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THE ENTIRE CYCLIC COHOMOLOGY OF NONCOMMUTATIVE 2-TORI

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ABSTRACT. Our aim in this paper is to compute the entire cyclic cohomology of noncommutative 2-tori. First of all, we clarify their algebraic structure of noncommutative 2-tori as an F^*-algebra, according to the idea of Elliott–Evans. Actually, they are the inductive limit of subhomogeneous F^*-algebras. Using such a result, we compute their entire cyclic cohomology, which is isomorphic to their periodic one as a complex vector space.

KEYWORDS: *Entire cyclic cohomology, Fréchet $*$-algebras, noncommutative tori.*

MSC (2000): 19D55, 46L80, 46L87.

INTRODUCTION

Elliott and Evans [3] show that the irrational rotation C^*-algebras (or noncommutative 2-tori) T^2_θ are isomorphic to certain inductive limits, which are now called AT-algebras,

$$\varinjlim(C(T)\otimes(M_{q_{2n}}(\mathbb{C})\oplus M_{q_{2n-1}}(\mathbb{C})),\pi_n).$$

To compute the entire cyclic cohomology of their smooth parts $(T^2_\theta)^\infty$, we need to know their algebraic structure. In this paper, we elaborate Elliott and Evans' result cited above, and show that $(T^2_\theta)^\infty$ are isomorphic to inductive limits

$$\varinjlim(C^\infty(T)\otimes(M_{q_{2n}}(\mathbb{C})\oplus M_{q_{2n-1}}(\mathbb{C})),\pi^\infty_n)$$

as Fréchet $*$-algebras (or F^*-algebras). Using this fact, we can compute their entire cyclic cohomology quite easily.

In Section 1, we prepare the notations needed for $(T^2_\theta)^\infty$ and review the definition of entire cyclic cohomology. In Section 2, we determine the algebraic structure of $(T^2_\theta)^\infty$ by using appropriate smooth functions to construct projections instead of the original ones due to Rieffel [8]. In Section 3, it is shown that the functor of entire cyclic cohomology H^*_ε is continuous in some sense. More precisely,

$$H^*_\varepsilon(\varinjlim \mathfrak{A}_n) \simeq \varprojlim H^*_\varepsilon(\mathfrak{A}_n)$$

(cf. Meyer [7]), where the right hand side means the projective limit of $H^*_\varepsilon(\mathfrak{A}_n)$ which will be defined in the same section.

Our main result is stated in Section 4.

1. PRELIMINARIES

First of all, we define some notations for our discussion in this section.

Given an irrational number θ, let us treat the noncommutative 2-tori $(T^2_\theta)^\infty$ generated by two unitaries u, v with relation

$$uv = \mathrm{e}^{2\pi\mathrm{i}\theta} vu$$

as a Fréchet $*$-algebra (or F^*-algebra). In some cases, we regard each element of $(T^2_\theta)^\infty$ as an operator on the Hilbert space $L^2(T)$ of the square integrable complex valued functions on the 1-torus T. For instance, for $f \in L^2(T)$, $t \in T$,

$$(uf)(t) = tf(t), \quad (vf)(t) = f(\mathrm{e}^{-2\pi\mathrm{i}\theta} t).$$

There is a smooth action α of T^2 on $(T^2_\theta)^\infty$ defined by

$$\alpha_{t,s}(u) = tu, \quad \alpha_{t,s}(v) = sv$$

for $t, s \in T$. Moreover, we have the two $*$-derivations δ_1, δ_2 on $(T^2_\theta)^\infty$ associated with α satisfying

$$\delta_1(u) = \mathrm{i}u, \quad \delta_2(u) = 0, \quad \delta_1(v) = 0, \quad \delta_2(v) = \mathrm{i}v.$$

Using these derivations, we define seminorms $\|\cdot\|_{k,l}$ on $(T^2_\theta)^\infty$ by

$$\|x\|_{k,l} = \|\delta_1^k \circ \delta_2^l(x)\|,$$

where $\|\cdot\|$ is the usual C^*-norm on T^2_θ.

Here, we briefly review the definition of entire cyclic cohomology. For any unital F^*-algebra $\mathfrak{A}$ and any integer $n \geqslant 0$, we put C^n be the set of all $(n+1)$-linear functionals on $\mathfrak{A}$. For $n < 0$, let $C^n = \{0\}$. Moreover, we define

$$C^{\mathrm{ev}} = \{(\varphi_{2n})_n : \varphi_{2n} \in C^{2n}\ (n \geqslant 0)\}, \quad C^{\mathrm{od}} = \{(\varphi_{2n+1})_n : \varphi_{2n+1} \in C^{2n+1}\ (n \geqslant 0)\}.$$

We call (φ_{2n}) an entire even cochain if for each bounded subset $\Sigma \subset \mathfrak{A}$, we can find a constant $C > 0$ such that

$$|\varphi_{2n}(a_0, \ldots, a_{2n})| \leqslant C \cdot n!$$

for all $n \geqslant 1$ and $a_j \in \Sigma$. In odd case, we define entire odd cochains by the same way as in even case. We denote by $C^{\mathrm{ev}}_\varepsilon$ (respectively, $C^{\mathrm{od}}_\varepsilon$) the set of all entire even (respectively, odd) cochains. Then we define the entire cyclic cohomology of $\mathfrak{A}$ by the cohomology of the short complex

$$C_\varepsilon^{\mathrm{ev}} \overset{\partial}{\underset{\partial}{\leftrightarrows}} C_\varepsilon^{\mathrm{od}},$$

where ∂ are certain derivations defined by Connes [2].

