J. OPERATOR THEORY
68:1(2012), 3–17

FRACTIONAL INTEGRAL OPERATORS ON WEIGHTED ANISOTROPIC HARDY SPACES

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Communicated by Kenneth R. Davidson

ABSTRACT. In this article we study the fractional integral operators associated with anisotropic setting. We derive weighted norm inequalities for these operators on weighted anisotropic H^p spaces and weighted L^p spaces.

KEYWORDS: *$\mathcal{A}_p$ weight, anisotropic Hardy space, fractional integral operator, Hardy–Littlewood–Sobolev inequality.*

MSC (2000): Primary 47G10; Secondary 42B30.

1. INTRODUCTION

On the Euclidean space $\mathbb{R}^n$, for $0 < \alpha < n$, the fractional integral or the Riesz potential I_α is defined by

$$I_\alpha f(x) = \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \mathrm{d}y,$$

with $\gamma(\alpha) = \pi^{n/2}\Gamma(\alpha/2)/\Gamma(n/2 - \alpha/2)$. The celebrated result for I_α is the Hardy–Littlewood–Sobolev inequality (see [18]). The strong type (p,q), where $1 < p < \infty$ and $1/q = 1/p - \alpha/n$, was obtained by Hardy–Littlewood [14] when $n = 1$ and by Sobolev [17] for general n. The weak type $(1, n/(n-\alpha))$ first appeared in Zygmund [21]. In 1980, Taibleson and Weiss [20] extended the Hardy–Littlewood–Sobolev inequality to the standard Hardy spaces. The weighted (L^p, L^q) boundedness of I_α was established by Muckenhoupt and Wheeden [16] in 1974; the weighted (H^p, L^q) and weighted (H^p, H^q) boundedness of I_α was established by Strömberg and Wheeden [19] in 1985.

In 2003, Bownik [2] introduced the anisotropic Hardy spaces H^p_A associated with a dilation A. Many of the classical results arising from the real variable study of Hardy spaces of Fefferman–Stein [10] and also the parabolic Hardy spaces of

Calderón–Torchinsky [6], [7] are generalized. Recently, Bownik et al. [4] introduced weighted anisotropic Hardy spaces. In this article, we study the boundedness of fractional integral operator associated to a quasi-norm acting on weighted anisotropic Hardy spaces $H^p_{w,A}$.

We first recall the definition of fractional integral operator associated to a quasi-norm introduced by Ding and Lan [8]. Let $T : \mathscr{S}(\mathbb{R}^n) \mapsto \mathscr{S}'(\mathbb{R}^n)$ be a continuous linear operator. By the Schwartz kernel theorem there exists $S \in \mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\langle Tf, g\rangle = \langle S, g \otimes f\rangle \quad \text{for all } f, g \in \mathscr{S}(\mathbb{R}^n), \tag{1.1}$$

where $g \otimes f(x,y) = g(x)f(y)$. Let $\Omega = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \neq y\}$. We say that $S \in \mathscr{S}'$ is *regular* on Ω if there exists a locally integrable function $K(x,y)$ on Ω such that

$$S(h) = \int_\Omega K(x,y)h(x,y)\mathrm{d}x\mathrm{d}y \quad \text{for all } h \in \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n) \text{ supported on } \Omega.$$

We introduce the fractional operators associated to a quasi-norm ρ.

DEFINITION 1.1. Let $0 < \alpha < 1$, $\gamma > 0$, and $T : \mathscr{S}(\mathbb{R}^n) \mapsto \mathscr{S}'(\mathbb{R}^n)$ be a continuous linear operator. We say that T is a *fractional integral operator associated to a quasi-norm ρ of order α with regularity γ*, denoted by $T^\rho_{\alpha,\gamma}$, if there is a constant $C > 0$ such that a distribution S given by (1.1) is regular on Ω with kernel $K(x,y)$ satisfying:

(i) for all $(x,y) \in \Omega$, $|K(x,y)| \leqslant C\rho(x-y)^{-1+\alpha}$;

(ii) if $(x,y), (x',y) \in \Omega$ and $\rho(x-y) \geqslant b^{2\sigma}\rho(x'-x)$, then

$$|K(x',y) - K(x,y)| \leqslant C\frac{\rho(x'-x)^\gamma}{\rho(x-y)^{1-\alpha+\gamma}};$$

(iii) if $(x,y), (x,y') \in \Omega$ and $\rho(x-y) \geqslant b^{2\sigma}\rho(y'-y)$, then

$$|K(x,y') - K(x,y)| \leqslant C\frac{\rho(y'-y)^\gamma}{\rho(x-y)^{1-\alpha+\gamma}},$$

where $b > 1$ is a constant and σ is a fixed positive integer that will be defined later. $T^\rho_{\alpha,0}$ will denote a continuous linear operator T satisfying condition (i) only. If T is a convolution fractional integral operator with kernel $K(x)$, then the above conditions (ii) and (iii) reduce to

$$|K(x-y) - K(x)| \leqslant C\frac{\rho(y)^\gamma}{\rho(x)^{1-\alpha+\gamma}} \quad \text{when } \rho(x) \geqslant b^{2\sigma}\rho(y).$$

In particular, if $K(x,y) = \rho(x-y)^{-1+\alpha}$, we denote the corresponding fractional operator simply by T^ρ_α.

Ding and Lan showed that $T^\rho_{\alpha,0}$ is bounded from L^p to L^q.

LEMMA 1.2 ([8]). *Let* $0 < \alpha < 1$, $1 < p < 1/\alpha$, *and* $1/q = 1/p - \alpha$. *Then* $T^{\rho}_{\alpha,0}$ *is of type* (p,q), *and is of weak type* $(1, 1/(1-\alpha))$.

In this article, we first extend this result to weighted case. Here the weight belongs to $\mathcal{A}(p,q)$ or $\mathcal{A}_p$, the Muckenhoupt weight classes associated to a matrix A, which was introduced by Bownik and Ho [3] and will be defined later.

THEOREM 1.3. *If* $0 < \alpha < 1$, $1 < p < 1/\alpha$, $1/q = 1/p - \alpha$, *and* $w \in \mathcal{A}(p,q)$, *then there is a constant* C *independent of* f *such that*

$$\Big(\int\limits_{\mathbb{R}^n} |T^{\rho}_{\alpha,0} f(x) w(x)|^q \mathrm{d}x\Big)^{1/q} \leqslant C \Big(\int\limits_{\mathbb{R}^n} |f(x)w(x)|^p \mathrm{d}x\Big)^{1/p}.$$

Set $w(E) = \int\limits_E w(x)\mathrm{d}x$ for any subset $E \subseteq \mathbb{R}^n$. If we consider the operator T^{ρ}_{α}, we have the converse of Theorem 1.3, including $p = 1$.

