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# REALISING THE $C^*$-ALGEBRA OF A HIGHER-RANK GRAPH AS AN EXEL'S CROSSED PRODUCT

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ABSTRACT. We use the boundary-path space of a finitely-aligned $k$-graph $\Lambda$ to construct a compactly-aligned product system $X$, and we show that the graph algebra $C^*(\Lambda)$ is isomorphic to the Cuntz–Nica–Pimsner algebra $\mathcal{NO}(X)$. In this setting, we introduce the notion of a crossed product by a semigroup of partial endomorphisms and partially-defined transfer operators by defining it to be $\mathcal{NO}(X)$. We then compare this crossed product with other definitions in the literature.

KEYWORDS: *Cuntz–Pimsner algebra, Hilbert bimodule, k-graph, crossed product.*

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## INTRODUCTION

In [6], Exel proposed a new definition for a crossed product of a unital $C^*$-algebra $A$ by an endomorphism $\alpha$. Exel's definition depends not only on $\alpha$, but also on the choice of *transfer operator*: a positive continuous linear map $L : A \to A$ satisfying $L(\alpha(a)b) = aL(b)$. We call a triple $(A, \alpha, L)$ an *Exel system*. In his motivating example, Exel finds a family of Exel systems whose crossed products model the Cuntz–Krieger algebras [4]. This marked the first time a crossed product by an endomorphism could successfully model Cuntz–Krieger algebras.

There are two obvious extensions of Exel's construction. Firstly, to a theory of crossed products of *non-unital* $C^*$-algebras capable of modeling the directed-graph generalisation of the Cuntz–Krieger algebras [20]. In [2], the authors successfully built such a theory, and they realised the graph algebras of locally-finite graphs with no sources as Exel crossed products ([2], Theorem 5.1). The crossed product in question was built from the infinite-path space $E^\infty$ and the shift map $\sigma$ on $E^\infty$. The hypotheses on $E$ ensure that $E^\infty$ is locally compact, and $\sigma$ is everywhere defined, and this allows an Exel system to be defined. The other extension of Exel's work is to crossed products by *semigroups* of endomorphisms and transfer operators. In [17], Larsen has a crossed-product construction for dynamical systems $(A, P, \alpha, L)$ in which $P$ is an abelian semigroup, $\alpha$ is an action of $P$ by endomorphisms, and $L$ is an action of $P$ by transfer operators. Exel has also worked in this area with his theory of interaction groups [7], [8].

Motivated by these ideas, we construct a semigroup crossed product that can model the $C^*$-algebras of the higher-rank graphs, or $k$-graphs, of Kumjian and Pask [16]. The only restriction we place on the $k$-graphs $\Lambda$ whose $C^*$-algebras we model is a necessary finitely-aligned hypothesis, so our result applies in the fullest possible generality. This does come at a price, however, as without a locally-finite hypothesis, or a restriction on sources, the space of infinite paths is not locally compact. To get a locally-compact space we need to consider the bigger *boundary-path space* $\partial\Lambda$, and on this space the shift maps $\sigma_n$, $n \in \mathbb{N}^k$, will not in general be everywhere defined. This means we can not form Exel systems, or even a dynamical system in the sense of Larsen [17]. We overcome this problem by first ignoring the crossed-product construction, and focusing on building a *product system*.

A product system of Hilbert $A$-bimodules over a semigroup $P$ is a semigroup $X = \bigsqcup_{p \in P} X_p$ such that each $X_p$ is a Hilbert $A$-bimodule, and $x \otimes_A y \mapsto xy$ determines an isomorphism of $X_p \otimes_A X_q$ onto $X_{pq}$ for each $p, q \in P$. Fowler introduced such product systems in [11]. Fowler also defined a Cuntz–Pimsner covariance condition for representations of product systems, and introduced the universal $C^*$-algebra $\mathcal{O}(X)$ for Cuntz–Pimsner covariant representations of $X$. This generalised Pimsner's $C^*$-algebra for a single Hilbert bimodule [19]. In [23], Sims and Yeend looked at the problem of associating a $C^*$-algebra to product systems which satisfies a gauge-invariant uniqueness theorem, and noted in particular that Fowler's $\mathcal{O}(X)$ will not in general do the job. For a large class of semigroups, and a class of product systems called compactly-aligned, Sims and Yeend introduced a covariance condition for representations — called Cuntz–Nica–Pimsner covariance — and a $C^*$-algebra $\mathcal{NO}(X)$ universal for such representations. A gauge-invariant uniqueness theorem for $\mathcal{NO}(X)$ is proved in [3].

We build from $\partial\Lambda$ and the $\sigma_n$ topological graphs in the sense of Katsura [14], and then we apply the construction from [14] to get Hilbert $C_0(\partial\Lambda)$-bimodules $X_n$. We glue the bimodules together to form the *boundary-path product system* $X$ over $\mathbb{N}^k$. This gives a new class of product systems for which the Cuntz–Nica–Pimsner algebra $\mathcal{NO}(X)$ is tractable. The main result in this paper says that for $\Lambda$ a finitely-aligned $k$-graph, the graph algebra $C^*(\Lambda)$ is isomorphic to $\mathcal{NO}(X)$. A result, we feel, that gives extra credence to Sims and Yeend's construction, at least in the case for the semigroup $\mathbb{N}^k$. We then construct for each $n \in \mathbb{N}^k$ a partial endomorphism $\alpha_n$ on $C_0(\partial\Lambda)$ and a partially-defined transfer operator $L_n$, and we *define* the crossed product $C_0(\partial\Lambda) \rtimes_{\alpha,L} \mathbb{N}^k$ to be $\mathcal{NO}(X)$. This gives us our desired result: $C_0(\partial\Lambda) \rtimes_{\alpha,L} \mathbb{N}^k \cong C^*(\Lambda)$.

We begin with some preliminaries in Section 1. We state some necessary definitions from the $k$-graph literature, and we state the definition of the Cuntz–Krieger algebra of a $k$-graph. We then state the definitions from [23] needed to make sense of the notion of Cuntz–Nica–Pimsner covariance, and the Cuntz–Nica–Pimsner algebra of a compactly-aligned product system. In Section 2 we construct from a finitely-aligned $k$-graph $\Lambda$ the boundary-path product system $X$. The proof that $X$ is compactly-aligned requires substantial detail, so we leave this result for the appendix. In Section 3 we prove the existence of a canonical isomorphism $C^*(\Lambda) \to \mathcal{NO}(X)$. In Section 4 we introduce the crossed product $C_0(\partial\Lambda) \rtimes_{\alpha,L} \mathbb{N}^k$, and we discuss the relationship between this crossed product and the crossed product in [2]; Exel and Royer's crossed product by a partial endomorphism [10]; and Larsen's semigroup crossed product [17].

## 1. PRELIMINARIES

### 1.1. $k$-GRAPHS AND THEIR CUNTZ–KRIEGER ALGEBRAS.

A higher-rank graph, or $k$-graph, is a pair $(\Lambda, d)$ consisting of a countable category $\Lambda$ and a degree functor $d : \Lambda \to \mathbb{N}^k$ satisfying the unique factorisation property: for all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$. We now recall some definitions from the $k$-graph literature; for more details see [5].

For $\lambda, \mu \in \Lambda$ we denote

$$\Lambda^{\min}(\lambda, \mu) := \{(\alpha, \beta) \in \Lambda \times \Lambda : \lambda\alpha = \mu\beta \text{ and } d(\lambda\alpha) = d(\lambda) \vee d(\mu)\}.$$

A $k$-graph $\Lambda$ is *finitely-aligned* if $\Lambda^{\min}(\lambda, \mu)$ is at most finite for all $\lambda, \mu \in \Lambda$. For each $v \in \Lambda^0$ we denote by $v\Lambda := \{\lambda \in \Lambda : r(\lambda) = v\}$. A subset $E \subseteq v\Lambda$ is *exhaustive* if for every $\mu \in v\Lambda$ there exists a $\lambda \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$. We denote the set of all *finite* exhaustive subsets of $\Lambda$ by $\mathcal{FE}(\Lambda)$. We denote by $v\mathcal{FE}(\Lambda)$ the set $\{E \in \mathcal{FE}(\Lambda) : E \subseteq v\Lambda\}$.

For each $m \in (\mathbb{N} \cup \{\infty\})^k$ we get a $k$-graph $\Omega_{k,m}$ through the following construction. The set $\Omega_{k,m}^0 := \{p \in \mathbb{N}^k : p \leqslant m\}$, and

$$\Omega_k^* := \{(p, q) \in \Omega_{k,m}^0 \times \Omega_{k,m}^0 : p \leqslant q\}.$$

The range map is given by $r(p, q) = p$; the source map by $s(p, q) = q$; and the degree functor by $d(p, q) = q - p$. Composition is given by $(p, q)(q, r) = (p, r)$.

For $k$-graph $\Lambda$ we define a *graph morphism* $x$ to be a degree-preserving functor from $\Omega_{k,m}$ to $\Lambda$. The range and degree maps are extended to all graph morphisms $x : \Omega_{k,m} \to \Lambda$ by setting $r(x) := x(0)$ and $d(x) := m$. We define the *boundary-path space* $\partial\Lambda$ to be the set of all graph morphisms $x$ such that for all $n \in \mathbb{N}^k$ with $n \leqslant d(x)$, and for all $E \in x(n)\mathcal{FE}(\Lambda)$, there exists $\lambda \in E$ such that $x(n, n + d(\lambda)) = \lambda$. We know from Lemmas 5.13 of [5] that if $\lambda \in \Lambda x(0)$, then

$\lambda x \in \partial\Lambda$. We know from Lemma 5.15 of [5] that for each $v \in \Lambda^0$ there exists $x \in v\partial\Lambda = \{x \in \partial\Lambda : r(x) = v\}$.

We recall from [23] the following definition.

DEFINITION 1.1. Let $\Lambda$ be a finitely-aligned $k$-graph. A *Cuntz–Krieger family* in a $C^*$-algebra $B$ is a collection $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries in $B$ satisfying:
(CK1) $\{t_v : v \in \Lambda^0\}$ consists of mutually orthogonal projections;
(CK2) $t_\lambda t_\mu = t_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;
(CK3) $t_\lambda^* t_\mu = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} t_\alpha t_\beta^*$; and
(CK4) $\prod_{\lambda\in E} (t_v - t_\lambda t_\lambda^*) = 0$ for every $v \in \Lambda^0$ and $E \in v\mathcal{FE}(\Lambda)$.

The *Cuntz–Krieger algebra*, or *graph algebra*, $C^*(\Lambda)$ is the universal $C^*$-algebra generated by a Cuntz–Krieger $\Lambda$-family.

## 1.2. PRODUCT SYSTEMS AND THEIR CUNTZ–NICA–PIMSNER ALGEBRAS.

In this subsection we state some key definitions from Sections 2 and 3 of [23]; see [23] for more details.

Suppose $A$ is a $C^*$-algebra, and $(G, P)$ is a quasi-lattice ordered group in the sense that: $G$ is a discrete group and $P$ is a subsemigroup of $G$; $P \cap P^{-1} = \{e\}$; and with respect to the partial order $p \leqslant q \iff p^{-1}q \in P$, any two elements $p, q \in G$ which have a common upper bound in $P$ have a least upper bound $p \vee q \in P$. Suppose $X := \bigcup_{p\in P} X_p$ is a product system of Hilbert $A$-bimodules. For each $p \in P$ and each $x, y \in X_p$ the operator $\Theta_{x,y} : X_p \to X_p$ defined by $\Theta_{x,y}(z) := x \cdot \langle y, z\rangle_A$ is adjointable with $\Theta_{x,y}^* = \Theta_{y,x}$. The span $\mathcal{K}(X_p) := \overline{\operatorname{span}}\{\Theta_{x,y} : x, y \in X_p\}$ is a closed two-sided ideal in $\mathcal{L}(X_p)$ called the *algebra of compact operators on* $X_p$. For $p, q \in P$ with $e < p \leqslant q$ there is a homomorphism $\iota_p^q : \mathcal{L}(X_p) \to \mathcal{L}(X_q)$ characterised by

$$\iota_p^q(S)(xy) = (Sx)y \quad \text{for all } x \in X_p, y \in X_{p^{-1}q}. \tag{1.1}$$

For $p \not\leqslant q$ we define $\iota_p^q(S) = 0_{\mathcal{L}(X_q)}$ for all $S \in \mathcal{L}(X_p)$. The product system $X$ is called *compactly-aligned* if for all $p, q \in P$ such that $p \vee q < \infty$, and for all $S \in \mathcal{K}(X_p)$ and $T \in \mathcal{K}(X_q)$, we have $\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T) \in \mathcal{K}(X_{p\vee q})$.

A *representation* $\psi$ of $X$ in a $C^*$-algebra $B$ is a map $X \to B$ such that:

(1) each $\psi|_{X_p} := \psi_p : X_p \to B$ is linear, and $\psi_e : A \to B$ is a homomorphism;
(2) $\psi_p(x)\psi_y(q) = \psi_{pq}(xy)$ for all $p, q \in P$, $x \in X_p$, and $y \in X_q$; and
(3) $\psi_e(\langle x, y\rangle_A^p) = \psi_p(x)^*\psi_p(y)$ for all $p \in P$, and $x, y \in X_p$.

It follows from Pimsner's results [19] that for each $p \in P$ there is a homomorphism $\psi^{(p)} : \mathcal{K}(X_p) \to B$ satisfying $\psi^{(p)}(\Theta_{x,y}) = \psi_p(x)\psi_p(y)^*$ for all $x, y \in X_p$. A representation $\psi$ of $X$ is *Nica-covariant* if for all $p, q \in P$ and all

$S \in \mathcal{K}(X_p), T \in \mathcal{K}(X_q)$ we have

$$\psi^{(p)}(S)\psi^{(q)}(T) = \begin{cases} \psi^{(p\vee q)}(\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T)) & \text{if } p \vee q < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

We denote by $\phi_p$ the homomorphism $A \to \mathcal{L}(X_p)$ implementing the left action of $A$ on $X_p$. We define $I_e = A$, and for each $q \in P \setminus \{e\}$ we write $I_q := \bigcap_{e<p\leqslant q} \ker \phi_p$. We then denote by $\widetilde{X}_q$ the Hilbert $A$-bimodule

$$\widetilde{X}_q := \bigoplus_{p\leqslant q} X_p \cdot I_{p^{-1}q},$$

and we denote by $\widetilde{\phi}_q$ the homomorphism implementing the left action of $A$ on $\widetilde{X}_q$. The product system $X$ is said to be *$\widetilde{\phi}$-injective* if every $\widetilde{\phi}_q$ is injective.

For $p, q \in P$ with $p \neq e$ there is a homomorphism $\widetilde{\iota}_p^{\,q} : \mathcal{L}(X_p) \to \mathcal{L}(\widetilde{X}_q)$ determined by $S \mapsto \bigoplus_{r\leqslant q} \iota_p^r(S)$ for all $S \in \mathcal{L}(X_p)$; and characterised by

$$(\widetilde{\iota}_p^{\,q}(S)x)(r) = \iota_p^r(S)x(r) \quad \text{for all } x \in \widetilde{X}_q. \tag{1.2}$$

A representation $\psi$ of a $\widetilde{\phi}$-injective product system $X$ in a $C^*$-algebra $B$ is *Cuntz–Pimsner covariant* if $\sum_{p\in F} \psi^{(p)}(T_p) = 0_B$ whenever $F \subset P$ is finite, $T_p \in \mathcal{K}(X_p)$ for each $p \in F$, and $\sum_{p\in F} \widetilde{\iota}_p^{\,s}(T_p) = 0$ for large $s$ (see Definition 3.8 of [23] for the meaning of "for large $s$"). A representation $\psi$ of a $\widetilde{\phi}$-injective product system $X$ is *Cuntz–Nica–Pimsner covariant* if it is both Nica covariant and Cuntz–Pimsner covariant. It is proved in Proposition 3.12 of [23] that there exists a $C^*$-algebra $\mathcal{NO}(X)$, called the *Cuntz–Nica–Pimsner algebra of* $X$, which is universal for Cuntz–Nica–Pimsner covariant representations of $X$. We denote the universal Cuntz–Nica–Pimsner representation by $j_X : X \to \mathcal{NO}(X)$.