2. $(T_\theta^2)^\infty$ IS A FRÉCHET INDUCTIVE LIMIT

In this section, we prove the key lemma which states that noncommutative 2-tori $(T_\theta^2)^\infty$ as F^*-algebras are isomorphic to inductive limits

$$\varinjlim (C^\infty(T) \otimes (M_{q_{2n}}(\mathbb{C}) \oplus M_{q_{2n-1}}(\mathbb{C})), \pi_n^\infty),$$

where the sequence $\{q_{2n-1}\}_n$ appears in the continued fraction expansion of θ.

Let $\begin{pmatrix} p' & p \\ q' & q \end{pmatrix} \in SL(2,\mathbb{Z})$ with $p/q < \theta < p'/q'$, $q > 0$ and $q' > 0$ for each fixed $\theta \in (0,1)$. We write $\beta = p' - q'\theta, \beta' = q\theta - p$. First of all, we construct two projections e_β and $e_{\beta'}$ in $(T_\theta^2)^\infty$ with traces β and β' respectively using the functions f_β and g_β defined below. We regard the 1-torus T as the interval $[0,1]$. Since $\begin{pmatrix} p' & p \\ q' & q \end{pmatrix} \in SL(2,\mathbb{Z})$, we note that $q\beta + q'\beta' = 1$. In particular, we have $0 < \beta < 1/q, 0 < \beta' < 1/q'$. When $\beta \geqslant 1/2q$, we put

$$f_1(x) = \mathrm{e}^{-\alpha/x}, \qquad f_2(x) = 1 - f_1(1/q - \beta - x),$$
$$f_3(x) = f_2(1/q - x), \qquad f_4(x) = f_1(1/q - x),$$

where $\alpha = (1/q - \beta)\log\sqrt{2}$. Using the functions described above, we define the functions f, g defined by

$$f_\beta(x) = \begin{cases} f_1(x) & (0 \leqslant x \leqslant 1/2q - \beta/2), \\ f_2(x) & (1/2q - \beta/2 \leqslant x \leqslant 1/q - \beta), \\ 1 & (1/q - \beta \leqslant x \leqslant \beta), \\ f_3(x) & (\beta \leqslant x \leqslant \beta/2 + 1/2q), \\ f_4(x) & (\beta/2 + 1/2q \leqslant x \leqslant 1/q), \\ 0 & (1/q \leqslant x < 1), \end{cases}$$

$$g_\beta(x) = \chi_{[\beta,1/q]}(x)\sqrt{f(x) - f(x)^2},$$

where χ stands for the characteristic function. In the case when $\beta < 1/2q$, we put

$$f_1(x) = \mathrm{e}^{-\alpha'/x} \qquad f_2(x) = 1 - f_1(1/q - \beta - x),$$
$$f_3(x) = f_2(\beta - x), \qquad f_4(x) = f_1(\beta - x),$$

where $\alpha' = \beta \log \sqrt{2}$, and define

$$f_\beta(x) = \begin{cases} f_1(x) & (1/2q - \beta \leqslant x \leqslant 1/2q - \beta/2), \\ f_2(x) & (1/2q - \beta/2 \leqslant x \leqslant 1/2q), \\ f_3(x) & (1/2q \leqslant x \leqslant 1/2q + \beta/2), \\ f_4(x) & (1/2q + \beta/2 \leqslant x \leqslant 1/2q + \beta), \\ 0 & (\text{otherwise}), \end{cases}$$

$$g_\beta(x) = \chi_{[1/2q, 1/2q+\beta]}(x)\sqrt{f(x) - f(x)^2}.$$

We note that, in either case, f and g are infinitely differentiable functions. Putting e_β by

$$e_\beta = v^{-q'} g(u) + f(u) + g(u) v^{q'},$$

where $f(u)$ and $g(u)$ belong to the Fréchet $*$-algebra $F^*(u)$ generated by u, we have the following lemma:

LEMMA 2.1. *e_β cited above is a projection in $(T^2_\theta)^\infty$.*

The proof follows from Connes [1].

Another projection $e_{\beta'}$ is constructed by the similar way as v and u^{-1} in place of u and v, and as q' and β' in place of q and β respectively.