THEOREM 1.4. *For* $0 < \alpha < 1$, $1 \leqslant p < 1/\alpha$, *and* $1/q = 1/p - \alpha$, *if* w *is a nonnegative function on* $\mathbb{R}^n$ *such that*

$$[w^q(\{x \in \mathbb{R}^n : |T^{\rho}_{\alpha} f(x)| > \lambda\})]^{1/q} \leqslant \frac{C}{\lambda}\Big(\int\limits_{\mathbb{R}^n} |f(x)w(x)|^p \mathrm{d}x\Big)^{1/p} \quad \text{for all } \lambda > 0, \tag{1.2}$$

where C *is independent of* λ *and* f, *then* $w \in \mathcal{A}(p,q)$.

Finally, we show that $T^{\rho}_{\alpha,\gamma}$ is bounded from weighted H^p_A to weighted L^q.

THEOREM 1.5. *Let* $0 < \alpha < 1$, $1/(1+\alpha) \leqslant p \leqslant 1$, $1/q = 1/p - \alpha$. *If there exists* $\overline{p} > 1$ *such that* $\alpha - 1 - \gamma + \overline{p}/p < 0$ *and* $w^{p/\overline{p}} \in \mathcal{A}(\overline{p}, \overline{q})$, *where* $1/\overline{q} = 1/\overline{p} - \alpha$, *then* $T^{\rho}_{\alpha,\gamma}$ *is bounded from* $H^p_{w^p,A}$ *to* $L^q_{w^q}$.

We immediately have the following corollary.

COROLLARY 1.6. *Let* $0 < \alpha < 1$, $1/(1+\alpha) \leqslant p \leqslant 1$, *and* $1/q = 1/p - \alpha$. *If* $\alpha - 1 - \gamma + 1/p < 0$ *and* $w^{p/(1-\alpha)} \in \mathcal{A}_1$, *then* $T^{\rho}_{\alpha,\gamma}$ *is bounded from* $H^p_{w^p,A}$ *to* $L^q_{w^q}$.

Throughout the article C denotes a positive constant not necessarily the same at each occurrence. The conjugate exponent of $p > 1$ is denoted by $p' = p/(p-1)$.

2. PRELIMIMINARIES

In this section, we review some facts about the weighted anisotropic Hardy spaces. For more details, we refer the reader to [2], [4]. An $n \times n$ real matrix A is called an *expansive matrix*, sometimes called a *dilation*, if $|\lambda| > 1$ for all eigenvalues λ's of A. Suppose $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A (taken according to the multiplicity) so that $1 < |\lambda_1| \leqslant \cdots \leqslant |\lambda_n|$. A set $\Delta \subseteq \mathbb{R}^n$ is said to be an *ellipsoid* if $\Delta = \{x \in \mathbb{R}^n : |Px| < 1\}$, for some nondegenerate $n \times n$

matrix P. For a dilation A, there exists an ellipsoid Δ and $r > 1$ such that $\Delta \subseteq r\Delta \subseteq A\Delta$, where $|\Delta|$, the Lebesgue measure of Δ, equals to 1. Set $B_k := A^k\Delta$ for $k \in \mathbb{Z}$. We have $B_k \subseteq rB_k \subseteq B_{k+1}$, and $|B_k| = b^k$, where $b = |\det A| > 1$. Let $\mathcal{B}$ denote the collection of dilated balls associated with the dilation A, i.e., $\mathcal{B} = \{x + B_k : x \in \mathbb{R}^n, k \in \mathbb{Z}\}$. Let σ be the smallest positive integer so that $2B_0 \subseteq B_\sigma := A^\sigma B_0$. A *homogeneous quasi-norm* associated with an expansive matrix A is a measurable mapping $\rho_A : \mathbb{R}^n \mapsto [0, \infty)$ satisfying

$$\begin{aligned} \rho_A(x) &> 0 && \text{for } x \neq 0, \\ \rho_A(Ax) &= |\det A| \rho_A(x) && \text{for } x \in \mathbb{R}^n, \\ \rho_A(x+y) &\leqslant C_A(\rho_A(x) + \rho_A(y)) && \text{for } x, y \in \mathbb{R}^n, \end{aligned}$$

where $C_A \geqslant 1$ is a constant. One can show that all homogeneous quasi-norms associated with a fixed dilation A are equivalent (see Lemma 2.4 of [2]). Define the *step homogeneous quasi-norm* ρ on $\mathbb{R}^n$ induced by dilation A as

$$\rho(x) = \begin{cases} b^j & \text{if } x \in B_{j+1} \backslash B_j, \\ 0 & \text{if } x = 0. \end{cases}$$

Then for any $x, y \in \mathbb{R}^n$, $\rho(x+y) \leqslant b^\sigma(\rho(x) + \rho(y))$. Let λ_-, λ_+ be any numbers satisfying $1 < \lambda_- < |\lambda_1| \leqslant |\lambda_n| < \lambda_+$. Then there exists a constant $c > 0$ such that, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} c^{-1}\rho(x)^{\ln\lambda_-/\ln b} \leqslant |x| \leqslant c\rho(x)^{\ln\lambda_+/\ln b} \quad \text{for } \rho(x) \geqslant 1, \\ c^{-1}\rho(x)^{\ln\lambda_+/\ln b} \leqslant |x| \leqslant c\rho(x)^{\ln\lambda_-/\ln b} \quad \text{for } \rho(x) \leqslant 1. \end{aligned}$$

We say that a C^∞ complex valued function φ on $\mathbb{R}^n$ belongs to the *Schwartz class* $\mathscr{S}$ if

$$\|\varphi\|_{\beta,m} := \sup_{x\in\mathbb{R}^n} \rho(x)^m |\partial^\beta \varphi(x)| < \infty \quad \text{for every multi-index } \beta \text{ and integer } m \geqslant 0.$$

The dual of $\mathscr{S}$, the *space of tempered distributions* on $\mathbb{R}^n$, is denoted by $\mathscr{S}'$. For $N \in \mathbb{N} \cup \{0\}$, denote

$$\mathscr{S}_N = \{\varphi \in \mathscr{S} : \|\varphi\|_{\beta,m} \leqslant 1 \text{ for } |\beta| \leqslant N, m \leqslant N\}.$$