## 2. THE BOUNDARY-PATH PRODUCT SYSTEM OF A $k$-GRAPH

Let $\Lambda$ be a finitely-aligned $k$-graph. For $\lambda \in \Lambda$ we denote the set $D_\lambda := \{x \in \partial\Lambda : x(0, d(\lambda)) = \lambda\}$. For $n \in \mathbb{N}^k$ we denote

$$\mathcal{A}^n := \{(\lambda, F) : \lambda \in \Lambda \text{ with } d(\lambda) \geqslant n,\ F \subseteq s(\lambda)\Lambda \text{ a finite set}\},$$

and $\mathcal{A} := \bigcup_{n\in\mathbb{N}^k} \mathcal{A}^n$. For $(\lambda, F) \in \mathcal{A}$ we denote $D_{\lambda F} := \bigcup_{\nu\in F} D_{\lambda\nu}$. It is proved in Section 5 of [5] that the family of sets $\{D_\lambda \setminus D_{\lambda F} : (\lambda, F) \in \mathcal{A}\}$ is a basis of compact and open sets for a Hausdorff topology on $\partial\Lambda$, and $\partial\Lambda$ is a locally compact Hausdorff space. For each $n \in \mathbb{N}^k$ we denote $\partial\Lambda^{\geqslant n} := \{x \in \partial\Lambda : d(x) \geqslant n\}$ and $\partial\Lambda^{\ngeqslant n} := \partial\Lambda \setminus \partial\Lambda^{\geqslant n}$. We now use the subsets $\partial\Lambda^{\geqslant n}$ to construct topological graphs in the sense of Katsura [14], [15].

PROPOSITION 2.1. *Let $n \in \mathbb{N}^k$ with $\partial\Lambda^{\geqslant n} \neq \emptyset$. Denote by $\sigma_n$ the shift on $\partial\Lambda^{\geqslant n}$ given by $\sigma_n(x)(m) = x(m+n)$, and $\iota : \partial\Lambda^{\geqslant n} \to \partial\Lambda$ the inclusion mapping. Then $E_n := (\partial\Lambda, \partial\Lambda^{\geqslant n}, \sigma_n, \iota)$ is a topological graph.*

*Proof.* We use the definition of convergence given in Remark 5.6 of [5]. Let $(x_i)$ be a sequence in $\partial\Lambda^{\ngeqslant n}$ converging to $x$. If $x \in \partial\Lambda^{\geqslant n}$, then there exists $j \in \{1, \ldots, k\}$ and a subsequence $(x_{ik})$ of $(x_i)$ such that $d(x_{ik})_j < d(x)_j$ for all $x_{ik}$. This contradicts that $(x_{ik})$ converges to $x$, so we must have $x \in \partial\Lambda^{\ngeqslant n}$, and hence $\partial\Lambda^{\ngeqslant n}$ is closed in $\partial\Lambda$. Hence $\partial\Lambda^{\geqslant n}$ is locally compact.

Let $x \in \partial\Lambda^{\geqslant n}$. Then $D_{x(0,n)}$ is an open neighbourhood of $x$, with $D_{x(0,n)} \subseteq \partial\Lambda^{\geqslant n}$. The map $\sigma_n|_{D_{x(0,n)}} : D_{x(0,n)} \to D_{s(x(0,n))}$ is a bijection, and $\sigma_n(D_{x(0,n)}) = D_{s(x(0,n))}$ is open in $\partial\Lambda$. Now suppose $\lambda \in s(x(0,n))\Lambda$ and $F \subseteq s(\lambda)\Lambda$. Then

$$\sigma_n|_{D_{x(0,n)}}(D_{x(0,n)\lambda} \setminus D_{x(0,n)\lambda F}) = D_\lambda \setminus D_{\lambda F}$$

is open in $D_{s(x(0,n))}$, and

$$(\sigma_n|_{D_{x(0,n)}})^{-1}(D_\lambda \setminus D_{\lambda F}) = D_{x(0,n)\lambda} \setminus D_{x(0,n)\lambda F}$$

is open in $D_{x(0,n)}$. Hence, $\sigma_n|_{D_{x(0,n)}}$ is continuous and open, and so it is a homeomorphism of $D_{x(0,n)}$ onto $D_{s(x(0,n))}$. Hence $\sigma_n$ is a local homeomorphism. We know that $\iota$ is continuous, so the result follows. ∎

We now use Katsura's construction [14] to form Hilbert bimodules. For $f, g \in C_c(\partial\Lambda^{\geqslant n})$ and $a \in C_0(\partial\Lambda)$, we define

$$(f \cdot a)(x) := f(x)a(\sigma_n(x)), \text{ and} \tag{2.1}$$

$$\langle f, g\rangle_n(x) := \sum_{\sigma_n(y)=x} \overline{f(y)}g(y). \tag{2.2}$$

We complete $C_c(\partial\Lambda^{\geqslant n})$ under the norm $\|\cdot\|_n$ given by $\langle\cdot,\cdot\rangle_n$ to get a Hilbert $C_0(\partial\Lambda)$-module $X_n = X(E_n)$. The formula

$$(a \cdot f)(x) := a(\iota(x))f(x) = a(x)f(x), \tag{2.3}$$

defines an action of $C_0(\partial\Lambda)$ by adjointable operators on $X_n$, which we denote by $\phi_n : C_0(\partial\Lambda) \to \mathcal{L}(X_n)$, and then $X_n$ becomes a Hilbert $C_0(\partial\Lambda)$-bimodule. For $n \in \mathbb{N}^k$ with $\partial\Lambda^{\geqslant n} = \emptyset$ we set $X_n := \{0\}$. Note that $X_0 = C_0(\partial\Lambda)$.

PROPOSITION 2.2. *Let $m, n \in \mathbb{N}^k$ with $\partial\Lambda^{\geqslant m}, \partial\Lambda^{\geqslant n} \neq \emptyset$. Then the map*

$$\pi : C_c(\partial\Lambda^{\geqslant m}) \times C_c(\partial\Lambda^{\geqslant n}) \to C_c(\partial\Lambda^{\geqslant m+n})$$

*given by $\pi(f,g)(x) = f(x)g(\sigma_m(x))$ is a surjective map which induces an isomorphism $\pi_{m,n} : X_m \otimes X_n \to X_{m+n}$ satisfying $\pi_{m,n}(f \otimes g) = f(g \circ \sigma_m)$.*

To prove this proposition we need some results. To state these results we use the following notation.

NOTATION 2.3. (i) Recall from Definition 3.10 of [5] that given $\lambda \in \Lambda$ and $E \subseteq r(\lambda)\Lambda$ we denote

$$\mathrm{Ext}(\lambda; E) := \bigcup_{\nu \in E} \{\alpha \in \Lambda : (\alpha, \beta) \in \Lambda^{\min}(\lambda, \nu) \text{ for some } \beta \in \Lambda\}.$$

For $\lambda, \mu \in \Lambda$ we denote $F(\lambda, \mu) := \mathrm{Ext}(\lambda; \{\mu\})$. Since $\Lambda$ is finitely-aligned, $F(\lambda, \mu)$ is a finite subset of $s(\lambda)\Lambda$, and so $(\lambda, F(\lambda, \mu)) \in \mathcal{A}$. We have

$$D_{\lambda F(\lambda,\mu)} = D_{\mu F(\mu,\lambda)}. \tag{2.4}$$

(ii) Let $\lambda, \mu \in \Lambda$ and $x \in \partial\Lambda$ with $d(x) \geqslant d(\lambda) \vee d(\mu)$. Then we denote by $x_\lambda^\mu$ the path

$$x_\lambda^\mu := x(d(\lambda), d(\lambda) \vee d(\mu)).$$

LEMMA 2.4. *Let $(\lambda, F), (\mu, G) \in \mathcal{A}$. Then we have*

$$(D_\lambda \setminus D_{\lambda F}) \cap (D_\mu \setminus D_{\mu G}) = \bigsqcup_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} D_{\lambda\alpha} \setminus D_{\lambda\alpha F_\alpha}, \tag{2.5}$$

*where*

$$F_\alpha := \Big(\bigcup_{\nu\in F} F(\lambda\alpha, \lambda\nu)\Big) \cup \Big(\bigcup_{\xi\in G} F(\lambda\alpha, \mu\xi)\Big).$$

*Proof.* The factorisation property ensures that the union in (2.5) is disjoint.

Let $x \in (D_\lambda \setminus D_{\lambda F}) \cap (D_\mu \setminus D_{\mu G})$. Then $d(x) \geqslant d(\lambda) \vee d(\mu)$; the pair $(x_\lambda^\mu, x_\mu^\lambda) \in \Lambda^{\min}(\lambda, \mu)$; and $x \in D_{\lambda x_\lambda^\mu}$. Using (2.4) we have

$$x \in D_{\lambda x_\lambda^\mu F(\lambda x_\lambda^\mu, \lambda\nu)} = D_{\lambda\nu F(\lambda\nu, \lambda x_\lambda^\mu)} \Longrightarrow x \in D_{\lambda F},$$

which contradicts $x \in D_\lambda \setminus D_{\lambda F}$, so we must have $x \notin D_{\lambda x_\lambda^\mu F(\lambda x_\lambda^\mu, \lambda\nu)}$ for all $\nu \in F$. By symmetry, we also have $x \notin D_{\lambda x_\lambda^\mu F(\lambda x_\lambda^\mu, \mu\xi)}$ for all $\xi \in G$. Hence $x \in D_{\lambda x_\lambda^\mu} \setminus D_{\lambda x_\lambda^\mu F_{x_\lambda^\mu}}$.

Now suppose $y$ is an element of the right-hand-side of (2.5). So there exists $(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)$ with $y \in D_{\lambda\alpha} \setminus D_{\lambda\alpha F_\alpha}$. We have $y \in D_{\lambda\alpha} \subseteq D_\lambda$. Assume $y \in D_{\lambda\nu}$ for some $\nu \in F$. Then $d(y) \geqslant d(\lambda\alpha) \vee d(\lambda\nu)$; the pair $(y_{\lambda\alpha}^{\lambda\nu}, y_{\lambda\nu}^{\lambda\alpha}) \in \Lambda^{\min}(\lambda\alpha, \lambda\nu)$; and $y \in D_{\lambda\alpha F(\lambda\alpha,\lambda\nu)} \subseteq D_{\lambda\alpha F_\alpha}$. This is a contradiction, and so $y \notin D_{\lambda\nu}$ for all $\nu \in F$. Hence $y \in D_\lambda \setminus D_{\lambda F}$. By symmetry, we also have $y \in D_\mu \setminus D_{\mu G}$. Hence $y \in (D_\lambda \setminus D_{\lambda F}) \cap (D_\mu \setminus D_{\mu G})$. ∎

LEMMA 2.5. *Let $n \in \mathbb{N}^k$ and $(\lambda, F) \in \mathcal{A}$ with $D_\lambda \setminus D_{\lambda F} \subseteq \partial\Lambda^{\geqslant n}$. Then we have*

$$D_\lambda \setminus D_{\lambda F} = \bigsqcup_{\mu\in s(\lambda)\Lambda^{d(\lambda)\vee n - d(\lambda)}} D_{\lambda\mu} \setminus D_{\lambda\mu\,\mathrm{Ext}(\mu;F)}, \tag{2.6}$$

*where $(\lambda\mu, \mathrm{Ext}(\mu; F)) \in \mathcal{A}^n$ for each $\mu \in s(\lambda)\Lambda^{d(\lambda)\vee n - d(\lambda)}$.*

*Proof.* The factorisation property ensures that the union in (2.6) is disjoint.

Suppose $x \in D_\lambda \setminus D_{\lambda F}$, and consider the path $\mu := x(d(\lambda), d(\lambda) \vee n) \in s(\lambda)\Lambda^{d(\lambda)\vee n - d(\lambda)}$. Then $x \in D_{\lambda\mu}$. If $x \in D_{\lambda\mu \operatorname{Ext}(\mu;F)}$, then there exists $\nu \in F$ and $(\alpha, \beta) \in \Lambda^{\min}(\mu, \nu)$ with $x \in D_{\lambda\mu\alpha} = D_{\lambda\nu\beta} \subseteq D_{\lambda\nu} \subseteq D_{\lambda F}$. But this is a contradiction, and so we must have $x \in D_{\lambda\mu} \setminus D_{\lambda\mu \operatorname{Ext}(\mu;F)}$.

Now, let $y \in D_{\lambda\mu} \setminus D_{\lambda\mu \operatorname{Ext}(\mu;F)}$ for some $\mu \in s(\lambda)\Lambda^{d(\lambda)\vee n - d(\lambda)}$. Then $y \in D_\lambda$. If $y \in D_{\lambda\nu}$ for some $\nu \in F$, then the pair

$$(y^{\lambda\nu}_{\lambda\mu}(d(\lambda), d(\lambda) + d(\mu) \vee d(\nu)), y^{\lambda\mu}_{\lambda\nu}(d(\lambda), d(\lambda) + d(\mu) \vee d(\nu))) \in \Lambda^{\min}(\mu, \nu),$$

and $y \in D_{\lambda\mu \operatorname{Ext}(\mu;F)}$. This is a contradiction, and so we must have $y \in D_\lambda \setminus D_{\lambda F}$.

Finally, for each $\mu \in s(\lambda)\Lambda^{d(\lambda)\vee n - d(\lambda)}$ the set $\operatorname{Ext}(\mu; F)$ is finite because $F$ is finite and $\Lambda$ is finitely-aligned. We obviously have $d(\lambda\mu) \geqslant n$, and so $(\lambda\mu, \operatorname{Ext}(\mu; F)) \in \mathcal{A}^n$. ∎

*Proof of Proposition* 2.2. To show that $\pi$ is surjective we let $f \in C_c(\partial\Lambda^{\geqslant m+n})$. For each $x \in \operatorname{supp} f$ there exists $(\lambda, F) \in \mathcal{A}$ with $x \in D_\lambda \setminus D_{\lambda F} \subseteq \partial\Lambda^{\geqslant m+n}$. So there exists a subset $\mathcal{J} \subseteq \mathcal{A}$ such that $\operatorname{supp} f \subseteq \bigcup_{(\lambda,F)\in\mathcal{J}} D_\lambda \setminus D_{\lambda F}$, where $D_\lambda \setminus D_{\lambda F} \subseteq \partial\Lambda^{\geqslant m+n}$ for each $(\lambda, F) \in \mathcal{J}$. It follows from Lemma 2.5 that each $D_\lambda \setminus D_{\lambda F}$ is a disjoint union of sets of the form $D_\mu \setminus D_{\mu G}$ with $(\mu, G) \in \mathcal{A}^{m+n}$, and so there exists a subset $\mathcal{J}' \subseteq \mathcal{A}^{m+n}$ such that $\operatorname{supp} f \subseteq \bigcup_{(\mu,G)\in\mathcal{J}'} D_\mu \setminus D_{\mu G}$, where $D_\mu \setminus D_{\mu G} \subseteq \partial\Lambda^{\geqslant m+n}$ for each $(\mu, G) \in \mathcal{J}'$. Since $\operatorname{supp} f$ is compact, there exists a finite number of pairs $(\mu_j, G_j) \in \mathcal{J}'$ with $\operatorname{supp} f \subseteq \bigcup_{j=1}^{h} D_{\mu_j} \setminus D_{\mu_j G_j}$. Now for each $1 \leqslant j \leqslant h$ let $\lambda_j := \mu_j(m, d(\mu_j))$, and consider the function $\mathcal{X}_{\cup_j D_{\lambda_j} \setminus D_{\lambda_j G_j}} \in C_c(\partial\Lambda^{\geqslant n})$. Consider also $\widetilde{f} \in C_c(\partial\Lambda^{\geqslant m})$ which is equal to $f$ on $\partial\Lambda^{\geqslant m+n}$ and zero on the complement. Then we have $\pi(\widetilde{f}, \mathcal{X}_{\cup_j D_{\lambda_j} \setminus D_{\lambda_j G_j}}) = f$, and so $\pi$ maps onto $C_c(\partial\Lambda^{\geqslant m+n})$.

