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NON-COMMUTATIVE L_p-SPACES ASSOCIATED WITH A MAHARAM TRACE

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ABSTRACT. Non-commutative L_p-spaces $L^p(M, \Phi)$ associated with the Maharam trace are defined and their dual spaces are described.

KEYWORDS: *Von Neumann algebra, measurable operator, Dedekind complete Riesz space, integration with respect to a vector-valued trace.*

MSC (2000): 28B15, 46L50.

INTRODUCTION

Development of the theory of integration for measures μ with the values in Dedekind complete Riesz spaces has inspired the study of (bo)-complete lattice-normed spaces $L^p(\mu)$ (see, for example, 6.1.8 of [7]). Note that, if the measure μ satisfies the Maharam property, then the spaces $L^p(\mu)$ are Banach–Kantorovich spaces.

The existence of center-valued traces on finite von Neumann algebras naturally leads to a study of the integration for traces with the values in a complex Dedekind complete Riesz space $F_{\mathbb{C}} = F \oplus \mathrm{i}F$. For commutative von Neumann algebras, the development of $F_{\mathbb{C}}$-valued integration is a part of the study of the properties of order continuous positive maps of Riesz spaces, for which we refer to the treatise by A.G. Kusraev [7]. The operators possessing the Maharam property provide important examples of such mappings, while the L^p-spaces associated with such operators are non-trivial examples of Banach–Kantorovich Riesz spaces.

Let M be a non-commutative von Neumann algebra, let $F_{\mathbb{C}}$ be a von Neumann subalgebra in the center of M, and let $\Phi : M \to F_{\mathbb{C}}$ be a trace such that $\Phi(zx) = z\Phi(x)$ for all $z \in F_{\mathbb{C}}$, $x \in M$. Then the non-commutative L^p-space $L^p(M, \Phi)$ is a Banach–Kantorovich space [1], [6], and the trace Φ satisfies the Maharam property, that is, if $0 \leqslant z \leqslant \Phi(x)$, $z \in F_{\mathbb{C}}$, $0 \leqslant x \in M$, then there exists $y \in M$, $0 \leqslant y \leqslant x$ such that $\Phi(y) = z$ (compare with 3.4.1 of [7]).

In [2], a faithful normal trace Φ on M with the values in an arbitrary complex Dedekind complete Riesz space was considered. In particular, a complete description of such traces in the case when Φ is a Maharam trace was given. In the same paper, utilizing the locally measure topology on the algebra $S(M)$ of all measurable operators affiliated with M, the Banach–Kantorovich space $L^1(M, \Phi) \subset S(M)$ was constructed and a version of Radon–Nikodym-type theorem for Maharam traces was established.

In the present article, we define a new class of Banach–Kantorovich spaces, non-commutative L_p-spaces $L^p(M, \Phi)$ associated with a Maharam trace; also, we give a description of their dual spaces.

We use the terminology and results of the theory of von Neumann algebras [10], [11], the theory of measurable operators [9], [8], and of the theory of Dedekind complete Riesz space and Banach–Kantorovich spaces [7].

1. PRELIMINARIES

Let X be a vector space over the field $\mathbb{C}$ of complex numbers, and let F be a Riesz space. A mapping $\|\cdot\| : X \to F$ is said to be a *vector (F-valued) norm* if it satisfies the following axioms:

(i) $\|x\| \geqslant 0$, $\|x\| = 0 \Leftrightarrow x = 0$ $(x \in X)$;
(ii) $\|\lambda x\| = |\lambda| \|x\|$ $(\lambda \in \mathbb{C}, x \in X)$;
(iii) $\|x + y\| \leqslant \|x\| + \|y\|$ $(x, y \in X)$.

A *norm* $\|\cdot\|$ is called *decomposable* if the following property holds:

PROPERTY 1. If $f_1, f_2 \geqslant 0$ and $\|x\| = f_1 + f_2$, then there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $\|x_k\| = f_k$ $(k = 1, 2)$.

If property 1 is valid only for disjoint elements $f_1, f_2 \in F$, the norm is called *disjointly decomposable* or, briefly, *d-decomposable*.

The pair $(X, \|\cdot\|)$ is called a *lattice-normed space* (shortly, LNS). If the norm $\|\cdot\|$ is decomposable (d-decomposable), then so is the space $(X, \|\cdot\|)$.

A net $\{x_\alpha\}_{\alpha \in A} \subset X$ (bo)- converges to $x \in X$ if the net $\{\|x_\alpha - x\|\}_{\alpha \in A}$ (o)-converges to zero in the Riesz space F. A net $\{x_\alpha\}_{\alpha \in A}$ is said to be a *(bo)- Cauchy net* if $\sup\limits_{\alpha, \beta \geqslant \gamma} \|x_\alpha - x_\beta\| \downarrow 0$. An LNS is called *(bo)- complete* if any (bo)-Cauchy net (bo)-converges. A Banach–Kantorovich space (shortly, BKS) is a d-decomposable (bo)-complete LNS. It is well known that every BKS is a decomposable LNS.

Let F be a Dedekind complete Riesz space with a weak identity $\mathbf{1}_F$, and let $F_{\mathbb{C}} = F \oplus \mathrm{i}F$ be the complexification of F. If $z = \alpha + \mathrm{i}\beta \in F_{\mathbb{C}}$, $\alpha, \beta \in F$, then $\overline{z} := \alpha - \mathrm{i}\beta$, and $|z| := \sup\{\mathrm{Re}(e^{\mathrm{i}\theta} z) : 0 \leqslant \theta < 2\pi\}$ (see 1.3.13 of [7]).

Let $(X, \|\cdot\|_X)$ be the BKS over F. A linear operator $T : X \to F_{\mathbb{C}}$ is said to be *F-bounded* if there exists $0 \leqslant c \in F$ such that $|T(x)| \leqslant c\|x\|_X$ for all $x \in X$. For any F-bounded operator T, define the element $\|T\| = \sup\{|T(x)| : x \in X,$

$\|x\|_X \leqslant \mathbf{1}_F\}$, which is called the *abstract F-norm of the operator T* ([7], 4.1.3). It is known that $|T(x)| \leqslant \|T\| \, \|x\|_X$ for all $x \in X$ ([7], 4.1.1).

The set X^* of all F-bounded linear mappings from X into $F_{\mathbb{C}}$ is called the *F-dual space to the BKS X*. For $T, S \in X^*$, we set $(T+S)(x) = Tx + Sx$, $(\lambda T)(x) = \lambda Tx$, where $x \in X$, $\lambda \in \mathbb{C}$. It is clear that X^* is a linear space with respect to the introduced algebraic operations. Moreover, $(X^*, \|\cdot\|)$ is a BKS ([7], 4.2.6).

Let H be a Hilbert space, let $B(H)$ be the $*$-algebra of all bounded linear operators on H, and let $\mathbf{1}$ be the identity operator on H. Given a von Neumann algebra M acting on H, denote by $Z(M)$ the center of M and by $P(M)$ the lattice of all projections in M. Let $P_{\text{fin}}(M)$ be the set of all finite projections in M.

A densely-defined closed linear operator x (possibly unbounded) affiliated with M is said to be *measurable* if there exists a sequence $\{p_n\}_{n=1}^{\infty} \subset P(M)$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset \mathfrak{D}(x)$ and $p_n^{\perp} = \mathbf{1} - p_n \in P_{\text{fin}}(M)$ for every $n = 1, 2, \dots$ (here $\mathfrak{D}(x)$ is the domain of x). Let us denote by $S(M)$ the set of all measurable operators.

Let x, y be measurable operators. Then $x+y$, xy and x^* are densely-defined and preclosed. Moreover, the closures $\overline{x+y}$ (strong sum), $\overline{xy}$ (strong product) and x^* are also measurable, and $S(M)$ is a $*$-algebra with respect to the strong sum, strong product, and the adjoint operation (see [9]). For any subset $E \subset S(M)$ we denote by E_h (respectively E_+) the set of all self-adjoint (respectively positive) operators from E.

For $x \in S(M)$ let $x = u|x|$ be the polar decomposition, where $|x| = (x^*x)^{1/2}$, u is a partial isometry in $B(H)$ such that u^*u is a right support of x. Then $u \in M$ and $|x| \in S(M)$. If $x \in S_h(M)$ and $\{E_\lambda(x)\}$ are the spectral projections of x, then $\{E_\lambda(x)\} \subset P(M)$.

Let M be a commutative von Neumann algebra. Then M is $*$-isomorphic to the $*$-algebra $L^\infty(\Omega, \Sigma, \mu)$ of all essentially bounded complex measurable functions with the identification almost everywhere, where (Ω, Σ, μ) is a measurable space. In addition $S(M) \cong L^0(\Omega, \Sigma, \mu)$, where $L^0(\Omega, \Sigma, \mu)$ is the $*$-algebra of all complex measurable functions with the identification almost everywhere [9].

The locally measure topology $t(M)$ on $L^0(\Omega, \Sigma, \mu)$ is by definition the linear (Hausdorff) topology whose base of neighborhoods of zero is given by

$$\begin{aligned} W(B, \varepsilon, \delta) = \{f \in L^0(\Omega,\, \Sigma,\, \mu) :\ & \text{there exists a set } E \in \Sigma, \text{ such that } E \subseteq B, \\ & \mu(B \setminus E) \leqslant \delta,\ f\chi_E \in L^\infty(\Omega, \Sigma, \mu),\ \|f\chi_E\|_{L_\infty(\Omega,\Sigma,\mu)} \leqslant \varepsilon\}. \end{aligned}$$

Here ε, δ run over all strictly positive numbers and $B \in \Sigma$, $\mu(B) < \infty$. It is known that $(S(M), t(M))$ is a complete topological $*$-algebra.