For $\varphi \in \mathscr{S}$ and $k \in \mathbb{Z}$, define the dilation of φ to the scale k by

$$\varphi_k(x) = b^{-k}\varphi(A^{-k}x).$$

In particular, if we take $A = 2I$ where I is the identity matrix, then the dilations associated with A are the usual isotropic dyadic dilations. Suppose $f \in \mathscr{S}'$. The *nontangential maximal function* of f with respect to φ is defined as

$$M_\varphi f(x) := \sup\{|f * \varphi_k(y)| : x - y \in B_k, k \in \mathbb{Z}\}.$$

For given $N \in \mathbb{N} \cup \{0\}$ we define the *nontangential grand maximal function* of f as

$$M_N f(x) := \sup_{\varphi\in\mathscr{S}_N} M_\varphi f(x).$$

We say that a nonnegative measurable function w belongs to the Muckenhoupt weight class associated to A, denoted by $w \in \mathcal{A}_p$, $p > 1$, if there is a constant $C > 0$ such that

$$\sup_{B\in\mathcal{B}} \Big(\frac{1}{|B|}\int_B w(y)\mathrm{d}y\Big)\Big(\frac{1}{|B|}\int_B w(y)^{-1/(p-1)}\mathrm{d}y\Big)^{p-1} \leqslant C.$$

Here we define $0\cdot\infty$ to be 0. For $p = 1$, we say $w \in \mathcal{A}_1$ if

$$\sup_{B\in\mathcal{B}} \Big(\frac{1}{|B|}\int_B w(y)\mathrm{d}y\Big)\Big(\operatorname*{ess\,sup}_{y\in B} w(y)^{-1}\Big) \leqslant C.$$

Finally, $\mathcal{A}_\infty := \bigcup\limits_{p>1} \mathcal{A}_p$. It is known that if $w \in \mathcal{A}_p$ for $1 < p < \infty$, then $w \in \mathcal{A}_r$ for all $r > p$ and $w \in \mathcal{A}_q$ for some $1 < q < p$. We denote $q_w = \inf\{q > 1 : w \in \mathcal{A}_q\}$ the *critical index* of $w \in \mathcal{A}_\infty$. For $1 < p, q < \infty$, a nonnegative measurable function w is said to belong to $\mathcal{A}(p, q)$, if there exists constant $C > 0$ such that

$$\sup_{B\in\mathcal{B}} \Big(\frac{1}{|B|}\int_B w(y)^q\mathrm{d}y\Big)^{1/q}\Big(\frac{1}{|B|}\int_B w(y)^{-p'}\mathrm{d}y\Big)^{1/p'} \leqslant C. \tag{2.1}$$

In the case $p = 1$, (2.1) should be interpreted to mean

$$\sup_{B\in\mathcal{B}} \Big(\frac{1}{|B|}\int_B w(y)^q\mathrm{d}y\Big)^{1/q}\Big(\operatorname*{ess\,sup}_{y\in B} \frac{1}{w(y)}\Big) \leqslant C. \tag{2.2}$$

A closely related notion to $\mathcal{A}_p$ is the reverse Hölder condition. If there exist $r > 1$ and a fixed constant $C > 0$ such that

$$\Big(\frac{1}{|B|}\int_B w(y)^r\mathrm{d}y\Big) \leqslant C\Big(\frac{1}{|B|}\int_B w(y)\mathrm{d}y\Big) \quad \text{for every } B \in \mathcal{B},$$

then w is said to satisfy the *reverse Hölder condition of order r* and written by $w \in \mathcal{RH}_r$. It follows from Hölder's inequality that $w \in \mathcal{RH}_r$ implies $w \in \mathcal{RH}_s$ for $s < r$. It is well-known that $w \in \mathcal{A}_\infty$ if and only if $w \in \mathcal{RH}_r$ for some $r > 1$. Moreover, if $w \in \mathcal{RH}_r$ for $r > 1$, then $w \in \mathcal{RH}_{r+\varepsilon}$ for some $\varepsilon > 0$. Thus we write $r_w = \sup\{r > 1 : w \in \mathcal{RH}_r\}$ to denote the *critical index of w for the reverse Hölder condition*.

We summarize some properties about $\mathcal{A}_p$, $\mathcal{A}(p, q)$ and $\mathcal{RH}_r$ (cf. [3], [4], [9], [11], [16]), which will be used in the sequel.

PROPOSITION 2.1. (i) *If $w \in \mathcal{A}_q \cap \mathcal{RH}_r$ for some $q \geqslant 1$ and $r > 1$, then there exist constants $C_1, C_2 > 0$ such that*

$$C_1\Big(\frac{|E|}{|B|}\Big)^q \leqslant \frac{w(E)}{w(B)} \leqslant C_2\Big(\frac{|E|}{|B|}\Big)^{(r-1)/r}$$

for any measurable subset E of $B \in \mathcal{B}$.

(ii) *$w \in \mathcal{A}_p$ if and only if $w^{1-p'} \in \mathcal{A}_{p'}$.*

(iii) *If* $w \in \mathcal{A}_p$ *with* $1 \leqslant p < \infty$*, then there exists a small enough* $\varepsilon > 0$ *such that* $w^{1+\varepsilon} \in \mathcal{A}_p$.

(iv) $w \in \mathcal{A}(p,q)$ *implies that* $w^p \in \mathcal{A}_p$ *and* $w^q \in \mathcal{A}_q$.

(v) *Suppose that* $0 < \alpha < 1$, $1 < p < 1/\alpha$*, and* $1/q = 1/p - \alpha$*. Then* $w \in \mathcal{A}(p,q)$ *if and only if* $w^q \in \mathcal{A}_{q(1-\alpha)}$.

We recall the definition of weighted anisotropic Hardy spaces introduced in [4]. Let $0 < p < \infty$ and $w \in \mathcal{A}_\infty$ with critical index q_w. Write

$$N_{p,w} := \begin{cases} [(q_w/p - 1)\ln b/\ln \lambda_-] + 2 & \text{if } 0 < p \leqslant q_w, \\ 2 & \text{if } p > q_w, \end{cases}$$

where $[\cdot]$ denotes the integer function. For each $N \geqslant N_{p,w}$, the *weighted anisotropic Hardy space* associated with a dilation A is defined by

$$H^p_{w,A} := \{f \in \mathscr{S}' : M_N f \in L^p_w\},$$

with the quasi-norm $\|f\|_{H^p_{w,A}} = \|M_N f\|_{L^p_w}$. The definition of $H^p_{w,A}$ does not depend on the choice of N provided $N \geqslant N_{p,w}$ (cf. [4]).