LEMMA 2.2. *The projections $e_\beta, \alpha_{e^{2\pi i p/q},1}(e_\beta), \ldots, \alpha^{q-1}_{e^{2\pi i p/q},1}(e_\beta)$ are mutually orthogonal. So are the projections $e_{\beta'}, \alpha_{1,e^{-2\pi i p'/q'}}(e_{\beta'}), \ldots, \alpha^{q'-1}_{1,e^{-2\pi i p'/q'}}(e_{\beta'})$.*

Proof. We have that

$$\alpha_{e^{2\pi i p/q},1}(e_\beta) = v^{-q'} g(e^{2\pi i p/q} u) + f(e^{2\pi i p/q} u) + g(e^{2\pi i p/q} u) v^{q'}.$$

Since the supports of g and $g(e^{2\pi i p/q}\cdot)$ are disjoint, we see for example that

$$\begin{aligned} e_\beta \alpha_{e^{2\pi i p/q},1}(e_\beta) &= v^{-q'} g(u) v^{-q'} g(e^{2\pi i p/q} u) + f(u) v^{-q'} g(e^{2\pi i p/q} u) \\ &\quad + g(u) v^{q'} f(e^{2\pi i p/q} u) + g(u) v^{q'} g(e^{2\pi i p/q} u) v^{q'} \\ &= v^{-2q'} g(e^{-2\pi i q'\theta} u) g(e^{2\pi i p/q} u) + v^{q'} g(e^{2\pi i q'\theta} u) f(e^{2\pi i p/q} u) \\ &\quad + v^{-q'} f(e^{-2\pi i p/q} u) + v^{q'} g(e^{2\pi i q'\theta} u) g(e^{2\pi i p/q} u) v^{q'} \\ &= v^{-2q'} g(e^{2\pi i \beta} u) g(e^{2\pi i p/q} u) + v^{-q'} f(e^{2\pi i \beta} u) g(e^{2\pi i p/q} u) \\ &\quad + v^{-q'} g(e^{-2\pi i \beta} u) f(e^{2\pi i p/q} u) + v^{q'} g(e^{-2\pi i \beta} u) g(e^{2\pi i p/q} u) v^{q'}. \end{aligned}$$

When $\beta \geqslant 1/2q$, since $\operatorname{supp} f = [0, 1/q]$ and $\operatorname{supp} g = [\beta, 1/q]$, we have

$$\operatorname{supp} g(e^{2\pi i \beta}\cdot) = [2\beta, 1/q + \beta], \qquad \operatorname{supp} g(e^{-2\pi i \beta}\cdot) = [0, 1/q - \beta],$$
$$\operatorname{supp} g(e^{-2\pi i p/q}\cdot) = [\beta + p/q, (p+1)/q], \qquad \operatorname{supp} f(e^{2\pi i \beta}\cdot) = [\beta, \beta + 1/q],$$
$$\operatorname{supp} f(e^{2\pi i p/q}\cdot) = [p/q, (p+1)/q].$$

Using the fact that p and q are mutually prime, we conclude that the supports of $g(\mathrm{e}^{2\pi \mathrm{i}\beta}\cdot)$ and $g(\mathrm{e}^{2\pi \mathrm{i}p/q}\cdot)$ are disjoint and so on, which implies that

$$e_\beta \alpha_{\mathrm{e}^{2\pi \mathrm{i}p/q},1}(e_\beta) = 0.$$

By the analogous argument, we also have that the above equation holds when $\beta < 1/2q$. By the same way, we see that

$$\alpha^k_{\mathrm{e}^{2\pi \mathrm{i}p/q},1}(e_\beta)\alpha^l_{\mathrm{e}^{2\pi \mathrm{i}p/q},1}(e_\beta) = 0$$

for $k, l \in \{0, 1, \ldots, q-1\}$ with $k \neq l$, as desired. Similarly, we can prove that the projections $e_{\beta'}, \alpha_{1,\mathrm{e}^{-2\pi \mathrm{i}p'/q'}}(e_{\beta'}), \ldots, \alpha^{q'-1}_{1,\mathrm{e}^{-2\pi \mathrm{i}p'/q'}}(e_{\beta'})$ are also mutually orthogonal. ∎

Now we define the elements e_1 and e_2 by

$$e_1 = \sum_{k=0}^{q'-1} (\alpha')^k(e_{\beta'}), \quad e_2 = 1 - \sum_{k=0}^{q-1} \alpha^k(e_\beta),$$

where $\alpha = \alpha_{\mathrm{e}^{2\pi \mathrm{i}p/q},1}, \alpha' = \alpha_{1,\mathrm{e}^{-2\pi \mathrm{i}p'/q'}}$. By the previous proposition, both e_1 and e_2 are projections in $(T^2_\theta)^\infty$. Furthermore, we have that $\tau(e_\beta) = \beta, \tau(e_{\beta'}) = \beta'$, where $\tau(x)$ is the canonical trace of $x \in T^2_\theta$.