It is clear that zero neighborhoods $W(B, \varepsilon, \delta)$ are closed and have the following property: if $f \in W(B, \varepsilon, \delta)$, $g \in L^\infty(\Omega, \Sigma, \mu)$, $\|g\|_{L\infty(\Omega,\Sigma,\mu)} \leqslant 1$, then $gf \in W(B, \varepsilon, \delta)$.

A net $\{f_\alpha\}$ converges locally in measure to f (notation: $f_\alpha \xrightarrow{t(M)} f$) if and only if $f_\alpha \chi_B$ converges in μ-measure to $f\chi_B$ for each $B \in \Sigma$ with $\mu(B) < \infty$.

Let now M be an arbitrary finite von Neumann algebra, $\Phi_M : M \to Z(M)$ be a center-valued trace on M ([10], 7.11). Let $Z(M) \cong L^\infty(\Omega, \Sigma, \mu)$. The locally measure topology $t(M)$ on $S(M)$ is the linear (Hausdorff) topology whose base of neighborhoods of zero is given by

$$V(B, \varepsilon, \delta) = \{x \in S(M) : \text{ there exists } p \in P(M), z \in P(Z(M)) \\ \text{such that } xp \in M, \|xp\|_M \leqslant \varepsilon, z^\perp \in W(B, \varepsilon, \delta), \Phi_M(zp^\perp) \leqslant \varepsilon z\},$$

where $\|\cdot\|_M$ is the C^*-norm in M. It is known that $(S(M), t(M))$ is a complete topological $*$-algebra [13].

From Section 3.5 of [8], we have the following criterion for convergence in the topology $t(M)$.

PROPOSITION 1.1. *A net* $\{x_\alpha\}_{\alpha\in A} \subset S(M)$ *converges to zero in the topology* $t(M)$ *if and only if* $\Phi_M(E_\lambda^\perp(|x_\alpha|) \xrightarrow{t(M)} 0$ *for any* $\lambda > 0$.

Let M be an arbitrary von Neumann algebra, and let F be a Dedekind complete Riesz space. An $F_{\mathbb{C}}$*-valued trace* on the von Neumann algebra M is a linear mapping $\Phi : M \to F_{\mathbb{C}}$ with $\Phi(x^*x) = \Phi(xx^*) \geqslant 0$ for all $x \in M$. It is clear that $\Phi(M_h) \subset F$, $\Phi(M_+) \subset F_+ = \{a \in F : a \geqslant 0\}$. A trace Φ is said to be *faithful* if the equality $\Phi(x^*x) = 0$ implies $x = 0$, *normal* if $\Phi(x_\alpha) \uparrow \Phi(x)$ for every $x_\alpha, x \in M_h$, $x_\alpha \uparrow x$.

If M is a finite von Neumann algebra, then its canonical center-valued trace $\Phi_M : M \to Z(M)$ is an example of a $Z(M)$-valued faithful normal trace.

Let us list some properties of the trace $\Phi : M \to F_{\mathbb{C}}$.

PROPOSITION 1.2 ([2]). (i) *Let* $x, y, a, b \in M$. *Then*

$$\Phi(x^*) = \overline{\Phi(x)}, \quad \Phi(xy) = \Phi(yx), \quad \Phi(|x^*|) = \Phi(|x|), \\ |\Phi(axb)| \leqslant \|a\|_M \|b\|_M \Phi(|x|);$$

(ii) *If* Φ *is a faithful trace, then* M *is finite;*

(iii) *If* $x_n, x \in M$ *and* $\|x_n - x\|_M \to 0$, *then* $|\Phi(x_n) - \Phi(x)|$ *relative uniform converges to zero;*

(iv) $\Phi(|x + y|) \leqslant \Phi(|x|) + \Phi(|y|)$ *for all* $x, y \in M$.

The trace $\Phi : M \to F_{\mathbb{C}}$ possesses *the Maharam property* if for any $x \in M_+$, $0 \leqslant f \leqslant \Phi(x)$, $f \in F$, there exists $y \in M_+$, $y \leqslant x$ such that $\Phi(y) = f$. A faithful normal $F_{\mathbb{C}}$-valued trace Φ with the Maharam property is called *a Maharam trace* (compare with III, 3.4.1 of [7]). Obviously, any faithful finite numerical trace on M is a $\mathbb{C}$-valued Maharam trace.

Let us give another examples of Maharam traces. Let M be a finite von Neumann algebra, let $\mathcal{A}$ be a von Neumann subalgebra in $Z(M)$, and let $T : Z(M) \to \mathcal{A}$ be a linear positive normal operator, $T(x^*x) = 0 \Leftrightarrow x = 0$. If $f \in S(\mathcal{A})$ is a reversible positive element, then $\Phi(T, f)(x) = fT(\Phi_M(x))$ is an

$S(\mathcal{A})$-valued faithful normal trace on M. In addition, if $T(ab) = aT(b)$ for all $a \in \mathcal{A}, b \in Z(M)$, then $\Phi(T, f)$ is a Maharam trace on M.

If τ is a faithful normal finite numerical trace on M and $\dim(Z(M)) > 1$, then $\Phi(x) = \tau(x)\mathbf{1}$ is a $Z(M)$-valued faithful normal trace, which does not possess the Maharam property (see [2]).

Let F have a weak order unit $\mathbf{1}_F$. Denote by $B(F)$ the complete Boolean algebra of unitary elements with respect to $\mathbf{1}_F$, and let Q be the Stone compact space of the Boolean algebra $B(F)$. Let $C_\infty(Q)$ be the Dedekind complete Riesz space of all continuous functions $a : Q \to [-\infty, +\infty]$ such that $a^{-1}(\{\pm\infty\})$ is a nowhere dense subset of Q. We identify F with the order-dense ideal in $C_\infty(Q)$ containing algebra $C(Q)$ of all continuous real functions on Q. In addition, $\mathbf{1}_F$ is identified with the function equal to 1 identically on Q ([7], 1.4.4).

We need the following theorem from [2].

THEOREM 1.3. *Let Φ be an $F_{\mathbb{C}}$-valued Maharam trace on a von Neumann algebra M. Then there exists a von Neumann subalgebra $\mathcal{A}$ in $Z(M)$, a $*$-isomorphism ψ from $\mathcal{A}$ onto the $*$-algebra $C(Q)_{\mathbb{C}}$, a positive linear normal operator $\mathcal{E}$ from $Z(M)$ onto $\mathcal{A}$ with $\mathcal{E}(\mathbf{1}) = \mathbf{1}$, $\mathcal{E}^2 = \mathcal{E}$, such that:*

(i) $\Phi(x) = \Phi(\mathbf{1})\psi(\mathcal{E}(\Phi_M(x)))$ *for all* $x \in M$;

(ii) $\Phi(zy) = \Phi(z\mathcal{E}(y))$ *for all* $z, y \in Z(M)$;

(iii) $\Phi(zy) = \psi(z)\Phi(y)$ *for all* $z \in \mathcal{A}, y \in M$.

Due to Theorem 1.3, the $*$-algebra $\mathcal{B} = C(Q)_{\mathbb{C}}$ is a commutative von Neumann algebra, and $*$-algebra $C_\infty(Q)_{\mathbb{C}}$ is identified with the $*$-algebra $S(\mathcal{B})$. It is clear that the $*$-isomorphism ψ from $\mathcal{A}$ onto $\mathcal{B}$ can be extended to a $*$-isomorphism from $S(\mathcal{A})$ onto $S(\mathcal{B})$. We denote this mapping also by ψ.

Let Φ be a $S(\mathcal{B})$-valued Maharam trace on a von Neumann algebra M. A net $\{x_\alpha\} \subset S(M)$ converges to $x \in S(M)$ with respect to the trace Φ (notation: $x_\alpha \xrightarrow{\Phi} x$) if $\Phi(E_\lambda^\perp(|x_\alpha - x|)) \xrightarrow{t(\mathcal{B})} 0$ for all $\lambda > 0$.

PROPOSITION 1.4 ([2]). *$x_\alpha \xrightarrow{\Phi} x$ if and only if $x_\alpha \xrightarrow{t(M)} x$.*

An operator $x \in S(M)$ is said to be *Φ-integrable* if there exists a sequence $\{x_n\} \subset M$ such that $x_n \xrightarrow{\Phi} x$ and $\|x_n - x_m\|_\Phi \xrightarrow{t(\mathcal{B})} 0$ as $n, m \to \infty$.

Let x be a Φ-integrable operator from $S(M)$. Then there exists a $\widehat{\Phi}(x) \in S(\mathcal{B})$ such that $\Phi(x_n) \xrightarrow{t(\mathcal{B})} \widehat{\Phi}(x)$. In addition $\widehat{\Phi}(x)$ does not depend on the choice of a sequence $\{x_n\} \subset M$, for which $x_n \xrightarrow{\Phi} x$, $\Phi(|x_n - x_m|) \xrightarrow{t(\mathcal{B})} 0$ [2]. It is clear that each operator $x \in M$ is Φ-integrable and $\widehat{\Phi}(x) = \Phi(x)$.

Denote by $L^1(M, \Phi)$ the set of all Φ-integrable operators from $S(M)$. If $x \in S(M)$ then $x \in L^1(M, \Phi)$ if and only if $|x| \in L^1(M, \Phi)$, in addition $|\widehat{\Phi}(x)| \leqslant \widehat{\Phi}(|x|)$ [2]. For any $x \in L^1(M, \Phi)$, set $\|x\|_{1,\Phi} = \widehat{\Phi}(|x|)$. It is known that $L^1(M, \Phi)$

is a linear subspace of $S(M)$, $ML^1(M,\Phi)M \subset L^1(M,\Phi)$, and $x^* \in L^1(M,\Phi)$ for all $x \in L^1(M,\Phi)$ [2]. Moreover, the following theorem is true.