3. COMPARISON WITH THE FRACTIONAL MAXIMAL FUNCTION

Consider the fractional maximal function M_α defined by

$$M_\alpha f(x) = \sup_{k \in \mathbb{Z}} \frac{1}{|B_k|^{1-\alpha}} \int_{x+B_k} |f(y)| \mathrm{d}y.$$

In particular, for $\alpha = 0$, M_α is just the *Hardy–Littlewood maximal function* M_{HL} (with respect to a dilation A with a quasi-norm ρ). The same techniques as in the case $\alpha = 0$ (see [5], for example) show that M_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for $1 < p < 1/\alpha$ and $1/q = 1/p - \alpha$, and is of weak type $(1, 1/(1-\alpha))$.

We may use M_α to majorize $T^\rho_{\alpha,0}$ as follows.

THEOREM 3.1. *If* $w \in \mathcal{A}_\infty$, $0 < q < \infty$*, and* $0 < \alpha < 1$*, then there is a constant* C*, independent of* f*, such that*

$$\int_{\mathbb{R}^n} |T^\rho_{\alpha,0} f(x)|^q w(x) \mathrm{d}x \leqslant C \int_{\mathbb{R}^n} [M_\alpha f(x)]^q w(x) \mathrm{d}x$$

and

$$\sup_{\lambda > 0} \lambda^q w(\{x \in \mathbb{R}^n : |T^\rho_{\alpha,0} f(x)| > \lambda\}) \leqslant C \sup_{\lambda > 0} \lambda^q w(\{x \in \mathbb{R}^n : |M_\alpha f(x)| > \lambda\}).$$

To prove Theorem 3.1, we need the following lemma.

LEMMA 3.2. *For $0 < \alpha < 1$, there exists a constant K depending only on α such that if $\widetilde{B} \in \mathcal{B}$ satisfies $\{x \in \widetilde{B} : T_\alpha^\rho(|f|)(x) \leqslant r\} \neq \varnothing$ for some $r > 0$, then*

$$|\{x \in \widetilde{B} : |T_\alpha^\rho f(x)| > r\lambda \text{ and } M_\alpha f(x) \leqslant r\beta\}| \leqslant K|\widetilde{B}|\Big(\frac{\beta}{\lambda}\Big)^{1/(1-\alpha)} \quad \forall\, \lambda \geqslant 4b^\sigma \text{ and } \beta > 0.$$

Proof. Suppose $\widetilde{B} = x_0 + B_j$. Let $g(x) = f(x)\chi_{x_0+B_{j+\sigma}}(x)$ and $h(x) = f(x) - g(x)$. Assume that there is an $x_1 \in \widetilde{B}$ such that $M_\alpha f(x_1) \leqslant r\beta$; otherwise, the conclusion is trivial. By Lemma 1.2, there is a constant C, depending only on α, such that

$$|\{x \in \mathbb{R}^n : |T_\alpha^\rho g(x)| > r\lambda/2\}| \leqslant C\Big(\frac{1}{r\lambda}\int_{\mathbb{R}^n} |g(x)|\mathrm{d}x\Big)^{1/(1-\alpha)} \quad \forall\, r > 0.$$

Let $P = x_1 + B_{j+2\sigma}$. Then $x_0 + B_{j+\sigma} \subseteq P$, and hence

$$\int_{\mathbb{R}^n} |g(x)|\mathrm{d}x \leqslant \int_P |f(x)|\mathrm{d}x \leqslant M_\alpha f(x_1) b^{(1-\alpha)(j+2\sigma)} \leqslant r\beta b^{(1-\alpha)(j+2\sigma)},$$

which implies

$$|\{x \in \mathbb{R}^n : |T_\alpha^\rho g(x)| > r\lambda/2\}| \leqslant Cb^{j+2\sigma}\Big(\frac{\beta}{\lambda}\Big)^{1/(1-\alpha)}. \tag{3.1}$$

Let $z \in \widetilde{B}$ satisfy $T_\alpha^\rho(|f|)(z) \leqslant r$. If $x \in \widetilde{B}$ and $y \notin x_0 + B_{j+\sigma}$, then $\rho(z-y) \leqslant b^\sigma(\rho(z-x) + \rho(x-y)) \leqslant 2b^\sigma \rho(x-y)$ since $\rho(z-x) \leqslant b^j \leqslant \rho(x-y)$. Therefore

$$\begin{aligned}|T_\alpha^\rho h(x)| &= \Big|\int_{x_0+B_{j+\sigma}^c} \frac{f(y)\mathrm{d}y}{\rho(x-y)^{1-\alpha}}\Big| \leqslant (2b^\sigma)^{1-\alpha}\int_{x_0+B_{j+\sigma}^c} \frac{|f(y)|\mathrm{d}y}{\rho(z-y)^{1-\alpha}}\\ &\leqslant (2b^\sigma)^{1-\alpha} T_\alpha^\rho(|f|)(z) \leqslant 2b^\sigma r \quad \text{for } x \in \widetilde{B}.\end{aligned}$$

For $\lambda \geqslant 4b^\sigma$, $\{x \in \widetilde{B} : |T_\alpha^\rho f(x)| > r\lambda \text{ and } M_\alpha f(x) \leqslant r\beta\}$ is a subset of $\{x \in \mathbb{R}^n : |T_\alpha^\rho g(x)| > r\lambda/2\}$. Then the proof follows from (3.1) by choosing $K = Cb^{2\sigma}$. ∎

We also need a Whitney type covering lemma.

LEMMA 3.3 ([4]). *Let Ω be an open proper subset of $\mathbb{R}^n$. For each integer $m \geqslant 0$, there exists a positive constant R depending only on m, a sequence $\{x_j\}_j \subseteq \Omega$ and a sequence $\{l_j\}_j \subseteq \mathbb{Z}$ such that:*

(i) *$\Omega = \bigcup_j (x_j + B_{l_j})$;*

(ii) *$(x_i + B_{l_i-2\sigma}) \cap (x_j + B_{l_j-2\sigma}) = \varnothing$ for all i, j with $i \neq j$;*

(iii) *$(x_j + B_{l_j+m}) \cap \Omega^c = \varnothing$ and $(x_j + B_{l_j+m+1}) \cap \Omega^c \neq \varnothing$ for all j;*

(iv) *$(x_i + B_{l_i+m-2\sigma}) \cap (x_j + B_{l_j+m-2\sigma}) \neq \varnothing$ implies that $|l_i - l_j| \leqslant \sigma$;*

(v) *for each j, the cardinality of $\{i : (x_i + B_{l_i+m-2\sigma}) \cap (x_j + B_{l_j+m-2\sigma}) \neq \varnothing\}$ is less than R.*