Routine calculations show that $\pi$ is bilinear, and so it induces a surjective linear map $\pi_{m,n} : C_c(\partial\Lambda^{\geqslant m}) \odot C_c(\partial\Lambda^{\geqslant n}) \to C_c(\partial\Lambda^{\geqslant m+n})$ satisfying $\pi_{m,n}(f \otimes g)(x) = f(x)g(\sigma_m(x))$. It follows immediately from the formulas (2.1) and (2.3) that $\pi$ preserves the left and right actions.

To see that $\pi_{m,n}$ preserves the inner product, we let $f, h \in C_c(\partial\Lambda^{\geqslant m})$ and $g, l \in C_c(\partial\Lambda^{\geqslant n})$. Then for $x \in \partial\Lambda^{\geqslant m+n}$ we have

$$\begin{aligned}\langle f \otimes g, h \otimes l\rangle(x) = \langle \langle h, f\rangle_m \cdot g, l\rangle_n(x) &= \sum_{\sigma_n(y)=x} \overline{\langle h, f\rangle_m(y) g(y)}\, l(y) \\ &= \sum_{\sigma_n(y)=x} \Big( \sum_{\sigma_m(z)=y} h(z)\overline{f(z)} \Big) \overline{g(y)}\, l(y)\end{aligned}$$

$$= \sum_{\sigma_{m+n}(z)=x} \overline{g(\sigma_m(z))} l(\sigma_m(z)) \overline{f(z)} h(z). \tag{2.7}$$

Now

$$\begin{aligned}
\langle \pi_{m,n}(f \otimes g), \pi_{m,n}(h \otimes l) \rangle_{m+n}(x) &= \sum_{\sigma_{m+n}(z)=x} \overline{\pi_{m,n}(f \otimes g)(z)} \pi_{m,n}(h \otimes l)(z) \\
&= \sum_{\sigma_{m+n}(z)=x} \overline{f(z) g(\sigma_m(z))} h(z) l(\sigma_m(z)) \\
&= \langle f \otimes g, h \otimes l \rangle (x),
\end{aligned}$$

and so $\pi_{m,n}$ preserves the inner product. Hence it extends to an isomorphism $\pi_{m,n} : X_m \otimes X_n \to X_{m+n}$. ∎

REMARK 2.6. Suppose $\partial\Lambda^{\geqslant m}, \partial\Lambda^{\geqslant n} \neq \emptyset$ and $\partial\Lambda^{\geqslant m+n} = \emptyset$. We claim that $X_m \otimes X_n = \{0\}$. To see this is true, we assume the contrary. Then there exists $f \in C_c(\partial\Lambda^{\geqslant m})$ and $g \in C_c(\partial\Lambda^{\geqslant n})$ with $f \otimes g \neq 0$. It follows from equation (2.7) that

$$\langle f \otimes g, f \otimes g \rangle (x) = \sum_{\sigma_{m+n}(z)=x} |f(z)|^2 |g(\sigma_m(z))|^2,$$

and this implies

$$\begin{aligned}
\langle f \otimes g, f \otimes g \rangle \neq 0 &\iff \sigma_{m+n}^{-1}(x) \neq \emptyset \text{ for some } x \in \partial\Lambda \\
&\iff \partial\Lambda^{\geqslant m+n} \neq \emptyset.
\end{aligned}$$

This is a contradiction, and so we must have $X_m \otimes X_n = \{0\} = X_{m+n}$.

Now suppose that $\partial\Lambda^{\geqslant m} \neq \emptyset$ and $\partial\Lambda^{\geqslant n} = \emptyset$. Then we have $\partial\Lambda^{\geqslant m+n} = \emptyset$, and so $X_n = \{0\} = X_{m+n}$. Then $X_m \otimes X_n = X_m \otimes \{0\} = \{0\} = X_{m+n}$. So we can extend Proposition 2.2 to include all $m, n \in \mathbb{N}^k$, and we think of $\pi_{m,n}$ for $m, n$ as in this remark as the trivial map from $\{0\}$ to itself.

PROPOSITION 2.7. *The family* $X := \bigsqcup_{n \in \mathbb{N}^k} X_n$ *of Hilbert bimodules over* $C_0(\partial\Lambda)$ *with multiplication given by*

$$xy := \pi_{m,n}(x \otimes y) \tag{2.8}$$

*is a product system over* $\mathbb{N}^k$.

*Proof.* We just need to check that $ax = a \cdot x$ and $xa = x \cdot a$ for all $x \in X_n$, $n \in \mathbb{N}^k$ and $a \in C_0(\partial\Lambda)$, but this follows from (2.1), (2.3) and the definition of multiplication (2.8). ∎

We prove that $X$ is compactly-aligned in the Appendix.

Given the definition (2.8) of multiplication within $X$, we now have the following restatement of Proposition 2.2. This corollary plays an important role in subsequent sections.

COROLLARY 2.8. *Let $n \in \mathbb{N}^k$ and $h \in C_c(\partial\Lambda^{\geqslant n})$. Then for every $l, m \in \mathbb{N}^k$ with $n = l + m$, there exists $f \in C_c(\partial\Lambda^{\geqslant l})$ and $g \in C_c(\partial\Lambda^{\geqslant m})$ with $h = fg$.*

## 3. THE CUNTZ-NICA-PIMSNER ALGEBRA $\mathcal{NO}(X)$

Recall that we denote by $j_X : X \to \mathcal{NO}(X)$ the universal Cuntz–Nica–Pimsner representation of $X$. For each $m \in \mathbb{N}^k$ we denote by $j_{X,m}$ the restriction of $j_X$ to $X_m$. For each $\lambda \in \Lambda$ the set $D_\lambda$ is closed and open, and so the characteristic function $\mathcal{X}_{D_\lambda} \in C_c(\partial\Lambda^{\geqslant d(\lambda)}) \subset X_{d(\lambda)}$.

THEOREM 3.1. *Let $\Lambda$ be a finitely-aligned k-graph and $X$ be the associated product system of Hilbert bimodules given in Proposition 2.7. Denote by $\{s_\lambda : \lambda \in \Lambda\}$ the universal Cuntz–Krieger $\Lambda$-family in $C^*(\Lambda)$. There exists an isomorphism $\pi : C^*(\Lambda) \to \mathcal{NO}(X)$ such that $\pi(s_\lambda) = j_{X,d(\lambda)}(\mathcal{X}_{D_\lambda})$.*

To prove this result we first show that $S := \{S_\lambda := j_{X,d(\lambda)}(\mathcal{X}_{D_\lambda}) : \lambda \in \Lambda\}$ is a set of partial isometries in $\mathcal{NO}(X)$ satisfying (CK1) and (CK2). We use the Nica covariance of $j_X$ to show that $S$ satisfies (CK3), and the Cuntz–Pimsner covariance of $j_X$ to show that $S$ satisfies (CK4). The universal property of $C^*(\Lambda)$ then gives us a map $\pi : C^*(\Lambda) \to \mathcal{NO}(X)$ with $\pi(s_\lambda) = j_X(\mathcal{X}_{D_\lambda})$ for each $\lambda \in \Lambda$. We show that $S$ generates $\mathcal{NO}(X)$, and we use the gauge-invariant uniqueness theorem for $C^*(\Lambda)$ ([22], Theorem 4.2) to prove that $\pi$ is injective.

PROPOSITION 3.2. *The set $S = \{S_\lambda : \lambda \in \Lambda\}$ is a family of partial isometries satisfying* (CK1) *and* (CK2).

*Proof.* Let $\lambda \in \Lambda$. Using (2.1) and (2.2) we get $\mathcal{X}_{D_\lambda} \cdot \langle \mathcal{X}_{D_\lambda}, \mathcal{X}_{D_\lambda} \rangle_{d(\lambda)} = \mathcal{X}_{D_\lambda}$, and it follows that $S_\lambda S_\lambda^* S_\lambda = S_\lambda$. It follows from the properties of characteristic functions that $\{S_v = j_{X,0}(\mathcal{X}_{D_v})\}$ is a set of mutually orthogonal projections, thus (CK1) is satisfied. Relation (CK2) follows from the calculation

$$\begin{aligned}\mathcal{X}_{D_\lambda}\mathcal{X}_{D_\mu}(x) &= \pi_{d(\lambda),d(\mu)}(\mathcal{X}_{D_\lambda} \otimes \mathcal{X}_{D_\mu})(x) = \mathcal{X}_{D_\lambda}(x)\mathcal{X}_{D_\mu}(\sigma_{d(\lambda)}(x)) \\ &= \begin{cases} 1 & \text{if } x(0,d(\lambda)) = \lambda \text{ and } x(d(\lambda), d(\lambda)+d(\mu)) = \mu, \\ 0 & \text{otherwise,} \end{cases} \\ &= \mathcal{X}_{D_{\lambda\mu}}(x). \qquad \blacksquare\end{aligned}$$

PROPOSITION 3.3. *The set $S$ satisfies relation* (CK3):

$$S_\lambda^* S_\mu = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} S_\alpha S_\beta^* \quad \textit{for all } \lambda, \mu \in \Lambda.$$

To prove this proposition we need the next result. For $\lambda, \mu \in \Lambda$ with $d(\lambda) = d(\mu)$ we denote by $\Theta_{\lambda,\mu}$ the rank-one operator $\Theta_{\mathcal{X}_{D_\lambda},\mathcal{X}_{D_\mu}} \in \mathcal{K}(X_{d(\lambda)})$.

LEMMA 3.4. *Let $\lambda, \mu \in \Lambda$. Then we have*

$$\iota_{d(\lambda)}^{d(\lambda)\vee d(\mu)}(\Theta_{\lambda,\lambda})\iota_{d(\mu)}^{d(\lambda)\vee d(\mu)}(\Theta_{\mu,\mu}) = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} \Theta_{\lambda\alpha,\mu\beta}. \tag{3.1}$$

*Proof.* Let $f \in C_c(\partial\Lambda^{\geqslant d(\mu)})$ and $g \in C_c(\partial\Lambda^{\geqslant d(\lambda)\vee d(\mu)-d(\mu)})$. We show that the operators in (3.1) agree on the product $fg \in C_c(\partial\Lambda^{\geqslant d(\lambda)\vee d(\mu)})$, and then the result will follow from Corollary 2.8 and the fact that $C_c(\partial\Lambda^{\geqslant d(\lambda)\vee d(\mu)})$ is dense in $X_{d(\lambda)\vee d(\mu)}$.

We know that for each $\mu \in \Lambda$ we have $\Theta_{\mu,\mu}(f) = \mathcal{X}_{D_\mu} \cdot \langle \mathcal{X}_{D_\mu}, f\rangle_{d(\mu)}$. It follows from a routine calculation using (2.1) and (2.2) that $\Theta_{\mu,\mu}(f) = \mathcal{X}_{D_\mu} f$, where $\mathcal{X}_{D_\mu} f$ is a product of functions in $C_c(\partial\Lambda^{\geqslant d(\mu)})$. It now follows from (1.1) that

$$\iota_{d(\mu)}^{d(\lambda)\vee d(\mu)}(\Theta_{\mu,\mu})(fg) = (\Theta_{\mu,\mu}(f))g = (\mathcal{X}_{D_\mu} f)g. \tag{3.2}$$

We now use Corollary 2.8 to factor $(\mathcal{X}_{D_\mu} f)g = hl$, where $h \in C_c(\partial\Lambda^{\geqslant d(\lambda)})$ and $l \in C_c(\partial\Lambda^{\geqslant d(\lambda)\vee d(\mu)-d(\lambda)})$. For $x \in \partial\Lambda^{\geqslant d(\lambda)\vee d(\mu)}$ we have

$$\begin{aligned}\iota_{d(\lambda)}^{d(\lambda)\vee d(\mu)}(\Theta_{\lambda,\lambda})\iota_{d(\mu)}^{d(\lambda)\vee d(\mu)}(\Theta_{\mu,\mu})(fg)(x) &= \iota_{d(\lambda)}^{d(\lambda)\vee d(\mu)}(\Theta_{\lambda,\lambda})(hl)(x) = (\mathcal{X}_{D_\lambda} h)l(x)\\ &= \begin{cases} hl(x) & \text{if } x \in D_\lambda,\\ 0 & \text{otherwise}\end{cases} = \begin{cases} fg(x) & \text{if } x \in D_\lambda \cap D_\mu,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

We know from Lemma 2.4 that $D_\lambda \cap D_\mu = \bigsqcup_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} D_{\lambda\alpha}$. So we have the following and the result follows:

$$\begin{aligned}\iota_{d(\lambda)}^{d(\lambda)\vee d(\mu)}(\Theta_{\lambda,\lambda})\iota_{d(\mu)}^{d(\lambda)\vee d(\mu)}(\Theta_{\mu,\mu})(fg)(x) &= \begin{cases} fg(x) & \text{if } x \in \bigsqcup_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} D_{\lambda\alpha},\\ 0 & \text{otherwise.}\end{cases}\\ &= \Big(\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} \Theta_{\lambda\alpha,\mu\beta}\Big)(fg)(x). \quad \blacksquare\end{aligned}$$

*Proof of Proposition* 3.3. It follows from the Nica covariance of $j_X$ that

$$S_\lambda S_\lambda^* S_\mu S_\mu^* = j_X^{(d(\lambda))}(\Theta_{\lambda,\lambda}) j_X^{(d(\mu))}(\Theta_{\mu,\mu}) = j_X^{(d(\lambda)\vee d(\mu))}(\iota_{d(\lambda)}^{d(\lambda)\vee d(\mu)}(\Theta_{\lambda,\lambda})\iota_{d(\mu)}^{d(\lambda)\vee d(\mu)}(\Theta_{\mu,\mu})).$$

It follows from this equation and Lemma 3.4 that

$$\begin{aligned} S_\lambda\Big(\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} S_\alpha S_\beta^*\Big)S_\mu^* &= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} S_{\lambda\alpha}S_{\mu\beta}{}^* = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} j_X^{(d(\lambda)\vee d(\mu))}(\Theta_{\lambda\alpha,\mu\beta})\\ &= j_X^{(d(\lambda)\vee d(\mu))}\Big(\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} \Theta_{\lambda\alpha,\mu\beta}\Big) = S_\lambda S_\lambda^* S_\mu S_\mu^*.\end{aligned}$$

It then follows that

$$S_\lambda^* S_\mu = (S_\lambda^* S_\lambda S_\lambda^*)(S_\mu S_\mu^* S_\mu) = S_\lambda^*(S_\lambda S_\lambda^* S_\mu S_\mu^*) S_\mu = S_\lambda^* S_\lambda \Big( \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} S_\alpha S_\beta^* \Big) S_\mu^* S_\mu$$

$$= \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} S_{s(\lambda)} S_\alpha (S_{s(\mu)} S_\beta)^* = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)} S_\alpha S_\beta^*. \quad ∎$$

Recall from Section 1.2 that $I_n$ is given by $I_n := \bigcap_{0<m\leqslant n} \ker \phi_m$. To prove that $S$ satisfies (CK4), we need to find families which span dense subspaces of the Hilbert bimodules $X_m \cdot I_{n-m}$, for $m, n \in \mathbb{N}^k$ with $m \leqslant n$. To do this, we must first find families which span dense subspaces of the bimodules $X_n$ and the ideals $I_n$.

