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# WEIGHTED BMO AND TOEPLITZ OPERATORS ON THE BERGMAN SPACE $A^1$

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ABSTRACT. We generalize our recent results on the boundedness and compactness of Toeplitz operators to the Bergman space $A^1$ on the unit disc. While the natural condition for the symbols leads to $BMO_\partial$-type symbol classes, the inevitable logarithmic correction for the nonreflexive case requires a separate treatment of $BA$-and $BO$-type symbols.

KEYWORDS: *Toeplitz operator, boundedness, compactness, bounded mean oscillation, Bergman space $A^1$.*

MSC (2000): 47B35, 30H20.

## 1. INTRODUCTION

In this note we generalize to the case $p = 1$ our recent results in [8], describing a large class of unbounded symbols that generate bounded Toeplitz operators $T_a : A^p \to A^p$, $1 < p < \infty$ (see the end of the section for notation). Due to the discontinuity of the Bergman projection for $L^1$-norm, a logarithmic correction will be needed, in a way or another. Recall that the sufficient condition for the boundedness of $T_a : A^p \to A^p$ in [8] is a generalized bounded average ($BA$-) condition on hyperbolic discs. For the $p = 1$ case a rather obvious modification is to add a weight (logarithm of the boundary distance, see (3.3)) to the condition. However, this direct approach would lead to an unsatisfactory situation, since the constant functions do not have logarithmically vanishing averages over hyperbolic discs which approach the boundary, so they do not satisfy the emerging sufficient condition. The solution to this unexpected problem is, in addition to the $BA$-conditions, to consider separately also bounded oscillation ($BO$-) type conditions, see Proposition 2.4.

Section 2 is a generalization of [11], containing the necessary decomposition results of the weighted $BMO_{\partial,\log}$-functions to the $BA_{\log}$- and $BO_{\log}$-components. While the boundedness of a Toeplitz-operator $T_a : A^1 \to A^1$ for $a \in BMO_{\partial,\log} \cap$

$L^\infty$ is known, see e.g. [7], we give a new, very short proof in the special case $a \in BO_{\log} \cap L^\infty$. The main consideration of this work, the generalization of [8] to the case $p = 1$, is contained in Section 3. The main result is presented in Theorem 3.2.

As for the literature, the following works are related to Toeplitz and Hankel operators on $A^1$: [1], [2], [5], [6], [7], [9], [10], [11]. In particular we mention that [9] contains a sufficient condition for the boundedness of $T_a : A^1 \to A^1$ in terms of a new integral operator $R_\alpha$, which is however not so easy to compare with our more geometric condition.

We follow the notation and terminology of [12]. The spaces are defined over the open unit disc $\mathbb{D} = \{|z| < 1\}$ of the complex plane. The $L^p$-spaces, $1 \leqslant p \leqslant \infty$, with norm $\|\cdot\|_p$, are defined with respect to the normalized two-dimensional Lebesgue measure $\mathrm{d}A$. The Bergman space $A^p$ is the closed subspace of $L^p$ consisting of analytic functions. Bergman projection is the operator $P : f \mapsto \int_{\mathbb{D}} f(w)(1 - z\overline{w})^{-2}\mathrm{d}A(w)$. Given a *locally integrable* function $a : \mathbb{D} \to \mathbb{C}$, the Toeplitz operator $T_a$ with symbol $a$ is defined as $PM_a$, where $M_a$ is the pointwise multiplication with $a$. The Hankel operator $H_a$ is defined as $M_a - T_a$.

## 2. WEIGHTED BMO SPACES

The Bergman ball $D(z,r)$ with center $z$ and radius $r$ is defined by $D(z,r) = \{w \in \mathbb{D} : \beta(z,w) < r\}$, where $\beta(z,w)$ is the Bergman, or hyperbolic, metric. For a locally integrable function $f : \mathbb{D} \to \mathbb{C}$, the averaging function $\widehat{f}_r$ is defined by

$$\widehat{f}_r(z) = \frac{1}{|D(z,r)|} \int_{D(z,r)} f(w)\mathrm{d}A(w) \quad (z \in \mathbb{D}),$$

where $|D(z,r)|$ is the (Euclidean) area of $D(z,r)$. The space of bounded mean oscillation $BMO_r^p$ in the Bergman metric consists of all locally $L^p$ integrable functions for which its norm $\sup_{z\in\mathbb{D}} MO_r^p(f)(z)$ is finite, where

$$MO_r^p(f)(z) = \Big(\frac{1}{|D(z,r)|} \int_{D(z,r)} |f(w) - \widehat{f}_r(z)|^p \mathrm{d}A(w)\Big)^{1/p}.$$

