strict geometric unitary if and only if j^u_U is a complete isometry.*

If $\mathbb{V}$ and $\mathbb{W}$ are operator spaces, an operator space matrix norm $\|\cdot\|_\mu$ on the algebraic tensor product $\mathbb{V} \odot \mathbb{W}$ is called a *subcross matrix norm* if $\|v \otimes w\|_\mu \leqslant \|v\|\|w\|$ for any $p, q \in \mathbb{N}$, $v \in M_p(\mathbb{V})$ and $w \in M_q(\mathbb{W})$.

DEFINITION 3.4. Let $\mathbb{V}$ and $\mathbb{W}$ be unital operator spaces with identity e and f respectively. A *unital operator space subcross matrix norm* on the algebraic tensor product $\mathbb{V} \odot \mathbb{W}$ is an operator space subcross matrix norm such that $e \otimes f$ is a complete strict geometric unitary.

One may wonder whether there are many unital operator space subcross norm around. The following result shows that it is the case. Moreover, one can actually construct, in a functorial way, a unital operator space subcross norm from a given operator space subcross norm.

PROPOSITION 3.5. *Let $\mathbb{V}$ and $\mathbb{W}$ be unital operator spaces with identity e and f respectively. Suppose that $\|\cdot\|_\mu$ is an operator space subcross matrix norm on $\mathbb{V}\odot\mathbb{W}$ with $\|\cdot\|_\vee \leqslant \|\cdot\|_\mu$, where $\|\cdot\|_\vee$ is the injective operator space tensor norm.*

(i) *$j^{e\otimes f}_{V\odot_\mu W}$ is injective and induces a unital operator space subcross matrix norm, $\|\cdot\|^u_\mu$, on $V\odot W$. Moreover, $\|\cdot\|_\mu = \|\cdot\|^u_\mu$ if and only if $\|\cdot\|_\mu$ is unital.*

(ii) *$\|\cdot\|_\vee = \|\cdot\|^u_\vee \leqslant \|\cdot\|^u_\mu \leqslant \|\cdot\|^u_\wedge$ (where $\|\cdot\|_\wedge$ is the projective operator space tensor norm).*

Proof. (i) It is easy to check that $j^{e\otimes f}_{V\check{\odot}W}$ is a complete isometry (i.e. $e\otimes f$ is the identity of $V\check{\odot}W$). Let $\Phi : V\odot_\mu W \to V\check{\odot}W$ be the canonical map and $\bar{\Phi} : A^{e\otimes f}(V\odot_\mu W) \to A^{e\otimes f}(V\check{\odot}W)$ be the corresponding $*$-homomorphism. Since $\bar{\Phi}\circ j^{e\otimes f}_{V\odot_\mu W} = j^{e\otimes f}_{V\check{\odot}W}\circ\Phi$ and is injective, $j^{e\otimes f}_{V\odot_\mu W}$ is also injective. The second statement is clear.

(ii) The first inequality is clear, and the second one follows from Lemma 3.3(i). ∎

If $\|\cdot\|_\mu$ is an operator space subcross matrix norm on $V\odot W$, we denote by $V\odot^u_\mu W$ the unital operator space $j^{e\otimes f}_{V\odot_\mu W}(V\odot_\mu W)$.

REMARK 3.6. (i) If $\mathbb{U}$ is another unital operator space with an identity g,

$$\mathrm{Mor}^{e\otimes f}_g(\mathbb{V}\widehat{\odot}^u\mathbb{W};\mathbb{U}) = \{\Psi\in \mathrm{CB}(\mathbb{V};\mathrm{CB}(\mathbb{W};\mathbb{U})) : \|\Psi\|_{\mathrm{cb}}\leqslant 1; \Psi(e)(f) = g\}.$$

(ii) By part (i), for any $n,p,q\in\mathbb{N}$, $v\in M_p(\mathbb{V})$, $w\in M_q(\mathbb{W})$, $\alpha\in M_{n,pq}$ and $\beta\in M_{pq,n}$, we know that $\|\alpha(v\otimes w)\beta\|^u_\wedge$ equals

$$\sup\{\|\alpha(\varphi_{p,q}(v,w))\beta\| : \varphi\in \mathrm{CB}(\mathbb{V}\times\mathbb{W};M_k); \|\varphi\|_{\mathrm{cb}}\leqslant 1; \varphi(e,f) = I_k; k\in\mathbb{N}\}.$$

(iii) If B is a completely contractive unital Banach algebra (respectively, unital operator algebra) with an algebraic identity e, then the product on B induces a map in $\mathrm{Mor}^{e\otimes e}_e(B^u\widehat{\otimes}^u B^u; B^u)$ (respectively, $\mathrm{Mor}^{e\otimes e}_e(B\otimes^u_h B; B)$).

3.3. CB(A) AND CB($\mathcal{M}_*$) ARE UNITAL OPERATOR SPACES. Our last two subsections concern with the existence of complete geometric unitaries in $\mathrm{CB}(\mathbb{V};\mathbb{W})$. Let us first recall the well-known fact that $T\mapsto T^*$ is a complete isometry from $\mathrm{CB}(\mathbb{V};\mathbb{W})$ to $\mathrm{CB}(\mathbb{W}^*;\mathbb{V}^*)$. This gives the following lemma.