THEOREM 1.5 ([2]). (i) *$(L^1(M,\Phi), \|\cdot\|_{1,\Phi})$ is a Banach–Kantorovich space;*

(ii) *$S(\mathcal{A})L^1(M,\Phi) \subset L^1(M,\Phi)$, in addition $\widehat{\Phi}(zx) = \psi(z)\widehat{\Phi}(x)$ for all $z \in S(\mathcal{A})$, $x \in L^1(M,\Phi)$.*

2. L_p-SPACES ASSOCIATED WITH A MAHARAM TRACE

Let $\mathcal{B}$ be a commutative von Neumann algebra, which is $*$-isomorphic to a von Neumann subalgebra $\mathcal{A}$ in $Z(M)$, and let $\Phi : M \to S(\mathcal{B})$ be a Maharam trace on M (see Theorem 1.3). For any $p > 1$, set $L^p(M,\Phi) = \{x \in S(M) : |x|^p \in L^1(M,\Phi)\}$ and $\|x\|_{p,\Phi} = \widehat{\Phi}(|x|^p)^{1/p}$. It is clear that $M \subset L^p(M,\Phi)$.

Let e be a nonzero projection in $\mathcal{B}$, and put $\Phi_e(a) = \Phi(a)e$, $a \in M$. A mapping $\Phi_e : M \to S(\mathcal{B}e)$ is a normal (not necessarily faithful) $S(\mathcal{B}e)$-valued trace on M. Denote by $s(\Phi_e) := \mathbf{1} - \sup\{p \in P(M) : \Phi_e(p) = 0\}$ the support of the trace Φ_e. It is clear that $s(\Phi_e) \in P(Z(M))$ and $\Phi_e(a) = \Phi(as(\Phi_e))$ is a faithful normal $S(\mathcal{B}e)$-valued trace on $Ms(\Phi_e)$ (compare 5.15 of [10]). Moreover Φ_e possesses the Maharam property.

If e and g are orthogonal nonzero projections in $P(\mathcal{B})$, then $\Phi_g(s(\Phi_e)) = \Phi(s(\Phi_e))g = \Phi_e(\mathbf{1})g = \Phi(\mathbf{1})eg = 0$, i.e. $s(\Phi_e)s(\Phi_g) = 0$. Let $\{e_i\}_{i\in I}$ be a family of nonzero mutually orthogonal projections in $P(\mathcal{B})$ with $\sup\limits_{i\in I} e_i = \mathbf{1}_{\mathcal{B}}$, where $\mathbf{1}_{\mathcal{B}}$ is the unit of the algebra $\mathcal{B}$. If $z = \mathbf{1} - \sup\limits_{i\in I} s(\Phi_{e_i})$ then $\Phi(z)e_i = \Phi_{e_i}(z) = 0$ for all $i \in I$. Therefore $\Phi(z) = 0$, i.e. $z = 0$, or $\sup\limits_{i\in I} s(\Phi_{e_i}) = \mathbf{1}$.

Further, we need the following

PROPOSITION 2.1. *Let $x \in S(M)$ and let $\{e_i\}_{i\in I}$ be the family of nonzero mutually orthogonal projections in $P(\mathcal{B})$ with $\sup\limits_{i\in I} e_i = \mathbf{1}_{\mathcal{B}}$. Then $x \in L^p(M,\Phi)$ if and only if $xs(\Phi_e) \in L^p(Ms(\Phi_{e_i}), \Phi_{e_i})$ for all $i \in I$. In addition $\|x\|_{p,\Phi}e_i = \|xs(\Phi_{e_i})\|_{p,\Phi_{e_i}}$.*

Proof. Let $x \in L^p(M,\Phi)$, $a_n = E_n(|x|^p)|x|^p$ where $E_n(|x|^p)$ is the spectral projection of $|x|^p$ corresponding to the interval $(-\infty, n]$. It is clear that $a_n \xrightarrow{\Phi} |x|^p$ and $\Phi(|a_n - a_m|) \xrightarrow{t(\mathcal{B})} 0$ as $n, m \to \infty$. Hence, $a_n s(\Phi_{e_i}) \xrightarrow{\Phi_{e_i}} |x|^p s(\Phi_{e_i})$ (see Proposition 1.4). In addition, from the inequality $\Phi_{e_i}(|a_n s(\Phi_{e_i}) - a_m s(\Phi_{e_i})|) = \Phi(|a_n - a_m|s(\Phi_{e_i})) \leqslant \Phi(|a_n - a_m|)$, we have $\Phi_{e_i}(|a_n s(\Phi_{e_i}) - a_m s(\Phi_{e_i})|) \xrightarrow{t(\mathcal{B}e_i)} 0$. This means that $|xs(\Phi_{e_i})|^p = |x|^p s(\Phi_{e_i}) \in L^1(Ms(\Phi_{e_i}), \Phi_{e_i})$ and $\|xs(\Phi_{e_i})\|_{p,\Phi_{e_i}} = \widehat{\Phi}_{e_i}(|x|^p s(\Phi_{e_i}))^{1/p} = (\widehat{\Phi}(|x|^p)e_i)^{1/p} = \|x\|_{p,\Phi}e_i$.

Conversely, let $xs(\Phi_{e_i}) \in L^p(Ms(\Phi_{e_i}), \Phi_{e_i})$ for all $i \in I$. Set $a_{n,i} = E_n(|xs(\Phi_{e_i})|^p)|xs(\Phi_{e_i})|^p$. It is clear that $a_{n,i} \uparrow |xs(\Phi_{e_i})|^p = |x|^p s(\Phi_{e_i})$ as $n \to \infty$ for any fixed

$i \in I$. Therefore $a_{n,i} \xrightarrow{t(Ms(\Phi_{e_i}))} |x|^p s(\Phi_{e_i})$, $\Phi_{e_i}(|a_{n,i} - a_{m,i}|) \xrightarrow{t(\mathcal{B}e_i)} 0$ as $n, m \to \infty$. Since $0 \leqslant \Phi(\sqrt{a_{n,i}} a_{m,j} \sqrt{a_{n,i}}) = \Phi(a_{n,i} a_{m,j}) \leqslant \|a_{m,j}\|_M \Phi(a_{n,i}) = \|a_{m,j}\|_M \Phi(a_{n,i}) e_i$ and $\Phi(a_{n,i} a_{m,j}) \leqslant \|a_{n,i}\|_M \Phi(a_{m,j}) e_j$, we have $\Phi(a_{n,i} a_{m,j}) = 0$. Hence, $a_{n,i} a_{m,j} = 0$ for all n, m, $i \neq j$. Since $0 \leqslant a_{n,i} \leqslant ns(\Phi_{e_i})$, $s(\Phi_{e_i}) s(\Phi_{e_j}) = 0$, $i \neq j$, there is an $x_n \in M_+$ such that $x_n s(\Phi_{e_i}) = a_{n,i}$. Using the equality $\sup_{i \in I} s(\Phi_{e_i}) = \mathbf{1}$, we obtain $x_n \xrightarrow{t(M)} |x|^p$ ([16]), moreover $\Phi(|x_n - x_m|) \xrightarrow{t(\mathcal{B})} 0$. Therefore $x \in L^p(M, \Phi)$. ∎

Similar to the case of the space $L^1(M, \Phi)$, the subset $L^p(M, \Phi)$ is invariant with respect to the action of involution in $S(M)$. The following proposition is devoted to this fact.

PROPOSITION 2.2. *If $x \in L^p(M, \Phi)$, then $x^* \in L^p(M, \Phi)$ and $\|x\|_{p,\Phi} = \|x^*\|_{p,\Phi}$.*

Proof. Let $x = u|x|$ be the polar decomposition of x. Since an algebra M has a finite type, we can suppose that u is a unitary operator in M. For each $y \in S(M)$, we set $U(y) = uyu^*$. Then the mapping $U : S(M) \to S(M)$ is a $*$-isomorphism, and therefore $U(\varphi(y)) = \varphi(U(y))$ for any continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ and $y \in S_+(M)$ [16]. If $\varphi(t) = t^p$, $p > 1$, $t \geqslant 0$, and $y \in S_+(M)$ then $uy^p u^* = (uyu^*)^p$. In particular, we obtain the equality $|x^*|^p = u|x|^p u^*$. Hence, $x^* \in L^p(M, \Phi)$. Moreover $\|x^*\|_{p,\Phi} = \widehat{\Phi}(|x^*|^p)^{1/p} = \widehat{\Phi}(u|x|^p u^*)^{1/p} = \widehat{\Phi}(|x|^p)^{1/p} = \|x\|_{p,\Phi}$. ∎

Now we need a version of the Hölder inequality for Maharam traces. In the proof of this inequality for numerical traces, properties of decreasing rearrangements of integrable operators are used [5]. For Maharam traces such theory of decreasing rearrangements does not exist. Therefore we use another approach connected with the concept of a bitrace on a C^*-algebra.

Let $\mathcal{N}$ be a C^*-algebra. A function $s : \mathcal{N} \times \mathcal{N} \to \mathbb{C}$ is called a *bitrace* on $\mathcal{N}$ ([3], 6.2.1) if the following relations hold:

(i) $s(x, y)$ is a positively defined sesquilinear Hermitian form on $\mathcal{N}$;
(ii) $s(x, y) = s(y^*, x^*)$ for all $x, y \in \mathcal{N}$;
(iii) $s(zx, y) = s(x, z^* y)$ for all $x, y, z \in \mathcal{N}$;
(iv) for any $z \in \mathcal{N}$, the mapping $x \to zx$ is continuous on $(\mathcal{N}, \|\cdot\|_s)$ where $\|x\|_s = \sqrt{s(x, x)}$, $x \in \mathcal{N}$;
(v) the set $\{xy : x, y \in \mathcal{N}\}$ is dense in $(\mathcal{N}, \|\cdot\|_s)$.

If $\mathcal{N}$ has a unit, then condition (v) holds automatically.

Let us list examples of bitraces associated with the Maharam trace.