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Since $|T^\rho_{\alpha,0}f(x)| \leqslant CT^\rho_\alpha(|f|)(x)$, it suffices to show

$$\int_{\mathbb{R}^n}[T^\rho_\alpha(|f|)(x)]^q w(x)\mathrm{d}x \leqslant C\int_{\mathbb{R}^n}[M_\alpha f(x)]^q w(x)\mathrm{d}x, \tag{3.2}$$

$$\sup_{\lambda>0}\lambda^q w(\{x\in\mathbb{R}^n : T^\rho_\alpha(|f|)(x)>\lambda\})\leqslant C\sup_{\lambda>0}\lambda^q w(\{x\in\mathbb{R}^n : M_\alpha f(x)>\lambda\}). \tag{3.3}$$

Without loss of generality, we may assume $f \geqslant 0$ for replacing f by $|f|$. We also assume that $f(x)$ and $w(x)$ are locally integrable, otherwise the conclusions are trivial.

Given $r > 0$, by Lemma 3.3 (by taking $m = 2\sigma$) we decompose the set $\{x \in \mathbb{R}^n : T^\rho_\alpha f(x) > r\}$ into dilated balls $\{x_j + B_{l_j}\}_j$ with the following properties:

(a) $\{x_j + B_{l_j-2\sigma}\}$ are mutual disjoint,
(b) for every j, $\{x_j + B_{l_j}\}_j$ have at most R overlaps,
(c) $T^\rho_\alpha f(x) \leqslant r$ at some point of $x_j + B_{l_j+2\sigma+1}$.

Let K be as in Lemma 3.2 and $\lambda = 4b^\sigma$. Since $w \in \mathcal{A}_\infty$, it follows from Proposition 2.1 that, by choosing $\varepsilon = (1/2R)\lambda^{-q}$, there exists a $\delta > 0$ such that if $B \in \mathcal{B}$, E is a subset of B and $|E| \leqslant \delta|B|$, then $w(E) \leqslant \varepsilon w(B)$. Set $D = \lambda\delta^{1-\alpha}(Kb^{2\sigma+1})^{\alpha-1}$. For $\beta \in (0, D]$, let $Q_j = x_j + B_{l_j}$ and $E_j = \{x \in Q_j : T^\rho_\alpha f(x) > r\lambda$ and $M_\alpha f(x) \leqslant r\beta\}$. By Lemma 3.2, $|E_j| \leqslant K(\beta/\lambda)^{1/(1-\alpha)}|x_j + B_{l_j+2\sigma+1}| = \delta|Q_j|$ which yields $w(E_j) \leqslant (1/2R)\lambda^{-q}w(Q_j)$. Summing on j shows that

$$w(\{T^\rho_\alpha f>r\lambda \text{ and } M_\alpha f\leqslant r\beta\})\leqslant w\Big(\bigcup_j E_j\Big)\leqslant\sum_j w(E_j)\leqslant\frac{1}{2R}\lambda^{-q}w(Q_j)\leqslant\frac{1}{2}\lambda^{-q}w(\{T^\rho_\alpha f>r\}).$$

This implies

$$w(\{T^\rho_\alpha f > r\lambda\}) \leqslant w(\{M_\alpha f > r\beta\}) + \frac{1}{2}\lambda^{-q}w(\{T^\rho_\alpha f > r\}) \quad \forall\, \beta \in (0, D]. \tag{3.4}$$

We assume that f has compact support and write $\mathrm{supp}(f) \subseteq B_{l_0}$. Given $x \in B^c_{l_0+2\sigma}$, choose a point $u \in B_{l_0}$ such that $\rho(x-u) = \min\{\rho(x-y) : y \in B_{l_0}\}$. (The existence of u is due to the discrete values of ρ.) Let l_1 be the smallest integer satisfying $B_{l_0} \subseteq x + B_{l_1} := P$. We may write $x \in B_{l_0+2\sigma+m+1} \setminus B_{l_0+2\sigma+m}$ for some integer $m \geqslant 0$. Since $B_{l_0} \nsubseteq x + B_{l_1-1}$, there exists a point $y_1 \in B_{l_0}$ but $y_1 \notin x + B_{l_1-1}$, which implies $b^{l_1-1} \leqslant \rho(x - y_1) \leqslant b^\sigma(\rho(x) + \rho(y_1)) \leqslant b^{l_0+3\sigma+m} + b^{l_0+\sigma-1}$. We also note that $\rho(x - u) \geqslant b^{-\sigma}\rho(x) - \rho(u) \geqslant b^{l_0+\sigma+m} - b^{l_0-1}$. Thus, if we take $N_1 = (b^{3\sigma+m+2} + b^{\sigma+1})/(b^{\sigma+m+1} - 1) > 1$, then $|P| = b^{l_1} \leqslant N_1\rho(x - u)$. Therefore,

$$T^\rho_\alpha f(x)\leqslant\frac{1}{\rho(x-u)^{1-\alpha}}\int_{B_{l_0}}f(y)\mathrm{d}y\leqslant\frac{|P|^{1-\alpha}}{\rho(x-u)^{1-\alpha}}M_\alpha f(x)\leqslant N_1^{1-\alpha}M_\alpha f(x) \quad \text{for } x\in B^c_{l_0+2\sigma}.$$

Choose $\beta = \min(D, N_1^{\alpha-1})$. The above estimate yields

$$\{T_\alpha^\rho f > r\} \cap B_{l_0+2\sigma}^c \subseteq \{M_\alpha f > r\beta\}. \tag{3.5}$$

Both (3.4) and (3.5) show that

$$w(\{T_\alpha^\rho f > r\lambda\}) \leqslant 2w(\{M_\alpha f > r\beta\}) + \frac{1}{2}\lambda^{-q} w(\{T_\alpha^\rho f > r\} \cap B_{l_0+2\sigma}). \tag{3.6}$$

Multiply both sides of (3.6) by r^{q-1} and integrate from 0 to some positive N and a change of variables, the left side becomes

$$\lambda^{-q} \int_0^{N\lambda} r^{q-1} w(\{T_\alpha^\rho f > r\}) \mathrm{d}r. \tag{3.7}$$

With a change of variables for the first integral on the right, the right side becomes

$$2\beta^{-q} \int_0^{N\beta} r^{q-1} w(\{M_\alpha f > r\}) \mathrm{d}r + \frac{1}{2}\lambda^{-q} \int_0^{N} r^{q-1} w(\{T_\alpha^\rho f > r\} \cap B_{l_0+2\sigma}) \mathrm{d}r. \tag{3.8}$$