PROPOSITION 3.5. *For each $n \in \mathbb{N}^k$ we have $X_n = \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{A}^n\}$.*

*Proof.* Let $f \in C_c(\partial\Lambda^{\geqslant n})$. We can use the same argument as in the beginning of the proof of Proposition 2.2 to write $\operatorname{supp} f \subseteq \bigcup_{j=1}^{h} D_{\mu_j} \setminus D_{\mu_j G_j}$, where $(\mu_j, G_j) \in \mathcal{A}^n$ and $D_{\mu_j} \setminus D_{\mu_j G_j} \subseteq \partial\Lambda^{\geqslant n}$ for each $1 \leqslant j \leqslant h$. We now take a partition of unity $\rho_1, \ldots, \rho_h$ subordinate to $\{D_{\mu_j} \setminus D_{\mu_j G_j} : 1 \leqslant j \leqslant h\}$, and for $f_j := f\rho_j \in C(D_{\mu_j} \setminus D_{\mu_j G_j})$ we have

$$f = \sum_{j=1}^{h} f_j. \tag{3.3}$$

Now for each $1 \leqslant j \leqslant h$ we have $d(\mu_j) \geqslant n$. So $\sigma_n$ is injective on $D_{\mu_j} \setminus D_{\mu_j G_j}$, and hence

$$\|f_j\|_n = \sup\{|f_j(x)| : x \in D_{\mu_j} \setminus D_{\mu_j G_j}\} = \|f_j\|_\infty. \tag{3.4}$$

Now, it follows from Lemma 2.4 that for each $(\lambda, F) \in \mathcal{A}$ the set

$$\operatorname{span}\{\mathcal{X}_{D_\mu \setminus D_{\mu G}} : (\mu, G) \in \mathcal{A} \text{ and } D_\mu \setminus D_{\mu G} \subseteq D_\lambda \setminus D_{\lambda F}\}$$

is a subalgebra of $C(D_\lambda \setminus D_{\lambda F})$. An application of the Stone–Weierstrass Theorem shows that the closure of that span is equal to $C(D_\lambda \setminus D_{\lambda F})$, and hence each $f_j$ can be uniformly approximated by elements in $\operatorname{span}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : d(\lambda) \geqslant n\}$. It now follows from (3.4) that $f_j$ can be uniformly approximated by elements in $\operatorname{span}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : d(\lambda) \geqslant n\}$ with respect to $\|\cdot\|_n$, and then (3.3) says that $f$ can be approximated by elements in $\operatorname{span}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : d(\lambda) \geqslant n\}$ with respect to $\|\cdot\|_n$. The result follows because $C_c(\partial\Lambda \setminus \partial_n)$ is dense in $X_n$ with respect to $\|\cdot\|_n$. ∎

DEFINITION 3.6. Let $i \in \{1, \ldots, k\}$ and $e_i$ denote the standard basis element of $\mathbb{N}^k$. We say that $(\lambda, F) \in \mathcal{A}$ satisfies condition $\mathrm{K}(i)$ if

$$\mu \in s(\lambda)\Lambda \text{ with } d(\mu) \geqslant e_i \Longrightarrow D_\mu \subseteq D_\nu \text{ for some } \nu \in F.$$

PROPOSITION 3.7. *For each $n \in \mathbb{N}^k$ we have*

$$I_n = \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : n_i > 0 \Longrightarrow d(\lambda)_i = 0 \textit{ and } (\lambda, F) \textit{ satisfies condition } \mathrm{K}(i)\}.$$

To prove this proposition we need the following result.

LEMMA 3.8. *Let $i \in \{1, \dots, k\}$ and $(\lambda, F) \in \mathcal{A}$. Then $D_\lambda \setminus D_{\lambda F} \subseteq \partial\Lambda^{\not\geqslant e_i}$ if and only if $d(\lambda)_i = 0$ and $(\lambda, F)$ satisfies condition* K($i$)*. Moreover, we have*

$$\ker \phi_{e_i} = \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{A},\, d(\lambda)_i = 0 \text{ and } (\lambda, F) \text{ satisfies condition } \mathrm{K}(i)\}. \tag{3.5}$$

*Proof.* Suppose $D_\lambda \setminus D_{\lambda F} \subseteq \partial\Lambda^{\not\geqslant e_i}$. Then we obviously have $d(\lambda)_i = 0$. Suppose that $(\lambda, F)$ does not satisfy condition K($i$). Then there exists $\mu \in s(\lambda)\Lambda$ with $d(\mu) \geqslant e_i$, and $x \in D_\mu$ with $x \notin D_\nu$ for all $\nu \in F$. Consider the boundary path $\lambda x$. We have $d(\lambda x)_i > 0$ and $\lambda x \in D_\lambda \setminus D_{\lambda F}$. But $d(\lambda x)_i > 0 \Longrightarrow \lambda x \in \partial\Lambda^{\geqslant e_i}$, and this is a contradiction, so $(\lambda, F)$ satisfies condition K($i$).

Now suppose that $d(\lambda)_i = 0$ and $(\lambda, F)$ satisfies condition K($i$). Assume that $D_\lambda \setminus D_{\lambda F} \not\subseteq \partial\Lambda^{\not\geqslant e_i}$, so there exists $x \in D_\lambda \setminus D_{\lambda F}$ with $x \in \partial\Lambda^{\geqslant e_i}$. This implies that $d(x)_i > 0$. Consider the edge $\mu := x(d(\lambda), d(\lambda) + e_i)$, which we know exists because $d(\lambda)_i = 0$. We have $\mu \in s(\lambda)\Lambda$ and $d(\mu) = e_i$. The boundary path $\sigma_{d(\lambda)}(x)$ satisfies $\sigma_{d(\lambda)}(x) \in D_\mu$ and $\sigma_{d(\lambda)}(x) \notin D_\nu$ for all $\nu \in F$, and so $D_\mu \not\subseteq D_\nu$ for all $\nu \in F$. But this contradicts that $(\lambda, F)$ satisfies condition K($i$), so we must have $D_\lambda \setminus D_{\lambda F} \subseteq \partial\Lambda^{\not\geqslant e_i}$.

Now, it follows from Lemma 2.4 and an application of the Stone-Weierstrass Theorem for locally compact spaces that for any open subset $U$ of $\partial\Lambda$ we have

$$C_0(U) = \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{A} \text{ and } D_\lambda \setminus D_{\lambda F} \subseteq U\}.$$

It follows that

$$\begin{aligned}
\ker \phi_{e_i} &= \{a \in C_0(\partial\Lambda) : a|_{\partial\Lambda^{\geqslant e_i}} = 0\} = \{a \in C_0(\partial\Lambda) : a|_{\overline{\partial\Lambda^{\geqslant e_i}}} = 0\} \\
&= C_0(\operatorname{int} \partial\Lambda^{\not\geqslant e_i}) = \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{A} \text{ and } D_\lambda \setminus D_{\lambda F} \subseteq \operatorname{int} \partial\Lambda^{\not\geqslant e_i}\} \\
&= \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{A} \text{ and } D_\lambda \setminus D_{\lambda F} \subseteq \partial\Lambda^{\not\geqslant e_i}\} \\
&= \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{A},\, d(\lambda)_i = 0,\, (\lambda, F) \text{ satisfies condition } \mathrm{K}(i)\}. \qquad \blacksquare
\end{aligned}$$

*Proof of Proposition* 3.7. We have

$$\ker \phi_n = \{a \in C_0(\partial\Lambda) : a|_{\partial\Lambda^{\geqslant n}} = 0\} = \{a \in C_0(\partial\Lambda) : a|_{\overline{\partial\Lambda^{\geqslant n}}} = 0\} = C_0(\operatorname{int} \partial\Lambda^{\not\geqslant n}).$$

Since $m \leqslant n \Longrightarrow \partial\Lambda^{\not\geqslant m} \subseteq \partial\Lambda^{\not\geqslant n}$, it follows that $m \leqslant n \Rightarrow \ker \phi_m \subseteq \ker \phi_n$. Hence $I_n = \bigcap_{\{i : n_i > 0\}} \ker \phi_{e_i}$, and the result now follows from Lemma 3.8. $\blacksquare$

NOTATION 3.9. Let $n \in \mathbb{N}^k$. We define

$$\mathcal{I}(I_n) := \{(\lambda, F) \in \mathcal{A} : D_\lambda \setminus D_{\lambda F} \neq \emptyset \text{ and } n_i > 0 \Longrightarrow d(\lambda)_i = 0 \text{ and } (\lambda, F) \text{ satisfies condition } \mathrm{K}(i)\},$$

and for $\mu \in \Lambda$ we write $\mu\mathcal{I}(I_n) := \{(\mu\lambda, F) : (\lambda, F) \in \mathcal{I}(I_n) \text{ with } s(\mu) = r(\lambda)\}$. The reason for introducing this notation is that we can now write

$$I_n = \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{I}(I_n)\}.$$

PROPOSITION 3.10. *Let $m, n \in \mathbb{N}^k$ with $m \leqslant n$. Then we have*

$$X_m \cdot I_{n-m} = \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{A}^m \cap \lambda(0, m)\mathcal{I}(I_{n-m})\}. \tag{3.6}$$

*Proof.* We have $X_m \cdot I_{n-m} = \overline{\operatorname{span}}\{x \cdot a : x \in X_m,\ a \in I_{n-m}\}$. To prove that the right-hand side of (3.6) is contained in the left-hand side, we let $m, n \in \mathbb{N}^k$ with $m \leqslant n$, and suppose $(\lambda, F) \in \mathcal{A}^m \cap \lambda(0, m)\mathcal{I}(I_{n-m})$. Then $(\lambda(m, d(\lambda)), F) \in \mathcal{I}(I_{n-m})$, and for $x \in \partial\Lambda^{\geqslant m}$ we have

$$\begin{aligned}\mathcal{X}_{D_\lambda \setminus D_{\lambda F}}(x) &= \begin{cases} 1 & \text{if } x(0, d(\lambda)) = \lambda \text{ and } x(0, d(\lambda\nu)) \neq \lambda\nu, \text{for all } \nu \in F, \\ 0 & \text{otherwise;} \end{cases} \\ &= \mathcal{X}_{D_\lambda}(x)\mathcal{X}_{D_{\lambda(m,d(\lambda))} \setminus D_{\lambda(m,d(\lambda))F}}(\sigma_m(x)) = (\mathcal{X}_{D_\lambda} \cdot \mathcal{X}_{D_{\lambda(m,d(\lambda))} \setminus D_{\lambda(m,d(\lambda))F}})(x).\end{aligned}$$

So $\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} = \mathcal{X}_{D_\lambda} \cdot \mathcal{X}_{D_{\lambda(m,d(\lambda))} \setminus D_{\lambda(m,d(\lambda))F}} \in X_m \cdot I_{n-m}$, and it follows that

$$\overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{A}^m \cap \lambda(0, m)\mathcal{I}(I_{n-m})\} \subset X_m \cdot I_{n-m}.$$

It follows from Proposition 3.7 and Proposition 3.5 that

$$X_m \cdot I_{n-m} = \overline{\operatorname{span}}\{\mathcal{X}_{D_\rho \setminus D_{\rho F}} \cdot \mathcal{X}_{D_\tau \setminus D_{\tau G}} : (\rho, F) \in \mathcal{A}^m \text{ and } (\tau, G) \in \mathcal{I}(I_{n-m})\}.$$

So to prove that the left-hand side of (3.6) is contained in the right-hand side, it suffices to show that for $(\rho, F) \in \mathcal{A}^m$ and $(\tau, G) \in \mathcal{I}(I_{n-m})$ the product $\mathcal{X}_{D_\rho \setminus D_{\rho F}} \cdot \mathcal{X}_{D_\tau \setminus D_{\tau G}}$ is an element of the right-hand side. Since $\sigma_m^{-1}$ is continuous, the intersection

$$(D_\rho \setminus D_{\rho F}) \cap \sigma_m^{-1}(D_\tau \setminus D_{\tau G}) \tag{3.7}$$

is an open and compact subset of $D_\rho \setminus D_{\rho F}$. Since it is open, we know there exists a subset $\mathcal{J} \subseteq \mathcal{A}^m$ such that $(D_\rho \setminus D_{\rho F}) \cap \sigma_m^{-1}(D_\tau \setminus D_{\tau G}) = \bigcup_{(\eta,H) \in \mathcal{J}} D_\eta \setminus D_{\eta H}$; since it is compact, there is a finite number, say $h$, of pairs $(\eta_j, H_j) \in \mathcal{J}$ with

$$(D_\rho \setminus D_{\rho F}) \cap \sigma_m^{-1}(D_\tau \setminus D_{\tau G}) = \bigcup_{j=1}^{h} D_{\eta_j} \setminus D_{\eta_j H_j}.$$

We know from Lemma 2.4 that the intersection of sets in the above finite union is a finite, disjoint union of sets of the same form. So it follows that there is a finite number, say $l$, of pairs $(\mu_j, L_j) \in \mathcal{A}^m$ and constants $c_j$ such that

$$\mathcal{X}_{D_\rho \setminus D_{\rho F}} \cdot \mathcal{X}_{D_\tau \setminus D_{\tau G}} = \mathcal{X}_{(D_\rho \setminus D_{\rho F}) \cap \sigma_m^{-1}(D_\tau \setminus D_{\tau G})} = \sum_{j=1}^{l} c_j \mathcal{X}_{D_{\mu_j} \setminus D_{\mu_j L_j}}. \tag{3.8}$$

To finish the proof, we need to show that each $(\mu_j, L_j) \in \mu_j(0, m)\mathcal{I}(I_{n-m})$. Suppose $n_i > m_i$ and $d(\mu_j)_i > m_i$. Then for $x \in D_{\mu_j} \setminus D_{\mu_j L_j}$ we have $\sigma_m(x) \in D_\tau \setminus D_{\tau G}$

and $\sigma_m(x)_i > 0$. Since $d(\tau)_i = 0$, there exists a path $\alpha := \sigma_m(x)(d(\tau), d(\tau) + e_i)$ satisfying $\alpha \in s(\tau)\Lambda^{e_i}$. Since $(\tau, G)$ satisfies condition K($i$), we have $D_\alpha \subseteq D_\xi$ for some $\xi \in G$. But this implies that $\sigma_m(x) = \tau\alpha\sigma_m(x)(d(\tau) + e_i, d(x)) \in D_{\tau\xi} \subseteq D_{\tau G}$, which contradicts $\sigma_m(x) \in D_\tau \setminus D_{\tau G}$. So we must have $d(\mu_j)_i = m_i$.

Now suppose $n_i > m_i$ and there exists an edge $\zeta \in s(\mu_j)\Lambda^{e_i}$ with $D_\zeta \not\subseteq D_\nu$ for any $\nu \in L_j$. Let $x \in s(\zeta)\partial\Lambda$. Then $\mu_j\zeta x \in D_{\mu_j} \setminus D_{\mu_j L_j}$, which implies

$$\sigma_m(\mu_j\zeta x) \in D_\tau \setminus D_{\tau G}. \tag{3.9}$$

Since $d(\tau)_i = 0$, there exists a path $\beta := \sigma_m(\mu_j\zeta x)(d(\tau), d(\tau) + e_i)$ satisfying $\beta \in s(\tau)\Lambda^{e_i}$. Since $(\tau, G)$ satisfies condition K($i$), we have $D_\beta \subseteq D_\xi$ for some $\xi \in G$. But this implies that $\sigma_m(\mu_j\zeta x) = \tau\beta\sigma_m(\mu_j\zeta x)(d(\tau) + e_i, d(x)) \in D_{\tau\xi} \subseteq D_{\tau G}$, which contradicts (3.9). So $D_\zeta \subseteq D_\nu$ for some $\nu \in L_j$, and hence $(\mu_j, L_j)$ satisfies condition K($i$). ∎

NOTATION 3.11. Let $m, n \in \mathbb{N}^k$ with $m \leqslant n$. We denote

$$\mathcal{I}(X_m \cdot I_{n-m}) := \{(\lambda, F) : D_\lambda \setminus D_{\lambda F} \neq \emptyset,\ (\lambda, F) \in \mathcal{A}^m \cap \lambda(0, m)\mathcal{I}(I_{n-m})\}.$$

So we have

$$X_m \cdot I_{n-m} = \overline{\operatorname{span}}\{\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} : (\lambda, F) \in \mathcal{I}(X_m \cdot I_{n-m})\}.$$

PROPOSITION 3.12. *The set* $S = \{S_\lambda : \lambda \in \Lambda\}$ *satisfies* (CK4)

$$\prod_{\mu \in \mathcal{F}} (S_v - S_\mu S_\mu^*) = 0$$

*for all* $v \in \Lambda^0$ *and all nonempty finite exhaustive sets* $\mathcal{F} \subset r^{-1}(v)$.

