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REALIZATION OF CONDITIONALLY MONOTONE INDEPENDENCE AND MONOTONE PRODUCTS OF COMPLETELY POSITIVE MAPS

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ABSTRACT. The paper gives an operator algebras model for the conditional monotone independence, introduced by T. Hasebe. The construction is used to prove an embedding result for the N. Muraki's monotone product of C^*-algebras. Also, the formulas from the definition of conditional monotone independence are used to define the monotone product of maps which is shown to preserve complete positivity, similarly to the results from the case of free products.

KEYWORDS: *Completely positive maps, monotone and conditional monotone independence, monotone Fock spaces.*

MSC (2000): 46L53, 47L55.

1. INTRODUCTION

This material presents some results in monotone probability, a non-unital and non-symmetric type of non-commutative probability. More precisely, if $(\mathcal{A}, \psi)$ is a non-commutative probability space and I a totally ordered set, then a family of subalgebras $\{\mathcal{A}_i\}_{i\in I}$ of $\mathcal{A}$ is said to be monotone independent with respect to ψ if for any $a_1, \dots, a_n$ with $a_k \in \mathcal{A}_{i_k}$ such that $i_s \neq i_{s+1}$, the following properties are satisfied (for a more functorial treatment of monotone independence, see [19] [12]):

(1) $\psi(a_1 \cdots a_n) = \psi(a_1)\psi(a_2 \cdots a_n)$ if $i_1 > i_2$;
(2) $\psi(a_1 \cdots a_n) = \psi(a_1 \cdots a_{n-1})\psi(a_n)$ if $i_n > i_{n-1}$;
(3) $\psi(a_1 \cdots a_n) = \psi(a_1 \cdots a_{k-1}\psi(a_k)a_{k+1} \cdots a_n)$, if $i_{k-1} < i_k > i_{k+1}$.

Many results from the free probabilities theory have non-trivial monotone independence analogues - the monotone Fock space, respectively bimodule of [18] and [21] are counterparts to the full Fock space ([23], [22]), the H- and K-transforms from [18] and [13] are analogue to the Voiculescu's R- and S-transforms etc. In [14], T. Hasebe introduced the notion of conditionally monotone independence, in analogy to the notion of conditionally freeness from [5], [6]. More precisely, if $\mathcal{A}$ is a unital algebra endowed with two normalized linear functionals φ and ψ, a family of subalgebras $\{\mathcal{A}_i\}_{i\in I}$ of $\mathcal{A}$ is said to be conditionally monotone independent if they are monotone independent with respect to ψ and for any $a_1, \dots, a_n$ with $a_k \in \mathcal{A}_{i_k}$ such that $i_s \neq i_{s+1}$, we have that:

(1′) $\varphi(a_1 \cdots a_n) = \varphi(a_1)\varphi(a_2 \cdots a_n)$ if $i_1 > i_2$;
(2′) $\varphi(a_1 \cdots a_n) = \varphi(a_1 \cdots a_{n-1})\varphi(a_n)$ if $i_n > i_{n-1}$;
(3′) $\varphi(a_1 \cdots a_n) = \varphi(a_1 \cdots a_{k-1})[\varphi(a_k) - \psi(a_k)]\varphi(a_{k+1} \cdots a_n) + \varphi(a_1 \cdots a_{k-1} \psi(a_k)a_{k+1} \cdots a_n)$, if $i_{k-1} < i_k > i_{k+1}$.

A Fock space model for the theory of conditional freeness is presented in [5]. Also, there is an important connection between conditional freeness and complete positive maps: in [1], [2] and [8], it is shown how the relations from the definition of the conditional freeness appear in the construction of the free product of completely positive maps, which turns to also be complete positive. The present material addresses all these topics for the case of conditionally monotone independence.

The paper is organized in 4 sections, including the Introduction. In Section 2 we will present a operator algebraic model for the conditionally monotone independence using the "monotone Fock space" introduced in [18] and the ideas from the Fock model for conditionally freeness from [5], thus completing the construction from [14]. In Section 3 we construct the monotone product of maps, and using the results and some techniques from [1] and [14] we prove that a monotone product of completely positive maps is completely positive. In Section 4 we define the montone product of C^*-algebras with conditional expectations, refining the construction from [18], and prove some embedding results similar to the ones presented in Sections 1 and 2 of [8] for the free products.

2. REALIZATION OF CONDITIONALLY MONOTONE INDEPENDENCE

Let $\{(\mathcal{A}_i, \varphi_i, \psi_i)\}_{i\in I}$ be a family of $*$-algebras, each endowed with two states (throughout the paper I will always be a totally ordered set). If I has a minimal element, 0_I, since in the definition of the conditional monotone independence the functional ψ_{0_i} does not appear, we will also suppose that $\varphi_{0_I} = \psi_{0_I}$.

As in [5], for each $j \in I$ we consider $*$-representations $\pi_j, \sigma_j : \mathcal{A}_j \longrightarrow B(\mathcal{H}_j)$ given by the GNS-constructions with states φ_j, ψ_j, respectively, i. e.

$$\varphi_j(a_j) = \langle \pi_j(a_j)\xi_j, \xi_j\rangle \quad \text{and} \quad \psi_j(a_j) = \langle \sigma_j(a_j)\xi_j, \xi_j\rangle$$

with $a_j \in \mathcal{A}_j$ and $\|\xi_j\| = 1$. As remarked in [5], we can always choose the same vector ξ_j for both states, but by doing so we may lose the cyclicity of ξ_j.

Let $(\mathcal{H},\xi)$ be the monotone product of the family $\{(\mathcal{H}_j,\xi_j)\}_{j\geqslant 0}$ (see [18], [21]):

$$\mathcal{H}=\mathbb{C}\xi\oplus\bigoplus_{n=1}^{\infty}\Big(\bigoplus_{i_1>\cdots>i_n}\mathcal{H}_{i_1}^{\circ}\otimes\cdots\otimes\mathcal{H}_{i_n}^{\circ}\Big)$$

where $\mathcal{H}_j^{\circ}=\mathcal{H}_j\ominus\mathbb{C}\xi_j$. We also define

$$\mathcal{H}(k)=\mathbb{C}\xi\oplus\bigoplus_{n=1}^{\infty}\Big(\bigoplus_{i_1>\cdots>i_n,\,i_1\leqslant k}\mathcal{H}_{i_1}^{\circ}\otimes\cdots\otimes\mathcal{H}_{i_n}^{\circ}\Big)$$

and consider the adjointable partial isometries $V_k:\mathcal{H}\longrightarrow\mathcal{H}_k\otimes\mathcal{H}(k-1)$ given by $V_k\xi=\xi_k\otimes\xi$ and, for $f_1\otimes\cdots\otimes f_n\in\mathcal{H}_{i_1}^{\circ}\otimes\cdots\otimes\mathcal{H}_{i_n}^{\circ}$,

$$V_kf_1\otimes\cdots\otimes f_n=\begin{cases}0 & \text{if } i_1>k,\\ f_1\otimes\cdots\otimes f_n & \text{if } i_1=k,\\ \xi_k\otimes f_1\otimes\cdots\otimes f_n & \text{if } i_1<k.\end{cases}$$

For $T\in B(\mathcal{H}_k)$, we define $\omega_k(T)=V_k^*(T\otimes\mathrm{Id}_{\mathcal{H}(k)})V_k$; a trivial computation gives that $\omega_k(T_1T_2)=\omega_k(T_1)\omega_k(T_2)$. We will consider the $*$-representation $j_k:\mathcal{A}_k\longrightarrow B(\mathcal{H})$

$$j_k(a)=\omega_k(\pi_k(a))P_k\oplus\omega_k(\sigma_k(a))P_k^{\perp}$$

where P_k is the orthogonal projection on $\mathbb{C}\xi\oplus\mathcal{H}_k$.

Finally, let Φ be the state on $B(\mathcal{H})$ given by $\Phi(T)=\langle T\xi,\xi\rangle$.

REMARK 2.1. From the definition of ω_i and π_i, we have that $\Phi\circ j_i=\varphi_i$ for all $i\in I$.

LEMMA 2.2. *Suppose that $i_k\neq i_{k+1}$ for $1\leqslant k<n$, that $a_k\in\mathcal{A}_{i_k}$ and $A_k=j_{i_k}(a_k)$. Then*

$$A_1\cdots A_n\xi=\Phi(A_1\cdots A_n)\xi+\eta\quad\textit{for some } \eta\in\mathcal{H}(k)^{\circ}=\mathcal{H}(k)\ominus\mathbb{C}\xi.$$

Proof. Induction on n. For $n=1$, we have

$$\begin{aligned}A_1\xi&=\omega_{i_1}(\pi_{i_1}(a_1))P_{i_1}\xi\oplus\omega_{i_1}(\sigma_{i_1}(a_1))P_{i_1}^{\perp}\xi=\omega_{i_1}(\pi_{i_1}(a_1))\xi,\ \text{since}P_{i_1}^{\perp}\xi=0\\&=V_{i_1}^*[\pi_{i_1}(a_1)\xi_{i_1}]\otimes\xi=V_{i_1}^*[\langle\pi_{i_1}(a_1)\xi_{i_1},\xi_{i_1}\rangle\xi_{i_1}+P_{\xi_{i_1}}^{\perp}\pi_{i_1}(a_1)\xi_{i_1}]\otimes\xi\\&=V_{i_1}^*[\varphi_{i_1}\xi_{i_1}+P_{\xi_{i_1}}^{\perp}\pi_{i_1}(a_1)\xi_{i_1}]\otimes\xi=\varphi_{i_1}\xi+P_{\xi_{i_1}}^{\perp}\pi_{i_1}(a_1)\xi_{i_1}\end{aligned}$$

where $P_{\xi_{i_1}}$ is the orthogonal projection on $\mathbb{C}\xi_{i_1}$. The conclusion follows now from Remark 2.1.