As pointed out by K. Zhu [11], the definition of $BMO_r^p$ depends on $p$ (unlike in the case of the classical $BMO$ for the unit circle) and $BMO_r^p \subset BMO_r^q$ properly for $q < p$.

We also introduce the space of *logarithmic mean oscillation* $BMO^p_{r,\log}$ that consists of all functions $f$ in $BMO_r^p$ with finite norm

$$\sup_{z\in\mathbb{D}} \mathcal{W}(z) MO_r^p(f)(z) < \infty, \quad \text{where } \mathcal{W}(z) := 1 + \log \frac{1}{1 - |z|}.$$

For a continuous function $f$ on $\mathbb{D}$, define the oscillation $\omega_r(f)$ of $f$ by

$$\omega_r(f)(z) = \sup_{w\in D(z,r)} |f(z) - f(w)|, \quad z \in \mathbb{D}. \tag{2.1}$$

By $BO_r$ we denote the space of all continuous functions of bounded oscillation with the semi-norm $\sup_{z\in\mathbb{D}} |\omega_r(f)(z)|$. For the weight $\mathcal{W}$, the weighted $BO$ space, denoted by $BO_{r,\log}$, consists of all $f \in BO$ for which $\mathcal{W}(z)\omega_r(f)(z)$ is bounded on $\mathbb{D}$; the norm is defined by

$$\|f\|_{BO_{r,\log}} = \sup_{z\in\mathbb{D}} \mathcal{W}(z)\omega_r(f)(z)\,. \tag{2.2}$$

Concerning functions with bounded averages, we say $f \in BA_r^p$ if $\widehat{|f|_r^p}$ is bounded. We say that $f \in BA^p_{r,\log}$, if $\mathcal{W}^p\widehat{|f|_r^p}$ is bounded, and the corresponding norm is defined by

$$\|f\|_{BA^p_{r,\log}} = \sup_{z\in\mathbb{D}} \mathcal{W}(z)(\widehat{|f|_r^p})^{1/p}$$

which is comparable to the expression

$$\sup_{z\in\mathbb{D}} \Big(\frac{1}{|D(z,r)|}\int_{D(z,r)} (\mathcal{W}(w)|f(w)|)^p \mathrm{d}A(w)\Big)^{1/p}.$$

All of the spaces defined above can be shown to be independent of the number $r$, so that we can use the notations

$$BMO^p_\partial, \quad BMO^p_{\partial,\log}, \quad BA^p, \quad BA^p_{\log}, \quad BO, \quad BO_{\log},$$

instead of $BMO_r^p$, $BMO^p_{r,\log}$, $BA_r^p$, $BA^p_{r,\log}$, $BO_r$, $BO_{r,\log}$, respectively. As for the norms, any $r$ can be used, since the norms corresponding to different $r$ are equivalent.

PROPOSITION 2.1. *If $f \in BMO^p_{\partial,\log}$, then there exist functions $f_1 \in BO_{\log}$ and $f_2 \in BA^p_{\log}$ such that $f = f_1 + f_2$. These can be chosen such that $\|f_1\|_{BO_{\log}} + \|f_2\|_{BA^p_{\log}} \leqslant C\|f\|_{BMO^p_{\partial,\log}}$.*

*Proof.* This is similar to the proof of Theorem 5 in [11]. Let us fix $r > 0$ and let $f_1 = \widehat{f}_r$ and $f_2 = f - \widehat{f}_r$. Since $|D(z,r)|$, $|D(w,r)|$, and $|D(z,2r)|$ are all comparable for $w \in D(z,r)$ (see Lemma 4 of [11]) and since $D(z,r)$ and $D(w,r)$ are both contained in $D(z,2r)$ for $w \in D(z,r)$, we have, for $\beta(z,w) \leqslant r$,

$$\begin{aligned}\mathcal{W}(z)|\widehat{f}_r(z) - \widehat{f}_r(w)| &\leqslant \mathcal{W}(z)|\widehat{f}_r(z) - \widehat{f}_{2r}(z)| + \mathcal{W}(z)|\widehat{f}_r(w) - \widehat{f}_{2r}(z)| \\ &\leqslant \frac{C\mathcal{W}(z)}{|D(z,2r)|}\int_{D(z,2r)} |f(v) - \widehat{f}_{2r}(z)|\mathrm{d}A(v).\end{aligned}$$