To prove this proposition we need the following results. For a finite subset $G \subset \Lambda$ we denote by $\bigvee d(G)$ the element $\bigvee_{\mu \in G} d(\mu)$ of $\mathbb{N}^k$.

LEMMA 3.13. *Let* $v \in \Lambda^0$ *and* $\mathcal{F} \subseteq v\Lambda$ *a finite exhaustive set;* $n \in \mathbb{N}^k$ *with* $n \geqslant \vee d(\mathcal{F})$ *and* $m \in \mathbb{N}^k$ *with* $m \leqslant n$*; and* $\lambda \in v\Lambda$ *and* $F \subseteq s(\lambda)\Lambda$ *with* $(\lambda, F) \in \mathcal{I}(X_m \cdot I_{n-m})$*. Then there exists* $\eta \in \mathcal{F}$ *such that* $\lambda$ *extends* $\eta$*.*

*Proof.* Suppose $\lambda$ does not extend any element of $\mathcal{F}$. Since $D_\lambda \setminus D_{\lambda F} \neq \emptyset$, there exists a boundary path $x \in D_\lambda \setminus D_{\lambda F}$. Since $\mathcal{F}$ is exhaustive, there exists $\eta \in \mathcal{F}$ with $x(0, d(\eta)) = \eta$. So $x \in D_\eta \cap (D_\lambda \setminus D_{\lambda F})$, and the pair $(x_\lambda^\eta, x_\eta^\lambda) \in \Lambda^{\min}(\lambda, \eta)$. Since $\lambda$ does not extend $\eta$, there exists $i \in \{1, \ldots, k\}$ with $d(\lambda)_i < d(\eta)_i$, and hence $d(x_\lambda^\eta) \geqslant e_i$. Since $m_i \leqslant d(\lambda)_i < d(\eta)_i \leqslant n_i$, we know $(\lambda, F)$ satisfies condition K($i$), and hence $D_{x_\lambda^\eta} \subseteq D_\nu$ for some $\nu \in F$. But this implies that $x \in D_{\lambda x_\lambda^\eta} \subseteq D_{\lambda\nu}$, which contradicts the fact $x \notin D_{\lambda F}$. So $\lambda$ must extend an element of $\mathcal{F}$. ∎

LEMMA 3.14. *Suppose* $n \in \mathbb{N}^k$ *and* $\mu \in \Lambda$ *with* $d(\mu) \leqslant n$*. Consider the element* $\widetilde{x}$ *given by* $\widetilde{x} := (0, \ldots, 0, \mathcal{X}_{D_\lambda \setminus D_{\lambda F}}, 0, \ldots, 0) \in \widetilde{X}_n$*, where* $(\lambda, F) \in \mathcal{I}(X_m \cdot I_{n-m})$ *for*

*$m \leqslant n$. Then we have*

$$\widetilde{\iota}^{n}_{d(\mu)}(\Theta_{\mu,\mu})(\widetilde{x}) = \begin{cases} \widetilde{x} & \text{if } \lambda \text{ extends } \mu, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* It follows from (1.2) that for $r \leqslant n$ we have

$$\widetilde{\iota}^{n}_{d(\mu)}(\Theta_{\mu,\mu})(\widetilde{x})(r) = \iota^{r}_{d(\mu)}(\Theta_{\mu,\mu})(\widetilde{x}(r)) = \begin{cases} \iota^{m}_{d(\mu)}(\Theta_{\mu,\mu})(\mathcal{X}_{D_\lambda \setminus D_{\lambda F}}) & \text{if } r = m, \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Now assume $m \geqslant d(\mu)$. A straightforward calculation shows that

$$\mathcal{X}_{D_\lambda \setminus D_{\lambda F}} = \mathcal{X}_{D_{\lambda(d(\mu),d(\lambda))} \setminus D_{\lambda(0,d(\mu))}} \mathcal{X}_{D_{\lambda(d(\mu),d(\lambda))F}}. \tag{3.11}$$

We also have

$$\begin{aligned} \Theta_{\mu,\mu}(\mathcal{X}_{D_{\lambda(0,d(\mu))}})(x) &= (\mathcal{X}_{D_\mu} \cdot \langle \mathcal{X}^*_{D_\mu}, \mathcal{X}_{D_{\lambda(0,d(\mu))}} \rangle_{d(\mu)})(x) \\ &= \mathcal{X}_{D_\mu}(x) \langle \mathcal{X}^*_{D_\mu}, \mathcal{X}_{D_{\lambda(0,d(\mu))}} \rangle_{d(\mu)}(\sigma_{d(\mu)}(x)) \\ &= \begin{cases} \sum_{\sigma_{d(\mu)}(y) = \sigma_{d(\mu)}(x)} \mathcal{X}_{D_\mu}(y) \mathcal{X}_{D_{\lambda(0,d(\mu))}}(y) & \text{if } x(0,d(\mu)) = \mu, \\ 0 & \text{otherwise;} \end{cases} \\ &= \begin{cases} 1 & \text{if } \lambda(0,d(\mu)) = \mu \text{ and } x(0,d(\mu)) = \mu, \\ 0 & \text{otherwise;} \end{cases} \\ &= \begin{cases} \mathcal{X}_{D_\mu}(x) & \text{if } \lambda \text{ extends } \mu, \\ 0 & \text{otherwise;} \end{cases} = \begin{cases} \mathcal{X}_{D_{\lambda(0,d(\mu))}}(x) & \text{if } \lambda \text{ extends } \mu, \\ 0 & \text{otherwise.} \end{cases} \qquad (3.12) \end{aligned}$$

It now follows from equations (3.11) and (3.12) that

$$\begin{aligned} \iota^{m}_{d(\mu)}(\Theta_{\mu,\mu})(\mathcal{X}_{D_\lambda \setminus D_{\lambda F}}) &= \iota^{m}_{d(\mu)}(\Theta_{\mu,\mu})(\mathcal{X}_{D_{\lambda(0,d(\mu))}} \mathcal{X}_{D_{\lambda(d(\mu),d(\lambda))} \setminus D_{\lambda(d(\mu),d(\lambda))F}}) \\ &= \Theta_{\mu,\mu}(\mathcal{X}_{D_{\lambda(0,d(\mu))}}) \mathcal{X}_{D_{\lambda(d(\mu),d(\lambda))} \setminus D_{\lambda(d(\mu),d(\lambda))F}} \\ &= \begin{cases} \mathcal{X}_{D_{\lambda(0,d(\mu))}} \mathcal{X}_{D_{\lambda(d(\mu),d(\lambda))} \setminus D_{\lambda(d(\mu),d(\lambda))F}} & \text{if } \lambda \text{ extends } \mu, \\ 0 & \text{otherwise;} \end{cases} \\ &= \begin{cases} \mathcal{X}_{D_\lambda \setminus D_{\lambda F}} & \text{if } \lambda \text{ extends } \mu, \\ 0 & \text{otherwise.} \end{cases} \qquad (3.13) \end{aligned}$$

Equations (3.10) and (3.13) now give the result. ∎

We are now ready to prove that $S$ satisfies relation (CK4). The proof runs through the main argument from the proof of Proposition 5.4 of [23].

*Proof of Proposition* 3.12. Fix $v \in \Lambda^0$ and a finite exhaustive set $\mathcal{F} \subset v\Lambda$. We must show that

$$\prod_{\mu \in \mathcal{F}} (S_v - S_\mu S^*_\mu) = 0.$$

Recall from [21] that for a nonempty subset $G$ of $\mathcal{F}$, $\Lambda^{\min}(G)$ denotes the set $\{\lambda \in \Lambda : d(\lambda) = \bigvee d(G), \lambda \text{ extends } \mu \text{ for all } \mu \in G\}$. Recall also that $\bigvee \mathcal{F} := \bigcup_{G \subset \mathcal{F}} \Lambda^{\min}(G)$ is finite and is closed under minimal common extensions. We have

$$\prod_{\mu\in\mathcal{F}}(S_v - S_\mu S_\mu^*) = S_v + \sum_{\substack{\emptyset\neq G\subset F\\ \lambda\in\Lambda^{\min}(G)}} (-1)^{|G|} S_\lambda S_\lambda^* = j_X^{(0)}(\Theta_{v,v}) + \sum_{\substack{\emptyset\neq G\subset F\\ \lambda\in\Lambda^{\min}(G)}} (-1)^{|G|} j_X^{(\bigvee d(G))}(\Theta_{\lambda,\lambda}),$$

where the first equation can be obtained through repeated application of (CK3). Since $j_X$ is Cuntz–Pimsner covariant, it suffices to show that for each $q \in \mathbb{N}^k$ there exists $r \geqslant q$ such that for all $s \geqslant r$, we have

$$\widetilde{\iota}_0^s(\Theta_{v,v}) + \sum_{\substack{\emptyset\neq G\subset F\\ \lambda\in\Lambda^{\min}(G)}} (-1)^{|G|} \widetilde{\iota}^s_{\bigvee d(G)}(\Theta_{\lambda,\lambda}) = 0.$$

For this, fix $q \in \mathbb{N}^k$, let $r = q \vee (\bigvee d(\mathcal{F}))$ and fix $s \geqslant r$. It suffices to show that

$$\Big(\widetilde{\iota}_0^s(\Theta_{v,v}) + \sum_{\substack{\emptyset\neq G\subset F\\ \lambda\in\Lambda^{\min}(G)}} (-1)^{|G|} \widetilde{\iota}^s_{\bigvee d(G)}(\Theta_{\lambda,\lambda})\Big)(\widetilde{x}) = 0, \tag{3.14}$$

where $\widetilde{x} \in \widetilde{X}_s$ is given by $\widetilde{x} := (0, \ldots, 0, \mathcal{X}_{D_\rho \setminus D_{\rho F}}, 0, \ldots, 0)$, for $(\rho, F) \in \mathcal{I}(X_t \cdot I_{s-t})$, $t \leqslant s$. For any $\mu \in \mathcal{F}$ we have $s \geqslant d(\mu)$. It then follows from Lemma 3.14 that

$$\widetilde{\iota}^s_{d(\mu)}(\Theta_{\mu,\mu})(\widetilde{x}) = \begin{cases} \widetilde{x} & \text{if } \rho \text{ extends } \mu, \\ 0 & \text{otherwise.} \end{cases} \tag{3.15}$$

Fix a nonempty subset $G$ of $\mathcal{F}$. Then

$$\Big(\prod_{\mu\in G} \widetilde{\iota}^s_{d(\mu)}(\Theta_{\mu,\mu})\Big)(\widetilde{x}) = \begin{cases} \widetilde{x} & \text{if } \rho \text{ extends each } \mu \in G, \\ 0 & \text{otherwise.} \end{cases}$$

The factorisation property implies that $\rho$ extends each $\mu \in G$ if and only if there exists $\lambda \in \Lambda^{\min}(G)$ such that $\rho$ extends $\lambda$. The factorisation property also implies that if there does exist such a $\lambda \in \Lambda^{\min}(G)$, then it is necessarily unique. We therefore have

$$\Big(\prod_{\mu\in G} \widetilde{\iota}^s_{d(\mu)}(\Theta_{\mu,\mu})\Big)(\widetilde{x}) = \Big(\sum_{\lambda\in\Lambda^{\min}(G)} \widetilde{\iota}^s_{\bigvee d(G)}(\Theta_{\mu,\mu})\Big)(\widetilde{x}).$$

Since $G$ was an arbitrary subset of $\mathcal{F}$, we have

$$\begin{aligned}\Big(\prod_{\mu\in\mathcal{F}} \big(\widetilde{\iota}_0^s(\Theta_{v,v}) - \widetilde{\iota}^s_{d(\mu)}(\Theta_{\mu,\mu})\big)\Big)(\widetilde{x}) &= \Big(\widetilde{\iota}_0^s(\Theta_{v,v}) + \sum_{\emptyset\neq G\subset\mathcal{F}}\Big((-1)^{|G|}\prod_{\mu\in G}\widetilde{\iota}^s_{d(\mu)}(\Theta_{\mu,\mu})\Big)\Big)(\widetilde{x}) \\ &= \Big(\widetilde{\iota}_0^s(\Theta_{v,v}) + \sum_{\substack{\emptyset\neq G\subset F\\ \lambda\in\Lambda^{\min}(G)}} (-1)^{|G|} \widetilde{\iota}^s_{\bigvee d(G)}(\Theta_{\lambda,\lambda})\Big)(\widetilde{x}).\end{aligned}$$

Now we can apply Lemma 3.13 to see that there exists $\eta \in \mathcal{F}$ such that $\rho$ extends $\eta$. It now follows from equation (3.15) that

$$\Big(\prod_{\mu\in\mathcal{F}}(\widetilde{\iota}^s_0(\Theta_{v,v})-\widetilde{\iota}^s_{d(\mu)}(\Theta_{\mu,\mu}))\Big)(\widetilde{x})$$
$$=\Big(\prod_{\mu\in\mathcal{F}\setminus\{\eta\}}(\widetilde{\iota}^s_0(\Theta_{v,v})-\widetilde{\iota}^s_{d(\mu)}(\Theta_{\mu,\mu}))\Big)((\widetilde{\iota}^s_0(\Theta_{v,v})-\widetilde{\iota}^s_{d(\eta)}(\Theta_{\eta,\eta})))(\widetilde{x})=0,$$

and hence equation (3.14) is established. ∎

*Proof of Theorem* 3.1. Lemma 3.2, Proposition 3.3 and Proposition 3.12 show that the set $S := \{S_\lambda = j_X(\mathcal{X}_{D_\lambda}) : \lambda \in \Lambda\}$ is a family of partial isometries satisfying the Cuntz–Krieger relations (CK1)–(CK4). It follows from the universal property of $C^*(\Lambda)$ that there exists a homomorphism $\pi : C^*(\Lambda) \to \mathcal{NO}(X)$ such that $\pi(s_\lambda) = j_X(\mathcal{X}_{D_\lambda})$ for each $\lambda \in \Lambda$. We know from Proposition 3.12 of [23] that $\mathcal{NO}(X) = \overline{\operatorname{span}}\{j_X(x)j_X(y)^* : x, y \in X\}$. For each $\lambda \in \Lambda$ and $F \subseteq s(\lambda)\Lambda$ we have $\mathcal{X}_{D_\lambda\setminus D_{\lambda F}} = \mathcal{X}_{D_\lambda} - \sum_{\nu\in F}\mathcal{X}_{D_{\lambda\nu}}$, and so

$$j_X(\mathcal{X}_{D_\lambda\setminus D_{\lambda F}}) = j_X(\mathcal{X}_{D_\lambda}) - j_X\Big(\sum_{\nu\in F}\mathcal{X}_{D_{\lambda\nu}}\Big) = S_\lambda - \sum_{\nu\in F} S_{\lambda\nu}.$$

It then follows from Proposition 3.5 that $S$ generates $\mathcal{NO}(X)$, and hence $\pi$ is surjective. It follows from Lemma 5.13(2) and Lemma 5.15 of [5] that each $D_\lambda \neq \emptyset$, and hence each $\mathcal{X}_{D_\lambda} \neq 0$. It then follows from Theorem 4.1 of [23] that each $S_\lambda \neq 0$. (Note that the quasi-lattice ordered group $(\mathbb{N}^k, \mathbb{Z}^k)$ satisfies condition (3.5) of [23], and so Theorem 4.1 of [23] can indeed be applied.) Since $\pi$ intertwines the gauge actions of $\mathbb{T}^k$ on $\mathcal{NO}(X)$ and $C^*(\Lambda)$, the gauge-invariant uniqueness theorem for $C^*(\Lambda)$ ([22], Theorem 4.2) implies that $\pi$ is an isomorphism. ∎