LEMMA 2.3. *The projections e_1 and e_2 are unitarily equivalent in $(T^2_\theta)^\infty$.*

Proof. First of all, we show that $(T^2_\theta)^\infty$ is algebraically simple. Let $\mathfrak{I}$ be a non-zero $*$-ideal of $(T^2_\theta)^\infty$. Since the closure $\overline{\mathfrak{I}}$ of $\mathfrak{I}$ in T^2_θ is a closed $*$-ideal of T^2_θ, it follows by the algebraic simplicity of T^2_θ that $\overline{\mathfrak{I}}$ must be equal to T^2_θ. Then, there is an element $x \in \mathfrak{I}$ such that $\|1 - x\| < 1$, so that the spectrum of x does not include the origin of $\mathbb{C}$. Since the function $h(t) = 1/t$ is holomorphic on the spectrum of x, it follows that $h(x) = x^{-1} \in (T^2_\theta)^\infty$. Hence, $1 = x^{-1}x \in \mathfrak{I}$, which implies that $\mathfrak{I} = T^2_\theta$, as claimed.

Next, we have to verify that the stable rank of $(T^2_\theta)^\infty$ is equal to one, i.e., the set of all invertible elements of $(T^2_\theta)^\infty$ is dense in $(T^2_\theta)^\infty$. If we would have this fact, $(T^2_\theta)^\infty$ has cancellation property (cf. Rieffel [9], [10]). Take any element $a \in (T^2_\theta)^\infty$. We may assume that $a \geqslant 0$. Then, for $\forall \varepsilon > 0$, there exists an invertible element $b \geqslant 0$ in T^2_θ such that $\|a - b\| < \varepsilon/2$ (note that T^2_θ is of stable rank one.). By the density of $(T^2_\theta)^\infty$, we can find an element $c \in (T^2_\theta)^\infty$ with $c \geqslant 0$ and $\|b - c\| < \varepsilon/2$. We act $(T^2_\theta)^\infty$ on $L^2(T)$ defined before. Let us show that c is invertible as an operator on $L^2(T)$. If $\xi \in \ker c$ and $\|b - c\| < \varepsilon/2$, we have

$$\|(b - c)\xi\| = \|b\xi\| < \frac{\varepsilon}{2}\|\xi\|.$$

Since ε is arbitrary, we see that $\xi = 0$, which means that c is an injective operator. We note that we can find a positive number $\varepsilon/2 > \delta > 0$ such that $\|b\xi\| \geqslant \delta\|\xi\|$ for any $\xi \in L^2(T)$. We then have for any $\xi \in L^2(T)$,

$$\|c\xi\| \geqslant |\|(b - c)\xi\| - \|b\xi\|| \geqslant \left|\delta - \frac{\varepsilon}{2}\right|\|\xi\|,$$

which implies that c^{-1} is bounded. By the triangle inequality, $\|a-c\| \leqslant \|a-b\| + \|b-c\| < \varepsilon$. Consequently, the stable rank of $(T_\theta^2)^\infty$ is one.

Now recall that $\tau(e_1) = \tau(e_2)$, we thus have $[e_1] = [e_2] \in K_0((T_\theta^2)^\infty)$. Since $(T_\theta^2)^\infty$ has cancellation property, they are unitarily equivalent in $(T_\theta^2)^\infty$. ∎

Let $\theta = [a_0, a_1, \ldots, a_n, \ldots]$ be the continued fraction expansion and define the matrices $P_1, P_2, \ldots$ by

$$P_n = \begin{pmatrix} a_{4n} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{4n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{4n-2} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{4n-3} & 1 \\ 1 & 0 \end{pmatrix}$$

for $n \geqslant 1$. Moreover, we put

$$\begin{pmatrix} q_{2n} \\ q_{2n-1} \end{pmatrix} = P_n P_{n-1} \cdots P_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\mathfrak{A}_n = M_{q_{2n}}(C^\infty(T)) \oplus M_{q_{2n-1}}(C^\infty(T)).$$

For each $n \geqslant 1$, we construct homomorphisms $\pi_n^\infty : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ as follows: we write $P_{n+1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Let $z \in C^\infty(T)$ be the canonical unitary generator of $C^\infty(T)$. The element

$$\begin{pmatrix} z & & \\ & \ddots & \\ & & z \end{pmatrix} \oplus O_{q_{2n-1}} \in \mathfrak{A}_n = M_{q_{2n}}(C^\infty(T)) \oplus M_{q_{2n-1}}(C^\infty(T))$$

should be mapped to the element

$$\begin{pmatrix} J_a & & & & & \\ & \ddots & & & & \\ & & J_a & & & \\ & & & O_b & & \\ & & & & \ddots & \\ & & & & & O_b \end{pmatrix} \oplus \begin{pmatrix} J'_c & & & & & \\ & \ddots & & & & \\ & & J'_c & & & \\ & & & O_d & & \\ & & & & \ddots & \\ & & & & & O_d \end{pmatrix} \in \mathfrak{A}_{n+1}$$