Let M be a von Neumann algebra, let $\Phi : M \to S(\mathcal{B})$ be a Maharam trace and let $Q = Q(P(\mathcal{B}))$ be the Stone compact space of the Boolean algebra $P(\mathcal{B})$. We claim that $s(\Phi(\mathbf{1})) = \mathbf{1}_{\mathcal{B}}$. If it is not the case, then $e = \mathbf{1}_{\mathcal{B}} - s(\Phi(\mathbf{1})) \neq 0$

and $z = \psi^{-1}(e) \neq 0$ where ψ is a $*$-isomorphism from Theorem 1.3. By Theorem 1.5(ii), we haves $\Phi(z) = e\Phi(\mathbf{1}) = 0$, which contradicts to the faithfulness of the trace Φ. Thus, $s(\Phi(\mathbf{1})) = \mathbf{1}_{\mathcal{B}}$, and therefore the following elements are defined: $(\Phi(\mathbf{1}))^{-1} \in S_+(\mathcal{B})$ and $(\Phi(\mathbf{1}))^{-1}\Phi(x) \in C(Q)$ where $x \in M$. For any $t \in Q$, set $\varphi_t(x) = (\Phi(\mathbf{1})^{-1}\Phi(x))(t)$. It is clear that φ_t is a finite numerical trace on M. The function $s_t(x,y) = \varphi_t(y^*x) = \varphi_t(xy^*)$ is a bitrace on M. In fact, the conditions (i)–(iii) are obvious; (iv) follows from the inequality $\|zx\|_{s_t} = \sqrt{\varphi_t((zx)^*(zx))} = \sqrt{\varphi_t(x^*z^*zx)} \leqslant \|z\|_M\|x\|_{s_t}$.

Let $s(x,y)$ be an arbitrary bitrace on a von Neumann algebra M. Set $N_s = \{x \in M : s(x,x) = 0\}$. It follows from 6.2.2 of [3], that N_s is a self-adjoint two-sided ideal in M. We consider the factor-space M/N_s with the scalar product $([x],[y])_s = s(x,y)$ where $[x],[y]$ are the equivalence classes from M/N_s with representatives x and y, respectively. Denote by $(H_s, (\cdot,\cdot)_s)$ the Hilbert space which is the completion of $(M/N_s, (\cdot,\cdot)_s)$. By the formula $\pi_s(x)([y]) = [xy]$, $x, y \in M$, one defines a $*$-homomorphism $\pi_s : M \to B(H_s)$. In addition $\pi_s(\mathbf{1}_M) = \mathbf{1}_{B(H_s)}$.

Denote by $U_s(M)$ the von Neumann subalgebra in $B(H_s)$ generated by operators $\pi_s(x)$, i.e. $U_s(M)$ is the closure of the $*$-subalgebra $\pi_s(M)$ in $B(H_s)$ with respect to the weak operator topology. According to 6.2 of [4], there exists a faithful normal semifinite numerical trace τ_s on $(U_s(M))_+$ such that $\tau_s(\pi(x^2)) = ([x],[x]) = s(x,x)$ for all $x \in M_+$. If φ is a trace on M and $s(x,y) = \varphi(y^*x)$ then $\tau_s(\pi_s(x^2)) = \varphi(x^2)$ for all $x \in M_+$. This means that $\tau_s(\pi_s(x)) = \varphi(x)$ for any $x \in M_+$. In addition, if $\varphi(\mathbf{1}_M) < \infty$, then $\tau_s(\mathbf{1}_{B(H_s)}) < \infty$. Consequently, τ_s is a faithful normal finite trace on $U_s(M)$.

THEOREM 2.3. *Let Φ be a $S(\mathcal{B})$-valued Maharam trace on the von Neumann algebra M, $p,q > 1$, $1/p + 1/q = 1$. If $x \in L^p(M,\Phi)$, $y \in L^q(M,\Phi)$, then $xy \in L^1(M,\Phi)$ and $\|xy\|_{1,\Phi} \leqslant \|x\|_{p,\Phi}\|y\|_{q,\Phi}$.*

Proof. We consider the bitrace $s_t(x,y) = \varphi_t(y^*x)$ on M where $\varphi_t(x) = ((\Phi(\mathbf{1}))^{-1}\Phi(x))(t)$, $t \in Q(P(\mathcal{B}))$. Denote by τ_t a faithful normal finite trace on $(U_{s_t}(M))_+$ such that $\tau_t(\pi_{s_t}(x)) = \varphi_t(x)$ for all $x \in M_+$. Since the trace τ_t is finite, $\tau_t(\pi_{s_t}(x)) = \varphi_t(x)$ for any $x \in M$. Let $L^p(U_{s_t}(M), \tau_t)$ be the non-commutative L^p-space associated with the numerical trace τ_t. It follows from [5] that

$$\tau_t(|\pi_{s_t}(xy)|) \leqslant \tau_t(|\pi_{s_t}(x)|^p)^{1/p}\tau_t(|\pi_{s_t}(y)|^q)^{1/q}.$$

Since $\pi_{s_t}(|x|) = |\pi_{s_t}(x)|$, $x \in M$, we get $\pi_{s_t}(|x|^p) = (\pi_{s_t}(|x|))^p$ ([3], 1.5.3).

Thus, $\varphi_t(|xy|) \leqslant \varphi_t(|x|^p)^{1/p}\varphi_t(|y|^q)^{1/q}$, or

$$(\Phi(\mathbf{1}))^{-1}\Phi(|xy|)(t) \leqslant [((\Phi(\mathbf{1}))^{-1}\Phi(|x|^p))(t)]^{1/p}[((\Phi(\mathbf{1}))^{-1}\Phi(|y|^q))(t)]^{1/q}$$

for all $t \in Q(P(\mathcal{B}))$. This means that

$$(\Phi(\mathbf{1}))^{-1}\Phi(|xy|) \leqslant [((\Phi(\mathbf{1}))^{-1}\Phi(|x|^p))]^{1/p}[((\Phi(\mathbf{1}))^{-1}\Phi(|y|^q))]^{1/q}.$$

Multiplying this inequality by $\Phi(\mathbf{1})$, we get $\|xy\|_{1,\Phi} \leqslant \|x\|_{p,\Phi}\|y\|_{q,\Phi}$.

Let now $x \in L^p_+(M,\Phi)$, $y \in L^q_+(M,\Phi)$. We claim that $xy \in L^1(M,\Phi)$. Set $a_n = E_n(x)x$, $b_n = E_n(y)y$. We have $a_n, b_n \in M_+$ and $a_n \uparrow x$, $b_n \uparrow y$, in particular, $a_n \xrightarrow{\Phi} x$, $b_n \xrightarrow{\Phi} y$. Hence, $a_n b_n \in M$ and $a_n b_n \xrightarrow{\Phi} xy$. In addition, $\|a_n b_n - a_m b_m\|_{1,\Phi} \leqslant \|a_n\|_{p,\Phi}\|b_n - b_m\|_{q,\Phi} + \|a_n - a_m\|_{p,\Phi}\|b_m\|_{q,\Phi}$. Since $\|a_n\|_{p,\Phi} \leqslant \|x\|_{p,\Phi}$, $\|b_m\|_{q,\Phi} \leqslant \|y\|_{q,\Phi}$, and for $n > m$, $\|a_n - a_m\|^p_{p,\Phi} = \widehat{\Phi}(x^p E_n(x)E^\perp_m(x)) \xrightarrow{t(\mathcal{B})} 0$, $\|b_n - b_m\|^q_{q,\Phi} = \widehat{\Phi}(y^q E_n(y)E^\perp_m(y)) \xrightarrow{t(\mathcal{B})} 0$, we get $\|a_n b_n - a_m b_m\|_{1,\Phi} \xrightarrow{t(\mathcal{B})} 0$ as $n, m \to \infty$. This means that $xy \in L^1(M,\Phi)$ and $\|a_n b_n - xy\|_{1,\Phi} \xrightarrow{t(\mathcal{B})} 0$. The inequality $|\|xy\|_{1,\Phi} - \|a_n b_n\|_{1,\Phi}| \leqslant \|xy - a_n b_n\|_{1,\Phi}$ implies $\|a_n b_n\|_{1,\Phi} \xrightarrow{t(\mathcal{B})} \|xy\|_{1,\Phi}$. Since

$$\|a_n b_n\|_{1,\Phi} \leqslant \|a_n\|_{p,\Phi}\|b_n\|_{q,\Phi} \xrightarrow{t(\mathcal{B})} \|x\|_{p,\Phi}\|y\|_{q,\Phi},$$

we obtain $\|xy\|_{1,\Phi} \leqslant \|x\|_{p,\Phi}\|y\|_{q,\Phi}$.

If $x \in L^p(M,\Phi)$ is arbitrary, $y \in L^q_+(M,\Phi)$ and $x = u|x|$ is the polar decomposition of x with the unitary $u \in M$, then $xy = u(|x|y) \in L^1(M,\Phi)$ and $\|xy\|_{1,\Phi} = \||x|y\|_{1,\Phi} \leqslant \|x\|_{p,\Phi}\|y\|_{q,\Phi}$.