Thus, Hölder's inequality shows that

$$\mathcal{W}(z)|\widehat{f}_r(z) - \widehat{f}_r(w)| \leqslant C\mathcal{W}(z)MO^p_{2r}(f)(z),$$

and so $f_1 \in BO_{r,\log}$. Concerning $f_2$, let $D := D(z,r)$, and estimate

$$\begin{aligned}
&\mathcal{W}(z)(\widehat{|f_2|^p_r}(z))^{1/p}\\
&\leqslant \mathcal{W}(z)\Big(\frac{1}{|D|}\int_D |f(v)-\widehat{f}_r(z)|^p \mathrm{d}A(v)\Big)^{1/p} + \Big(\frac{1}{|D|}\int_D (\mathcal{W}(z)|\widehat{f}_r(v)-\widehat{f}_r(z)|)^p \mathrm{d}A(v)\Big)^{1/p}\\
&\leqslant \mathcal{W}(z)MO^p_r(f)(z) + \mathcal{W}(z)\omega_r(\widehat{f}_r)(z),
\end{aligned}$$

which is bounded since $f \in BMO^p_{r,\log}$ and $\widehat{f}_r \in BO_{r,\log}$. ∎

We show that on the other hand $BA^p_{\log}$ and $BO_{\log}$ are contained in $BMO^p_{\partial,\log}$. If $f \in BA^p_{r,\log}$, then, just by the triangle inequality,

$$\begin{aligned}
\mathcal{W}(z)MO^p_r(f)(z) &= \mathcal{W}(z)\Big(\frac{1}{|D|}\int_D |f(v)-\widehat{f}_r(z)|^p \mathrm{d}A(v)\Big)^{1/p}\\
&\leqslant \mathcal{W}(z)\Big(\frac{1}{|D|}\int_D |f(v)|^p \mathrm{d}A(v)\Big)^{1/p} + \mathcal{W}(z)\Big(\frac{1}{|D|}\int_D |\widehat{f}_r(z)|^p \mathrm{d}A(v)\Big)^{1/p}\\
&= \mathcal{W}(z)(\widehat{|f|^p_r})^{1/p} + \mathcal{W}(z)|\widehat{f}_r|)^{1/p} \leqslant 2\|f\|_{BA^p_{\log}}, \quad \text{where } D := D(z,r).
\end{aligned}$$

So $BA^p_{\log} \subset BMO^p_{\partial,\log}$ with $\|f\|_{BMO^p_{\partial,\log}} \leqslant 2\|f\|_{BA^p_{\log}}$.

PROPOSITION 2.2. *For $1 \leqslant p < \infty$, we have $BMO^p_{\partial,\log} = BO_{\log} + BA^p_{\log}$. Moreover, if $f = f_1 + f_2$ with $f_1 \in BO_{\log}$ and $f_2 \in BA^p_{\log}$, then $\|f\|_{BMO^p_{\log}} \leqslant C\|f_1\|_{BO_{\log}} + C\|f_2\|_{BA^p_{\log}}$.*

*Proof.* By the above, we only need to consider $f \in BO_{\log}$:

$$\begin{aligned}
\mathcal{W}(z)MO^2_r(f)(z) &= \mathcal{W}(z)\Big(\frac{1}{|D|}\int_D \Big(\Big|f(w) - \frac{1}{|D|}\int_D f(v)\mathrm{d}A(v)\Big|\Big)^p \mathrm{d}A(w)\Big)^{1/p}\\
&= \Big(\frac{1}{|D|}\int_D \Big(\frac{1}{|D|}\Big|\int_D \mathcal{W}(z)(f(w) - f(v))\mathrm{d}A(v)\Big|\Big)^p \mathrm{d}A(w)\Big)^{1/p}\\
&\leqslant \|f\|_{BO_{\log}}, \quad \text{with } D := D(z,r),
\end{aligned}$$

by (2.2), hence, $BO_{\log} \subset BMO^p_{\partial,\log}$ and $\|f\|_{BMO^p_{\partial,\log}} \leqslant \|f\|_{BO_{\log}}$. ∎