## 4. CONNECTIONS TO SEMIGROUP CROSSED PRODUCTS

We begin this section by building a crossed product from a finitely-aligned $k$-graph $\Lambda$. For each $n \in \mathbb{N}^k$ we define a *partial endomorphism* $\alpha_n : C_0(\partial\Lambda) \to C_0(\partial\Lambda^{\geqslant n})$ given by $\alpha_n(f) = f \circ \sigma_n$. We claim that for $f \in C_c(\partial\Lambda^{\geqslant n})$ the function $L_n(f)$ given by

$$L_n(f)(x) = \begin{cases} \sum\limits_{\sigma_n(y)=x} f(y) & \text{if } x \in \sigma_n(\partial\Lambda), \\ 0 & \text{otherwise,}\end{cases}$$

is well-defined and is an element of $C_c(\partial\Lambda)$. We can cover $\operatorname{supp} f$ with finitely many sets $U_i$ such that $\sigma_n(U_i)$ is open, $\overline{\sigma_n(U_i)}$ is compact, and $\sigma_n|_{U_i}$ is a homeomorphism. The function $f$ must be zero on all but a finite number of points in $\sigma_n^{-1}(x)$. Then near any $x \in \sigma_n(\partial\Lambda)$, $L_n(f) = \sum_{\{i:x\in\sigma_n(U_i)\}} f \circ (\sigma_n|_{U_i})^{-1}$ is a finite sum of continuous functions with compact support. Since $\sigma_n(x)$ is open, $L_n(f) \in C_c(\partial\Lambda)$, and the claim is proved. Routine calculations show that each $L_n$ satisfies the transfer-operator identity: $L_n(\alpha_n(f)g) = fL_n(g)$ for all $f \in C_0(\partial\Lambda)$, $g \in C_c(\partial\Lambda^{\geqslant n})$. Adapting Exel's construction of a Hilbert bimodule [6] to accommodate the partial maps, and applying it to $(C_0(\partial\Lambda), \alpha_n, L_n)$, gives the Hilbert $C_0(\partial\Lambda)$-bimodule $X_n$ from Section 2. So we consider the boundary-path product system $X$, and take the suggested route of Section 9 of [2] for defining a crossed product for the system $(C_0(\partial\Lambda), \mathbb{N}^k, \alpha, L)$:

DEFINITION 4.1. Let $\Lambda$ be a finitely-aligned $k$-graph, and consider the product system $X$ given in Proposition 2.7. We define the *crossed product* $C_0(\partial\Lambda) \rtimes_{\alpha,L} \mathbb{N}^k$ to be the Cuntz–Nica–Pimsner algebra $\mathcal{NO}(X)$.

COROLLARY 4.2. *Let $\Lambda$ be a finitely-aligned k-graph. Then $C_0(\partial\Lambda) \rtimes_{\alpha,L} \mathbb{N}^k \cong C^*(\Lambda)$.*

For the remainder of this section we discuss the relationship between the crossed product $C_0(\partial\Lambda) \rtimes_{\alpha,L} \mathbb{N}^k$ and the other crossed products in the literature which are given via transfer operators; namely, the non-unital version of Exel's crossed product [2], Exel and Royer's crossed product by a partial endomorphism [10], and Larsen's crossed product for semigroups [17]. The upshot of this discussion is that, when these crossed products can be defined, they coincide with $C_0(\partial\Lambda) \rtimes_{\alpha,L} \mathbb{N}^k$. To be make things clear, we use the following notation.

NOTATION 4.3. (i) For $(A, \beta, \mathcal{L})$ a dynamical system in the sense of Exel and Royer [10] we denote by $A \rtimes^{\mathrm{ER}}_{\beta,\mathcal{L}} \mathbb{N}$ the crossed product given in Definition 1.6 of [10].

(ii) For $(A, \beta, \mathcal{L})$ a dynamical system in the sense of [2], [6] we denote by $A \rtimes^{\mathrm{BRV}}_{\beta,\mathcal{L}} \mathbb{N}$ the crossed product given in Section 4 of [2].

(iii) For $P$ an abelian semigroup and $(A, P, \beta, \mathcal{L})$ a dynamical system in the sense of Larsen [17] we denote by $A \rtimes^{\mathrm{Lar}}_{\beta,\mathcal{L}} P$ the crossed product given in Definition 2.2 of [17].

4.1. DIRECTED GRAPHS. Suppose $\Lambda$ is a 1-graph. Then for each $\lambda, \mu \in \Lambda$ we have $|\Lambda^{\min}(\lambda, \mu)| \in \{0, 1\}$, and so $\Lambda$ is finitely aligned. As shown in Examples 10.1–10.2 of [20], $\Lambda$ is the path category of the directed graph $E := (\Lambda^0, d^{-1}(1), r, s)$. We know from Proposition B.1 of [22] that $C^*(\Lambda)$ coincides with the graph algebra $C^*(E)$ as given in [12]. We denote by $E^*$ the set of finite paths in $E$ and by $E^\infty$ the set of infinite paths in $E$. We define $E^*_{\mathrm{inf}} := \{\mu \in E^* : |r^{-1}(s(\mu))| = \infty\}$ and $E^*_{\mathrm{s}} := \{\mu \in E^* : r^{-1}(s(\mu)) = \emptyset\}$, so $E^*_{\mathrm{inf}}$ is the set of paths whose source is an infinite receiver, and $E^*_{\mathrm{s}}$ is the set of paths whose source is a source in $E$. Then the boundary-path space $\partial\Lambda$ coincides with $\partial E := E^\infty \cup E^*_{\mathrm{inf}} \cup E^*_{\mathrm{s}}$. We now freely use directed graphs $E$ in place of 1-graphs $\Lambda$ in Definition 4.1.

PROPOSITION 4.4. *Let $E$ be a directed graph. Then $(C_0(\partial E), \alpha, L)$ is a dynamical system in the sense of* [10], *and we have $C_0(\partial E) \rtimes_{\alpha,L} \mathbb{N} \cong C_0(\partial E) \rtimes^{\mathrm{ER}}_{\alpha,L} \mathbb{N}$.*

To prove this proposition we need the following result.

PROPOSITION 4.5. *Let $(A, \beta, \mathcal{L})$ be a dynamical system in the sense of* [10], *and consider the Hilbert $A$-bimodule $M$ constructed in Section* 1 *of* [10]. *Then $A \rtimes^{\mathrm{ER}}_{\beta,\mathcal{L}} \mathbb{N}$ is isomorphic to Katsura's Cuntz–Pimsner algebra $\mathcal{O}_M$* [13].

*Proof.* The arguments in Section 3 of [1] (or Section 4 of [2]) extend across to this setting, except $A \rtimes^{\mathrm{ER}}_{\beta,\mathcal{L}} \mathbb{N}$ is defined by modding out redundancies $(a, k)$ with $a \in (\ker \phi)^{\perp} \cap \phi^{-1}(\mathcal{K}(M))$ instead of $\overline{A\alpha(A)A} \cap \phi^{-1}(\mathcal{K}(M))$. But $(\ker \phi)^{\perp} \cap \phi^{-1}(\mathcal{K}(M))$ is precisely the ideal involved in Katsura's definition of $\mathcal{O}_M$ ([13], Definition 3.5). ∎

*Proof of Proposition* 4.4. The construction of the Hilbert $A$-bimodule $M$ from [10] gives $X_1$. We know from Proposition 5.3 of [23] that $\mathcal{NO}(X)$ is isomorphic to Katsura's $\mathcal{O}_{X_1}$. We know from Proposition 4.5 that $C_0(\partial E) \rtimes^{\mathrm{ER}}_{\alpha,L} \mathbb{N} \cong \mathcal{O}_M$. So we have

$$C_0(\partial E) \rtimes_{\alpha,L} \mathbb{N} = \mathcal{NO}(X) \cong \mathcal{O}_{X_1} = \mathcal{O}_M \cong C_0(\partial E) \rtimes^{\mathrm{ER}}_{\alpha,L} \mathbb{N}. \qquad \blacksquare$$

4.2. LOCALLY-FINITE DIRECTED GRAPHS WITH NO SOURCES. For a locally-finite directed graph $\Lambda := E$ with no sources we have $\partial E = E^{\infty}$. We denote by $\sigma$ the backward shift on $E^{\infty}$, and $\alpha_E$ the endomorphism of $C_0(E^{\infty})$ given by $\alpha_E(f) = f \circ \sigma$. So $\alpha_E = \alpha_1$. For each $f \in C_0(E^{\infty})$ we denote by $L_E(f)$ the function given by

$$L_E(f)(x) = \begin{cases} \frac{1}{|\sigma^{-1}(x)|} \sum\limits_{\sigma(y)=x} f(y) & \text{if } x \in \sigma(E^{\infty}), \\ 0 & \text{otherwise.} \end{cases}$$

So $L_E$ is the normalised version of $L_1$. It is proved in Section 2.1 of [2] that $L_E$ is a transfer operator for $(C_0(E^{\infty}), \alpha_E)$.

PROPOSITION 4.6. *Let $E$ be a locally-finite directed graph with no sources. Then we have $C_0(E^{\infty}) \rtimes_{\alpha,L} \mathbb{N} \cong C_0(E^{\infty}) \rtimes^{\mathrm{BRV}}_{\alpha_E,L_E} \mathbb{N}$.*

*Proof.* Recall the construction of the Hilbert $C_0(E^{\infty})$-bimodule $M_{L_E}$ ([2], Section 3), and in particular that $q : C_0(E^{\infty}) \to M_{L_E}$ denotes the quotient map. Since $E$ is locally finite, the shift $\sigma$ is proper. We can use this fact to find for each $x \in E^{\infty}$ an open neighbourhood $V$ of $\sigma(x)$ such that $|\sigma^{-1}(v)| = |\sigma^{-1}(\sigma(x))|$ for each $v \in V$, and it follows that the map $d : E^{\infty} \to \mathbb{C}$ given by $d(x) = \sqrt{|\sigma^{-1}(\sigma(x))|}$ is continuous. Straightforward calculations show that $U : C_c(E^{\infty}) \to M_{L_E}$ given by $U(f) = q(df)$ extends to an isomorphism of $X_1$ onto $M_{L_E}$. So $\mathcal{O}_{X_1} \cong \mathcal{O}_{M_{L_E}}$. Since $E$ has no sources, the homomorphism $\phi : C_0(E^{\infty}) \to \mathcal{L}(M_{L_E})$ giving the left action on $M_{L_E}$ is injective, and so $(\ker \phi)^{\perp} = C_0(E^{\infty})$. It then follows from

Corollary 4.2 of [2] that $C_0(E^\infty) \rtimes^{\mathrm{BRV}}_{\alpha_E, L_E} \mathbb{N} \cong \mathcal{O}_{M_{L_E}}$. Finally, we know from Proposition 5.3 of [23] that $\mathcal{NO}(X) \cong \mathcal{O}_{X_1}$, so we have

$$C_0(E^\infty) \rtimes_{\alpha,L} \mathbb{N} = \mathcal{NO}(X) \cong \mathcal{O}_{X_1} \cong \mathcal{O}_{M_{L_E}} \cong C_0(E^\infty) \rtimes^{\mathrm{BRV}}_{\alpha_E, L_E} \mathbb{N}. \qquad \blacksquare$$

4.3. REGULAR $k$-GRAPHS. We now examine how $C_0(\partial\Lambda) \rtimes_{\alpha,L} \mathbb{N}^k$ fits in with the theory of Larsen's semigroup crossed products [17].

If $\Lambda$ is a row-finite $k$-graph with no sources, then $\partial\Lambda$ is the set $\Lambda^\infty$ of all graph morphisms from $\Omega_{k,(\infty,\ldots,\infty)}$ to $\Lambda$, and the shift maps are everywhere defined. So $\alpha$ is an action by endomorphisms. We say a $k$-graph $\Lambda$ is *regular* if it is row-finite with no sources, and there exists $M_1, \ldots, M_k \in \mathbb{N} \setminus \{0\}$ such that for each $i \in \{1, \ldots, k\}$ we have $|\Lambda^{e_i} v| = M_i$ for all $v \in \Lambda^0$. For each $x \in \Lambda^\infty$ and $n \in \mathbb{N}^k$ define

$$\omega(n, x) := \left|\sigma_n^{-1}(\sigma_n(x))\right|^{-1} = \prod_{i=1}^{k} M_i^{-n_i}.$$

Then for each $f \in C_0(\Lambda^\infty)$ the map $\mathcal{L}_n(f)$ given by

$$\mathcal{L}_n(f)(x) = \begin{cases} \sum\limits_{\sigma_n(y)=x} \omega(n,y) f(y) & \text{if } x \in \sigma_n(\Lambda^\infty), \\ 0 & \text{otherwise,} \end{cases}$$

is a transfer operator for $(C_0(\Lambda^\infty), \alpha_n)$. Simple calculations show that

$$\sum_{\sigma_n(y)=x} \omega(n,y) = 1$$

for all $x \in \Lambda^\infty, n \in \mathbb{N}^k$, and that $\omega(m+n, x) = \omega(m,x)\omega(n, \sigma_m(x))$ for all $x \in \Lambda^\infty, m, n \in \mathbb{N}^k$. Hence Proposition 2.2 of [9], which still holds in the non-unital setting, gives an action $\mathcal{L}$ of $\mathbb{N}^k$ of transfer operators on $C_0(\Lambda^\infty)$. It follows that $(C_0(\Lambda^\infty), \mathbb{N}^k, \alpha, \mathcal{L})$ is a dynamical system in the sense of Larsen ([17], Section 2).

PROPOSITION 4.7. *Let $\Lambda$ be a regular $k$-graph. Then we have $C_0(\Lambda^\infty) \rtimes_{\alpha,L} \mathbb{N}^k \cong C_0(\Lambda^\infty) \rtimes^{\mathrm{Lar}}_{\alpha,\mathcal{L}} \mathbb{N}^k$.*

*Proof.* We apply the construction in Section 3.2 of [17] to $(C_0(\Lambda^\infty), \mathbb{N}^k, \alpha, \mathcal{L})$ to form a product system $M = \bigcup\limits_{n \in \mathbb{N}^k} M_{\mathcal{L}_n}$, and then Proposition 4.3 of [17] says $C_0(\Lambda^\infty) \rtimes^{\mathrm{Lar}}_{\alpha,\mathcal{L}} \mathbb{N}^k$ is isomorphic to Fowler's Cuntz–Pimsner algebra $\mathcal{O}(M)$ ([11], Proposition 2.9). Suppose $M_1, \ldots, M_k \in \mathbb{N} \setminus \{0\}$ such that for each $i \in \{1, \ldots, k\}$ we have $|\Lambda^{e_i} v| = M_i$ for all $v \in \Lambda^0$. For each $n \in \mathbb{N}^k$ denote $M_n := \prod\limits_{i=1}^{k} M_i^{-n_i}$. Then the map $f \mapsto q_n(\sqrt{M_n} f)$ from $C_c(\Lambda^\infty)$ to $M_{\mathcal{L}_n}$ extends to an isomorphism of $X_n$ onto $M_{\mathcal{L}_n}$. These maps induce an isomorphism of the product systems $X$ and $M$ (observe the formulae for multiplication within $X$ (Proposition 2.2) and $M$ ([17], Equation 3.8). So $\mathcal{O}(X) \cong \mathcal{O}(M)$.

Recall that each $X_n$ is constructed from the topological graph $(\Lambda^\infty, \Lambda^\infty, \sigma_n, \iota)$, where $\iota$ is the inclusion map. It then follows from Proposition 1.24 of [14] that each $\phi_n$ is injective and acts by compact operators. So we can apply Corollary 5.2 of [23] to see that $\mathcal{NO}(X)$ coincides with $\mathcal{O}(X)$. So we have

$$C_0(\Lambda^\infty) \rtimes_{\alpha,L} \mathbb{N}^k = \mathcal{NO}(X) = \mathcal{O}(X) \cong \mathcal{O}(M) \cong C_0(\Lambda^\infty) \rtimes^{\mathrm{Lar}}_{\alpha,\mathcal{L}} \mathbb{N}^k. \quad \blacksquare$$

4.4. CONCLUSION. The results in this section justify our decision to define the crossed product $C_0(\partial\Lambda) \rtimes_{\alpha,L} \mathbb{N}^k$ to be the Cuntz–Nica–Pimsner algebra $\mathcal{NO}(X)$, and we propose that the same definition is made for a general crossed product by a quasi-lattice ordered semigroup of partial endomorphisms and partially-defined transfer operators. The problem is that Sims and Yeend's Cuntz–Nica–Pimsner algebra is only appropriate for a particular family (containing $\mathbb{N}^k$) of quasi-lattice ordered semigroups. The "correct" definition of a Cuntz–Pimsner algebra of a product system over an arbitrary quasi-lattice ordered semigroup is yet to be found. (See [23], [3] for more discussion.)