Let now $x \in L^p(M,\Phi)$, $y \in L^q(M,\Phi)$ be arbitrary and let $y^* = v|y^*|$ be the polar decomposition of y^* with the unitary $v \in M$. According to Proposition 2.2, $|y^*| \in L^q(M,\Phi)$ and $\|y^*\|_{q,\Phi} = \|y\|_{q,\Phi}$. Therefore $xy = (x|y^*|)v^* \in L^1(M,\Phi)$ and $\|xy\|_{1,\Phi} = \|x|y^*|\|_{1,\Phi} \leqslant \|x\|_{p,\Phi}\|y\|_{q,\Phi}$. ∎

THEOREM 2.4. *Let Φ, M, p, and q be the same as in Theorem* 2.3. *If $x \in S(M)$, $xy \in L^1(M,\Phi)$ for all $y \in L^q(M,\Phi)$ and the set $D(x) = \{|\widehat{\Phi}(xy)| : y \in L^q(M,\Phi)$, $\|y\|_{q,\Phi} \leqslant \mathbf{1}_{\mathcal{B}}\}$ is bounded in $S_h(\mathcal{B})$, then $x \in L^p(M,\Phi)$ and $\|x\|_{p,\Phi} = \sup D(x)$.*

Proof. Let $x \neq 0$, and let $x = u|x|$ be the polar decomposition of x with the unitary $u \in M$. Set $y_n = |x|^{p-1}E_n(|x|)E^\perp_{1/n}(|x|)u^*$, $n = 1, 2, \ldots$ It is clear that $y_n \in M$ and

$$xy_n = u|x|^p E_n(|x|)E^\perp_{1/n}(|x|)u^* = uE_n(|x|)E^\perp_{1/n}(|x|)|x|^p E_n(|x|)E^\perp_{1/n}(|x|)u^* \geqslant 0.$$

On the other hand,

$$\begin{aligned}|y_n|^2 &= uE_n(|x|)E^\perp_{1/n}(|x|)|x|^{2p-2}E_n(|x|)E^\perp_{1/n}(|x|)u^* \\ &= uE_n(|x|)E^\perp_{1/n}(|x|)|x|^{2p/q}E_n(|x|)E^\perp_{1/n}(|x|)u^*,\end{aligned}$$

and therefore $0 \leqslant |y_n|^q = (|y_n|^2)^{q/2} = xy_n$, in particular, $\|y_n\|_{q,\Phi} = \Phi(xy_n)^{1/q}$.

Since $xy_n \xrightarrow{t(M)} u|x|^p u^* \neq 0$, we have $xy_n \neq 0$ for all $n \geqslant n_0$. Set $e_n = s(\Phi(xy_n))$ as $n \geqslant n_0$. Since $S_h(\mathcal{B}) = C_\infty(Q(P(\mathcal{B})))$, there exists a unique $b_n \in S_+(\mathcal{B})e_n$ such that $b_n\Phi(xy_n) = e_n$. It is clear that $b_n^{1/q}\Phi^{1/q}(xy_n) = e_n$. If $z_n = \psi^{-1}(e_n)$, $a_n = \psi^{-1}(b_n^{1/q}) \in S(\mathcal{A}z_n)$, then by Theorem 1.5(ii), $a_n y_n \in L^q(M,\Phi)$ and $\|a_n y_n\|^q_{q,\Phi} = \widehat{\Phi}(a_n^q |y_n|^q) = b_n\widehat{\Phi}(xy_n) = e_n \leqslant \mathbf{1}_{\mathcal{B}}$. Hence, $|\widehat{\Phi}(a_n x y_n)| =$

$|\widehat{\Phi}(x(a_n y_n))| \leqslant \sup D(x)$ for all $n \geqslant n_0$. On the other hand,

$$\widehat{\Phi}(a_n x y_n) = b_n^{1/q}\widehat{\Phi}(xy_n) = (b_n\widehat{\Phi}(xy_n))^{1/q}\widehat{\Phi}(xy_n)^{1-1/q} = \widehat{\Phi}(xy_n)^{1/p}$$
$$= \widehat{\Phi}(u|x|^p E_n(|x|)E^{\perp}_{1/n}(|x|)u^*)^{1/p} = \widehat{\Phi}(|x|^p E_n(|x|)E^{\perp}_{1/n}(|x|))^{1/p}.$$

Since $(|x|^p E_n(|x|)E^{\perp}_{1/n}(|x|)) \uparrow |x|^p$, $|x|^p(E_n(|x|)E^{\perp}_{1/n}(|x|)) \in M_+$ and $\widehat{\Phi}(|x|^p E_n(|x|)E^{\perp}_{1/n}(|x|)) \leqslant (\sup D(x))^p$, we have $|x|^p \in L^1(M,\Phi)$ and $\widehat{\Phi}(|x|^p) = \sup_{n\geqslant 1}\widehat{\Phi}(|x|^p E_n(|x|)E^{\perp}_{1/n}(|x|))$ [15]. This means that $x \in L^p(M,\Phi)$ and $\|x\|_{p,\Phi} \leqslant \sup D(x)$. Theorem 2.3 implies $\sup D(x) \leqslant \|x\|_{p,\Phi}$, and therefore $\|x\|_{p,\Phi} = \sup D(x)$. ∎

With the help of Theorem 2.4, it is not difficult to show that $L^p(M,\Phi)$ is disjointly decomposable LNS over $S_h(\mathcal{B})$ for all $p > 1$.

THEOREM 2.5. (i) *$L^p(M,\Phi)$ is a linear subspace in $S(M)$, and $\|\cdot\|_{p,\Phi}$ is the disjointly decomposable $S_h(\mathcal{B})$-valued norm on $L^p(M,\Phi)$;*

(ii) *$ML^p(M,\Phi)M \subset L^p(M,\Phi)$, and $\|axb\|_{p,\Phi} \leqslant \|a\|_M\|b\|_M\|x\|_{p,\Phi}$ for all $a,b \in M$, $x \in L^p(M,\Phi)$;*

(iii) *If $0 \leqslant x \leqslant y \in L^p(M,\Phi)$, $x \in S(M)$, then $x \in L^p(M,\Phi)$ and $\|x\|_{p,\Phi} \leqslant \|y\|_{p,\Phi}$.*

Proof. (i) It is clear that $\lambda x \in L^p(M,\Phi)$ and $\|\lambda x\|_{p,\Phi} = |\lambda|\|x\|_{p,\Phi}$ for all $x \in L^p(M,\Phi)$, $\lambda \in \mathbb{C}$. Moreover, $\|x\|_{p,\Phi} \geqslant 0$ and $\widehat{\Phi}(|x|^p) = \|x\|^p_{p,\Phi} = 0$ if and only if $x = 0$.

We claim that $x + y \in L^p(M,\Phi)$ and $\|x+y\|_{p,\Phi} \leqslant \|x\|_{p,\Phi} + \|y\|_{p,\Phi}$ for each $x, y \in L^p(M,\Phi)$. By theorem 2.3, $(x+y)z = xz + yz \in L^1(M,\Phi)$ for all $z \in L^q(M,\Phi)$, in addition

$$|\widehat{\Phi}((x+y)z)| \leqslant |\widehat{\Phi}(xz)| + |\widehat{\Phi}(yz)|.$$

If $\|z\|_{q,\Phi} \leqslant \mathbf{1}_{\mathcal{B}}$, then by Theorem 2.4,

$$|\widehat{\Phi}((x+y)z)| \leqslant \|x\|_{p,\Phi} + \|y\|_{p,\Phi}.$$

Using Theorem 2.4 again, we obtain $x+y \in L^p(M,\Phi)$ and $\|x+y\|_{p,\Phi} \leqslant \|x\|_{p,\Phi} + \|y\|_{p,\Phi}$. Thus, $L^p(M,\Phi)$ is a linear subspace in $S(M)$, and $\|\cdot\|_{p,\Phi}$ is a $S_h(\mathcal{B})$-valued norm on $L^p(M,\Phi)$.

Let us now show that the norm $\|\cdot\|_{p,\Phi}$ is d-decomposable. It is known ([2]) that, if $x \in L^1(M,\Phi)$, $\|x\|_{1,\Phi} = f_1 + f_2$, where $f_1, f_2 \in S_+(\mathcal{B})$, $f_1 f_2 = 0$, then, setting $x_i = xp_i$ for $p_i = \psi^{-1}(s(f_i))$, $i = 1,2$, we get $x = x_1 + x_2$ and $\|x_i\|_{\Phi} = f_i$, $i = 1,2$.

Let $y \in L^p_+(M,\Phi)$, $\|y\|_{p,\Phi} = g_1 + g_2$ where $g_1, g_2 \in S_+(\mathcal{B})$, $g_1 g_2 = 0$, i.e. $\|y^p\|_{1,\Phi} = \|y\|^p_{p,\Phi} = g_1^p + g_2^p$. Set $q_i = \psi^{-1}(s(g_i^p)) \in P(\mathcal{A}) \subset P(Z(M))$ and $y_i = yq_i$. Then $y_i^p = y^p q_i$ and using [2] for $x = y^p$, $f_i = g_i^p$, $i = 1,2$ we obtain that $y^p q_1 + y^p q_2 = y^p$ and $\|yq_i\|_{p,\Phi} = g_i$, $i = 1,2$. Since $q_1 q_2 = 0$, $q_1, q_2 \in P(Z(M))$, we have $yq_1 + yq_2 = y$.

Let now y be an arbitrary element from $L^p(M,\Phi)$ and let $y = u|y|$ be the polar decomposition of y with the unitary $u \in M$. Let $\| |y| \|_{p,\Phi} = \|y\|_{p,\Phi} = f_1 + f_2$ where $f_1, f_2 \in S_+(\mathcal{B}),\ f_1 f_2 = 0$. It follows from above that for $q_i = \psi^{-1}(s(f_i^p)) \in P(\mathcal{A})$, we have $|y| = |y|q_1 + |y|q_2$ and $\| |y|q_i \|_{p,\Phi} = f_i$. Consequently, $y = u|y| = yq_1 + yq_2$ and $\|yq_i\|_{p,\Phi} = \| |y|q_i \|_{p,\Phi} = f_i,\ i = 1,2$.

(ii) Let v be a unitary operator in M, $x \in L^p(M,\Phi)$. Then $|vx| = (x^* v^* v x)^{1/2} = |x|$, and therefore $vx \in L^p(M,\Phi)$. Since any operator $a \in M$ is a linear combination of four unitary operators, we have $ax \in L^p(M,\Phi)$, due to (i).