$(= (\underbrace{J_a \oplus \cdots \oplus J_a}_{q_{2n}} \oplus \underbrace{O_b \oplus \cdots \oplus O_b}_{q_{2n-1}}) \oplus (\underbrace{J'_c \oplus \cdots \oplus J'_c}_{q_{2n}} \oplus \underbrace{O_d \oplus \cdots \oplus O_d}_{q_{2n-1}}))$, where

$$J_k = \begin{pmatrix} 0 & & & z \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}, \quad J'_k = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \in M_k(C^\infty(T))$$

and O_l means the $l \times l$ zero matrix. Any element $(a_{ij}) \oplus O_{q_{2n-1}} \in M_{q_{2n}}(\mathbb{C}) \oplus M_{q_{2n-1}}(\mathbb{C}) \subset \mathfrak{A}_n$ should be mapped to

$$\begin{pmatrix} a_{11}I_a & \cdots & a_{1q_{2n}}I_a & \\ \vdots & & \vdots & \\ a_{q_{2n},1}I_a & \cdots & a_{q_{2n},q_{2n}}I_a & \\ & & & O_{bq_{2n-1}} \end{pmatrix} \oplus \begin{pmatrix} a_{11}I_c & \cdots & a_{1q_{2n}}I_c & \\ \vdots & & \vdots & \\ a_{q_{2n},1}I_c & \cdots & a_{q_{2n},q_{2n}}I_c & \\ & & & O_{dq_{2n-1}} \end{pmatrix},$$

where I_a, I_c are the $a \times a, c \times c$ identity matrices respectively. The second direct summand of $\mathfrak{A}_n$ should be mapped into $\mathfrak{A}_{n+1}$ by the similar way as q_{2n} replaced by q_{2n-1}, a and c by b and d respectively, and interchanging the places to whose elements are mapped from upper left-hand side to lower right-hand side. It is easily verified that these π_n^∞ are smooth inclusions.

Next, we need the following proposition. We define

$$e_{kk} = \alpha^{k-1}(e_\beta) \quad (k=1,2,\ldots,q-1) \quad \text{and} \quad e'_{kk} = (\alpha')^{k-1}(e_{\beta'}) \quad (k=1,2,\ldots,q'-1).$$

LEMMA 2.4. *Let $e_{22}ve_{11} = e_{21}|e_{22}ve_{11}|$ be the polar decomposition of $e_{22}ve_{11}$. Then, $e_{21} = e_{22}ve_{11}$.*

Proof. We write $x = ve_{11}$. Since $x^*x = e_{11}v^*ve_{11} = e_{11}$, we have $|x| = e_{11}$. Thus, $x = ve_{11}$ is the polar decomposition of x, which implies that it is a surjective operator since v is unitary. Hence, it follows that $\overline{\operatorname{Ran} e_{22}} = \overline{\operatorname{Ran} e_{22}ve_{11}}$, where $\overline{V}$ is the closure of a linear subspace V of the Hilbert space $L^2(T)$. Furthermore, it is also verified that $\overline{\operatorname{Ran} e_{11}} = \overline{\operatorname{Ran} |e_{22}ve_{11}|}$. Note that $e_{22}ve_{11} = (e_{22}ve_{11})e_{11}$. By uniqueness of polar decomposition, we deduce that $e_{21} = e_{22}ve_{11}$, as desired. ∎

By the similar way, we put $e'_{21} = e'_{22}ue'_{11}$. Our goal in this section is to construct the F^*-subalgebras generated by some unitaries, which are isomorphic to $M_{q_{2n}}(C^\infty(T)) \oplus M_{q_{2n-1}}(C^\infty(T))$. For this, since q_{2n-1} and q_{2n} are mutually prime, we can find an integer p_{2n-1}, p_{2n} with $\left(\begin{smallmatrix} p_{2n-1} & p_{2n} \\ q_{2n-1} & q_{2n} \end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ and $p_n/q_n \to \theta$ as $n \to \infty$. With the same notations as above, we set

$$\begin{pmatrix} p' & p \\ q' & q \end{pmatrix} = \begin{pmatrix} p_{2n} & p_{2n-1} \\ q_{2n} & q_{2n-1} \end{pmatrix}$$

and $\beta = \beta_n = p_{2n-1} - q_{2n-1}\theta, \beta' = \beta'_n = q_{2n}\theta - p_{2n}$, and so on. First of all, we check the following fact although it seems to be known:

LEMMA 2.5. *For arbitrary $h \in C^\infty(T)$, $\delta_j(h(u)) = h'(u)\delta_j(u)$ $(j = 1, 2)$, where h' is the first derivative of h.*

Proof. If $h(x) = \sum_{\nu=-m}^{n} a_\nu x^\nu$ is a Laurent polynomial, we have

$$\delta_1(h(u)) = \delta_1\Big(\sum_{\nu=-m}^{n} a_\nu u^\nu\Big) = \sum_{\nu=-m}^{n} a_\nu \nu \mathrm{i} u^\nu = \Big(\sum_{\nu=-m}^{n} a_\nu \nu u^{\nu-1}\Big)\mathrm{i}u = h'(u)\delta_1(u).$$

For any $h \in C^\infty(T)$, we can find a family of Laurent polynomials $\{p_n\}_{n\geqslant 1}$ such that $p_n \to h$ with respect to the seminorms $\{\|\cdot\|_{k,l}\}$. For $m, n \geqslant 1$, we have

$$\delta_1(p_n(u) - p_m(u)) = (p_n'(u) - p_m'(u))\delta_1(u) = (p_n'(u) - p_m'(u))u.$$