## 5. APPENDIX

Recall that for $(G, P)$ a quasi-lattice ordered group, and $X$ a product system over $P$ of Hilbert bimodules, we say that $X$ is *compactly-aligned* if for all $p, q \in P$ such that $p \vee q < \infty$, and for all $S \in \mathcal{K}(X_p)$ and $T \in \mathcal{K}(X_q)$, we have $\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T) \in \mathcal{K}(X_{p\vee q})$.

PROPOSITION 5.1. *The product system $X$ constructed in Section 2 is compactly-aligned.*

We start with a definition and some notation.

DEFINITION 5.2. Let $n \in \mathbb{N}^k$. We say that a subset $\mathcal{J} \subseteq \mathcal{A}^n$ is *disjoint* if

$$(\lambda, F), (\mu, G) \in \mathcal{J} \text{ with } (\lambda, F) \neq (\mu, G) \Longrightarrow (D_\lambda \setminus D_{\lambda F}) \cap (D_\mu \setminus D_{\mu G}) = \emptyset.$$

For $(\lambda, F), (\mu, G) \in \mathcal{A}^n$ we write

$$\Theta_{(\lambda,F),(\mu,G)} := \Theta_{\chi_{D_\lambda \setminus D_{\lambda F}}, \chi_{D_\mu \setminus D_{\mu G}}} \in \mathcal{K}(X_n).$$

Let $m, n \in \mathbb{N}^k$. To prove Proposition 5.1 we first need to show that for each $(\lambda_1, F_1), (\lambda_2, F_2) \in \mathcal{A}^m$ and $(\mu_1, G_1), (\mu_2, G_2) \in \mathcal{A}^n$ we have

$$\iota_m^{m\vee n}(\Theta_{(\lambda_1,F_1),(\lambda_2,F_2)})\iota_n^{m\vee n}(\Theta_{(\mu_1,G_1),(\mu_2,G_2)}) \in \mathcal{K}(X_{m\vee n}).$$

We do this by finding for each $(\alpha,\beta) \in \Lambda^{\min}(\lambda_2,\mu_1)$ finite subsets $\mathcal{H}_{(\alpha,\beta)}, \mathcal{J}_{(\alpha,\beta)} \subseteq \mathcal{A}^{m\vee n}$ such that $\bigsqcup_{(\alpha,\beta)} \mathcal{H}_{(\alpha,\beta)}$ and $\bigsqcup_{(\alpha,\beta)} \mathcal{J}_{(\alpha,\beta)}$ are disjoint, and

$$\iota_m^{m\vee n}(\Theta_{(\lambda_1,F_1),(\lambda_2,F_2)})\iota_n^{m\vee n}(\Theta_{(\mu_1,G_1),(\mu_2,G_2)}) = \sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda_2,\mu_1)} \sum_{\substack{(\kappa,H)\in\mathcal{H}_{(\alpha,\beta)} \\ (\omega,J)\in\mathcal{J}_{(\alpha,\beta)}}} \Theta_{(\kappa,H),(\omega,J)}. \tag{5.1}$$

To find the correct $\mathcal{H}_{(\alpha,\beta)}$ and $\mathcal{J}_{(\alpha,\beta)}$, we evaluate both sides of (5.1) on products $fg$, where $f \in C_c(\partial\Lambda^{\geqslant n})$ and $g \in C_c(\partial\Lambda^{\geqslant m\vee n-n})$. For the left-hand-side of (5.1) we use (1.1) and Corollary 2.8 to factor

$$\iota_n^{m\vee n}(\Theta_{(\mu_1,G_1),(\mu_2,G_2)})(fg) = \Theta_{(\mu_1,G_1),(\mu_2,G_2)}(f)g = hl,$$

where $h \in C_c(\partial\Lambda^{\geqslant m})$ and $l \in C_c(\partial\Lambda^{\geqslant m\vee n-m})$. Then for $x \in \partial\Lambda^{\geqslant m\vee n}$ we have

$$\begin{aligned}
&\iota_m^{m\vee n}(\Theta_{(\lambda_1,F_1),(\lambda_2,F_2)})\iota_n^{m\vee n}(\Theta_{(\mu_1,G_1),(\mu_2,G_2)})(fg)(x)\\
&= \iota_m^{m\vee n}(\Theta_{(\lambda_1,F_1),(\lambda_2,F_2)})(hl)(x) = \Theta_{(\lambda_1,F_1),(\lambda_2,F_2)}(h)l(x)\\
&= \mathcal{X}_{D_{\lambda_1}\setminus D_{\lambda_1 F_1}}(x)\langle \mathcal{X}_{D_{\lambda_2}\setminus D_{\lambda_2 F_2}}, h\rangle_m(\sigma_m(x))l(\sigma_m(x))\\
&= \mathcal{X}_{D_{\lambda_1}\setminus D_{\lambda_1 F_1}}(x)\Big(\sum_{\sigma_m(y)=\sigma_m(x)} \overline{\mathcal{X}_{D_{\lambda_2}\setminus D_{\lambda_2 F_2}}(y)}h(y)\Big)l(\sigma_m(x))\\
&= \begin{cases} hl(\lambda_2(0,m)\sigma_m(x)) & \text{if } x \in (D_{\lambda_1}\setminus D_{\lambda_1 F_1}) \cap \sigma_m^{-1}(D_{\lambda_2(m,d(\lambda_2))}\setminus D_{\lambda_2(m,d(\lambda_2))F_2}),\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

A similar calculation to the one above gives

$$\begin{aligned}
&hl(\lambda_2(0,m)\sigma_m(x))\\
&= \Theta_{(\mu_1,G_1),(\mu_2,G_2)}(f)g(\lambda_2(0,m)\sigma_m(x))\\
&= \begin{cases} fg(\mu_2(0,n)\sigma_n(\lambda_2(0,m)\sigma_m(x))) & \text{if } \lambda_2(0,m)\sigma_m(x) \in (D_{\mu_1}\setminus D_{\mu_1 G_1})\cap \\ & \qquad \sigma_n^{-1}(D_{\mu_2(n,d(\mu_2))}\setminus D_{\mu_2(n,d(\mu_2))G_2}),\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

So we label conditions

$$x \in (D_{\lambda_1}\setminus D_{\lambda_1 F_1}) \cap \sigma_m^{-1}(D_{\lambda_2(m,d(\lambda_2))}\setminus D_{\lambda_2(m,d(\lambda_2))F_2}), \text{ and} \tag{5.2}$$

$$\lambda_2(0,m)\sigma_m(x) \in (D_{\mu_1}\setminus D_{\mu_1 G_1}) \cap \sigma_n^{-1}(D_{\mu_2(n,d(\mu_2))}\setminus D_{\mu_2(n,d(\mu_2))G_2}), \tag{5.3}$$

and then we have

$$\begin{aligned}
&\iota_m^{m\vee n}(\Theta_{(\lambda_1,F_1),(\lambda_2,F_2)})\iota_n^{m\vee n}(\Theta_{(\mu_1,G_1),(\mu_2,G_2)})(fg)(x)\\
&\qquad = \begin{cases} fg(\mu_2(0,n)\sigma_n(\lambda_2(0,m)\sigma_m(x))) & \text{if } x \text{ satisfies (5.2) and(5.3)},\\ 0 & \text{otherwise.}\end{cases}
\end{aligned} \tag{5.4}$$

Now, for each $(\alpha,\beta)\in\Lambda^{\min}(\lambda_2,\mu_1)$, $(\kappa,H)\in\mathcal{H}_{(\alpha,\beta)}$ and $(\omega,J)\in\mathcal{J}_{(\alpha,\beta)}$ we have

$$\begin{aligned}
&\Theta_{(\kappa,H),(\omega,J)}(fg)(x)\\
&\qquad=\mathcal{X}_{D_\kappa\setminus D_{\kappa H}}(x)\langle\mathcal{X}_{D_\omega\setminus D_{\omega J}},fg\rangle_{m\vee n}(\sigma_{m\vee n}(x))\\
&\qquad=\mathcal{X}_{D_\kappa\setminus D_{\kappa H}}(x)\Big(\sum_{\sigma_{m\vee n}(y)=\sigma_{m\vee n}(x)}\overline{\mathcal{X}_{D_\omega\setminus D_{\omega J}}(y)}fg(y)\Big)\\
&\qquad=\begin{cases} fg(\tau(0,m\vee n)\sigma_{m\vee n}(x)) & \text{if } x\in(D_\kappa\setminus D_{\kappa H})\cap\sigma_{m\vee n}^{-1}(D_\omega\setminus D_{\omega J}),\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Since $\bigsqcup_{(\alpha,\beta)}\mathcal{H}_{(\alpha,\beta)}$ and $\bigsqcup_{(\alpha,\beta)}\mathcal{J}_{(\alpha,\beta)}$ are disjoint, we have

$$(5.5)\quad \Big(\sum_{(\alpha,\beta)\in\Lambda^{\min}(\lambda_2,\mu_1)}\sum_{\substack{(\kappa,H)\in\mathcal{H}\\(\tau,J)\in\mathcal{J}}}\Theta_{(\kappa,H),(\omega,J)}\Big)(fg)(x)$$
$$=\begin{cases} fg(\tau(0,m\vee n)\sigma_{m\vee n}(x)) & \text{if } x\in\bigsqcup_{\substack{(\alpha,\beta)\\(\kappa,H),(\omega,J)}}(D_\kappa\setminus D_{\rho H})\cap\sigma_{m\vee n}^{-1}(D_\omega\setminus D_{\omega J}),\\ 0 & \text{otherwise.}\end{cases}$$

Equation (5.1) now follows from (5.4), (5.5) and the following lemma.

LEMMA 5.3. *Let $m,n\in\mathbb{N}^k$, and suppose the pairs $(\lambda_1,F_1),(\lambda_2,F_2)\in\mathcal{A}^m$ and $(\mu_1,G_1),(\mu_2,G_2)\in\mathcal{A}^n$. Then for each pair $(\alpha,\beta)\in\Lambda^{\min}(\lambda_2,\mu_1)$ there exists finite and disjoint subsets $\mathcal{H}_{(\alpha,\beta)},\mathcal{J}_{(\alpha,\beta)}\subseteq\mathcal{A}^{m\vee n}$ such that $x\in\partial\Lambda^{\geqslant m\vee n}$ satisfies equations* (5.2) *and* (5.3) *if and only if*

$$(5.6)\qquad x\in\bigsqcup_{(\alpha,\beta)\in\Lambda^{\min}(\lambda_2,\mu_1)}\ \bigsqcup_{\substack{(\kappa,H)\in\mathcal{H}_{(\alpha,\beta)}\\(\omega,J)\in\mathcal{J}_{(\alpha,\beta)}}}(D_\kappa\setminus D_{\kappa H})\cap\sigma_{m\vee n}^{-1}(D_\omega\setminus D_{\omega J}).$$

*Moreover, if $x$ satisfies* (5.2) *and* (5.3) *and $x\in(D_\kappa\setminus D_{\kappa H})\cap\sigma_{m\vee n}^{-1}(D_\omega\setminus D_{\omega J})$, then we have*

$$\mu_2(0,n)\sigma_n(\lambda_2(0,m)\sigma_m(x))=\omega(0,m\vee n)\sigma_{m\vee n}(x).$$

*Proof.* Recall that for $\lambda,\mu\in\Lambda$ we denote by

$$F(\lambda,\mu)=\{\alpha\in\Lambda:(\alpha,\beta)\in\Lambda^{\min}(\lambda,\mu)\text{ for some }\beta\in\Lambda\}.$$

Let $(\alpha,\beta)\in\Lambda^{\min}(\lambda_2,\mu_1)$. For each $(\gamma,\delta)\in\Lambda^{\min}(\lambda_1(m,d(\lambda_1)),\lambda_2(m,d(\lambda_2))\alpha)$ we define

$$\begin{aligned}H_{\gamma,\alpha}:=\Big(\bigcup_{\nu\in F_1}F(\lambda_1\gamma,\lambda_1\nu)\Big)&\cup\Big(\bigcup_{\zeta\in F_2}F(\lambda_2(m,d(\lambda_2))\alpha\delta,\lambda_2(m,d(\lambda_2))\zeta)\Big)\\
&\cup\Big(\bigcup_{\eta\in G_1}F(\mu_1\beta\delta,\mu_1\eta)\Big),\end{aligned}$$

and

$$\mathcal{H}_{(\alpha,\beta)} := \{(\lambda_1\gamma, H_{\gamma,\alpha}) \in \mathcal{A}^{m\vee n} : (\gamma,\delta) \in \Lambda^{\min}(\lambda_1(m,d(\lambda_1)), \lambda_2(m,d(\lambda_2))\alpha)\}.$$

For each $(\rho,\tau) \in \Lambda^{\min}(\mu_2(n,d(\mu_2)), \mu_1(n,d(\mu_1))\beta)$ we define

$$J_{\rho,\beta} := \Big(\bigcup_{\xi\in G_2} F(\mu_2\rho, \mu_2\xi)\Big) \cup \Big(\bigcup_{\eta\in G_1} F(\mu_1(n,d(\mu_1))\beta\tau, \mu_1(n,d(\mu_1))\eta)\Big) \cup \Big(\bigcup_{\zeta\in F_2} F(\lambda_2\alpha\tau, \lambda_2\zeta)\Big),$$

and

$$\mathcal{J}_{(\alpha,\beta)} := \{(\mu_2\rho, H_{\rho,\beta}) \in \mathcal{A}^{m\vee n} : (\rho,\tau) \in \Lambda^{\min}(\mu_2(n,d(\mu_2)), \mu_1(n,d(\mu_1))\beta)\}.$$

The sets $\mathcal{H}_{(\alpha,\beta)}$ and $\mathcal{J}_{(\alpha,\beta)}$ are finite sets because $\Lambda$ is finitely-aligned. Since the paths in the elements of $\mathcal{H}_{(\alpha,\beta)}$ are of the same length, the factorisation property ensures that each $\mathcal{H}_{(\alpha,\beta)}$ is disjoint. For the same reason, each $\mathcal{J}_{(\alpha,\beta)}$ is disjoint. This explains why the second union in (5.6) is a disjoint union. Moreover, the sets $\bigsqcup_{(\alpha,\beta)} \mathcal{H}_{(\alpha,\beta)}$ and $\bigsqcup_{(\alpha,\beta)} \mathcal{J}_{(\alpha,\beta)}$ are disjoint, and hence why the first union in (5.6) is a disjoint union.

To prove the 'only if' part of the statement, we assume $x \in \partial\Lambda^{\geqslant m\vee n}$ satisfies (5.2) and (5.3). We have to find pairs

$$\begin{aligned}&(\alpha,\beta) \in \Lambda^{\min}(\lambda_2,\mu_1),\\ &(\gamma,\delta) \in \Lambda^{\min}(\lambda_1(m,d(\lambda_1)), \lambda_2(m,d(\lambda_2))\alpha), \text{ and}\\ &(\rho,\tau) \in \Lambda^{\min}(\mu_2(n,d(\mu_2)), \mu_1(n,d(\mu_1))\beta)\end{aligned}$$

such that

(a) $x \in D_{\lambda_1\gamma} \setminus D_{\lambda_1\gamma H_{\gamma,\alpha}}$, and
(b) $\sigma_{m\vee n}(x) \in D_{\mu_2\rho(m\vee n, d(\mu_2\rho))} \setminus D_{\mu_2\rho(m\vee n,d(\mu_2\rho))J_{\rho,\beta}}$.