We claim that $\|ax\|_{p,\Phi} \leqslant \|a\|_M \|x\|_{p,\Phi}$ for $a \in M,\ x \in L^p(M,\Phi)$. Let ν be a faithful normal semifinite numerical trace on $\mathcal{B}$. If for some $a \in M$, $x \in L^p(M,\Phi)$ the previous inequality is not true, then there are $\varepsilon > 0,\ 0 \neq e \in P(\mathcal{B}),\ \nu(e) < \infty$ such that

$$e\|ax\|_{p,\Phi} \geqslant e\|a\|_M \|x\|_{p,\Phi} + \varepsilon e.$$

By the formula

$$\tau(b) = \nu(e\Phi(b)(\mathbf{1}_{\mathcal{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(|x|^p))^{-1}), \quad b \in Ms(\Phi_e)$$

one defines a faithful normal finite numerical trace on $Ms(\Phi_e)$. If $z = \psi^{-1}(e) \in P(\mathcal{A})$, then $\Phi_e(\mathbf{1} - z) = (\mathbf{1}_{\mathcal{B}} - e)e\Phi(\mathbf{1}) = 0$, i.e. $s(\Phi_e) \leqslant z$. Since $\Phi(z - s(\Phi_e)) = \Phi(z(\mathbf{1} - s(\Phi_e))) = e\Phi(\mathbf{1} - s(\Phi_e)) = 0$, we get $z = s(\Phi_e)$. We consider the L^p-space $L^p(Ms(\Phi_e),\tau)$ associated with the numerical trace τ, and let us show that $xz \in L^p(Ms(\Phi_e),\tau)$. Let $x_n = E_n(|x|)|x|$. It is clear that $0 \leqslant x_n^p z \uparrow |x|^p z$ and $\tau(x_n^p z) \leqslant \nu(e) < \infty$. Hence, $|xz|^p = |x|^p z \in L^1(Ms(\Phi_e),\tau)$ and $\|xz\|_{p,\tau}^p = \lim\limits_{n\to\infty} \|x_n^p z\|_{p,\tau}^p = \nu(e\widehat{\Phi}(|x|^p z)(\mathbf{1}_{\mathcal{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(|x|^p))^{-1})$. Thus, if $a \in M$ then $axz \in L^p(Ms(\Phi_e),\tau)$, in addition

$$\begin{aligned}
\|a\|_M \|xz\|_{p,\tau}^p \geqslant \|axz\|_{p,\tau}^p &= \nu(e\widehat{\Phi}(|axz|^p)(\mathbf{1}_{\mathcal{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(|x|^p))^{-1}) \\
&= \nu((e\|ax\|_{p,\Phi}^p)(\mathbf{1}_{\mathcal{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(|x|^p))^{-1}) \\
&\geqslant \nu(e(\|a\|_M \|x\|_{p,\Phi} + \varepsilon)^p(\mathbf{1}_{\mathcal{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(|x|^p))^{-1}) > \|a\|_M^p \|xz\|_{p,\tau}^p,
\end{aligned}$$

which is not the case. Consequently, $\|ax\|_{p,\Phi} \leqslant \|a\|_M \|x\|_{p,\Phi}$.

If $b \in M,\ x \in L^p(M,\Phi)$, then by Proposition 2.2 and from above, we have $b^* x^* \in L^p(M,\Phi)$. Using Proposition 2.2 again, we obtain $xb = (b^* x^*)^* \in L^p(M,\Phi)$ and $\|xb\|_{p,\Phi} = \|b^* x^*\|_{p,\Phi} \leqslant \|b^*\|_M \|x^*\|_{p,\Phi} = \|b\|_M \|x\|_{p,\Phi}$.

(iii) Let $0 \leqslant x \leqslant y \in L^p(M,\Phi),\ x \in S(M)$. It follows from Section 2.4 of [8] that $\sqrt{x} = a\sqrt{y}$ where $a \in M$ with $\|a\|_M \leqslant 1$. Hence, $x = \sqrt{x}(\sqrt{x})^* = aya^* \in L^p(M,\Phi)$ and $\|x\|_{p,\Phi} \leqslant \|a\|_M \|a^*\|_M \|y\|_{p,\Phi} \leqslant \|y\|_{p,\Phi}$. ∎

Using the Hölder inequality and the (bo)-completeness of the space $(L^1(M,\Phi), \|\cdot\|_\Phi)$ we can establish the (bo)-completeness of the space $(L^p(M,\Phi), \|\cdot\|_{p,\Phi})$.

THEOREM 2.6. *Let Φ, M, p be the same as in Theorem 2.3. Then $(L^p(M,\Phi), \|\cdot\|_{p,\Phi})$ is a the Banach–Kantorovich space.*

Proof. First, we assume that $\mathcal{B}$ is a σ-finite von Neumann algebra. Then there exists a faithful normal finite numerical trace ν on $\mathcal{B}$. The numerical function $\tau(a) = \nu(\Phi(a)(\mathbf{1}_{\mathcal{B}} + \Phi(\mathbf{1}))^{-1})$ is a faithful normal finite trace on M. Moreover, the topology $t(M)$ coincides with measure topology t_τ on $(S(M), \tau)$ ([8], Section 3.5).

Let $\{x_\alpha\}_{\alpha\in A} \subset (L^p(M,\Phi), \|\cdot\|_{p,\Phi})$ be a (bo)-Cauchy net i.e. $b_\gamma = \sup\limits_{\alpha,\beta\geqslant\gamma} \|x_\alpha - x_\beta\|_{p,\Phi} \downarrow 0$. According to the Hölder inequality, for each $x \in L^p(M,\Phi)$ we have $x \in L^1(M,\Phi)$ and $\|x\|_{1,\Phi} = \widehat{\Phi}(|x|\mathbf{1}) \leqslant (\Phi(\mathbf{1}))^{1/q}\|x\|_{p,\Phi}$. In particular, the set $\{\|x_\alpha - x_\beta\|_{1,\Phi}\}_{\alpha,\beta\geqslant\gamma}$ is bounded in $S_h(\mathcal{B})$, and $\sup\limits_{\alpha,\beta\geqslant\gamma} \|x_\alpha - x_\beta\|_{1,\Phi} \leqslant (\Phi(\mathbf{1}))^{1/q} b_\gamma$ for all $\gamma \in A$. Consequently ([2]), there exists $x \in L^1(M,\Phi)$ such that $\|x_\alpha - x\|_{1,\Phi} \xrightarrow{(o)} 0$ in particular, $x_\alpha \xrightarrow{t_\tau} x$ and $y_\alpha = |x_\alpha - x_\beta| \xrightarrow{t_\tau} |x - x_\beta|$. Since the function $\varphi(t) = t^p$ is continuous on $[0,\infty)$, the operator function $y \longmapsto y^p$ is continuous on $(S_+(M), t_\tau)$ [12]. Hence, $0 \leqslant y_\alpha^p \xrightarrow{t_\tau} |x - x_\beta|^p$, in addition $\widehat{\Phi}(y_\alpha^p) = \|x_\alpha - x_\beta\|_{p,\Phi}^p \leqslant b_\gamma^p$. Using the Fatou's theorem [15], we obtain $|x - x_\beta|^p \in L^1(M,\Phi)$ and $\widehat{\Phi}(|x - x_\beta|^p) \leqslant b_\gamma^p$. Thus, $(x - x_\beta) \in L^p(M,\Phi)$ for all $\beta \geqslant \gamma$ and $\sup\limits_{\beta\geqslant\gamma} \|x - x_\beta\|_{p,\Phi} \leqslant b_\gamma \downarrow 0$. This means that $x \in L^p(M,\Phi)$, and $\|x_\alpha - x\|_{p,\Phi} \xrightarrow{(o)} 0$.

Now let $\mathcal{B}$ be an arbitrary von Neumann algebra (not necessarily σ-finite), and let $\{x_\alpha\} \subset L^p(M,\Phi)$ be a (bo)-Cauchy net. It follows from the above that there exists $x \in L^1(M,\Phi)$ such that $\|x_\alpha - x\|_{1,\Phi} \xrightarrow{(o)} 0$. In particular $x_\alpha \xrightarrow{t(M)} x$. Let ν be a faithful normal semifinite numerical trace on $\mathcal{B}$, and let $\{e_i\}_{i\in I}$ be the family of nonzero mutually orthogonal projections in $\mathcal{B}$ such that $\sup\limits_{i\in I} e_i = \mathbf{1}_{\mathcal{B}}$, and $\nu(e_i) < \infty$ for all $i \in I$. It is clear that $\{x_\alpha s(\Phi_{e_i})\}_{\alpha\in A}$ is a (bo)-Cauchy net in $L^p(Ms(\Phi_{e_i}), \Phi_{e_i})$. Since the algebra $\mathcal{B}e_i$ is σ-finite, from the above there exists $x_i \in L^p(Ms(\Phi_{e_i}), \Phi_{e_i})$ such that $\|x_i - x_\alpha s(\Phi_{e_i})\|_{p,\Phi_{e_i}} \xrightarrow{(o)} 0$. In particular, $x_\alpha s(\Phi_{e_i}) \xrightarrow{t(M)} x_i = x_i s(\Phi_{e_i})$. On the other hand, convergence $x_\alpha \xrightarrow{t(M)} x$ implies $x_\alpha s(\Phi_{e_i}) \xrightarrow{t(M)} xs(\Phi_{e_i})$. Thus, $xs(\Phi_{e_i}) = x_i s(\Phi_{e_i})$ for all $i \in I$. By Proposition 2.1, we have $x \in L^p(M,\Phi)$ and $\|x - x_\alpha\|_{p,\Phi} e_i = \|xs(\Phi_{e_i}) - x_\alpha s(\Phi_{e_i})\|_{p,\Phi_{e_i}} \xrightarrow{(o)} 0$ for all $i \in I$ and therefore $\|x - x_\alpha\|_{p,\Phi} \xrightarrow{(o)} 0$. ∎

PROPOSITION 2.7. *If* $\{x_\alpha\}_{\alpha\in A} \subset L_h^p(M,\Phi)$ *and* $x_\alpha \downarrow 0$, *then* $\|x_\alpha\|_{p,\Phi} \downarrow 0$.