LEMMA 3.7. *Let $\mathbb{V}$ and $\mathbb{W}$ be operator spaces. For any $T\in\mathfrak{S}_1(\mathrm{CB}(\mathbb{V};\mathbb{W}))$, one has $n_{\mathrm{cb}}(\mathrm{CB}(\mathbb{V};\mathbb{W});T)\geqslant n_{\mathrm{cb}}(\mathrm{CB}(\mathbb{W}^*;\mathbb{V}^*);T^*)$.*

THEOREM 3.8. *Let A be a C^*-algebra and $\mathcal{M}$ be a von Neumann algebra. Then $n_{\mathrm{cb}}(\mathrm{CB}(A);I_A) = 1$ and $n_{\mathrm{cb}}(\mathrm{CB}(\mathcal{M}_*);I_{\mathcal{M}_*}) = 1$.*

Proof. Let e be the identity of $\mathcal{M}$, $n\in\mathbb{N}$ and $T\in\mathfrak{S}_1(\mathrm{CB}(\mathcal{M};M_n(\mathcal{M})))$. For any $\varepsilon\in(0,\frac{1}{2})$, there exists $k\in\mathbb{N}$ and $a\in B_{M_k(\mathcal{M})}$ with $1\leqslant\|T_k(a)\|+\varepsilon$. If $u = \begin{pmatrix} a & (1-aa^*)^{1/2} \\ (1-a^*a)^{1/2} & -a^* \end{pmatrix}$, then u is a unitary in $M_{2k}(\mathcal{M})$ and $\|T_k(a)\|\leqslant$

$\|T_{2k}(u)\|$. As e is a complete strict geometric unitary, there exists $m \in \mathbb{N}$ and $\varphi \in \mathcal{S}_m^{\mathcal{M};e}$ with

$$\|T_{2k}(u)\| = \|(u^* \otimes I_n)T_{2k}(u)\| \leqslant \|\varphi_{2kn}((u^* \otimes I_n)T_{2k}(u))\| + \varepsilon.$$

Now define $\Phi : \mathrm{CB}(\mathcal{M}) \to M_{2km}$ by $\Phi(S) = \varphi_{2k}(u^* S_{2k}(u))$ ($S \in \mathrm{CB}(\mathcal{M})$). It is clear that $\Phi(I_{\mathcal{M}}) = I_{2km}$, and $\|\Phi\|_{\mathrm{cb}} \leqslant 1$ because

$$\Phi_l(S) = \varphi_{2kl}((u^* \otimes I_l)S_{2k}(u)) \quad (l \in \mathbb{N}; S \in \mathrm{CB}(\mathcal{M}; M_l(\mathcal{M}))).$$

Consequently, $\Phi \in \mathcal{S}_{2km}^{\mathrm{CB}(\mathcal{M});I_{\mathcal{M}}}$ and we have $\|\Phi_n(T)\| \geqslant 1 - 2\varepsilon$. This shows that $n_{\mathrm{cb}}(\mathrm{CB}(\mathcal{M}); I_{\mathcal{M}}) = 1$ and so, $n_{\mathrm{cb}}(\mathrm{CB}(\mathcal{M}_*); I_{\mathcal{M}_*}) = 1$ by Lemma 3.7. Finally, $n_{\mathrm{cb}}(\mathrm{CB}(A); I_A) = 1$ because A^{**} is a von Neumann algebra. ∎

3.4. A SITUATION WHEN $\mathrm{CB}(\max \mathbb{X}; \min \mathbb{X})$ IS UNITAL. For a normed space X, we denoted by $\max \mathbb{X}$ and $\min \mathbb{X}$ the two operator space structures on X as defined in Section 3.3 of [12]. In this subsection, we will show that $n_{\mathrm{cb}}(\mathrm{CB}(\max \mathbb{X}; \min \mathbb{X}); I_X)$ is greater than the numerical index $n(X)$. Consequently, $\mathrm{CB}(\max \mathbb{X}; \min \mathbb{X})$ is quasi-unital if X is a Banach space (note that in this case, $n(X) > 0$), and $\mathrm{CB}(\max \mathbb{X}; \min \mathbb{X})$ is a unital operator space if $n(X) = 1$. We recall that $n(X)$ is defined as follows: $n(X) := \inf_{T \in \mathfrak{S}_1(\mathcal{L}(X))} \gamma(T)$, where

$$\gamma(T) := \sup\{|f(Tx)| : x \in \mathfrak{S}_1(X); f \in \mathcal{S}^{X;x}\} \quad (T \in \mathcal{L}(X)).$$

Please see [6], [7], [8], [11], [14] and [15] for general information and background on numerical indices.

THEOREM 3.9. *Let X be a normed space. Then* $n_{\mathrm{cb}}(\mathrm{CB}(\max \mathbb{X}, \min \mathbb{X}); I_X) \geqslant n(X)$.

Proof. It is well-known that $\mathrm{CB}(\max \mathbb{X}; \min \mathbb{X}) = \min \mathcal{L}(\mathbb{X})$ and we have $n(X) \leqslant n(\mathcal{L}(X); I_X)$. We can apply Proposition 2.10(i) to complete the proof. ∎

Acknowledgements. We would like to thank D. Blecher for some helpful comments on an earlier version of this work, especially for informing us of the results in [4]. We would also like to thank the referee for some helpful comments. This work is supported by the National Natural Science Foundation of China (10771126 and 11126248).

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Received May 11, 2009; revised January 4, 2010.