For the induction step, we first write $A_2\cdots A_n\xi=\eta_1+\eta_2+\alpha\xi$ with $\eta_1\in\mathcal{H}_{i_1}^{\circ}$, $\eta_2\in\mathcal{H}(i_2)^{\circ}\ominus\mathcal{H}_{i_1}^{\circ}$ and $\alpha\in\mathbb{C}$.

The argument above gives $A_1(\alpha\xi)=\alpha(\zeta_1+\alpha_0\xi)$ with $\zeta_1\in\mathcal{H}_{i_1}^{\circ}$ and $\alpha_0\in\mathbb{C}$.

On the other hand,

$$A_1\eta_1 = P^{\perp}_{\xi_{i_1}} A_1\eta_1 + \alpha_1\xi \quad \text{with } \alpha_1 \in \mathbb{C} \text{ and } P^{\perp}_{\xi_{i_1}} A_1\eta_1 \in \mathcal{H}^{\circ}_{i_1}, \text{ and}$$
$$A_1\eta_2 = V^*_{i_1}(\sigma_{i_1}(a_1) \otimes \mathrm{Id})V_{i_1}\eta_2 = V^*_{i_1}(\sigma_{i_1}(a_1)\xi_{i_1} \otimes \eta_2) \in \mathcal{H}(i_1)^{\circ}.$$

Summing, we obtain $A_1 \cdots A_n\xi = \eta + \beta\xi$, with $\beta \in \mathbb{C}$ and $\eta \in \mathcal{H}(i_1)^{\circ}$, and since $\Phi(T) = \langle T\xi, \xi\rangle$, the result is proved. ∎

THEOREM 2.3. *With the notations from above, if $i_k \neq i_{k+1}$, $(k = 1, \dots, n-1)$, and $a_k \in \mathcal{A}_{i_k}$, then:*

(i) *for $i_1 > i_2$, we have that*

$$\Phi(j_{i_1}(a_1) \cdots j_{i_n}(a_n)) = \varphi_{i_1}(a_1)\Phi(j_{i_2}(a_2) \cdots j_{i_n}(a_n));$$

(ii) *for $i_n > i_{n-1}$, we have that*

$$\Phi(j_{i_1}(a_1) \cdots j_{i_n}(a_n)) = \Phi(j_{k_1}(a_1) \cdots j_{k_{n-1}}(a_{n-1}))\varphi_{i_n}(a_n);$$

(iii) *for $i_{l-1} < i_l > i_{l+1}$ (for some $1 < l < n$), we have that*

$$\begin{aligned}
&\Phi(j_{k_1}(a_1) \cdots j_{k_n}(a_n)) \\
&= \Phi(j_{k_1}(a_1) \cdots j_{k_{l-1}}(a_{l-1})\psi(a_l)j_{l+1}(a_{l+1}) \cdots j_{k_n}(a_n))) + \Phi(j_{k_1}(a_1) \cdots j_{k_{l-1}}(a_{l-1})) \\
&\qquad [\varphi_{k_l}(a_l) - \psi_{k_l}(a_l)]\Phi(j_{l+1}(a_{l+1}) \cdots j_{k_n}(a_n))).
\end{aligned}$$

Proof. First, to simplify the notations, we will write A_l for $j_{i_l}(a_l)$, $1 \leqslant l \leqslant n$.

From Lemma 2.2 and Remark 2.1, we have that $A_n\xi = \eta + \varphi_{i_n}(a_n)\xi$ with $\eta \in \mathcal{H}^{\circ}_{i_n}$. Since $i_n > i_{n-1}$, the definition of $V_{i_{n-1}}$ gives

$$\Phi(A_1 \cdots A_n) = \langle A_1 \cdots A_{n-1}\varphi_{i_n}(a_n)\xi, \xi\rangle = \langle A_1 \cdots A_{n-1}\xi, \xi\rangle\varphi_{i_n}(a_n)$$

so part (ii) is done.

For part (i), Lemma 2.2 gives

$$A_2 \cdots A_n\xi = \eta + \alpha\xi,$$

with $\alpha = \Phi(A_2 \cdots A_n) \in \mathbb{C}$ and $\eta \in \mathcal{H}(i_2)^{\circ}$. Since

$$A_1\eta = V_{i_1}(\sigma_{i_1}(a_1) \otimes \mathrm{Id})\xi_{i_1} \otimes \eta = V_{i_1}(\sigma_{i_1}(a_1)\xi_{i_1} \otimes \eta) \in \mathcal{H}(i_1)^{\circ},$$

we have that

$$\Phi(A_1 \cdots A_n) = \langle A_1\alpha\xi, \xi\rangle = \langle A_1\xi, \xi\rangle\alpha = \varphi_{i_1}(A_1)\Phi(A_2 \cdots A_n).$$

For part (iii), write $A_{l+1} \cdots A_n\xi = \eta + \alpha\xi$, with $\eta \in \mathcal{H}(l+1)^{\circ}$ and $\alpha = \Phi(A_{l+1} \cdots A_n) \in \mathbb{C}$. Also write

$$\pi_{i_l}(a_l)\xi_{i_l} = \zeta_1 + \beta_1\xi_{i_l}, \quad \sigma_{i_l}(a_l)\xi_{i_l} = \zeta_2 + \beta_2\xi_{i_l}$$

with $\beta_1 = \varphi_{i_l}(a_l)$, $\beta_2 = \psi_{i_l}(a_l)$ and $\zeta_1, \zeta_2 \in \mathcal{H}^{\circ}_l$.

We have that

$$\begin{aligned} A_l A_{l+1} \cdots A_n \xi &= A_l(\eta + \alpha\xi) = V_{i_l}(\sigma_{i_l}(a_l)\xi_{i_l} \otimes \eta) + V_{i_l}(\pi_{i_l}(a_l)\alpha\xi_{i_l} \otimes \xi) \\ &= V_{i_l}([\zeta_2 + \beta_2\xi_{i_l}] \otimes \eta + [\zeta_1\alpha + \beta_1\alpha\xi_{i_l}] \otimes \xi) = \zeta_2 \otimes \eta + \beta_2\eta + \zeta_1\alpha + \beta_1\alpha\xi \\ &= \zeta_2 \otimes \eta + \zeta_2\alpha + \beta_2\eta + \beta_2\alpha\xi + (\zeta_1 - \zeta_2)\alpha + (\beta_1 - \beta_2)\alpha\xi \end{aligned}$$

Since $i_l > i_{l-1}$ and $\zeta_1, \zeta_2 \in \mathcal{H}_{i_l}^\circ$, it follows that $A_{l-1}(\zeta_2 \otimes \eta + \zeta_2\alpha + (\zeta_1 - \zeta_2)\alpha) = 0$, therefore

$$\begin{aligned} \Phi(A_1 \cdots A_n) &= \langle A_1 \cdots A_{l-1} A_l (A_{l+1} \cdots A_n \xi), \xi \rangle \\ &= \langle A_1 \cdots A_{l-1}(\beta_2[\eta + \alpha\xi] + (\beta_1 - \beta_2)\alpha\xi), \xi \rangle \\ &= \langle A_1 \cdots A_{l-1}\beta_2 A_{l+1} \cdots A_n \xi, \xi \rangle + \langle A_1 \cdots A_{l-1}\xi, \xi \rangle (\beta_1 - \beta_2)\alpha. \quad \blacksquare \end{aligned}$$

3. MONOTONE PRODUCTS OF COMPLETELY POSITIVE MAPS

Let $\{\mathfrak{A}_i\}_{i \in I}$ be a family of $*$-algebras containing a C^*-algebra $\mathfrak{B}$ as a common $*$-subalgebra and suppose that each $\mathfrak{A}_i$ is endowed with a projection $\psi_i : \mathfrak{A}_i \longrightarrow \mathfrak{B}$. Let now $\mathfrak{D}$ be another $*$-algebra containing $\mathfrak{B}$ as a $*$-subalgebra and suppose $\theta_i : \mathfrak{A}_i \longrightarrow \mathfrak{D}$ are a family of $\mathfrak{B}$-$\mathfrak{B}$ bimodule maps such that $\theta_{i|\mathfrak{B}} = \mathrm{Id}_{\mathfrak{B}}$.

We will write $\mathfrak{A}_i^\circ$ for the set $\ker(\psi_i) \subset \mathfrak{A}_i$ and denote be $\underset{i \in I}{*}\mathfrak{A}_i$ the free product of $*$-algebras $\{\mathfrak{A}_i\}_{i \geqslant 0}$ *with amalgamation* over $\mathfrak{B}$.

We fist need to briefly review a result from [1].