Since $\{p_n(u)\}_n$ is Cauchy, $\{\delta_1(p_n(u))\}_{n\geqslant 1}$ is also a Cauchy sequence. Using the fact that δ_1 is a closed operator, we get

$$\delta_1(h(u)) = \lim_{n\to\infty} \delta_1(p_n(u)) = \lim_{n\to\infty} p_n'(u)\delta_1(u) = h'(u)\delta_1(u).$$

As $\delta_2(u)=0$, it is clear that $\delta_2(h(u))=0=h'(u)\delta_2(u)$. This completes the proof. ∎

In what follows, we use the notations $e_{11}^{(n)} = e_{\beta_n}$, $(e_{11}')^{(n)} = e_{\beta_n}'$ and so on for $n \geqslant 1$. Denoting $r_m = p_m/q_m$ for any integer $m \geqslant 1$, we define $u_n = u_{n,1} + u_{n,2}$ and $v_n = v_{n,1} + v_{n,2}$, where

$$u_{n,1} = \sum_{j=0}^{q_{2n}-1} \mathrm{e}^{2\pi \mathrm{i} r_{2n} j} \alpha^j_{\mathrm{e}^{2\pi\mathrm{i}r_{2n}},1}(e_{11}^{(n)}), \qquad u_{n,2} = \sum_{j=0}^{q_{2n-1}-1} \alpha^j_{1,\mathrm{e}^{-2\pi\mathrm{i}r_{2n-1}}}((e_{21}')^{(n)}),$$

$$v_{n,1} = \sum_{j=0}^{q_{2n}-1} \alpha^j_{\mathrm{e}^{2\pi\mathrm{i}r_{2n}},1}(e_{21}^{(n)}), \qquad v_{n,2} = \sum_{j=0}^{q_{2n-1}-1} \mathrm{e}^{-2\pi\mathrm{i}r_{2n-1}j} \alpha^j_{1,\mathrm{e}^{-2\pi\mathrm{i}r_{2n-1}}}((e_{11}')^{(n)}).$$

We note that since

$\alpha^{q_{2n}-1}(e_{21}^{(n)}) \in e_{11}^{(n)}(T_\theta^2)^\infty e_{q_{2n}q_{2n}}^{(n)}$, $\quad (\alpha')^{q_{2n-1}-1}((e_{21}')^{(n)}) \in (e_{11}')^{(n)}(T_\theta^2)^\infty (e_{q_{2n-1}q_{2n-1}}')^{(n)}$,

where $e_{q_{2n}q_{2n}}^{(n)} = \alpha^{q_{2n}-1}_{\mathrm{e}^{2\pi\mathrm{i}r_{2n}},1}(e_{11}^{(n)})$ and $(e_{q_{2n-1}q_{2n-1}}')^{(n)} = \alpha^{q_{2n-1}-1}_{1,\mathrm{e}^{-2\pi\mathrm{i}r_{2n-1}}}((e_{11}')^{(n)})$, we can find a unitary $v_{1q_{2n}} \in e_{11}^{(n)}(T_\theta^2)^\infty e_{11}^{(n)}$ (respectively, $u_{1q_{2n-1}}' \in (e_{11}')^{(n)}(T_\theta^2)^\infty (e_{11}')^{(n)}$) such that $\alpha^{q_{2n}-1}(e_{21}^{(n)}) = v_{1q_{2n}} e_{1q_{2n}}^{(n)}$ (respectively, $(\alpha')^{q_{2n-1}-1}((e_{21}')^{(n)}) = u_{1q_{2n-1}}'(e_{1q_{2n-1}}')^{(n)}$). By Lemma 2.2, we have

$$u_{n,1}u_{n,1}^* = \Big(\sum_{j=0}^{q_{2n}-1} \mathrm{e}^{2\pi\mathrm{i}r_{2n}j}\alpha^j_{\mathrm{e}^{2\pi\mathrm{i}r_{2n}},1}(e_{11}^{(n)})\Big)\Big(\sum_{j=0}^{q_{2n}-1} \mathrm{e}^{-2\pi\mathrm{i}r_{2n}j}\alpha^j_{\mathrm{e}^{2\pi\mathrm{i}r_{2n}},1}(e_{11}^{(n)})\Big)$$

$$= \sum_{j,m} \mathrm{e}^{2\pi\mathrm{i}r_{2n}(j-m)}\alpha^j_{\mathrm{e}^{2\pi\mathrm{i}r_{2n}},1}(e_{11}^{(n)})\alpha^m_{\mathrm{e}^{2\pi\mathrm{i}r_{2n}},1}(e_{11}^{(n)}) = \sum_{j=0}^{q_{2n}-1}\alpha^j_{\mathrm{e}^{2\pi\mathrm{i}r_{2n}},1}(e_{11}^{(n)}) = 1 - e_2^{(n)}.$$