The second term in (3.8) is bounded by half of (3.7) since $\lambda \geqslant 1$, and is finite by the local integrablity of w. Therefore

$$\frac{1}{2}\lambda^{-q} \int_0^{N\lambda} r^{q-1} w(\{T_\alpha^\rho f > r\}) \mathrm{d}r \leqslant 2\beta^{-q} \int_0^{N\lambda} r^{q-1} w(\{M_\alpha f > r\}) \mathrm{d}r.$$

By letting $N \to \infty$, the above inequality reduces to

$$\frac{\lambda^{-q}}{2q} \int_{\mathbb{R}^n} (T_\alpha^\rho f(x))^q w(x) \mathrm{d}x \leqslant \frac{2\beta^{-q}}{q} \int_{\mathbb{R}^n} (M_\alpha f(x))^q w(x) \mathrm{d}x. \tag{3.9}$$

To prove (3.9) for f without compact support, let $f_i(x) = f(x)\chi_{B_i}(x)$. Then (3.9) can be applied to f_i. Taking the limit as $i \to \infty$ and using the monotone convergence theorem, we show (3.9) for general f. This proves (3.2).

To prove (3.3), multiplying both sides of (3.6) by r^q and taking supremum on $0 < r < N$ for a given N, we have

$$\sup_{0<r<N} r^q w(\{T_\alpha^\rho f > r\lambda\}) \leqslant \sup_{0<r<N} 2r^q w(\{M_\alpha f > r\beta\}) + \sup_{0<r<N} \frac{1}{2}\Big(\frac{r}{\lambda}\Big)^q w(\{T_\alpha^\rho f > r\} \cap B_{l_0+2\sigma}),$$

which is equivalent to

$$\sup_{0<r<\lambda N} \Big(\frac{r}{\lambda}\Big)^q w(\{T_\alpha^\rho f > r\}) \leqslant \sup_{0<r<N\beta} 2\Big(\frac{r}{\beta}\Big)^q w(\{M_\alpha f > r\}) + \sup_{0<r<N} \frac{1}{2}\Big(\frac{r}{\lambda}\Big)^q w(\{T_\alpha^\rho f > r\} \cap B_{l_0+2\sigma}).$$

Therefore

$$\frac{1}{2} \sup_{0<r<\lambda N} \Big(\frac{r}{\lambda}\Big)^q w(\{T_\alpha^\rho f > r\}) \leqslant 2 \sup_{0<r<N\beta} \Big(\frac{r}{\beta}\Big)^q w(\{M_\alpha f > r\}).$$

Let $N \to \infty$ and get

$$\sup_{r>0} r^q w(\{T_\alpha^\rho f > r\}) \leqslant 4\Big(\frac{\lambda}{\beta}\Big)^q \sup_{r>0} r^q w(\{M_\alpha f > r\}) \tag{3.10}$$

for f with compact support. For general f, let $f_i(x) = f(x)\chi_{B_i}(x)$. Then (3.10) can be applied to f_i. Taking the limit as $i \to \infty$ gives (3.10) for general f. This shows (3.3) and the proof of Theorem 3.1 is completed. ∎

4. PROOFS OF THEOREMS 1.3 AND 1.4

Bernardis and Salinas ([1], Theorem 1.6) established a weighted norm inequality for fractional maximal function M_α on spaces of homogeneous type as follows.

PROPOSITION 4.1. *Suppose* $0 \leqslant \alpha < 1$ *and* $1 < p \leqslant q < \infty$. *Let* (W, V) *be a pair of weights with* $V^{-1/(p-1)} \in \mathcal{A}_\infty$. *Then*

$$\|M_\alpha f\|_{L^q_W} \leqslant C\|f\|_{L^p_V} \quad \text{for all } f \in L^p_V(\mathbb{R}^n)$$

if and only if

$$\frac{W(B)^{p/q}[V^{-1/(p-1)}(B)]^{p-1}}{|B|^{(1-\alpha)p}} \leqslant C < \infty \quad \text{for all } B \in \mathcal{B}. \tag{4.1}$$

For $0 \leqslant \alpha < 1$, $1 \leqslant p < 1/\alpha$, $1/q = 1/p - \alpha$, $W = w^q$, and $V = w^p$, inequality (2.1) implies (4.1). Furthermore, by (ii) and (iv) of Proposition 2.1, if $w \in \mathcal{A}(p,q)$, then $V^{-1/(p-1)} = w^{-p/(p-1)} \in \mathcal{A}_{p'} \subseteq \mathcal{A}_\infty$. By Proposition 4.1, we obtain

THEOREM 4.2. *If* $0 < \alpha < 1$, $1 < p < 1/\alpha$, $1/q = 1/p - \alpha$, *and* $w \in \mathcal{A}(p,q)$, *then there is a constant* C, *independent of* f, *such that*

$$\Big(\int_{\mathbb{R}^n} [M_\alpha f(x) w(x)]^q \mathrm{d}x\Big)^{1/q} \leqslant C\Big(\int_{\mathbb{R}^n} |f(x)w(x)|^p \mathrm{d}x\Big)^{1/p}.$$

Theorem 1.3 follows immediately from Theorems 3.1 and 4.2 since $w^q \in \mathcal{A}_q \subseteq \mathcal{A}_\infty$ by Proposition 2.1(iv).

Proof of Theorem 1.4. For $p > 1$, fix $B \in \mathcal{B}$ and let $D = \int_B w(x)^{-p'}\mathrm{d}x$. If $D = 0$, there is nothing to prove. If $D = \infty$, then w^{-1} is not in $L^{p'}(B)$, and hence there is a nonnegative function $g \in L^p(B)$ such that $\int_B g(x)w(x)^{-1}\mathrm{d}x = \infty$. Let $f(x) = g(x)w(x)^{-1}$ on B and $f(x) = 0$ otherwise. Then $T_\alpha^\rho f \equiv \infty$ and $\int_B [f(x)w(x)]^p \mathrm{d}x = \int_B g(x)^p \mathrm{d}x$. By the assumption (1.2), $\int_B w(x)^q \mathrm{d}x \leqslant C\lambda^{-q}$ for all $\lambda > 0$. Thus, $\int_B w(x)^q \mathrm{d}x = 0$ and (2.1) follows. For the general case $0 < D < \infty$, let

$f(x) = w(x)^{-p'}\chi_B(x)$. Then $T^{\rho}_{\alpha} f(x) \geqslant D|B|^{\alpha-1}$ for $x \in B$. Plugging $\lambda = D|B|^{\alpha-1}$ into (1.2), we get

$$\int_B w(x)^q \mathrm{d}x \leqslant \frac{C}{(D|B|^{\alpha-1})^q}\Big(\int_B [w(x)^{-p'} w(x)]^p \mathrm{d}x\Big)^{q/p},$$

which reduces to (2.1).