REMARK 2.3. One can define spaces with *vanishing logarithmic* mean oscillation, average, or oscillation, analogously (which we denote by $VMO^p_{\partial,\log}$, $VA^p_{\log}$, and $VO_{\log}$, respectively). One can show that also for them the following decomposition holds:

$$VMO^p_{\partial,\log} = VO_{\log} + VA^p_{\log}.$$

We end this section with a couple of lemmas. It is known that Toeplitz operators with symbols in $BMO_{\partial,\log} \cap L^\infty$ are bounded on $A^1$ (see [7]). We give a very short proof in a special case, which complements our main proof in the next section. Let $\mathcal{B}$ be the Bloch space on the unit disc; for details, see [12].

PROPOSITION 2.4. *If $a \in BO_{\log} \cap L^\infty$, then $T_a : A^1 \to A^1$ is bounded .*

*Proof.* Let $f \in A^1$ and $h \in \mathcal{B}$ be arbitrary. Using the usual dual pairing $\langle f,h\rangle = \int_{\mathbb{D}} f\overline{h}$ between $A^1$ and $\mathcal{B}$, we write $\langle T_a f,h\rangle = \langle PM_{\overline{a}}Pf,h\rangle = \langle f,PM_{\overline{a}}h\rangle$. By [10], $M_{\overline{a}}$ is bounded $BMO_\partial \to BMO_\partial$ for $a \in BO_{\log} \cap L^\infty$, and also the projection $P : BMO_\partial \to \mathcal{B}$ is bounded. Hence, $|\langle T_a f,h\rangle| = |\langle f,PM_{\overline{a}}h\rangle| \leqslant C\|M_a\|\|f\|_{A^1}\|h\|_{\mathcal{B}}$. ∎

A similar proof gives the following, which we later apply for Hankel operators.

LEMMA 2.5. *If $a \in BA^1$ then $M_a : A^1 \to L^1$ is bounded.*

*Proof.* Let $g \in BA^1$, $f \in A^1$, and also $h \in L^\infty$, $\|h\|_\infty \leqslant 1$, and write $\langle M_a f,h\rangle = \langle Pf,\overline{a}h\rangle = \langle f,P(\overline{a}h)\rangle$. But we have $\overline{a}h \in BA^1$ with $\|\overline{a}h\|_{BA^1} \leqslant \|a\|_{BA^1}\|h\|_\infty \leqslant \|a\|_{BA^1}$. Moreover, $P : BA^1 \to \mathcal{B}$ is bounded (since it is bounded $BMO_\partial \to \mathcal{B}$). Hence, $P(\overline{a}h) \in \mathcal{B}$ with $\|P(\overline{a}h)\|_{\mathcal{B}} \leqslant C\|a\|_{BA^1}$. As a conclusion, using the duality of $A^1$ and $\mathcal{B}$,

$$\|M_a f\|_{L^1} = \sup_{\|h\|_\infty \leqslant 1}|\langle M_a f,h\rangle| = \sup_{\|h\|_\infty \leqslant 1}|\langle f,P(\overline{a}h)\rangle| \leqslant C\|f\|_{A^1}\|P(\overline{a}h)\|_{\mathcal{B}} \leqslant C\|f\|_{A^1}\|a\|_{BA^1}.$$

Thus, $M_a$ is bounded. ∎

## 3. ON THE BOUNDEDNESS OF TOEPLITZ OPERATORS

Let us introduce a collection $\mathcal{Q}$ of subsets $Q := Q(r,\theta)$ of $\mathbb{D}$ which are rectangles in polar coordinates and have a hyperbolic radius bounded from above and below:

$$Q = \left\{\rho e^{i\phi} : r \leqslant \rho \leqslant 1 - \frac{1}{2}(1-r)\ ,\ \theta \leqslant \phi \leqslant \theta + \pi(1-r)\right\} \tag{3.1}$$

for all $0 < r < 1$, $\theta \in [0,2\pi]$.