DEFINITION 3.1. The *free product* of the maps $\{\theta_i\}_{i \in I}$ is the map

$$\theta_{*} = \underset{i \in I}{*}\theta_i : \underset{i \in I}{*}\mathfrak{A}_i \longrightarrow \mathfrak{D}$$

given by

$$\theta_{*}(a_1 \cdots a_n) = \theta_{i_1}(a_1) \cdots \theta_{i_n}(a_n) \quad \text{whenever } a_k \in \mathfrak{A}_{i_k}^\circ, i_j \neq i_{j+1}; \quad \theta_{*|\mathfrak{B}} = \mathrm{Id}_{\mathfrak{B}}.$$

THEOREM 3.2 ([1], Theorem 3.2). *If, with the notations above, $\{\mathfrak{A}_i\}_{i \in I}$, $\mathfrak{B}$, and $\mathfrak{D}$ are unital C^*-algebras, ψ_i are projections of norm 1 and θ_i are completely positive unital maps, then θ_{*} extends to a unital completely positive map from the universal free product of the C^*-algebras $\mathfrak{A}_i (i \in I)$ to $\mathfrak{D}$.*

As in [14], for each $i \in I$ consider now $\widetilde{\mathfrak{A}}_i = \mathfrak{B}1 \oplus \mathfrak{A}_i$ (direct sum of $\mathfrak{B}$-bimodules). If $\mathfrak{B}$ is unital, then $\widetilde{\mathfrak{A}}_i$ is a unitalization of $\mathfrak{A}_i$, but $1_{\widetilde{\mathfrak{A}}_i} \neq 1_{\mathfrak{A}_i}$. let $\widetilde{\mathfrak{A}} = \underset{i \in I}{*}\widetilde{\mathfrak{A}}_i$ be the free product of $*$-algebras *with* amalgamation over $\mathfrak{B}$. Note that we have the natural decomposition $\widetilde{\mathfrak{A}} = \mathfrak{B}1 \oplus \mathfrak{A}$, where $\mathfrak{A} = \overline{*}\mathfrak{A}_i$, the free product of $*$-algebras *without* amalgamation over $\mathfrak{B}$. The algebra $\mathfrak{A}$ is still a $\mathfrak{B}$-ring, and we have the vector spaces identification

$$\mathfrak{A} \cong \bigoplus_{n=1}^{\infty} \bigoplus_{i_1 \neq i_2 \cdots \neq i_n} \mathfrak{A}_{i_1} \otimes_{\mathfrak{B}} \mathfrak{A}_{i_2} \otimes_{\mathfrak{B}} \cdots \mathfrak{A}_{i_n}.$$

DEFINITION 3.3. The monotone product of the maps $\{\theta_i\}_{i \geqslant 0}$ is the map $\theta = \underset{i \in I}{\triangleright} \theta_i : \mathfrak{A} \longrightarrow \mathfrak{D}$ given by:

$$\theta(a_1 \cdots a_n) = \theta_{i_1}(a_1)\theta(a_2 \cdots a_n) \text{ if } i_1 > i_2, \tag{3.1}$$

$$\theta(a_1 \cdots a_n) = \theta(a_1 \cdots a_{n-1})\theta_{i_n}(a_n) \text{ if } i_n > i_{n-1}, \tag{3.2}$$

$$\begin{aligned}\theta(a_1 \cdots a_n) = {} & \theta(a_1 \cdots a_{k-1}\psi_{i_k}(a_k)a_{k+1} \cdots a_n) \\ & + \theta(a_1 \cdots a_{k-1})[\theta_{i_k}(a_k) - \psi_{i_k}(a_k)]\theta(a_{k+1} \cdots a_n).\end{aligned} \tag{3.3}$$

PROPOSITION 3.4. *The monotone product of maps, defined above, is associative.*

The proof is just a trivial (though tedious) re-writing of the argument from the scalar case in Theorem 3.6 of [14].

PROPOSITION 3.5. *Let $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{D}$ be $*$-algebras that contain $\mathfrak{B}$ as a common $*$-subalgebra. Suppose that $\psi_2 : \mathfrak{A}_2 \longrightarrow \mathfrak{B}$ is a conditional expectation and that $\theta_k : \mathfrak{A}_k \longrightarrow \mathfrak{D}, k = 1,2$ are $\mathfrak{B}$-bimodule maps.*

Consider $\widetilde{\mathfrak{A}}_k = \mathfrak{B}1 \oplus \mathfrak{A}_k, k = 1,2$ and $\widetilde{\psi}_k, \widetilde{\theta}_k : \widetilde{\mathfrak{A}}_k \longrightarrow \mathfrak{B}$ given by ($b \in \mathfrak{B}, a_1 \in \mathfrak{A}_1$, and $a_2 \in \mathfrak{A}_2$):

$$\widetilde{\psi}_1(b1 + a_1) = b, \quad \widetilde{\psi}_2(b1 + a_2) = b + \psi_2(a_2), \quad \widetilde{\theta}_k(b1 + a_k) = b + \theta_k(a_k).$$

Then, with the above notations, we have that $\widetilde{\theta}_2 * \widetilde{\theta}_1{}_{|\mathfrak{A}_1 \overline{*} \mathfrak{A}_2} = \theta_2 \triangleright \theta_1$.

Proof. For simplicity, denote $\theta = \theta_2 \triangleright \theta_1$. We just need to show that

$$\theta(a_1 \cdots a_n) = \theta_{i_1}(a_1) \cdots \theta_{i_n}(a_n)$$

whenever $a_j \in \ker(\psi_{i_j}) \cap \mathfrak{A}_{i_j}, i_j \in \{1,2\}$ with $i_k \neq i_{k+1}$.

We will prove the assertion by induction on n. The case $n = 1$ is trivial. For the induction step, note that if a_1 or a_n are from $\mathfrak{A}_2$, then the conclusion follows from Definition 3.3, relations (3.1), (3.2).

If $a_1, a_n \in \mathfrak{A}_1$, then there exists $k \in \{2, \ldots, n-1\}$ such that $a_k \in \mathfrak{A}_2$. Then (3.3) implies

$$\begin{aligned}\theta(a_1 \cdots a_n) &= \theta(a_1 \cdots a_{k-1}\psi_2(a_k)a_{k+1} \cdots a_n) + \theta(a_1 \cdots a_{k-1})[\theta_2(a_2) - \psi_2(a_k)]\theta(a_{k+1} \cdots a_n) \\ &= \theta(a_1 \cdots a_{k-1})\theta_2(a_2)\theta(a_{k+1} \cdots a_n),\end{aligned}$$

since $\psi_2(a_k) = 0$, and the conclusion follows from the induction hypothesis. ∎

THEOREM 3.6. *Suppose now that $\{\mathfrak{A}_i\}_{i \in I}, \mathfrak{D}$ are unital C^*-algebras, and $\mathfrak{B}$ is a common C^*-subalgebra of theirs containing the unit. Suppose that $\psi_i : \mathfrak{A}_i \longrightarrow \mathfrak{B}$*

are positive conditional expectations and $\theta_i : \mathfrak{A}_i \longrightarrow \mathfrak{D}$ *are unital, completely positive* $\mathfrak{B}$*-bimodule maps. Then the map* $\underset{i\geqslant 0}{\triangleright}\theta_i$ *is also a completely positive* $\mathfrak{B}$*-bimodule map.*

Proof. The proof relies heavily on Theorem 3.2 and the associativity of the monotone product of maps. To simplify the writing, denote $\theta = \underset{i\in I}{\triangleright}\theta_i$ the monotone product map and $\mathfrak{A} = \overline{\underset{i\in I}{*}}\mathfrak{A}_i$ the free product $*$-algebra.

We need to show that, for any positive integer n, if $A = [a_{i,j}]_{i,j=1}^n$ is a positive element from $M_n(\mathfrak{A})$ then the matrix $\theta(A) = [\theta(a_{i,j})]_{i,j=1}^n$ is also positive in $M_n(\mathfrak{D})$. Each entry $a_{i,j}$ of A is a finite sum

$$a_{i,j} = \sum_{l=1}^{N(i,j)} \alpha_l(i,j)$$

where each $\alpha_l(i,j)$ is a reduced product from $\mathfrak{A}$, i. e. is written as a product of the form $a_1a_2\cdots a_m$ with $a_k \in \mathfrak{A}_{i_k}$, $i_s \neq i_{s+1}$.

Let $N(A) = \text{card}\{i \in I :$ there is a word in one of the entries of A that contains elements from $\mathfrak{A}_i\}$.

We will prove the assertion by induction on $N(A)$. For $N(A) = 1$, the conclusion is equivalent to the completely positivity of θ_1.

If $N(A) = 2$, for $k = 1, 2$, let $\widetilde{\mathfrak{A}}_k = \mathfrak{B}1 \oplus \mathfrak{A}_k$ and, as in Proposition 3.5, consider the maps $\widetilde{\psi}_k, \widetilde{\theta}_k : \widetilde{\mathfrak{A}}_k \longrightarrow \mathfrak{B}$ given by ($b \in \mathfrak{B}, a_1 \in \mathfrak{A}_1$ and $a_2 \in \mathfrak{A}_2$):

$$\widetilde{\psi}_1(b1 + a_1) = b, \quad \widetilde{\psi}_2(b1 + a_2) = b + \psi_2(a_2), \quad \widetilde{\theta}_k(b1 + a_k) = b + \theta_k(a_k).$$

Remark that $\widetilde{\theta}_k$ are unital completely positive $\mathfrak{B}$-bimodule maps from the $*$-algebras $\widetilde{\mathfrak{A}}_k$ to $\mathfrak{D}$. First note that $1_{\mathfrak{A}_k}$ are projections in $\widetilde{\mathfrak{A}}_k$, respectively, and so are $e_k = 1_{\widetilde{\mathfrak{A}}_k} - 1_{\mathfrak{A}_k}$. Moreover, $\widetilde{\mathfrak{A}}_k = \mathfrak{A}_k \oplus \mathfrak{B}e_k$ (direct sum of C^*-algebras). If $a_k \in \mathfrak{A}_k, b \in \mathfrak{B}$, then $b1 + a_k = be_k + (a_k + b1_{\mathfrak{A}_k})$ and $a_k + b1_{\mathfrak{A}_k} = \alpha_k \in \mathfrak{A}_k$. It follows that

$$\widetilde{\theta}_k(\alpha_k + be_k) = \widetilde{\theta}_k(b1 + a_k) = b + \theta_k(a_k) = \theta_k(\alpha_k).$$

Theorem 3.2 implies now that $\widetilde{\theta}_2 * \widetilde{\theta}_1$ is a completely positive map from $\widetilde{\mathfrak{A}_2} * \widetilde{\mathfrak{A}_1}$ to $\mathfrak{D}$, particularly from $\mathfrak{A}_2 \overline{*} \mathfrak{A}_1$ to $\mathfrak{D}$, and the assertion follows now from Proposition 3.5.