For $p = 1$, fix a dilated ball $B \in \mathcal{B}$ and let $D = \operatorname{esssup}_{y\in B} w(y)^{-1}$. If $D = 0$, then (2.2) holds. Otherwise, given $\varepsilon > 0$, there is a subset E of B with positive measure such that $w(x) < D^{-1} + \varepsilon$ for all $x \in E$. Set $f = \chi_E$. Then $T^{\rho}_{\alpha} f(x) \geqslant |E||B|^{-1+\alpha}$ for $x \in B$. Choose $\lambda = |E||B|^{\alpha-1}$and (1.2) shows that

$$\Big(\int_B w(x)^q \mathrm{d}x\Big)^{1/q} \leqslant C|B|^{1-\alpha}(D^{-1} + \varepsilon).$$

Since ε is arbitrary, this shows (2.2) and completes the proof of Theorem 1.4. ∎

5. PROOFS OF THEOREM 1.5 AND COROLLARY 1.6

We recall the definition of weighted atoms. Let $w \in \mathcal{A}_\infty$ with the critical index q_w. For $0 < p \leqslant 1$, $q_w < q \leqslant \infty$, and $s \in \mathbb{N} \cup \{0\}$ with $s \geqslant [(q_w/p - 1)] \ln b/ \ln \lambda_-]$, a function $a \in L^q_w(\mathbb{R}^n)$ is said to be a $(p, q, s)_w$*-atom* if (i) $\operatorname{supp}(a) \subseteq x_0 + B_j$ for some $x_0 \in \mathbb{R}^n$ and $j \in \mathbb{Z}$, (ii) $\|a\|_{L^q_w} \leqslant w(x_0 + B_j)^{1/q-1/p}$, (iii) $\int_{\mathbb{R}^n} a(x)x^\beta \mathrm{d}x = 0$ for $|\beta| \leqslant s$.

Proof of Theorem 1.5. Let $\alpha, \gamma, p, \overline{p}, q, \overline{q}$, and w be given as in Theorem 1.5. By Theorems 6.2 and 7.2 of [4], it suffice to show that

$$\|T^{\rho}_{\alpha,\gamma} a\|_{L^q_{w^q}} \leqslant C \quad \text{for all } (p, \overline{p}, s)_{w^p}\text{-atom } a,$$

where C is a constant independent of a. Let a be any $(p, \overline{p}, s)_{w^p}$-atom with $\operatorname{supp}(a) \subseteq x_0 + B_l$, $\|a\|_{L^{\overline{p}}_w} \leqslant w(x_0 + B_l)^{1/\overline{p}-1/p}$, and $\int_{\mathbb{R}^n} a(x)x^\beta \mathrm{d}x = 0$ for $|\beta| \leqslant s$. We write

$$\begin{aligned}\|T^{\rho}_{\alpha,\gamma} a\|_{L^q_{w^q}} &= \Big(\int_{\mathbb{R}^n} |T^{\rho}_{\alpha,\gamma} a(x)|^q w^q(x)\mathrm{d}x\Big)^{1/q}\\ &\leqslant \Big(\int_{x_0+B_{l+2\sigma}} |T^{\rho}_{\alpha,\gamma} a(x)|^q w^q(x)\mathrm{d}x\Big)^{1/q} + \Big(\int_{x_0+B^{\mathrm{c}}_{l+2\sigma}} |T^{\rho}_{\alpha,\gamma} a(x)|^q w^q(x)\mathrm{d}x\Big)^{1/q}\\ &:= K_1 + K_2.\end{aligned}$$

Hölder's inequality, Theorem 1.3, and the size condition of a yield

$$K_1 \leqslant \Big(\int_{x_0+B_{l+2\sigma}} |T^{\rho}_{\alpha,\gamma}a(x)|^{\overline{q}} w^{\overline{q}p/\overline{p}}(x)\mathrm{d}x \Big)^{1/\overline{q}} \Big(\int_{x_0+B_{l+2\sigma}} w^p(x)\mathrm{d}x \Big)^{1/q-1/\overline{q}}$$
$$\leqslant C\Big(\int_{\mathbb{R}^n} |a(x)|^{\overline{p}} w^p(x)\mathrm{d}x \Big)^{1/\overline{p}} [w^p(x_0+B_{l+2\sigma})]^{1/q-1/\overline{q}} \leqslant C.$$

For K_2, we need an estimate of $T^{\rho}_{\alpha,\gamma}a(x)$ for $x \in x_0 + B^{\mathrm{c}}_{l+2\sigma}$. If $x \in x_0 + B^{\mathrm{c}}_{l+2\sigma}$ and $y \in x_0 + B_l$, then $\rho(x-x_0) \geqslant b^{2\sigma}\rho(y-x_0)$. By the condition (iii) of Definition 1.1 and the vanishing moment condition of a,

$$|T^{\rho}_{\alpha,\gamma}a(x)| = \Big| \int_{x_0+B_l} a(y)(K(x,y)-K(x,x_0))\mathrm{d}y \Big|$$
$$\leqslant C\rho(x-x_0)^{\alpha-1-\gamma} \int_{x_0+B_l} |a(y)|\rho(y-x_0)^{\gamma}\mathrm{d}y$$
$$\leqslant Cb^{l\gamma}\rho(x-x_0)^{\alpha-1-\gamma} \int_{x_0+B_l} |a(y)|\mathrm{d}y \quad \text{for } x \in x_0+B^{\mathrm{c}}_{l+2\sigma}.$$

Thus,

$$K_2 \leqslant Cb^{l\gamma} \int_{x_0+B_l} |a(y)|\mathrm{d}y \Big(\int_{x_0+B^{\mathrm{c}}_{l+2\sigma}} \rho(x-x_0)^{(\alpha-1-\gamma)q} w^q(x)\mathrm{d}x \Big)^{1/q}$$
$$\leqslant Cb^{l\gamma} \int_{x_0+B_l} |a(y)|\mathrm{d}y \Big(\sum_{j=0}^{\infty} \int_{x_0+(B_{l+2\sigma+j+1}\setminus B_{l+2\sigma+j})} \rho(x-x_0)^{(\alpha-1-\gamma)q} w^q(x)\mathrm{d}x \Big)^{1/q} \tag{5.1}$$
$$\leqslant Cb^{l\gamma} \int_{x_0+B_l} |a(y)|\mathrm{d}y \Big(\sum_{j=0}^{\infty} b^{(l+2\sigma+j)(\alpha-1-\gamma)q} \int_{x_0+B_{l+2\sigma+j+1}} w^q(x)\mathrm{d}x \Big)^{1/q}$$
$$= Cb^{l(\alpha-1)} \int_{x_0+B_l} |a(y)|\mathrm{d}y \Big(\sum_{j=0}^{\infty} b^{j(\alpha-1-\gamma)q} \int_{x_0+B_{l+2\sigma+j+1}} w^q(x)\mathrm{d}x \Big)^{1/q}.$$