Given $Q = Q(r,\theta) \in \mathcal{Q}$ and $\zeta = \rho e^{i\phi} \in Q$, we denote

$$\widehat{a}_Q(\zeta) := \frac{1}{|Q|}\int_r^\rho \int_\theta^\phi a(\varrho e^{i\varphi})\varrho \, d\varphi \, d\varrho. \tag{3.2}$$

PROPOSITION 3.1. *Assume that $a \in L^1_{\rm loc}$ and that there exists a constant $C > 0$ such that*

$$|\widehat{a}_Q(\zeta)| \leqslant \frac{C}{\mathcal{W}(\zeta)} = \frac{C}{1 + |\log(1 - |\zeta|)|} \tag{3.3}$$

*for all $Q \in \mathcal{Q}$ and all $\zeta \in Q$. Then $T_a : A^1 \to A^1$ is well defined and bounded, and there is a constant C such that*

$$\|T_a\| \leqslant C\mathbf{C}_a \quad \textit{with} \ \ \mathbf{C}_a := \sup_{Q \in \mathcal{Q}, \zeta \in Q} |\widehat{a}_Q(\zeta)|\mathcal{W}(\zeta). \tag{3.4}$$

Since $Q$ is essentially a hyperbolic disc, the $\zeta$ on the right hand side of (3.3) can be replaced by an arbitrary point of $Q$.

*Proof.* The proof follows that of Theorem 2.3 of [8], so we shall be quite brief with the calculations here.

We use an explicit decomposition of $\mathbb{D}$ into sets belonging to the family $\mathcal{Q}$. Let us denote by $n := n(m, \mu)$ a bijection from the set $\{(m, \mu) : m \in \mathbb{N}, \mu = 1, \ldots, 2^{-m}\}$ onto $\mathbb{N}$ which preserves the order in the sense that $m < l \Rightarrow n(m, \mu) < n(l, \lambda)$ for all $\mu$ and $\lambda$, and $\mu < \lambda \Rightarrow n(m, \mu) < n(m, \lambda)$ for all $m$.

We define the sets

$$Q_n := Q_{n(m,\mu)} := \{z = r\mathrm{e}^{\mathrm{i}\theta} : r_n < r \leqslant r'_n, \theta_n < \theta \leqslant \theta'_n\}, \tag{3.5}$$

where $m \in \mathbb{N}$ and $\mu = 1, \ldots, 2^m$, and, for $n = n(m, \mu)$, $r_n := 1 - 2^{-m+1}$, $r'_n := 1 - 2^{-m}$, $\theta_n := \pi(\mu - 1)2^{-m+1}$, $\theta'_n := \pi\mu 2^{-m+1}$.

The assumption (3.3) and the definition of $r_n$ imply for all $n$ and all $\zeta = \rho\mathrm{e}^{\mathrm{i}\phi} \in Q_n$

$$|\widehat{a}_{Q_n}(\zeta)| = \frac{1}{|Q_n|} \Big| \int_{r_n}^{\rho} \int_{\theta_n}^{\phi} a(\varrho\mathrm{e}^{\mathrm{i}\varphi})\varrho \mathrm{d}\varrho \mathrm{d}\varphi \Big| \leqslant \frac{\mathbf{C}_a}{m}, \tag{3.6}$$

where $\mathbf{C}_a$ is as in (3.4).

Let $f \in A^1$ and fix an $n(m, \mu)$. We perform the same integration by parts as in [8] as follows