The induction step follows from the above argument and the associativity of the monotone product of maps. To see that, we will again need an argument from [1].

A $*$-algebra A is said to satisfy the *Combes axiom* if for each $x \in A$ there is an $\lambda(x) > 0$ such that $x^*x \leqslant \lambda(x)$. As mentioned in [1], [2], the Stinespring Dilation Theorem can be easily reformulated as follows:

Let $\mathcal{A}$ *be a unital* $*$*-algebra satisfying the Combes axiom and let* $\Phi : \mathcal{A} \longrightarrow \mathcal{L}(\mathcal{H})$ *be a unital completely positive linear map. Then there exist a Hilbert space* K*, a* $*$*-representation* $\pi : \mathcal{A} \longrightarrow \mathcal{L}(\mathcal{K})$ *and an isometry* $V \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ *such that:*

(i) $\Phi(x) = V^*\pi(x)V$ *for all* $x \in \mathcal{A}$;
(ii) $\mathcal{K}$ *is the closed linear span of* $\pi(\mathcal{A})V\mathcal{H}$.

Suppose now that $\theta(A)$ is positive whenever $N(A) \leqslant n$ and let A' be a matrix from some $M_m(\mathfrak{A})$ such that $N(A') = n+1$, that is the words summing in the entries of A' are contain only elements from the subalgebras $\mathfrak{A}_{i_1}, \mathfrak{A}_{i_2}, \dots, \mathfrak{A}_{i_{n+1}}$, with $i_1 < \cdots < i_{n+1}$. Let $\mathfrak{A}(n) = \overline{\underset{1\leqslant j\leqslant n}{\ast}} \mathfrak{A}_{i_j}$. From the induction hypothesis, we have that $\widetilde{\theta} = \underset{1\leqslant j\leqslant n}{\triangleright} \theta_{i_j}$ is a completely positive map from $\mathfrak{A}'$ to $\mathfrak{D}$. Take now $\widetilde{\mathfrak{A}(n)}$ the unitalization of $\mathfrak{A}'$ and extend $\widetilde{\theta}$ to a completely positive map on $\widetilde{\mathfrak{A}}'$ as above.

Since $\widetilde{\mathfrak{A}(n)}$ is spanned by 1 and the unitaries of the C^*-algebras $\{\mathfrak{A}_{i_j}\}_{j=1}^{n}$ (in $\widetilde{\mathfrak{A}(n)}$ they are only partial isometries), we have that it satisfies the Combes axiom. The existence of the Stinespring dilation yields the extension of $\widetilde{\theta}$ to the greatest C^*-algebra norm

$$\|a\| = \sup\{\|\pi(a)\| : \pi \ast\text{-representation of } \widetilde{\mathfrak{A}(n)}\}$$

completion of $\widetilde{\mathfrak{A}(n)}$. Let $\widehat{\mathfrak{A}(n)}$ be this C^*-algebra.

Therefore the entries of A' are words only in elements from $\mathfrak{A}_{i_{n+1}}$ and $\widehat{\mathfrak{A}(n)}$, which are unital C^*-algebras endowed with the completely positive maps $\theta_{i_{n+1}}$ and $\widetilde{\theta}$. The conclusion follows now from the argument in the case $N(A) = 2$ and the associativity of the monotone products. ∎

Remark that $\overline{\underset{i\in I}{\ast}} \mathfrak{A}_i$ satisfies the Combes axiom, since it is generated by the C^*-algebras $\{\mathfrak{A}_i\}_{i\in I}$. The argument from above gives then the following

COROLLARY 3.7. *With the notations from Theorem* 3.6*, the map* $\theta = \underset{i\geqslant 0}{\triangleright} \theta_i$ *extends to a completely positive map on the universal free product (without amalgamation over* $\mathfrak{B}$*)* C^**-algebra* $\widehat{\underset{i\in I}{\ast}} \mathfrak{A}_i$.

4. EMBEDDINGS OF MONOTONE PRODUCTS OF C^*-ALGEBRAS AND COMPLETELY POSITIVE MAPS

This section is in all regards very similar to the Sections 1 and 2 of [6]. Most of the techniques are similar and the results are almost a verbatim translation from the free case to the monotone case. This was to be expected, since the monotone product of Hilbert bimodules is a subspace of the free product and the partial isometries in the definition of the monotone product of C^*-algebras are restrictions of the unitaries from the definition of the free product. The main difference is that we will utilize the construction from Section 2, while [6] is using the construction of the conditionally monotone product from [3].

4.1. MONOTONE PRODUCTS OF C^*-ALGEBRAS. We will use the following version of N. Muraki's construction of the monotone product of C^*-algebras.

Let $\{(\mathfrak{A}_i, \psi_i)\}_{i\in I}$ be a family of unital C^*-algebras containing a common C^*-algebra $\mathfrak{B}$ with $1_{\mathfrak{A}_i} \in \mathfrak{B}$ and each $\mathfrak{A}_i$ endowed with a positive conditional expectation $\psi_i : \mathfrak{A}_i \longrightarrow \mathfrak{B}$ and having faithful GNS representations.

We let $E_i = L^2(\mathfrak{A}_i, \psi_i), \psi_i = \widehat{1}_{\mathfrak{A}_i} \in E_i, E_i = \xi_i\mathfrak{B} \oplus E_i^\circ$. Similarly to the previous section, consider the Hilbert $\mathfrak{B}$-bimodules

$$E = \xi\mathfrak{B} \oplus \bigoplus_{n=1}^{\infty} \Big(\bigoplus_{i_1 > \cdots > i_n} E_{i_1}^\circ \otimes_{\mathfrak{B}} \cdots \otimes_{\mathfrak{B}} E_{i_n}^\circ \Big), \text{ and}$$

$$E(k) = \xi\mathfrak{B} \oplus \bigoplus_{n=1}^{\infty} \Big(\bigoplus_{\substack{i_1 > \cdots > i_n \\ i_1 \leqslant k}} E_{i_1}^\circ \otimes_{\mathfrak{B}} \cdots \otimes_{\mathfrak{B}} E_{i_n}^\circ \Big).$$

Remark that we can define $\widetilde{V}_k : E \longrightarrow E_k \otimes_{\mathfrak{B}} E(k-1)$ similarly to the operators V_k from the previous sections; they are adjointable partial isometries (considering the norm induced by the C^*-norm of $\mathfrak{B}$). For $a \in \mathcal{A}_k$, define

$$j_k(a) = \widetilde{V}_k^*(a \otimes \mathrm{Id})\widetilde{V}_k \in \mathcal{L}(E).$$

Finally, let $\mathfrak{A}$ be the C^*-algebra generated by $\{j_i(\mathfrak{A}_i)\}_{i\in I}$ in $\mathcal{L}(E)$ and let ψ, $\psi : \mathcal{L}(E) \longrightarrow \mathcal{B}$, be the functional given by $\psi(T) = \langle T\xi, \xi\rangle$.

We will call the pair $(\mathfrak{A}, \psi) = \underset{i\in I}{\triangleright}(\mathfrak{A}_i, \psi_i)$ the monotone product of the family of C^*-algebras $\{(\mathfrak{A}_i, \psi_i)\}_{i\in I}$.

The following property was shown in [18] for the case $\mathfrak{B} = \mathbb{C}$ and in [21] for the general setting:

PROPOSITION 4.1. *The functional ψ from above is a conditional expectation with respect to which the subalgebras $\{j_i(\mathfrak{A}_i)\}_{i\in I}$ are monotone independent, i.e. for any $a_k \in j_{i_k}(\mathfrak{A}_{i_k}), 1 \leqslant i \leqslant n$ such that $i_s \neq i_{s+1}$, we have:*
(i) $\psi(a_1 \cdots a_n) = \psi_{i_1}(a_1)\psi(a_2 \cdots a_n)$ *if* $i_1 > i_2$;
(ii) $\psi(a_1 \cdots a_n) = \psi(a_1 \cdots a_{n-1})\psi_{i_n}(a_n)$ *if* $i_n > i_{n-1}$;
(iii) $\psi(a_1 \cdots a_n) = \psi(a_1 \cdots a_{k-1}\psi_{i_k}(a_k)a_{k+1} \cdots a_n)$ *if* $i_{k-1} < i_k > i_{k+1}$.

REMARK 4.2. Actually the subalgebras $\{j_i(\mathfrak{A}_i)\}_{i\in I}$ are satisfying a stronger condition than (ii) and (iii) from the above Proposition. If $k < l$ and $a \in j_k(\mathfrak{A}_k), b \in j_l(\mathfrak{A}_l)$, then

$$ab_{|E\ominus E_l^\circ \otimes E(l-1)} = a\psi(b)_{|E\ominus E_l^\circ \otimes E(l-1)}.$$

Particularly, $a_1a_2a_3 = a_1\psi(a_2)a_3$ whenever $a_i \in j_{k_i}(\mathfrak{A}_{k_i})$ with $k_1 < k_2 > k_3$.

Proof. It suffices to show that $ab\eta = a\psi(b)\eta$ for all $\eta = f_1 \otimes_{\mathfrak{B}} \cdots \otimes_{gB} f_n$, with $f_j \in E_{k_j}^\circ$ such that $l \neq k_1 > \cdots > k_n$.