Hölder's inequality and the size condition of a give us

$$\int_{x_0+B_l} |a(y)|\mathrm{d}y \leqslant \|a\|_{L^{\overline{p}}_{w^p}} \Big(\int_{x_0+B_l} w(y)^{-p/(\overline{p}-1)}\mathrm{d}y \Big)^{1-1/\overline{p}} \tag{5.2}$$
$$\leqslant [w^p(x_0+B_l)]^{1/\overline{p}-1/p} \Big(\int_{x_0+B_l} w(y)^{-p/(\overline{p}-1)}\mathrm{d}y \Big)^{1-1/\overline{p}}.$$

Proposition 2.1(iv) shows $w^p \in \mathcal{A}_{\overline{p}}$ and $w^{\overline{q}p/\overline{p}} \in \mathcal{A}_{\overline{q}}$. Using Hölder's inequality, Proposition 2.1(i), and applying (2.1) to $w^{\overline{q}p/\overline{p}}$

$$\begin{aligned}
&\int\limits_{x_0+B_{l+2\sigma+j+1}} w^q(x)\mathrm{d}x \\
&\qquad \leqslant \Big(\int\limits_{x_0+B_{l+2\sigma+j+1}} w^{\overline{q}p/\overline{p}}(x)\mathrm{d}x\Big)^{q/\overline{q}} \Big(\int\limits_{x_0+B_{l+2\sigma+j+1}} w^p(x)\mathrm{d}x\Big)^{1-q/\overline{q}} \\
&\qquad = C[w^{\overline{q}p/\overline{p}}(x_0+B_l)]^{q/\overline{q}}[w^p(x_0+B_l)]^{q(1/p-1/\overline{p})} \\
&\qquad\quad \times \Big\{ \frac{w^{\overline{q}p/\overline{p}}(x_0+B_{l+2\sigma+j+1})}{w^{\overline{q}p/\overline{p}}(x_0+B_l)} \Big\}^{q/\overline{q}} \Big\{ \frac{w^p(x_0+B_{l+2\sigma+j+1})}{w^p(x_0+B_l)} \Big\}^{q(1/p-1/\overline{p})} \\
&\qquad \leqslant Cb^{jq\overline{p}/p}[w^{\overline{q}p/\overline{p}}(x_0+B_l)]^{q/\overline{q}}[w^p(x_0+B_l)]^{q(1/p-1/\overline{p})} \\
&\qquad \leqslant Cb^{ql(1-\alpha)}b^{jq\overline{p}/p}[w^p(x_0+B_l)]^{q(1/p-1/\overline{p})} \Big(\int\limits_{x_0+B_l} w(y)^{-\frac{p}{\overline{p}-1}}\mathrm{d}y\Big)^{q(-1+1/\overline{p})}.
\end{aligned} \tag{5.3}$$

Combining (5.1)–(5.3) gives us

$$K_2 \leqslant C\Big(\sum_{j=0}^{\infty} b^{jq(\alpha-1-\gamma+\overline{p}/p)}\Big)^{1/q} \leqslant C.$$

This completes the proof of Theorem 1.5. ∎

Proof of Corollary 1.6. Since $w^{p/(1-\alpha)} \in \mathcal{A}_1$, it follows from Proposition 2.1(iii) that there exists an $\varepsilon > 0$ such that $w^{p(1+\varepsilon)/(1-\alpha)} \in \mathcal{A}_1$, which still holds for any smaller $\varepsilon' < \varepsilon$. Let $\overline{p} = (\alpha+\varepsilon)/(\alpha+\alpha\varepsilon) > 1$ and $1/\overline{q} = 1/\overline{p} - \alpha$. Then $(1+\varepsilon)/(1-\alpha) = \overline{q}/\overline{p}$. Since $w^{p\overline{q}/\overline{p}} \in \mathcal{A}_1 \subseteq \mathcal{A}_{\overline{q}(1-\alpha)}$, Proposition 2.1(v) yields $w^{p/\overline{p}} \in \mathcal{A}(\overline{p},\overline{q})$. On the other hand, noting that $\alpha - 1 - \gamma + 1/p < 0$ and $\overline{p} = (\alpha+\varepsilon)/(\alpha+\alpha\varepsilon) \to 1$ as $\varepsilon \to 0$, we can choose a small enough $\varepsilon > 0$ such that $\alpha - 1 - \gamma + \overline{p}/p < 0$. Then Corollary 1.6 follows from Theorem 1.5. ∎

REMARK 5.1. In the proof of Theorem 1.5, we use only zero vanishing moment of an atom. Readers might expect to get a wider range of p by using the higher moments of an atom. In reality, the higher moments cannot be applied since the kernel of $T^{\rho}_{\alpha,\gamma}$ does not have much regularity. As a consequence, Theorem 1.5 holds only for some subrange of $0 < p \leqslant 1$. In order to strengthen the regularity of the fractional integral operators (without weights), Ding and Lan used differentiation to define such operators in Definition 1.1 of [8]; however, their approach does not work for our situation.

REMARK 5.2. Gatto et al. [12], [13], [15] studied fractional integral operators on the more general setting of spaces of homogeneous type. They showed that many classical results about fractional integral operators are valid, and many of classical proofs can be adapted, if some mild conditions are imposed on the spaces. The weighted norm inequalities for fractional integral operators on spaces of homogeneous type were obtained by Bernardis and Salinas [1]. However, there is no suitable definition of anisotropic structure for spaces of homogeneous type as far as the authors know.

Acknowledgements. The authors are grateful to the referee for valuable comments. Research by the first author was supported by NNSF of China under Grant #11071200 and NSF of Zhejiang Province under Grant #LY12A01011. Research by the second and third authors was supported by NSC of Taiwan under Grant #NSC 99-2115-M-008-002-MY3 and Grant #NSC 100-2115-M-008-002-MY3, respectively.

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Received October 23, 2009.