Proof. Let ν be a faithful normal semifinite numerical trace on $\mathcal{B}$. If $b = \inf\limits_{\alpha\in I} \|x_\alpha\|_{p,\Phi} \neq 0$, then there are $\varepsilon > 0$, $0 \neq e \in P(\mathcal{B})$ with $\nu(e) < \infty$ such that $e\|x_\alpha\|_{p,\Phi} \geqslant eb \geqslant \varepsilon e$ for all $\alpha \in A$. Put $\Phi_e(x) = e\Phi(x)$, $x \in M$, and $\tau(y) = \nu(\Phi(y)(\mathbf{1}_{\mathcal{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(x_{\alpha_0}^p))^{-1})$, $y \in Ms(\Phi_e)$, where α_0 is a fixed element from A. Let us prove that $L^p(Ms(\Phi_e),\tau) \subset L^p(Ms(\Phi_e),\Phi_e)$ and $\|x\|_{p,\tau}^p = \nu(\widehat{\Phi}(|x|^p)(\mathbf{1}_{\mathcal{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(x_{\alpha_0}^p))^{-1})$ for all $x \in L^p(Ms(\Phi_e),\tau)$. It is sufficient to consider the case where $x \in L^p_+(Ms(\Phi_e), \tau)$. Set $x_n = E_n(x)xs(\Phi_e)$. It is clear that $x_n \in (Ms(\Phi_e))_+$, $x_n^p \uparrow x^p$, $x_n^p \xrightarrow{t_\tau} x^p$, and therefore $x_n^p \xrightarrow{t(M)} x^p$. Moreover, $\Phi(|x_n^p - x_m^p|) = \Phi(x^p E_n(x) E_m^\perp(x))$ as $m < n$. Since $\|x_n^p - x_m^p\|_{1,\tau} = \|x^p E_n(x) E_m^\perp(x)\|_{1,\tau} \to 0$ as $n, m \to \infty$, we get $\Phi(|x_n^p - x_m^p|) = e\Phi(|x_n^p - x_m^p|) \xrightarrow{t(\mathcal{B})} 0$. This means that $x^p \in L^1(M, \Phi)$ and $\Phi(x_n^p) \uparrow \widehat{\Phi}(x^p)$, i.e. $x \in L^p(Ms(\Phi_e), \Phi_e)$ and $\|x\|_{p,\Phi_e} = \sup_{n \geqslant 1} (\Phi(x_n^p))^{1/p}$. Using the inequality $\tau(x_n^p) \leqslant \tau(x^p)$ we obtain that $\widehat{\Phi}(x^p)(\mathbf{1}_{\mathcal{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(x_{\alpha_0}^p))^{-1} \in L_1(\mathcal{B}, \nu)$ and

$$\nu(\widehat{\Phi}(x^p)(\mathbf{1}_{\mathcal{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(x_{\alpha_0}^p))^{-1}) = \sup_{n \geqslant 1} \tau(x_n^p) = \tau(x^p) = \|x\|_{p,\tau}^p.$$

Let us show that $x = x_{\alpha_0} s(\Phi_e) \in L^p(Ms(\Phi_e), \tau)$. As above, we consider $x_n = E_n(x)x$. Since

$$0 \leqslant \Phi(x_n^p)(\mathbf{1}_{\mathcal{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(x_{\alpha_0}^p))^{-1} \uparrow \widehat{\Phi}(x^p)(\mathbf{1}_{\mathcal{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(x_{\alpha_0}^p))^{-1} \leqslant e,$$

we get $\tau(x_n^p) \leqslant \nu(e) < \infty$. Consequently, $x \in L^p(Ms(\Phi_e), \tau)$. The inequality $0 \leqslant x_\alpha \leqslant x_{\alpha_0}$, for $\alpha \geqslant \alpha_0$ implies $x_\alpha s(\Phi_e) \in L^p(Ms(\Phi_e), \tau)$ (see Theorem 2.5(iii)). Since $x_\alpha s(\Phi_e) \downarrow 0$ and the norm $\|\cdot\|_{p,\tau}$ is order continuous, we have $\|x_\alpha s(\Phi_e)\|_{p,\tau} \downarrow 0$, i.e. $\nu(e\widehat{\Phi}(x_\alpha^p)(\mathbf{1}_{\mathcal{B}} + \Phi(\mathbf{1}) + \widehat{\Phi}(x_{\alpha_0}^p))^{-1}) \downarrow 0$. Hence, $e\widehat{\Phi}(x_\alpha)^p \downarrow 0$, which contradicts the inequality $e\Phi(x_\alpha^p) \geqslant \varepsilon^p e$. ∎

3. DUALITY FOR SPACES $L^p(M, \Phi)$

Let us start with the following property of L^p-spaces $L^p(M, \Phi)$.

PROPOSITION 3.1. *If $x \in L^p(M, \Phi)$, $y \in L^q(M, \Phi)$, $1/p + 1/q = 1$, $p, q > 1$, then $xy, yx \in L^1(M, \Phi)$ and $\widehat{\Phi}(xy) = \widehat{\Phi}(yx)$.*

Proof. Without loss of generality, we can take $x \geqslant 0$, $y \geqslant 0$. It follows from Theorem 2.3 that $xy \in L^1(M, \Phi)$. Hence, $yx = y^* x^* = (xy)^* \in L^1(M, \Phi)$ and $\widehat{\Phi}(yx) = \widehat{\Phi}((xy)^*) = \overline{\widehat{\Phi}(xy)}$. Let $x_n = xE_n(x)$, $y_n = yE_n(y)$. Using the inequalities $|\widehat{\Phi}(xy) - \Phi(x_n y_n)| \leqslant \|x - x_n\|_{p,\Phi} \|y\|_{q,\Phi} + \|x_n\|_{p,\Phi} \|y - y_n\|_{q,\Phi}$, we obtain $\Phi(x_n y_n) \xrightarrow{t(\mathcal{B})} \widehat{\Phi}(xy)$. Since $\Phi(x_n y_n) = \Phi(\sqrt{x_n} y_n \sqrt{x_n}) \geqslant 0$ for all n, we get $\widehat{\Phi}(xy) \geqslant 0$. Therefore $\widehat{\Phi}(xy) = \overline{\widehat{\Phi}(xy)} = \widehat{\Phi}(yx)$. ∎

Let $L^p(M, \Phi)^*$ be a BKS of all $S_h(\mathcal{B})$-bounded linear mappings from $L^p(M, \Phi)$ into $S(\mathcal{B})$, i.e. $S_h(\mathcal{B})$ is the dual space to the BKS $L^p(M, \Phi)$. It is clear that any $S_h(\mathcal{B})$-bounded linear operator is a continuous mapping from $(L^p(M, \Phi), \|\cdot\|_{p,\Phi})$ into $(S(\mathcal{B}), t(\mathcal{B}))$.

PROPOSITION 3.2 (Compare with 5.1.9 of [7]). *Let $T \in L^p(M, \Phi)^*$, $\psi : S(\mathcal{A}) \to S(\mathcal{B})$ be a $*$-isomorphism from Theorem* 1.5(ii). *Then $T(ax) = \psi(a)T(x)$ for all $a \in S(\mathcal{A})$, $x \in L^p(M, \Phi)$.*

Proof. By Theorem 1.5(ii), for each $z \in P(\mathcal{A})$, $x \in L^p(M,\Phi)$ we have $\|zx\|_{p,\Phi} = \widehat{\Phi}(z|x|^p)^{1/p} = \psi(z)\widehat{\Phi}(|x|^p)^{1/p} = \psi(z)\|x\|_{p,\Phi}$. Since $T \in L^p(M,\Phi)^*$, $|Tx| \leqslant c\|x\|_{p,\Phi}$ for some $c \in S_+(\mathcal{B})$ and all $x \in L^p(M,\Phi)$. Hence $|T(zx)| \leqslant \psi(z)c\|x\|_{p,\Phi}$, i.e. the support $s(T(zx))$ is majorized by the projection $\psi(z)$. Multiplying the equality $T(x) = T(zx) + T((\mathbf{1}-z)x)$ by $\psi(z)$, we obtain

$$\psi(z)T(x) = \psi(z)T(zx) = T(zx).$$

If $a = \sum_{i=1}^{n} \lambda_i z_i$ is a simple element from $S(\mathcal{A})$, where $\lambda_i \in \mathbb{C}$, $z_i \in P(\mathcal{A})$, $i = 1,\dots,n$, then

$$T(ax) = \sum_{i=1}^{n} \lambda_i T(z_i x) = \Big(\sum_{i=1}^{n} \lambda_i \psi(z_i)\Big) T(x) = \psi(a)T(x).$$

Let a be an arbitrary element from $S(\mathcal{A})$ and let $\{a_n\}$ be a sequence of simple elements from $S(\mathcal{A})$ such that $a_n \xrightarrow{t(\mathcal{A})} a$. Then $0 \leqslant \psi(|a_n - a|) \xrightarrow{t(\mathcal{B})} 0$, $\psi(a_n) \xrightarrow{t(\mathcal{B})} \psi(a)$, and

$$\|a_n x - ax\|_{p,\Phi} = \widehat{\Phi}(|a_n - a|^p|x|^p)^{1/p} = \psi(|a_n - a|)\|x\|_{p,\Phi} \xrightarrow{t(\mathcal{B})} 0.$$

Since T is continuous, $\psi(a_n)T(x) = T(a_n x) \xrightarrow{t(\mathcal{B})} T(ax)$. Due to the convergence $\psi(a_n)T(x) \xrightarrow{t(\mathcal{B})} \psi(a)T(x)$, the proof is complete. ∎

Now we pass to description of the $S_h(\mathcal{B})$-dual space $L^p(M,\Phi)^*$.