If $k_1 > l$, then both sides are zero. If $k_1 < l$, then

$$ab\eta = a\widetilde{V}_l^{\,*}((b \otimes \mathrm{Id})\xi_l \otimes \eta) = a\widetilde{V}_l^{\,*}(\psi(b)\xi_l + P_{\xi_l}^{\perp} b\xi_l) \otimes \eta = a\psi(b)\eta.$$

The last part follows from the fact that $j_i(\mathfrak{A}_i)(E) \subseteq E(i)$. ∎

REMARK 4.3. The above construction can easily be modified to obtain a representation of the free product $*$-algebra of the family $\{\mathfrak{A}_i\}_{i\in I}$ that satisfies the relations (i)–(iii) from Proposition 4.1 without the more restrictive condition from Remark 4.2. With the above notations, let $(\mathcal{E}, \xi)$ be the free product bimodule of the family $\{E_i, \xi_i\}_{i\in I}$, that is

$$\mathcal{E} = \mathfrak{B}\xi \oplus \bigoplus_{n=1}^{\infty} \Big(\bigoplus_{i_1 \neq \cdots \neq i_n} E_{i_1}^{\circ} \otimes \cdots \otimes E_{i_n}^{\circ} \Big)$$

where $E_j = E_j^{\circ} \oplus \mathfrak{B}\xi_j$.

We also define

$$\mathcal{E}(k) = \mathfrak{B}\xi \oplus \bigoplus_{n=1}^{\infty} \Big(\bigoplus_{i_1 \neq \cdots \neq i_n, i_1 \leqslant k} E_{i_1}^{\circ} \otimes \cdots \otimes E_{i_n}^{\circ} \Big)$$

and consider the partial isometries $W_k : \mathcal{E} \longrightarrow E_k \otimes \mathcal{E}(k-1)$ given by $W_k\xi = \xi_k \otimes \xi$ and, for $f_1 \otimes \cdots \otimes f_n \in E_{i_1}^{\circ} \otimes \cdots \otimes E_{i_n}^{\circ}$,

$$W_k f_1 \otimes \cdots \otimes f_n = \begin{cases} 0 & \text{if } i_1 > k, \\ f_1 \otimes \cdots \otimes f_n & \text{if } i_1 = k, \\ \xi_k \otimes f_1 \otimes \cdots \otimes f_n & \text{if } i_1 < k. \end{cases}$$

For $T \in \mathfrak{A}_i \subseteq \mathcal{L}(E_i)$, define $u_i(T) = W_k^*(T \otimes \mathrm{Id}_{\mathcal{E}(k-1)})W_k$ and $\psi(\cdot) = \langle \cdot\xi, \xi\rangle$. Since $\underset{i\in I}{\rhd} E_i = E$ is a sub-bimodule of $\mathcal{E}$ and $u_i(a)_{|E} = j_i(a)$, it follows that Proposition 4.1 holds true also for the family $\{u_i(\mathfrak{A}_i)\}_{i\in I}$.

To see that $\{u_i(\mathfrak{A}_i)\}_{i\in I}$ do not satisfy the relations from Remark 4.2, consider $i_1 < i_2 > i_3$ from I and for $j = 1,2,3$ take $a_j \in \mathfrak{A}_{i_j}$ such that $\widehat{a}_j^* \in E_{i_j}^{\circ}$ (that is $\psi(u_{i_j}(a_j)) = 0$). Consider also $f_2 = \widehat{a}_2^*$ and $f_3 = \langle f_2, f_2\rangle \widehat{a}_3^*$.

Denoting $A_j = u_{i_j}(a_j)$, we have that $\psi(A_2) = 0$, hence $A_1\psi(A_2)A_3 = 0$. On the other hand, since $\langle a_3 f_3, \xi\rangle = \langle f_3, f_3\rangle \neq 0$, we have that $a_3 f_3 = \zeta + \langle f_3, f_3\rangle$ with $\zeta \in E_{i_3}^{\circ}$. Therefore

$$A_2A_3 f_3 \otimes f_2 = a_2(\zeta \otimes f_2) + \langle f_3, f_3\rangle f_2 = \widehat{a}_2 \otimes \zeta \otimes f_3 + A_2\langle f_3, f_3\rangle f_2.$$

Since $\widehat{a}_2 \in E_{i_2}^{\circ}$ and $i_1 < i_2$, we have that

$$A_1A_2A_3 f_3 \otimes f_2 = A_1A_2\langle f_3, f_3\rangle f_2 = \widehat{a}_1\langle A_2\langle f_3, f_3\rangle f_2, \xi\rangle = \widehat{a}_1\langle\langle f_2, f_2\rangle \widehat{a_3^*}, \langle f_2, f_2\rangle \widehat{a_3^*}\rangle \neq 0.$$

LEMMA 4.4 ([8], Lemma 1.1). *Let $\{\mathfrak{A}_i\}_{i\in I}$ be a family of unital C^*-algebras containing a common unital C^*-subalgebra $\mathfrak{B}$ and having conditional expectations $\psi_i : \mathfrak{A}_i \longrightarrow \mathfrak{B}$ whose* GNS *representations are faithful. Let*

$$(\mathfrak{A}, \psi) = \underset{i\in I}{\rhd} (A_i, \psi_i)$$

be their monotone product of C^*-algebras as defined in Section 2. Then for every $i_0 > 0$ there exists a conditional expectation $\Psi_{i_0} : \mathfrak{A} \longrightarrow \mathfrak{A}_{i_0}$ such that $\Psi_{i_0|\mathfrak{A}_i} = \psi_i$ for every $i \neq i_0$ and, if $a_k \in \mathfrak{A}_{i_k}, i_s \neq i_{s+1}$, then

$$\Psi_{i_0}(a_1 \cdots a_n) = \begin{cases} \Psi_{i_0}(a_1)\Psi_{i_0}(a_2 \cdots a_n) & \text{if } i_1 > i_2, \\ \Psi_{i_0}(a_1 \cdots \psi_{i_k}(a_k) \cdots a_n) & \text{if } i_{k-1} < i_k > i_{k+1}, \\ \Psi_{i_0}(a_1 \cdots a_{n-1})\Psi_{i_0}(a_n) & \text{if } i_n > i_{n-1}. \end{cases} \tag{4.1}$$

Proof. Let $E_i = L^2(\mathfrak{A}_i, \psi_i), \xi_i = \widehat{1}_{\mathfrak{A}_i} \in E_i$, $E_i = \xi_i \mathfrak{B} \oplus E_i^\circ$. By construction, the algebra $\mathfrak{A}$ acts on the Hilbert bimodule $(E, \xi) = \underset{i\in I}{\rhd}(E_i, \xi_i)$. Identify the submodule $\xi\mathfrak{B} \oplus E_{i_0}^\circ$ with E_{i_0} and let $Q_{i_0} : E \longrightarrow E_{i_0}$ be the projection. Then $\Psi_{i_0}(x) = Q_{i_o} x Q_{i_0}$ has the desired properties. ∎

REMARK 4.5. With the notations from Section 2, consider

$$F = \mathfrak{A}_{i_0} \oplus \bigoplus_{n\geqslant 1} \bigoplus_{i_1 > \cdots > i_n \neq i_0} E_{i_1}^\circ \otimes_{\mathfrak{B}} \cdots E_{i_n}^\circ \otimes_{\mathfrak{B}} \otimes_{\mathfrak{B}} \mathfrak{A}_{i_0}.$$

Then $\Psi_{i_0} = \langle \cdot 1_{\mathfrak{A}_{i_0}}, 1_{\mathfrak{A}_{i_0}} \rangle$.

Let $\rho : \mathfrak{A}_{i_0} \longrightarrow \mathcal{L}(K)$ be a unital $*$-homomorphism for some Hilbert space K. Then ρ induces a $*$-homomorphism $\rho|^{\mathfrak{A}} : \mathfrak{A} \longrightarrow \mathcal{L}(F \otimes_\rho K)$ determined by its restrictions $\rho_i = \rho_{|\mathfrak{A}_i} \longrightarrow \mathcal{L}(F \otimes_\rho K)$ given as follows.

Writing $\mathcal{K} = F \otimes_\rho K$, we have

$$\mathcal{K} = K \oplus \bigoplus_{n\geqslant 1} \bigoplus_{i_1 > \cdots > i_n \neq i_0} E_{i_1}^\circ \otimes_{\mathfrak{B}} \cdots E_{i_n}^\circ \otimes_{\mathfrak{B}} \otimes_\rho K.$$

Consider the Hilbert spaces

$$\mathcal{K}(i) = (\eta_i \mathfrak{B} \otimes_{\rho_{|\mathfrak{B}}} K) \oplus \bigoplus_{n\geqslant 1} \bigoplus_{i_1 > \cdots > i_n,\, i > i_1, i_n \neq i_0} E_{i_1}^\circ \otimes_{\mathfrak{B}} \cdots E_{i_n}^\circ \otimes_{\mathfrak{B}} \otimes_\rho K. \text{ if } i \neq i_0,$$

$$\mathcal{K}(i_0) = \bigoplus_{n\geqslant 1} \bigoplus_{i_1 > \cdots > i_n,\, i_0 > i_1, i_n \neq i_0} E_{i_1}^\circ \otimes_{\mathfrak{B}} \cdots E_{i_n}^\circ \otimes_{\mathfrak{B}} \otimes_\rho K,$$

where $\eta_i \mathfrak{B}$ is just the Hilbert $\mathfrak{B}$-bimodule $\mathfrak{B}$ with identity element denoted by η_i. If $i \neq i_0$, consider the partial isometry $W_i : E_i \otimes_{\mathfrak{B}} \mathcal{K}(i) \longrightarrow \mathcal{K}$ given by

$$\begin{aligned} \xi_i \otimes (\eta_i \otimes v) &\mapsto v \\ \zeta_i \otimes (\eta_i \otimes v) &\mapsto \zeta \otimes v \\ \xi_i \otimes (\zeta_1 \otimes \cdots \otimes \zeta_n \otimes v) &\mapsto \zeta_1 \otimes \cdots \otimes \zeta_n \otimes v \\ \zeta \otimes (\zeta_1 \otimes \cdots \otimes \zeta_n \otimes v) &\mapsto \zeta \otimes \zeta_1 \otimes \cdots \otimes \zeta_n \otimes v \end{aligned}$$

for all $v \in K, \zeta \in E_i^\circ, \zeta_j \in E_{i_j}^\circ$. Then for every $a \in \mathfrak{A}_i$ with $i \neq i_0$ we have

$$\rho_i(a) = W_i(a \otimes \mathrm{Id}_{\mathcal{K}(i)})W_i^*.$$

Similarly, for i_0, we define the partial isometry $W_{i_0} : K \oplus (E_{i_0} \otimes_{\mathfrak{B}} \mathcal{K}(i_0)) \longrightarrow \mathcal{K}$ given by

$$\begin{array}{rcl} v \oplus 0 & \mapsto & v \\ 0 \oplus (\xi_{i_0} \otimes (\zeta_1 \otimes \cdots \otimes \zeta_n \otimes v) & \mapsto & \zeta_1 \otimes \cdots \otimes \zeta_n \otimes v \\ 0 \oplus (\zeta \otimes (\zeta_1 \otimes \cdots \otimes \zeta_n \otimes v) & \mapsto & \zeta \otimes \zeta_1 \otimes \cdots \otimes \zeta_n \otimes v. \end{array}$$

Then $\rho_{i_0}(a) = W_{i_0}(\rho(a) \oplus (a \otimes 1_{\mathcal{K}(i_0)}))W_{i_0}^*$. Note that the above description is related to the construction from Section 2.