$$\begin{aligned}
&\int_{Q_n} \frac{a(\zeta)f(\zeta)}{(1 - z\overline{\zeta})^2} \mathrm{d}A(\zeta) \\
&= \int_{r_n}^{r'_n} \int_{\theta_n}^{\theta'_n} \frac{a(r\mathrm{e}^{\mathrm{i}\theta})f(r\mathrm{e}^{\mathrm{i}\theta})}{(1 - zr\mathrm{e}^{-\mathrm{i}\theta})^2} r\mathrm{d}r\mathrm{d}\theta \\
&= \Big( \int_{r_n}^{r'_n} \int_{\theta_n}^{\theta'_n} a(\varrho\mathrm{e}^{\mathrm{i}\varphi})\varrho\mathrm{d}\varphi\mathrm{d}\varrho \Big) \frac{f(r'_n\mathrm{e}^{\mathrm{i}\theta'_n})}{(1 - zr'_n\mathrm{e}^{-\mathrm{i}\theta'_n})^2} - \int_{r_n}^{r'_n} \Big( \int_{r_n}^{r} \int_{\theta_n}^{\theta'_n} a(\varrho\mathrm{e}^{\mathrm{i}\varphi})\varrho\mathrm{d}\varphi\mathrm{d}\varrho \Big) \partial_r \frac{f(r\mathrm{e}^{\mathrm{i}\theta'_n})}{(1 - zr\mathrm{e}^{-\mathrm{i}\theta'_n})^2} \mathrm{d}r
\end{aligned}$$

$$-\int_{\theta_n}^{\theta'_n}\Big(\int_{r_n}^{r'_n}\int_{\theta_n}^{\theta} a(\varrho e^{i\varphi})\varrho d\varphi d\varrho\Big)\partial_\theta \frac{f(r'_n e^{i\theta})}{(1-zr'_n e^{-i\theta})^2}d\theta$$

$$+\int_{r_n}^{r'_n}\int_{\theta_n}^{\theta'_n}\Big(\int_{r_n}^{r}\int_{\theta_n}^{\theta} a(\varrho e^{i\varphi})\varrho d\varphi d\varrho\Big)\partial_r\partial_\theta \frac{f(re^{i\theta})}{(1-zre^{-i\theta})^2}d\theta dr$$

$$=: F_{1,n}(z)+F_{2,n}(z)+F_{3,n}(z)+F_{4,n}(z) =: F_n(z). \tag{3.7}$$

Let us consider $F_{1,n}$. The integral of $a$ there has, by (3.6), the bound

$$\Big|\int\int a(\varrho e^{i\varphi})\varrho d\varphi d\varrho\Big| \leqslant \frac{\mathbf{C}_a|Q_n|}{m}. \tag{3.8}$$

Repeating the estimations made in [8] we find that

$$|F_{1,n}(z)| \leqslant \frac{C\mathbf{C}_a}{m}\int_{Q_n}\frac{|f(\zeta)|}{|1-z\overline{\zeta}|^2}dA(\zeta). \tag{3.9}$$

By the same methods we get for $j=2,3$

$$|F_{j,n}(z)| \leqslant \frac{C\mathbf{C}_a}{m}\int_{Q_n}\Big(\frac{|f(\zeta)|}{|1-z\overline{\zeta}|^2}+\frac{|f'(\zeta)|(1-|\zeta|^2)}{|1-z\overline{\zeta}|^2}\Big)dA(\zeta), \tag{3.10}$$

and moreover

$$|F_{4,n}(z)| \leqslant \frac{C\mathbf{C}_a}{m}\int_{Q_n}\Big(\frac{|f(\zeta)|}{|1-z\overline{\zeta}|^2}+\frac{|f'(\zeta)|(1-|\zeta|^2)}{|1-z\overline{\zeta}|^2}+\frac{|f''(\zeta)|(1-|\zeta|^2)^2}{|1-z\overline{\zeta}|^2}\Big)dA(\zeta). \tag{3.11}$$

Notice that $1/m$ can be estimated by $C/\mathcal{W}(\zeta)$ on $Q_n$, hence, we can write

$$\begin{aligned}\sum_{n=1}^{\infty}F_n(z) &\leqslant \sum_{n=1}^{\infty}|F_n(z)| \leqslant \sum_{n=1}^{\infty}\sum_{j=1}^{4}|F_{j,n}(z)| \\ &\leqslant C\mathbf{C}_a\int_{\mathbb{D}}\frac{|f(\zeta)|+|f'(\zeta)|(1-|\zeta|^2)+|f''(\zeta)|(1-|\zeta|^2)^2}{\mathcal{W}(\zeta)|1-z\overline{\zeta}|^2}dA(\zeta).\end{aligned} \tag{3.12}$$