THEOREM 3.3. *Let Φ be an $S(\mathcal{B})$-valued Maharam trace on the von Neumann algebra M, $p,q > 1, 1/p + 1/q = 1$.*

(i) *If $y \in L^q(M,\Phi)$, then the linear mapping $T_y(x) = \widehat{\Phi}(xy)$, $x \in L^p(M,\Phi)$, is $S(\mathcal{B})$-bounded and $\|T_y\| = \|y\|_{q,\Phi}$.*

(ii) *If $T \in L^p(M,\Phi)^*$, then there exists a unique $y \in L^q(M,\Phi)$ such that $T = T_y$.*

Proof. (i) By the Hölder inequality (Theorem 2.3), $xy \in L^1(M,\Phi)$ for all $x \in L^p(M,\Phi)$ and $|T_y(x)| = |\widehat{\Phi}(xy)| \leqslant \|y\|_{q,\Phi}\|x\|_{p,\Phi}$. Hence, T_y is $S_h(\mathcal{B})$-bounded linear mapping from $L^p(M,\Phi)$ into $S(\mathcal{B})$. Due to Proposition 3.1 and Theorem 2.4 we have

$$\|T_y\| = \sup\{|\widehat{\Phi}(yx)| : x \in L^p(M,\Phi),\ \|x\|_{p,\Phi} \leqslant \mathbf{1}_{\mathcal{B}}\} = \|y\|_{q,\Phi}.$$

(ii) Since $s(\Phi(\mathbf{1})) = \mathbf{1}_{\mathcal{B}}$, we can define the element $b = (\Phi(\mathbf{1}))^{-1} \in S_+(\mathcal{A})$. If $\Phi_1(x) = b\Phi(x)$, $x \in M$, then $L^p(M,\Phi_1) = L^p(M,\Phi)$ and $\|x\|_{p,\Phi_1} = b^{1/p}\|x\|_{p,\Phi}$ for all $x \in L^p(M,\Phi)$. Therefore, one can take $\Phi(\mathbf{1}) = \mathbf{1}_{\mathcal{B}}$.

Let $T \in L^p(M,\Phi)^*$. We choose $a \in S_+(\mathcal{B})$ with $a\|T\| = s(\|T\|)$. Set $T_1(x) = aT(x)$, $x \in L^p(M,\Phi)$. It is clear that $T_1 \in L^p(M,\Phi)^*$ and $\|T_1\| = a\|T\| = s(\|T\|) \leqslant \mathbf{1}_{\mathcal{B}}$. If we show that there exists $y_1 \in L^q(M,\Phi)$ such that $T_1 x = \Phi(xy_1)$, then by virtue of Proposition 3.2, $Tx = \|T\|T_1(xy_1) = T(x(\psi^{-1}(\|T\|)y_1)) = T(xy)$ where $y = \psi^{-1}(\|T\|)y_1 \in L^q(M,\Phi)$. Thus, one can also take that $\|T\| \leqslant \mathbf{1}_{\mathcal{B}}$.

At first, we assume that the algebra $\mathcal{B}$ is σ-finite. Let ν be a faithful normal finite numerical trace on $\mathcal{B}$. Since $|\Phi(x)| \leqslant \|x\|_M \Phi(\mathbf{1}) \leqslant \|x\|_M \mathbf{1}_{\mathcal{B}},\ x \in M$, we get $\Phi(x) \in L^1(\mathcal{B}, \nu)$. Consider on M the faithful normal finite trace $\tau(x) = \nu(\Phi(x)),\ x \in M$. Using the same trick as in the proof of Proposition 2.7, we can show that $L^p(M,\tau) \subset L^p(M,\Phi)$ and $\tau(|x|^p) = \|x\|_{p,\tau}^p = \nu(\widehat{\Phi}(|x|^p))$ for all $x \in L^p(M,\tau)$. Since $|T(x)| \leqslant \|x\|_{p,\Phi} = (\widehat{\Phi}(|x|^p))^{1/p}$, we have $T(x) \in L^1(\mathcal{B}, \nu)$ for all $x \in L^p(M,\tau)$.

We define on $L^p(M,\tau)$ the linear $\mathbb{C}$-valued functional $f(x) = \nu(Tx), x \in L^p(M,\tau)$. Since

$$|f(x)| \leqslant \nu(|T(x)|) \leqslant \nu(\widehat{\Phi}(|x|^p)^{1/p} \mathbf{1}_{\mathcal{B}}) \leqslant (\nu(\widehat{\Phi}(|x|^p)))^{1/p} (\nu(\mathbf{1}_{\mathcal{B}}))^{1/q} = (\nu(\mathbf{1}_{\mathcal{B}}))^{1/q} \|x\|_{p,\tau}$$

for all $x \in L^p(M,\tau)$, we have that f is a bounded linear functional on $(L^p(M,\tau), \|\cdot\|_{p,\tau})$. Hence there exists an operator $y \in L^q(M,\tau) \subset L^q(M,\Phi)$ such that $f(x) = \tau(xy)$ for all $x \in L^p(M,\tau)$ [14]. We claim that $\tau(xy) = \nu(\widehat{\Phi}(xy))$ for all $x \in L^p(M,\tau)$. If $z \in L^1_+(M,\tau)$, then $z^{1/p} \in L^p_+(M,\tau)$, and therefore $\tau(z) = \nu(\widehat{\Phi}(z)$. Hence, $\tau(z) = \nu(\widehat{\Phi}(z))$ for all $z \in L^1(M,\tau)$, in particular, $\tau(xy) = \nu(\widehat{\Phi}(xy))$ where $x \in L^p(M,\tau)$. Thus, $\nu(T(x)) = f(x) = \tau(xy) = \nu(\widehat{\Phi}(xy))$ for all $x \in L^p(M,\tau)$.

Let $T(x) - \widehat{\Phi}(xy) = v|T(x) - \widehat{\Phi}(xy)|$ be the polar decomposition of the element $(T(x) - \widehat{\Phi}(xy)) \in S(\mathcal{B})$ and take $a = \psi^{-1}(v^*)$. Since

$$0 = \nu(T(ax) - \widehat{\Phi}(axy)) = \nu(v^*(T(x) - \widehat{\Phi}(xy))) = \nu(|T(x) - \widehat{\Phi}(xy)|),$$

we have $T(x) = \widehat{\Phi}(xy)$ for all $x \in L^p(M,\tau)$.

Let $x \in L^p_+(M,\Phi), x_n = xE_n(x)$. Then $\|x_n - x\|_{p,\Phi} \xrightarrow{t(\mathcal{B})} 0$ and therefore

$$T(x_n) \xrightarrow{t(\mathcal{B})} T(x) \quad \text{and} \quad |\widehat{\Phi}(x_n y) - \widehat{\Phi}(xy)| \leqslant \|x_n - x\|_{p,\Phi} \|y\|_{q,\Phi} \xrightarrow{t(\mathcal{B})} 0.$$

Since $T(x_n) = \widehat{\Phi}(x_n y)$, $T(x) = \widehat{\Phi}(xy)$, i.e. $T = T_y$.

If z is another element from $L^q(M,\Phi)$ with $T(x) = \widehat{\Phi}(xz),\ x \in L^p(M,\Phi)$, then $\widehat{\Phi}(x(y-z)) = 0$ for all $x \in L^p(M,\Phi)$. Taking $x = u^*$ where u is the unitary operator from the polar decomposition $y - z = u|y-z|$, we obtain $\widehat{\Phi}(|y-z|) = 0$, i.e. $y = z$.

Now let $\mathcal{B}$ be a general (not necessarily a σ-finite) von Neumann algebra. Let ν be a faithful normal semifinite numerical trace on $\mathcal{B}$, and let $\{e_i\}_{i \in I}$ be a family of nonzero mutually orthogonal projections in $\mathcal{B}$ with $\sup\limits_{i \in I} e_i = \mathbf{1}_{\mathcal{B}}$ and $\nu(e_i) < \infty$ for all $i \in I$. It is clear that $\mathcal{B}e_i$ is a σ-finite algebra and $\Phi_{e_i}(x) = e_i \Phi(x)$ is $S(\mathcal{B}e_i)$-valued Maharam trace on $Ms(\Phi_{e_i})$. Since $T \in L^p(M,\Phi)^*$, $T_i(x) = e_i T(x)$ is $S_h(\mathcal{B}e_i)$-bounded linear mapping. By virtue of what we proved above, there exists the unique $y_i \in L^q(Ms(\Phi_{e_i}), \Phi_{e_i})$, such that

$$e_i T(xs(\Phi_{e_i})) = \widehat{\Phi}_{e_i}(xs(\Phi_{e_i}) y_i) = e_i \widehat{\Phi}(xs(\Phi_{e_i}) y_i)$$

for all $x \in L^p(M,\Phi)$, $i \in I$. Moreover, $\|y_i\|_{q,\Phi} = \|T_i\| = \|T\|e_i$. Since $\sup_{i\in I} s(\Phi_{e_i}) = \mathbf{1}$, $\{s(\Phi_{e_i})\}_{i\in I} \subset P(Z(M))$ and $s(\Phi_{e_i})s(\Phi_{e_j}) = 0$ as $i \neq j$, there exists a unique $y \in S(M)$ such that $ys(\Phi_{e_i}) = y_i$. We have $e_i\widehat{\Phi}(|y|^q) = \widehat{\Phi}(|y_i|^q) = \|T\|^q e_i$ for all $i \in I$. Hence, $y \in L^q(M,\Phi)$ and $\|y\|_{q,\Phi} = \|T\|$ (see Proposition 2.1). In addition

$$e_i\widehat{\Phi}(xy) = \widehat{\Phi}_{e_i}(xs(\Phi_{e_i})y_i) = e_iT(xs(\Phi_{e_i})) = e_iT(x),$$

for all $i \in I$, i.e. $T_y(x) = \widehat{\Phi}(xy) = T(x),\ x \in L^p(M,\Phi)$. ∎

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