4.2. EMBEDDINGS OF MONOTONE PRODUCTS OF C^*-ALGEBRAS AND COMPLETELY POSITIVE MAPS.

PROPOSITION 4.6 ([8], Theorem 1.3). *Let $\mathfrak{B} \subseteq \widetilde{\mathfrak{B}}$ be a (not necessarily unital) inclusion of unital C^*-algebras. For each $i \in I$, suppose*

$$\begin{array}{ccccc} 1_{\widetilde{\mathfrak{A}}_i} & \in & \widetilde{\mathfrak{B}} & \subseteq & \widetilde{\mathfrak{A}}_i \\ & & \cup & & \cup \\ 1_{\mathfrak{A}_i} & \in & \mathfrak{B} & \subseteq & \mathfrak{A}_i \end{array}$$

are inclusions of C^*-algebras. Suppose that $\widetilde{\psi}_i : \widetilde{\mathfrak{A}}_i \longrightarrow \widetilde{\mathfrak{B}}$ are conditional expectations such that $\widetilde{\psi}_i(\mathfrak{A}_i) \subseteq \mathfrak{B}$ and assume that $\widetilde{\psi}_i$ and the restrictions $\widetilde{\psi}_{i|\mathfrak{A}_i}$ have faithful GNS representations. Let

$$(\widetilde{\mathfrak{A}}, \widetilde{\psi}) = \mathop{\triangleright}_{i \in I} (\widetilde{\mathfrak{A}}_i, \widetilde{\psi}_i), \quad (\mathfrak{A}, \psi) = \mathop{\triangleright}_{i \in I} (\mathfrak{A}_i, \psi_i),$$

be the monotone products of C^*-algebras. Then there is a unique $*$-homomorphism $\kappa : \mathfrak{A} \longrightarrow \widetilde{\mathfrak{A}}$ such that for every $i \in I$ the diagram

$$\begin{array}{ccc} \widetilde{\mathfrak{A}}_i & \hookrightarrow & \widetilde{\mathfrak{A}} \\ \cup & & \uparrow \kappa \\ \mathfrak{A}_i & \hookrightarrow & \mathfrak{A} \end{array}$$

commutes, where the horizontal arrows are the inclusions arising from the monotone product construction. Moreover, κ is necessarily injective.

Proof. Note that, since $\mathfrak{A}$ is generated by $\bigcup \mathfrak{A}_i$, it is clear that κ will be unique if it exists. Also, we can suppose that the inclusions $\mathfrak{B} \subseteq \widetilde{\mathfrak{B}}$ and $\mathfrak{A} \subseteq \widetilde{\mathfrak{A}}$ are unital: if $1_{\mathfrak{B}} \neq 1_{\widetilde{\mathfrak{B}}}$, then we may replace $\mathfrak{B}$ by $\mathfrak{B} + \mathbb{C}(1_{\widetilde{\mathfrak{B}}} - 1_{\mathfrak{B}})$ and each $\mathfrak{A}_i$ by $\mathfrak{A}_i + \mathbb{C}(1_{\widetilde{\mathfrak{A}}_i} - 1_{\mathfrak{A}_i})$.

Let $(\widetilde{\pi}_i, \widetilde{E}_i, \widetilde{\xi}_i) = \mathrm{GNS}(\widetilde{\mathfrak{A}}_i, \widetilde{\psi}_i)$, $(\pi_i, \widetilde{E}_i, \xi_i) = \mathrm{GNS}(\mathfrak{A}_i, \psi_i)$ and $(\widetilde{E}, \widetilde{\xi}) = \mathop{\triangleright}_{i \in I} (\widetilde{E}_i, \widetilde{\xi}_i)$, respectively $(E, \xi) = \mathop{\triangleright}_{i \in I} (E_i, \xi_i)$.

The inclusion $\mathfrak{A}_i \hookrightarrow \widetilde{\mathfrak{A}}_i$ gives an inner-product-preserving isometry of Banach spaces $E_i \hookrightarrow \widetilde{E}_i$ sending ξ_i to $\widetilde{\xi}_i$ and E_i° to a subspace of $\widetilde{E}_i^\circ$ and allowing, for each $i_1 > \cdots > i_n$, a cannonical identification of

$$E_{i_1}^\circ \otimes_{\mathfrak{B}} \cdots \otimes_{\mathfrak{B}} E_{i_{p-1}}^\circ \otimes_{\mathfrak{B}} \widetilde{E}_{i_p}^\circ \otimes_{\widetilde{\mathfrak{B}}} \cdots \otimes_{\widetilde{\mathfrak{B}}} \widetilde{E}_{i_n}^\circ$$

with the a closed subspace of $\widetilde{E}_{i_1}^{\circ} \otimes_{\widetilde{\mathfrak{B}}} \cdots \otimes_{\widetilde{\mathfrak{B}}} \widetilde{E}_{i_n}^{\circ}$. We will identify E with a subspace of $\widetilde{E}$ as follows:

$$E \cong \widetilde{\xi}\mathfrak{B} \oplus \bigoplus_{n=1}^{\infty} \Big(\bigoplus_{i_1 > \cdots > i_n} E_{i_1}^{\circ} \otimes_{\mathcal{B}} \cdots \otimes_{\mathcal{B}} E_{i_n}^{\circ} \Big) \subset \widetilde{E}.$$

Let now $\mathfrak{U} = \underline{\ast}\mathfrak{A}_i$ be the universal algebraic free product *without* amalgamation. Let $\sigma : \mathfrak{U} \longrightarrow \mathcal{L}(E)$, respectively $\widetilde{\sigma} : \mathfrak{U} \longrightarrow \mathcal{L}(\widetilde{E})$ be the homomorphism extending the homomorphisms $\pi_i : \mathfrak{A}_i \longrightarrow \mathcal{L}(E)$, respectively $\widetilde{\pi}_{i|\mathfrak{A}_i} : \mathfrak{A}_i \longrightarrow \mathcal{L}(\widetilde{E})$ (particularly, $\overline{\sigma(\mathfrak{U})} = \mathfrak{A}$).

In order to show that κ exists, it suffices to show that $\|\widetilde{\sigma}(x)\| \leqslant \|\sigma(x)\|$, for all $x \in \mathfrak{U}$.

Note that $\|\widetilde{\sigma}(x)\| \geqslant \|\sigma(x)\|$ for all $x \in \mathfrak{U}$, since the subspace E of $\widetilde{E}$ is invariant under $\widetilde{\sigma}(\mathfrak{U})$ and $\widetilde{\sigma}_{|E} = \sigma$. Henceforth, if κ exists, then it is injective.

Let τ be a faithful representation of $\widetilde{\mathfrak{B}}$ on a Hilbert space $\mathcal{W}$, then consider the Hilbert space $\widetilde{E} \otimes_{\tau} \mathcal{W}$ and let $\widetilde{\lambda} : \mathcal{L}(\widetilde{E}) \longrightarrow \mathcal{L}(\widetilde{E} \otimes_{\tau} \mathcal{W})$ be the $\ast$-homomorphism given by $\widetilde{\lambda}(x) = x \otimes 1_{\mathcal{W}}$. $\widetilde{\lambda}$ is faithful, hence it will suffice to show that $\|\widetilde{\lambda} \circ \widetilde{\sigma}(x)\| \leqslant \|\sigma(x)\|$ for all $x \in \mathfrak{U}$.

We will show that $\widetilde{\lambda} \circ \widetilde{\sigma}$ decomposes as a direct sum of subrepresentations, each of which is of the form $(\nu|^{\mathfrak{A}}) \circ \sigma$, where $\nu|^{\mathfrak{A}}$ is the $\ast$-representation of $\mathfrak{A}$ induced from a representation ν of some $\mathfrak{A}_i$.