Now, we know from (5.2) and (5.3) that $\lambda_2(0,m)\sigma_m(x) \in D_{\lambda_2} \cap D_{\mu_1}$, so we take

$$(\alpha,\beta) := (\lambda_2(0,m)\sigma_m(x)^{\mu_1}_{\lambda_2}, \lambda_2(0,m)\sigma_m(x)^{\lambda_2}_{\mu_1}) \in \Lambda^{\min}(\lambda_2,\mu_1). \tag{5.7}$$

We know from (5.2) and (5.3) that $\sigma_m(x) \in D_{\lambda_1(m,d(\lambda_1))} \cap D_{\lambda_2(m,d(\lambda_2))\alpha}$, so we define $(\gamma,\delta)$ to be the pair

$$(\sigma_m(x)^{\lambda_2(m,d(\lambda_2))\alpha}_{\lambda_1(m,d(\lambda_1))}, \sigma_m(x)^{\lambda_1(m,d(\lambda_1))}_{\lambda_2(m,d(\lambda_2))\alpha}) \in \Lambda^{\min}(\lambda_1(m,d(\lambda_1)), \lambda_2(m,d(\lambda_2))\alpha). \tag{5.8}$$

We now have $\sigma_m(x) \in D_{\lambda_1(m,d(\lambda_1))\gamma}$, and this along with (5.2) implies that $x \in D_{\lambda_1\gamma}$. We also have

$$x \in D_{\lambda_1\gamma} \text{ and } x \notin D_{\lambda_1 F_1} \Longrightarrow x \notin D_{\lambda_1\gamma\nu'} \text{ for all } \nu' \in \bigcup_{\nu\in F_1} F(\lambda_1\gamma, \lambda_1\nu); \tag{5.9}$$

$$\begin{aligned}
&\sigma_m(x) \in D_{\lambda_2(m,d(\lambda_2))\alpha\delta} \text{ and } \sigma_m(x) \notin D_{\lambda_2(m,d(\lambda_2))F_2}\\
&\Longrightarrow \sigma_m(x) \notin D_{\lambda_2(m,d(\lambda_2))\alpha\delta\zeta'} \text{ for all } \zeta' \in \bigcup_{\zeta\in F_2} F(\lambda_2(m,d(\lambda_2))\alpha\delta, \lambda_2(m,d(\lambda_2))\zeta)\\
&\Longleftrightarrow \sigma_m(x) \notin D_{\lambda_1(m,d(\lambda_1))\gamma\zeta'} \text{ for all } \zeta' \in \bigcup_{\zeta\in F_2} F(\lambda_2(m,d(\lambda_2))\alpha\delta, \lambda_2(m,d(\lambda_2))\zeta)
\end{aligned}$$

$$\Longleftrightarrow x \notin D_{\lambda_1\gamma\zeta'} \text{ for all } \bigcup_{\zeta\in F_2} F(\lambda_2(m,d(\lambda_2))\alpha\delta, \lambda_2(m,d(\lambda_2))\zeta); \tag{5.10}$$

and

$$\begin{aligned}
\lambda_2(0,m)\sigma_m(x) \in & D_{\lambda_2\alpha\delta} \text{ and } \lambda_2(0,m)\sigma_m(x) \notin D_{\mu_1 G_1}\\
&\Longrightarrow \lambda_2(0,m)\sigma_m(x) \notin D_{\lambda_2\alpha\delta\eta'} \text{ for all } \eta' \in \bigcup_{\eta\in G_1} F(\mu_1\beta\delta, \mu_1\eta)\\
&\Longleftrightarrow \sigma_m(x) \notin D_{\lambda_2(m,d(\lambda_2))\alpha\delta\eta'} \text{ for all } \eta' \in \bigcup_{\eta\in G_1} F(\mu_1\beta\delta, \mu_1\eta)\\
&\Longleftrightarrow \sigma_m(x) \notin D_{\lambda_1(m,d(\lambda_1))\gamma\eta'} \text{ for all } \eta' \in \bigcup_{\eta\in G_1} F(\mu_1\beta\delta, \mu_1\eta)
\end{aligned}$$

$$\Longleftrightarrow x \notin D_{\lambda_1\gamma\eta'} \text{ for all } \eta' \in \bigcup_{\eta\in G_1} F(\mu_1\beta\delta, \mu_1\eta). \tag{5.11}$$

It follows from (5.9), (5.10) and (5.11) that $x \notin D_{\lambda_1\gamma H_{\gamma,\alpha}}$, and so (a) is satisfied.

We have $\sigma_n(\lambda_2(0,m)\sigma_m(x)) \in D_{\mu_1(n,d(\mu_1))\beta}$, and it follows from (5.3) that $\sigma_n(\lambda_2(0,m)\sigma_m(x)) \in D_{\mu_2(n,d(\mu_2))}$. So we take

$$\begin{aligned}
(\rho,\tau) &:= (\sigma_n(\lambda_2(0,m)\sigma_m(x))^{\mu_1(n,d(\mu_1))\beta}_{\mu_2(n,d(\mu_2))}, \sigma_n(\lambda_2(0,m)\sigma_m(x))^{\mu_2(n,d(\mu_2))}_{\mu_1(n,d(\mu_1))\beta})\\
&\in \Lambda^{\min}(\mu_2(n,d(\mu_2)), \mu_1(n,d(\mu_1))\beta),
\end{aligned} \tag{5.12}$$

and we have

$$\begin{aligned}
\sigma_n(\lambda_2(0,m)\sigma_m(x)) \in & D_{\mu_2(n,d(\mu_2))\rho}\\
&\Longrightarrow \sigma_{m\vee n}(x) = \sigma_{m\vee n}(\lambda_2(0,m)\sigma_m(x)) \in D_{\mu_2\rho(m\vee n, d(\mu_2\rho))}.
\end{aligned}$$

Suppose for contradiction that there exists $\xi \in G_2$ and a pair $(\xi', \xi'')$ in the set $\Lambda^{\min}(\mu_2\rho, \mu_2\xi)$ with $\sigma_{m\vee n}(x) \in D_{\mu_2\rho(m\vee n, d(\mu_2\rho))\xi'}$. Then it follows from (5.12) that

$$\begin{aligned}
\sigma_n(\lambda_2(0,m)\sigma_m(x)) &= \sigma_n(\lambda_2(0,m)\sigma_m(x))(0, m\vee n - n)\sigma_{m\vee n}(x)\\
&= \mu_2(n,d(\mu_2))\rho(0, m\vee n - n)\sigma_{m\vee n}(x) = \mu_2\rho(n, m\vee n)\sigma_{m\vee n}(x)\\
&\in D_{\mu_2(n,d(\mu_2))\rho\xi'} = D_{\mu_2(n,d(\mu_2))\xi\xi''} \subseteq D_{\mu_2(n,d(\mu_2))G_2}.
\end{aligned}$$

This contradicts equation (5.3), and so we must have

$$\sigma_{m\vee n}(x) \notin D_{\mu_2\rho(m\vee n,d(\mu_2\rho))\xi'} \text{ for all } \xi' \in \bigcup_{\xi\in G_2} F(\mu_2\rho,\mu_2\xi). \tag{5.13}$$

Similar arguments show that

$$\sigma_{m\vee n}(x)\notin D_{\mu_2\rho(m\vee n,d(\mu_2\rho))\eta'}, \text{ for all } \eta'\in\bigcup_{\eta\in G_1} F(\mu_1(n,d(\mu_1))\beta\tau,\mu_1(n,d(\mu_1))\eta), \tag{5.14}$$

and

$$\sigma_{m\vee n}(x) \notin D_{\mu_2\rho(m\vee n,d(\mu_2\rho))\eta'}, \text{ for all } \zeta' \in \bigcup_{\zeta\in F_2} F(\lambda_2\alpha\tau,\lambda_2\zeta). \tag{5.15}$$

It follows from (5.13), (5.14) and (5.15) that $\sigma_{m\vee n}(x) \notin D_{\mu_2\rho(m\vee n,d(\mu_2\rho))J_{\rho,\beta}}$, and so (b) is satisfied.

To prove the "if" part of the statement, we assume there exists

$$\begin{aligned}
&(\alpha,\beta) \in \Lambda^{\min}(\lambda_2,\mu_1),\\
&(\gamma,\delta) \in \Lambda^{\min}(\lambda_1(m,d(\lambda_1)),\lambda_2(m,d(\lambda_2))\alpha), \text{ and}\\
&(\rho,\tau) \in \Lambda^{\min}(\mu_2(n,d(\mu_2)),\mu_1(n,d(\mu_1))\beta),
\end{aligned}$$

such that

$$x \in (D_{\lambda_1\gamma} \setminus D_{\lambda_1\gamma H_{\gamma,\alpha}}) \cap \sigma^{-1}_{m\vee n}(D_{\mu_2\rho(m\vee n,d(\mu_2\rho))} \setminus D_{\mu_2\rho(m\vee n,d(\mu_2\rho))J_{\rho,\beta}}).$$

We have

$$\begin{aligned}
x \in D_{\lambda_1\gamma} \setminus D_{\lambda_1\gamma H_{\gamma,\alpha}} &\Longrightarrow x \in D_{\lambda_1} \setminus D_{\lambda_1 F_1}, \text{ and}\\
x \in D_{\lambda_1\gamma} \setminus D_{\lambda_1\gamma H_{\gamma,\alpha}} &\Longrightarrow \sigma_m(x) \in D_{\lambda_1(m,d(\lambda_1))\gamma} \setminus D_{\lambda_1(m,d(\lambda_1))\gamma H_{\gamma,\alpha}}\\
&\Longleftrightarrow \sigma_m(x) \in D_{\lambda_2(m,d(\lambda_2))\alpha\delta} \setminus D_{\lambda_2(m,d(\lambda_2))\alpha\delta H_{\gamma,\alpha}}\\
&\Longleftrightarrow \sigma_m(x) \in D_{\lambda_2(m,d(\lambda_2))} \setminus D_{\lambda_2(m,d(\lambda_2))F_2}.
\end{aligned}$$

So (5.2) is satisfied. We have

$$\begin{aligned}
x \in D_{\lambda_1\gamma} \setminus D_{\lambda_1\gamma H_{\gamma,\alpha}} &\Longrightarrow \sigma_m(x) \in D_{\lambda_1(m,d(\lambda_1))\gamma} \setminus D_{\lambda_1(m,d(\lambda_1))\gamma H_{\gamma,\alpha}}\\
&\Longleftrightarrow \sigma_m(x) \in D_{\lambda_2(m,d(\lambda_2))\alpha\delta} \setminus D_{\lambda_2(m,d(\lambda_2))\alpha\delta H_{\gamma,\alpha}}\\
&\Longrightarrow \lambda_2(0,m)\sigma_m(x) \in D_{\lambda_2\alpha\delta} \setminus D_{\lambda_2\alpha\delta H_{\gamma,\alpha}}\\
&\Longleftrightarrow \lambda_2(0,m)\sigma_m(x) \in D_{\mu_1\beta\delta} \setminus D_{\mu_1\beta\delta H_{\gamma,\alpha}}\\
&\Longleftrightarrow \lambda_2(0,m)\sigma_m(x) \in D_{\mu_1} \setminus D_{\mu_1 G_1}.
\end{aligned}$$

We have

$$\begin{aligned}
x \in D_{\lambda_1\gamma} \Longrightarrow\ & \lambda_2(0,m)\sigma_m(x)(n,m\vee n)\\
&= (\lambda_2(0,m)\lambda_1(m,d(\lambda_1))\gamma)(n,m\vee n)\\
&= (\lambda_2(0,m)\lambda_2(m,d(\lambda_2))\alpha\delta)(n,m\vee n)\\
&= \lambda_2\alpha\delta(n,m\vee n) = \lambda_2\alpha(n,m\vee n) = \mu_1\beta(n,m\vee n)\\
&= (\mu_1(n,d(\mu_1))\beta)(n,m\vee n) = (\mu_1(n,d(\mu_1))\beta\tau)(n,m\vee n)\\
&= (\mu_2(n,d(\mu_2))\rho)(n,m\vee n).
\end{aligned}$$

It follows that

$$\begin{aligned}
\sigma_n(\lambda_2(0,m)\sigma_m(x)) &= (\lambda_2(0,m)\sigma_m(x))(n,m\vee n)\sigma_{m\vee n}(\lambda_2(0,m)\sigma_m(x))\\
&= (\lambda_2(0,m)\sigma_m(x))(n,m\vee n)\sigma_{m\vee n}(x)\\
&= (\mu_2(n,d(\mu_2))\rho)(n,m\vee n)\sigma_{m\vee n}(x),
\end{aligned}$$

and then we have

$$\begin{aligned}
\sigma_{m\vee n}(x) \in & D_{\mu_2\rho(m\vee n,d(\mu_2\rho))} \setminus D_{\mu_2\rho(m\vee n,d(\mu_2\rho))J_{\rho,\beta}}\\
&\Longrightarrow \sigma_n(\lambda_2(0,m)\sigma_m(x)) \in D_{\mu_2\rho(n,d(\mu_2\rho))} \setminus D_{\mu_2\rho(n,d(\mu_2\rho))J_{\rho,\beta}}\\
&\Longrightarrow \sigma_n(\lambda_2(0,m)\sigma_m(x)) \in D_{\mu_2(n,d(\mu_2))} \setminus D_{\mu_2(n,d(\mu_2))G_2}.
\end{aligned}$$

So (5.3) is satisfied.

To prove the final part of the result, recall that, given $x \in \partial\Lambda^{\geqslant m\vee n}$ satisfying (5.2) and (5.3), we have the following formula for the pair $(\rho,\tau)$ in the set $\Lambda^{\min}(\mu_2(n,d(\mu_2)),\mu_1(n,d(\mu_1))\beta)$:

$$(\rho,\tau) = (\sigma_n(\lambda_2(0,m)\sigma_m(x))^{\mu_1(n,d(\mu_1))\beta}_{\mu_2(n,d(\mu_2))}, \sigma_n(\lambda_2(0,m)\sigma_m(x))^{\mu_2(n,d(\mu_2))}_{\mu_1(n,d(\mu_1))\beta}).$$

We then have

$$\begin{aligned}
\mu_2(0,n)\sigma_n(\lambda_2(0,m)\sigma_m(x)) &= \mu_2(0,n)(\sigma_n(\lambda_2(0,m)\sigma_m(x)))(0,m\vee n-n)\sigma_{m\vee n}(x)\\
&= \mu_2(0,n)(\mu_2(n,d(\mu_2))\rho)(0,m\vee n-n)\sigma_{m\vee n}(x)\\
&= \mu_2\rho(0,m\vee n)\sigma_{m\vee n}(x). \quad ∎
\end{aligned}$$

*Proof of Proposition* 5.1. We have already established equation (5.1). Since $\Lambda$ is finitely-aligned, the sums in (5.1) are finite, and so

$$\iota_m^{m\vee n}(\Theta_{(\lambda_1,F_1),(\lambda_2,F_2)})\iota_n^{m\vee n}(\Theta_{(\mu_1,G_1),(\mu_2,G_2)}) \in \mathcal{K}(X_{m\vee n}),$$

for every $m,n \in \mathbb{N}^k$, $(\lambda_1,F_1),(\lambda_2,F_2) \in \mathcal{A}^m$ and $(\mu_1,G_1),(\mu_2,G_2) \in \mathcal{A}^n$. It then follows from Proposition 3.5 that $\iota_m^{m\vee n}(\Theta_{x_1,x_2})\iota_n^{m\vee n}(\Theta_{y_1,y_2}) \in \mathcal{K}(X_{m\vee n})$, for every $x_1,x_2 \in X_m$ and $y_1,y_2 \in X_n$. Hence, $\iota_m^{m\vee n}(S)\iota_n^{m\vee n}(T) \in \mathcal{K}(X_{m\vee n})$, for every $S \in \mathcal{K}(X_m)$ and $(T \in \mathcal{K}(X_n)$. ∎

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