We show below that the maximal Bergman projection satisfies the following estimate:

$$\Big\|\int_{\mathbb{D}}\frac{|g(\zeta)|}{\mathcal{W}(\zeta)|1-z\overline{\zeta}|^2}dA(\zeta)\Big\|_1 \leqslant C\|g\|_1 \tag{3.13}$$

for all $g\in L^1$ (see Corollary 3.13 of [12]). Moreover, by Theorem 4.28 of [12], the functions $|f'(\zeta)|(1-|\zeta|^2)$ and $|f''(\zeta)|(1-|\zeta|^2)^2$ belong to $L^1$. Using (3.12) and (3.13) we thus find that the series $\sum_{n=1}^{\infty}|F_n(z)|$ can be pointwise bounded by an $L^1$ function, and thus it converges for almost all $z$. Hence, the Toeplitz operator can be defined as a bounded operator on $A^1$ (see Remark 2.4. of [8]), and its norm has the bound (3.4).

As for the inequality (3.13), using the duality of $L^1$ and $L^\infty \ni h$ and the Forelli–Rudin estimate ([12], Lemma 3.10), we have the following which completes the proof:

$$\begin{aligned}\Big\| \int_{\mathbb{D}} \frac{|g(\zeta)|}{\mathcal{W}(\zeta)|1-z\overline{\zeta}|^2}\mathrm{d}A(\zeta)\Big\|_1 &\leqslant \sup_{\|h\|_\infty \leqslant 1} \int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|g(\zeta)h(z)|}{\mathcal{W}(\zeta)|1-z\overline{\zeta}|^2}\mathrm{d}A(\zeta)\mathrm{d}A(z)\\ &\leqslant \int_{\mathbb{D}} \frac{|g(\zeta)|}{\mathcal{W}(\zeta)} \int_{\mathbb{D}} \frac{1}{|1-z\overline{\zeta}|^2}\mathrm{d}A(z)\mathrm{d}A(\zeta)\\ &\leqslant \int_{\mathbb{D}} \frac{C|g(\zeta)|}{\mathcal{W}(\zeta)}\mathcal{W}(\zeta)\mathrm{d}A(\zeta) = C\|g\|_1. \qquad \blacksquare\end{aligned}$$

By combining Propositions 2.4 and 3.1, and denoting by $X$ the space of symbols satisfying the condition (3.3), we obtain our most general result on the boundedness of $T_a$:

THEOREM 3.2. *If $a \in BO_{\log} \cap L^\infty + X$, then $T_a : A^1 \to A^1$ is bounded.*

Notice that the constants do not satisfy (3.3) though they correspond to the multiples of the identity operator, hence, Theorem 3.2 is well motivated in comparison with Proposition 3.1. Since $X$ contains the space $BA^1_{\log}$, we have the following strengthening of the result in [7]:

THEOREM 3.3. *If $a \in BO_{\log} \cap L^\infty + BA^1_{\log}$, then $T_a : A^1 \to A^1$ is bounded.*

Next we give the following sufficient condition for $T_a$ to be compact on $A^1$.

PROPOSITION 3.4. *Assume that $a \in L^1_{\mathrm{loc}}$ and that*

$$\lim_{d(Q)\to 0} \sup_{\zeta \in Q} |\widehat{a}_Q(\zeta)|\mathcal{W}(\zeta) = 0, \tag{3.14}$$

*where $d(Q) := \inf\{|z-w| : z \in Q, |w| = 1\}$. Then $T_a : A^1 \to A^1$ is compact.*

*Proof.* By definitions, we actually have

$$\lim_{n\to\infty} \sup_{\zeta \in Q_n} |\widehat{a}_{Q_n}(\zeta)|\mathcal{W}(\zeta) = 0. \tag{3.15}$$

We take an arbitrary sequence $(f_k)_{k=1}^\infty \subset A^1$ with $\|f_k\|_1 \leqslant 1$ for all $k$ such that $f_k \to 0$ uniformly on compact subsets of $\mathbb{D}$; we show that $\|T_a f_k\|_1 \to 0$.