For $n > 0$ and $i_1 > \cdots > i_n$ and $1 \leqslant p \leqslant n$, consider the Hilbert space $H_p^{(i_1,\ldots,i_n)} = E_{i_1}^{\circ} \otimes_{\mathfrak{B}} \cdots \otimes_{\mathfrak{B}} E_{i_{p-1}}^{\circ} \otimes_{\mathfrak{B}} K_{i_p} \otimes_{\widetilde{\mathfrak{B}}} \widetilde{E}_{i_{p+1}}^{\circ} \otimes_{\widetilde{\mathfrak{B}}} \cdots \otimes_{\widetilde{\mathfrak{B}}} \widetilde{E}_{i_n}^{\circ} \otimes_{\tau} \mathcal{W}$ defined as

$$\begin{aligned} E_{i_1}^{\circ} \otimes_{\mathfrak{B}} \cdots \otimes_{\mathfrak{B}} E_{i_{p-1}}^{\circ} \otimes_{\mathfrak{B}} \widetilde{E}_{i_p}^{\circ} \otimes_{\widetilde{\mathfrak{B}}} & \cdots \otimes_{\widetilde{\mathfrak{B}}} \widetilde{E}_{i_n}^{\circ} \otimes_{\tau} \mathcal{W} \ominus \\ E_{i_1}^{\circ} \otimes_{\mathfrak{B}} & \cdots \otimes_{\mathfrak{B}} E_{i_{p-1}}^{\circ} \otimes_{\mathfrak{B}} E_{i_p}^{\circ} \otimes_{\widetilde{\mathfrak{B}}} \cdots \otimes_{\widetilde{\mathfrak{B}}} \widetilde{E}_{i_n}^{\circ} \otimes_{\tau} \mathcal{W}. \end{aligned}$$

Then

$$\widetilde{E} \otimes_{\tau} \mathcal{W} = (E \otimes_{\tau_{|\mathfrak{B}}} \mathcal{W}) \oplus \Big(\bigoplus_{n=1}^{\infty} \bigoplus_{i_1 > \cdots > i_n, 1 \leqslant p \leqslant n} H_p^{(i_1,\ldots,i_n)} \Big).$$

As previously mentioned, $\widetilde{\sigma}(\mathfrak{U})E \subseteq E$ and $\widetilde{\sigma}_{|E} = \sigma$, so $E \otimes_{\tau_{|\mathfrak{B}}} \mathcal{W}$ is invariant under $\widetilde{\lambda} \circ \widetilde{\sigma}(\mathfrak{U})$, and $\|\widetilde{\lambda} \circ \widetilde{\sigma}(x)_{|E \otimes_{\tau} \mathcal{W}}\| = \|\sigma(x)\|$ for all $x \in \mathfrak{U}$.

Define $\widetilde{\mathcal{W}}(i_1, \ldots, i_n) = \overline{\widetilde{\lambda} \circ \widetilde{\sigma}(\mathfrak{U}) H_1^{(i_1,\ldots,i_n)}}$. Since $\widetilde{\pi}_i(\mathfrak{A}_i)E_i \subseteq E_i$ we have that

$$\widetilde{\mathcal{W}}(i_1, \cdots, i_n) = H_1^{(i_1,\ldots,i_n)} \oplus \Big(\bigoplus_{l \geqslant 1} \bigoplus_{k_1 > \cdots > k_l, k_l > i_1} E_{k_1}^{\circ} \otimes_{\mathfrak{B}} \cdots E_{k_s} \otimes_{\mathfrak{B}} H_1^{(i_1,\ldots,i_n)} \Big).$$

Thus,

$$\widetilde{E} \otimes_{\tau} \mathcal{W} = (E \otimes_{\tau_{|\mathfrak{B}}} \mathcal{W}) \oplus \bigoplus_{n \geqslant 1} \bigoplus_{i_1 > \cdots > i_n} \widetilde{\mathcal{W}}(i_1, \ldots, i_n);$$

Hence to prove the theorem it will suffice to show that for all $i_1 > \cdots > i_n$ and all $x \in \mathfrak{U}$,

$$\|\widetilde{\lambda} \circ \widetilde{\sigma}(x)_{|\widetilde{\mathcal{W}}(i_1,\dots,i_n)}\| \leqslant \|\sigma(x)\|. \tag{4.2}$$

But letting $\nu : \mathfrak{A}_{i_1} \longrightarrow \mathcal{L}(H_1^{(i_1,\dots,i_n)})$ be the $*$-homomorphism

$$\nu(a) = (\widetilde{\pi}_{i_1}(a) \otimes 1_{\widetilde{E}^{\circ}_{i_2} \otimes_{\mathfrak{B}} \cdots \otimes_{\mathfrak{B}} \widetilde{E}^{\circ}_{i_n} \otimes_{\tau} \mathcal{W}})_{|H_1^{(i_1,\dots,i_n)}}$$

and considering $\nu|^{\mathfrak{A}}$ be the representation of $\mathfrak{A}$ induced from ν with respect to the conditional expectation $\Psi_{i_1} : \mathfrak{A} \longrightarrow \mathfrak{A}_{i_1}$ found in Lemma 4.4, it is straightforward to check that

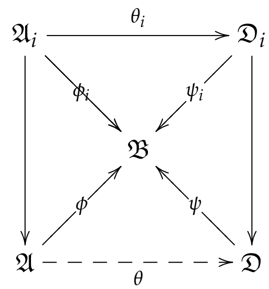
$$\widetilde{\lambda} \circ \widetilde{\sigma}_{|\widetilde{\mathcal{W}}(i_1,\cdots,i_n)} = (\nu|^{\mathfrak{A}}) \circ \sigma,$$

which, in turn implies (4.2). ∎

THEOREM 4.7 ([8], Theorem 2.2). *Let $\mathcal{B}$ be a unital C^*-algebra, and for every $i \geqslant 0$ let $\mathcal{A}_i$ and D_i be unital C^*-algebras containing copies of $\mathcal{B}$ as unital C^*-subalgebras and having conditional expectations $\phi_i : \mathcal{A}_i \longrightarrow \mathcal{B}$, respectively $\psi_i : D_i \longrightarrow \mathcal{B}$, whose GNS representations are faithful. Suppose that for each $i \geqslant 0$ there is a unital completely positive map $\theta_i : \mathcal{A}_i \longrightarrow D_i$ that is also a $\mathcal{B}$ bimodule map and satisfies $\psi_i \circ \theta_i = \phi_i$. Denote*

$$(\mathcal{A}, \phi) = \underset{i \in I}{\triangleright}(\mathcal{A}_i, \phi_i), \quad (D, \psi) = \underset{i \in I}{\triangleright}(D_i, \psi_i),$$

the monotone products of C^*-algebras. Then there is a unital completely positive map $\theta : \mathcal{A} \longrightarrow D$ such that for all $i \geqslant 0$ the diagram



commutes, where the vertical arrows are the (non-unital) inclusions arising from the monotone product construction. Moreover, the mapping θ satisfies:

(i) $\theta(a_1 \cdots a_n) = \theta(a_1)\theta(a_2 \cdots a_n)$, *if* $i_1 > i_2$;
(ii) $\theta(a_1 \cdots a_n) = \theta(a_1 \cdots a_{n-1})\theta(a_n)$, *if* $i_n > i_{n-1}$;
(iii) $\theta(a_1 \cdots a_n) = \theta(a_1 \cdots a_{l-1} \cdot \phi_{i_l}(a_l) \cdot a_{l+1} \cdots a_n)$ *if* $i_{l-1} < i_l > i_{l+1}$.

Proof. Let $(\pi_I, E_i, \xi_i) = \mathrm{GNS}(D_i, \psi_i)$, and $(E, \xi) = \triangleright(E_i, \xi_i)$, as in the previous section.

Consider the Hilbert $\mathcal{B}$-bimodule $F_i = \mathcal{A}_i \otimes_{\pi_i \circ \theta_i} E_i$ with the distinguished element $\eta_i = 1 \otimes \xi_i \in F_i$. The mapping θ_i restricts to the identity map on $\mathcal{B}$, so in F_i we have that $b \otimes \zeta = 1 \otimes (b\zeta)$ for every $b \in \mathcal{B}$. Consider the unital $*$-homomorphism $\sigma_i : \mathcal{A}_i \longrightarrow \mathcal{L}(F_i)$

$$\sigma_i(a_1)(a_2 \otimes \zeta) = (a_1 a) \otimes \zeta, \quad \text{for all } a_1, a \in \mathcal{A}_1, \zeta \in E_i$$

and the map $\rho_i : \mathcal{L}(F_i) \longrightarrow \mathcal{B}$ given by $\rho_i(x) = \langle \eta_i, x\eta_i \rangle$. As in [8] we have that, identifying $\mathcal{B}$ with $\sigma_i(\mathcal{B}) \subseteq \mathcal{L}(F_i)$, the map $\rho_i : \mathcal{L}(F_i) \longrightarrow \mathcal{B}$ is a conditional expectation, that $L^2(\mathcal{L}(F_i), \rho_i) \cong F_i$, that the GNS representation of ρ_i is faithful on $\mathcal{L}(F_i)$ and that $\rho_i \circ \sigma_i = \phi_i$.

Take now $(\mathcal{M}, \rho) = \triangleright(\mathcal{L}(F_i), \rho_i)$ and note that (see [21]) $\mathcal{M} \subseteq \mathcal{L}(F)$, where $(F, \eta) = \triangleright(F_i, \eta_i)$. By Proposition 4.6 there is a $*$-homomorphism $\sigma : \mathcal{A} \longrightarrow \mathcal{M}$ such that $\sigma_{|\mathcal{A}_i} = \sigma_i$.

Consider the operator $v_i : E_i \longrightarrow F_i$ given by $\zeta \longrightarrow 1 \otimes \zeta$. As shown in 4.4, proof of Theorem 2.2, we have that v_i is an adjointable (its adjoint being the operator $F_i \longrightarrow E_i$ sending $a \otimes \zeta$ to $\theta_i(a)\zeta$), that $v_i(E_i^\circ) \subseteq F_i^\circ$ and $v_i^* v_i = 1$. Since θ_i is a left $\mathfrak{B}$-bimodule map, $v_i(b\zeta) = 1 \otimes (b\zeta) = b \otimes \zeta = b(v(\zeta))$, for all $b \in \mathfrak{B}$, $\zeta \in E_i$.

Taking direct sum of operators $v_{i_1} \otimes \cdots \otimes v_{i_n}$, we get that $v \in \mathcal{L}(E)$ such that $\langle v\zeta, \zeta \rangle = \langle \zeta, \zeta \rangle$ for every $\zeta \in E$, that $v\xi = \eta$ and

$$v(\zeta_1 \otimes \cdots \otimes \zeta_n) = (v_{i_1}\zeta_1) \otimes \cdots \otimes (v_{i_n}\zeta_n), \quad \text{whenever } \zeta_j \in E_{i_j}, i_1 > \cdots > i_n.$$

Let $\theta : \mathfrak{A} \longrightarrow \mathcal{L}(E)$ be the unital completely positive map

$$\theta(x) = v^* \sigma(x) v.$$

We will show that θ satisfies the Theorem. In order to show that the diagram commutes, let $w_i : E \longrightarrow E_i \otimes_{\mathfrak{B}} E(i-1)$ and $y_i : F \longrightarrow F_i \otimes_{\mathfrak{B}} F(i-1)$ be the partial isometries that we used in the monotone product construction for the inclusions $\mathfrak{A}_i \hookrightarrow \mathfrak{A}$, respectively $\mathcal{L}(F_i) \hookrightarrow \mathcal{M}$. Exactly as in [8], note that $v_i(E(i-1)) \subseteq F(i-1)$ and that $y_i v = (v_i \otimes v_{|E_i}) w_i$.

Hence, for $a \in \mathfrak{A}_i$, we have that

$$\begin{aligned}\theta(a) &= v^*\sigma(a)v = v^*\sigma_i(a)v = v^* y_i[\sigma_i(a) \otimes 1_{F(i-1)}] y_i v \\ &= w_i^*[v_i \sigma_i(a) v_i \otimes (v_{|E(i-1)})^* v_{|E(i)}] w_i = w_i^*[\theta_i(a) \otimes 1 - E(i)] w_i = \theta_i(a).\end{aligned}$$

It suffices to show (i)–(iii), since they also imply that $\mathfrak{A} \subseteq \mathfrak{D}$, finishing the proof.

For (i) we need to show that

$$\theta(a_1 \cdots a_n)\widetilde{\zeta} = \theta(a_1 \cdots a_{n-1})\theta(a_n)\widetilde{\zeta} \tag{4.3}$$

for all $\widetilde{\zeta} \in E$ and all $a_k \in \mathfrak{A}_{i_k}$, $i_s \neq i_{s+1}$ and $i_n > i_{n-1}$.

To simplify the notations, for $a \in \mathfrak{A}_i$, we will write $a^\circ = a - \phi_i(a) \in \mathfrak{A}_i^\circ$.

If $\widetilde{\zeta} = \xi$, (4.3) becomes

$$v^*\sigma(a_1 \cdots a_n) v\xi = v^*\sigma(a_1 \cdots a_{n-1} v v^* \sigma(a_n) v\xi.$$

But

$$\begin{aligned}\sigma(a_1\cdots a_n)v\xi &= \sigma(a_1\cdots a_n)(1\otimes\xi) = \sigma(a_1)\cdots\sigma(a_n)(1\otimes\xi)\\ &= \sigma(a_1)\cdots\sigma(a_{n-1})(a_n\otimes\xi) = \sigma(a_1)\cdots\sigma(a_{n-1})[a_n^\circ + \phi_n(a_n))\otimes\xi]\\ &= \sigma(a_1)\cdots\sigma(a_{n-1})(1\otimes\phi_n(a_n)\xi)\end{aligned}$$

since $a_n^\circ\otimes\xi\in E_n^\circ$ and $\phi_n(a_n)\otimes\xi = 1\otimes\phi_n(a_n)\xi$. Note also that

$$\begin{aligned}\sigma(a_1\cdots a_{n-1})vv^*\sigma(a_n)v\xi &= \sigma(a_1\cdots a_{n-1})vv^*(a_n\otimes\xi) = \sigma(a_1\cdots a_{n-1})(1\otimes\widehat{\theta(a_n)})\\ &= \sigma(a_1\cdots a_{n-1})(1\otimes\widehat{\theta(a_n^\circ)} + 1\otimes\phi_n(a_n)\xi)\\ &= \sigma(a_1\cdots a_{n-1})(1\otimes\phi_n(a_n)\xi),\ \text{ since } \widehat{\theta(a_n^\circ)}\in E_n^\circ.\end{aligned}$$

Suppose now that $\widetilde{\zeta} = \zeta_1\otimes\cdots\otimes\zeta_m$, with $\zeta_j\in E_{l_j}^\circ$, $l_1>\cdots>l_m$; we will use the notation $\widetilde{\zeta}'$ for $\zeta_2\otimes\cdots\otimes\zeta_m$.

If $l_1 > i_n$, then both sides of (4.3) are zero. If $l_1 < i_n$, then

$$\sigma(a_n)v\widetilde{\zeta} = \sigma(a_n)(1\otimes\xi_n)\otimes v\widetilde{\zeta}$$

and the argument reduces to the case $\widetilde{\zeta} = \xi$.

If $l_1 = i_n$, then we have

$$\begin{aligned}\sigma(a_1\cdots a_{n-1})vv^*\sigma(a_n)v\xi &= \sigma(a_1\cdots a_{n-1})vv^*\sigma(a_n)(1\otimes\zeta_1)\otimes(v\widetilde{\zeta}')\\ &= \sigma(a_1\cdots a_{n-1})v(\widehat{\theta(a_n)\zeta_1}\otimes\widetilde{\zeta}')\\ &= \sigma(a_1\cdots a_{n-1})(1\otimes\widehat{\theta(a_n)\zeta_1})\otimes(v\widetilde{\zeta}')\\ &= \sigma(a_1\cdots a_{n-1})\langle\widehat{\theta(a_n)\zeta_1},\xi_{i_n}\rangle(v\widetilde{\zeta}').\end{aligned}$$

On the other hand,

$$\begin{aligned}\sigma(a_1\cdots a_n)v^*\widetilde{\zeta} &= \sigma(a_1\cdots a_{n-1})\sigma(a_n)((1\otimes\zeta_1)\otimes(v\widetilde{\zeta}'))\\ &= \sigma(a_1\cdots a_{n-1})((a_n\otimes\zeta_1)\otimes(v\widetilde{\zeta}')).\end{aligned}$$

$a_n\otimes\zeta_1$ decomposes as $\langle a_n\otimes\zeta_1, 1\otimes\xi_{i_n}\rangle\xi_{i_n}+\eta$, with $\eta\in F_{i_n}^\circ$, therefore the equality above becomes

$$\sigma(a_1\cdots a_n)v^*\widetilde{\zeta} = \sigma(a_1\cdots a_{n-1})\langle a_n\otimes\zeta_1, 1\otimes\xi_{i_n}\rangle(v\widetilde{\zeta}').$$

Since we defined F_i as $\mathfrak{A}_i\otimes_{\pi_i\circ\theta_i}E_i$, we have that

$$\langle a_n\otimes\zeta_1, 1\otimes\xi_{i_n}\rangle = \langle\widehat{\theta(a_n)}\zeta_1,\xi_{i_n}\rangle,$$

hence the proof of property (i) is complete.

For (ii), it suffices to prove the property for the biggest $k\in\{1,2,\ldots,n\}$ such that $i_{k-1}<i_k>i_{k+1}$; also, since (i) was proved, we can suppose that $i_{k+1}>\cdots>i_n$. In this framework, we need to show that

$$v^*\sigma(a_1\cdots a_n)v = v^*\sigma(a_1\cdots a_{k-1}\phi_{i_k}(a_k)a_{k+1}\cdots a_n)v,$$

that is

$$v^*\sigma(a_1\cdots a_{k-1}a_k)\sigma(a_{k+1}\cdots a_n)v = v^*\sigma(a_1\cdots a_{k-1})\phi_{i_k}(a_k)\sigma(a_{k+1}\cdots a_n)v.$$

Since $i_k > i-k+1 > \cdots > i_n$, it follows that $\sigma(a_{k+1}\cdots a_n)v\widetilde{\zeta} \in F(i_k)$, for all $\widetilde{\zeta} \in E$, hence the assertion is equivalent to the first three cases from the proof of property (i).

For part (iii), we need to show that

$$v^*\sigma(a_1\cdots a_n)v = v^*\sigma(a_1)vv^*\sigma(a_2\cdots a_n)v$$

whenever $i_1 > i_2$. Since (i) and (ii) are proved, we can suppose that $i_1 > i_2 > \cdots > i_n$. In this framework we have that $\sigma(a_2\cdots a_n)v\widetilde{\zeta} \in F(i_1)$ for all $\widetilde{\zeta} \in E$, therefore it suffices to show that

$$v^*\sigma(a_1)\eta = v^*\sigma(a_1)vv^*\eta \quad \text{for all } \eta \in F(i_1). \tag{4.4}$$

But $v^*\sigma(a_1)\eta = v^*(a_1 \otimes \xi_{i_1}) \otimes \eta = \theta(a_1)\xi_{i_1} \otimes v^*\eta$. Also, since $v^*\eta \in E(i_1)$, we have that $vv^*\eta \in F(i_1)$, hence

$$\begin{aligned} v^*\sigma(a_1)vv^*\eta &= v^*\sigma(a_1)[(1\otimes \xi_{i_1})\otimes vv^*\eta] \\ &= v^*[(a_1\otimes\xi_{i_1})\otimes vv^*\eta] = \theta(a_1)\xi_{i_1}\otimes v^*vv^*\eta \\ &= \theta(a_1)\xi_{i_1}\otimes v^*\eta, \text{ since } v^*v = \mathrm{Id}. \quad \blacksquare \end{aligned}$$

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