Let $\varepsilon > 0$ be arbitary. Let the expressions $F_{j,n}(z)$ be as above, and let $F_{j,n,k}(z)$ be equal to $F_{j,n}(z)$ with $f$ is replaced by $f_k$. Consider $F_{1,n,k}(z)$. Applying (3.15) to (3.8) and deducing as in (3.8)–(3.9), we obtain instead of (3.9) the estimate

$$|F_{1,n,k}(z)| \leqslant C\frac{\varepsilon_n}{m} \int_{Q_n} \frac{|f_k(\zeta)|}{|1-z\overline{\zeta}|^2}\mathrm{d}A(\zeta) \leqslant C'\varepsilon_n \int_{Q_n} \frac{|f_k(\zeta)|}{\mathcal{W}(\zeta)|1-z\overline{\zeta}|^2}\mathrm{d}A(\zeta), \tag{3.16}$$

where we may assume that $\varepsilon_n \to 0$. Choose $N \in \mathbb{N}$ such that $\varepsilon_n < \varepsilon$ for $n \geqslant N$.

Since the closure of the set $D^{(N)} := \bigcup_{n \leqslant N} Q_n$ is compact, there exists a constant $C_N > 0$ such that

$$\sum_{n \leqslant N} \int_{Q_n} \frac{1}{|1 - z\overline{\zeta}|^2} \mathrm{d}A(\zeta) \leqslant C_N$$

for all $z \in \mathbb{D}$. Since the sequence $(f_k)$ converges to 0 on compact subsets, we may choose $M \in \mathbb{N}$ such that $|f_k(\zeta)| \leqslant \varepsilon/(1 + C_N)$ for all $k \geqslant M$ and $\zeta \in D^{(N)}$. We get for all $k \geqslant M$,

$$\begin{aligned}
\sum_{n=1}^{\infty} |F_{1,n,k}(z)| &\leqslant \sum_{n \leqslant N} \int_{Q_n} \frac{|f_k(\zeta)|}{\mathcal{W}(\zeta)|1 - z\overline{\zeta}|^2} \mathrm{d}A(\zeta) + \sum_{n > N} \int_{Q_n} \varepsilon_n \frac{|f_k(\zeta)|}{\mathcal{W}(\zeta)|1 - z\overline{\zeta}|^2} \mathrm{d}A(\zeta) \\
&\leqslant \sum_{n \leqslant N} \frac{\varepsilon}{1 + C_N} \int_{Q_n} \frac{1}{|1 - z\overline{\zeta}|^2} \mathrm{d}A(\zeta) + \varepsilon \int_{\mathbb{D}} \frac{|f_k(\zeta)|}{\mathcal{W}(\zeta)|1 - z\overline{\zeta}|^2} \mathrm{d}A(\zeta) \\
&\leqslant \varepsilon + \varepsilon \int_{\mathbb{D}} \frac{|f_k(\zeta)|}{\mathcal{W}(\zeta)|1 - z\overline{\zeta}|^2} \mathrm{d}A(\zeta).
\end{aligned}$$

The other expressions $F_{j,n,k}(z)$, $j = 2, 3, 4$, are treated similarly, keeping in mind that also the derivatives of the functions $f_k$ converge to 0 uniformly on compact subsets of $\mathbb{D}$. Arguing as in the proof of Proposition 3.1 we find that $\|T_a f_k\|_1 \leqslant C\varepsilon$ for large enough $k$. ∎

The previous lemma contains the result that if $a \in VA^1_{\log}$, then $T_a : A^1 \to A^1$ is compact.

We end our paper by a direct application to Hankel operators.

COROLLARY 3.5. *Let $a \in BMO^1_{\log}$ be such that $a = f + g$ with $f \in BO_{\log} \cap L^{\infty}$ and $g \in BA^1_{\log}$. Then the Hankel operator $H_a : A^1 \to L^1$ is bounded.*

*Proof.* Since $H_a$ depends on the symbol $a$ linearly, it suffices to show that both $H_f$ and $H_g$ are bounded. Trivially, $M_f$ is bounded from $A^1$ into $L^1$ for $f \in L^{\infty}$. Hence, by Proposition 2.4, $H_f = M_f - T_f : A^1 \to L^1$ is bounded. Similarly, the boundedness of $H_g : A^1 \to L^1$ follows from Lemma 2.5 and Proposition 3.1. ∎

REMARK 3.6. The problem whether $H_a : A^1 \to L^1$ is bounded for every $a \in BMO^1_{\log}$ remains open.

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