Similarly, $u_{n,1}^*u_{n,1} = 1 - e_2^{(n)}$, $v_{n,2}v_{n,2}^* = v_{n,2}^*v_{n,2} = e_1^{(n)}$. Moreover, we have

$$u_{n,2}u_{n,2}^* = \Big(\sum_{j=0}^{q_{2n-1}-2} (e_{2+j,1+j}')^{(n)} + u_{1q_{2n-1}}'(e_{1q_{2n-1}}')^{(n)}\Big)$$

$$\cdot\Big(\sum_{j=0}^{q_{2n-1}-2} (e_{1+j,2+j}')^{(n)} + (e_{q_{2n-1}1}')^{(n)}(u_{1q_{2n-1}}')^*\Big)$$

$$= ((e_{21}')^{(n)} + \cdots + (e_{q_{2n-1},q_{2n-1}-1}')^{(n)})((e_{12}')^{(n)} + \cdots + (e_{q_{2n-1}-1,q_{2n-1}}')^{(n)})$$

$$+ ((e_{21}')^{(n)} + \cdots + (e_{q_{2n-1},q_{2n-1}-1}')^{(n)})u_{1q_{2n-1}}'(e_{1q_{2n-1}}')^{(n)}$$

$$+ (e'_{q_{2n-1},1})^{(n)} u'_{1q_{2n-1}} ((e'_{12})^{(n)} + \cdots + (e'_{q_{2n-1}-1,q_{2n-1}})^{(n)})$$
$$+ (e'_{q_{2n-1},1})^{(n)} (u'_{1q_{2n-1}})^* u'_{1q_{2n-1}} (e'_{1q_{2n-1}})^{(n)},$$

where

$$(e'_{k,k-1})^{(n)} = \alpha^{k-2}_{1,\mathrm{e}^{-2\pi\mathrm{i}r_{2n-1}}}((e'_{11})^{(n)}), \quad (e_{k-1,k})^{(n)} = ((e_{k,k-1})^{(n)})^*$$

for $k = 2, \ldots, q_{2n-1}$. Since $u'_{1q_{2n-1}}$ is a unitary in $(e'_{11})^{(n)}(T^2_\theta)^\infty(e'_{11})^{(n)}$, it follows that the second and the third terms above are 0 and

$$\begin{aligned}(e'_{q_{2n-1},1})^{(n)} (u'_{1q_{2n-1}})^* u'_{1q_{2n-1}} (e'_{1q_{2n-1}})^{(n)} &= (e'_{q_{2n-1}1})^{(n)} (e'_{11})^{(n)} (e'_{1q_{2n-1}})^{(n)} \\ &= (e'_{q_{2n-1}q_{2n-1}})^{(n)}.\end{aligned}$$

Thus we have

$$u_{n,2}u^*_{n,2} = (e'_{11})^{(n)} + \cdots + (e'_{q_{2n-1}-1,q_{2n-1}-1})^{(n)} + (e'_{q_{2n-1}q_{2n-1}})^{(n)} = e^{(n)}_1.$$

The same calculations show that

$$u^*_{n,2}u_{n,2} = e^{(n)}_1, \quad v_{n,1}v^*_{n,1} = v^*_{n,1}v_{n,1} = 1 - e^{(n)}_2.$$

Moreover, we have

$$\begin{aligned}v_{n,1}u_{n,1} &= (e^{(n)}_{21} + \cdots + e^{(n)}_{q_{2n},q_{2n}-1} + u_{1q_{2n}} e^{(n)}_{1q_{2n}})(e^{(n)}_{11} + \cdots + \omega^{q_{2n}-1} e^{(n)}_{q_{2n}q_{2n}}) \\ &= e^{(n)}_{21} + \cdots + \omega^{q_{2n}-2} e^{(n)}_{q_{2n}q_{2n}-1} + \omega^{q_{2n}-1} u_{1q_{2n}} e^{(n)}_{1q_{2n}}\end{aligned}$$

and

$$\begin{aligned}u_{n,1}v_{n,1} &= (e^{(n)}_{11} + \cdots + \omega^{q_{2n}-1} e^{(n)}_{q_{2n}q_{2n}})(e^{(n)}_{21} + \cdots + e^{(n)}_{q_{2n},q_{2n}-1} + u_{1q_{2n}} e^{(n)}_{1q_{2n}}) \\ &= e^{(n)}_{11} u_{1q_{2n}} e^{(n)}_{1q_{2n}} + \omega e^{(n)}_{21} + \cdots + \omega^{q_{2n}-1} e^{(n)}_{q_{2n}q_{2n}-1},\end{aligned}$$

where

$$\begin{aligned}e^{(n)}_{kk} &= \alpha^{k-1}_{\mathrm{e}^{2\pi\mathrm{i}r_{2n}},1}(e_{\beta_n}) \quad (k = 2, \ldots, q_{2n} - 1), \\ e^{(n)}_{k,k-1} &= \alpha^{k-2}_{\mathrm{e}^{2\pi\mathrm{i}r_{2n}},1}(e^{(n)}_{21}), \quad e^{(n)}_{k-1,k} = (e^{(n)}_{k,k-1})^* \quad (k = 2, \ldots, q_{2n}),\end{aligned}$$

and $\omega = \mathrm{e}^{2\pi\mathrm{i}r_{2n}}$. Using the fact that $u_{1q_{2n}} \in e^{(n)}_{11}(T^2_\theta)^\infty e^{(n)}_{11}$ and $\omega^{q_{2n}} = 1$, we have

$$v_{n,1}u_{n,1} = \mathrm{e}^{-2\pi\mathrm{i}r_{2n}} u_{n,1}v_{n,1}.$$

To sum up, we get the following:

LEMMA 2.6. *The following hold:*

(i) $u_{n,1}$ *and* $u_{n,2}$ *are unitaries in* $(1 - e^{(n)}_2)(T^2_\theta)^\infty(1 - e^{(n)}_2)$ *and so are* $u_{n,2}$ *and* $v_{n,2}$ *in* $e^{(n)}_1(T^2_\theta)^\infty e^{(n)}_1$*;*

(ii) $u_{n,1}v_{n,1} = \mathrm{e}^{2\pi\mathrm{i}r_{2n}} v_{n,1}u_{n,1}$, $u_{n,2}v_{n,2} = \mathrm{e}^{2\pi\mathrm{i}r_{2n-1}} v_{n,2}u_{n,2}$.

Now we construct subalgebras isomorphic to

$$M_{q_{2n}}(C^\infty(T)) \oplus M_{q_{2n-1}}(C^\infty(T)).$$

Let $\{e_{ij}^{(n)}\}_{1\leqslant i,j\leqslant q_{2n}}$ be the matrix units constructed by

$$\{e_{11}^{(n)}, e_{22}^{(n)}, \dots e_{q_{2n}q_{2n}}^{(n)}, e_{21}^{(n)}, \dots, e_{q_{2n},q_{2n}-1}^{(n)}\}.$$

We then see the following lemma:

LEMMA 2.7. *The F^*-algebras*

$$F^*(\{e_{ij}^{(n)}\}_{1\leqslant i,j\leqslant q_{2n}}, v_{1q_{2n}})$$

generated by $\{e_{ij}^{(n)}\}_{1\leqslant i,j\leqslant q_{2n}}$ and $v_{1q_{2n}}$ are isomorphic to $M_{q_{2n}}(C^\infty(T))$ for all integers $n \geqslant 1$.

Proof. Consider the continuous field $S \ni t \mapsto e_{\beta_n}$ defined by Elliott and Evans [3], where S is a closed subinterval in $(0,\infty)$. The functions f and g appeared in the construction of e_{β_n} are depend on $t \in S$, so that we write $f = f_t$, $g = g_t$. It is not difficult to verify that

$$\|f_t^{(\nu)} - f_{t_0}^{(\nu)}\|_\infty, \quad \|g_t^{(\nu)} - g_{t_0}^{(\nu)}\|_\infty \to 0.$$

as $t \to t_0$ for any integer $\nu \geqslant 0$, where $f^{(\nu)}$ stands for the ν-th derivatives of $f \in C^\infty(T)$ and $\|\cdot\|_\infty$ is the supremum norm on $C^\infty(T)$. Then our statement of this lemma follows immediately. ∎

By the same way, it follows that the F^*-algebra $F^*(\{(e'_{ij})^{(n)}\}, u'_{1q_{2n-1}})$ generated by $\{(e'_{ij})^{(n)}\}_{1\leqslant i,j\leqslant q_{2n-1}}$ and $u'_{1q_{2n-1}}$ is isomorphic to $M_{q_{2n-1}}(C^\infty(T))$, where $\{(e'_{ij})^{(n)}\}_{1\leqslant i,j\leqslant q_{2n-1}}$ are the matrix units generated by

$$\{(e'_{11})^{(n)}, \dots, (e'_{q_{2n-1}q_{2n-1}})^{(n)}, (e'_{21})^{(n)}, \dots, (e'_{q_{2n-1},q_{2n-1}-1})^{(n)}\}.$$

LEMMA 2.8. *For each $h \in C^\infty(T)$ and any integer $k \geqslant 1$, there exist $\{a_{\nu,k}\} \subset \mathbb{R}$ such that*

$$\delta_1^k(h(u)) = \sum_{\nu=1}^{k} a_{\nu,k} h^{(\nu)}(u) u^\nu \quad (\nu = 1, \dots, k).$$

Proof. For $k = 1$, by Proposition 2.5. If this statement holds for some $k \geqslant 1$, one has

$$\begin{aligned}\delta_1^{k+1}(h(u)) &= \delta_1\Big(\sum_{\nu=1}^{k} a_{\nu,k} h^{(\nu)}(u)u^\nu\Big) = \sum_{\nu=1}^{k} a_{\nu,k}\delta_1(h^{(\nu)}(u)u^\nu)\\ &= \sum_{\nu=1}^{k} a_{\nu,k}(h^{(\nu+1)}(u)u \cdot u^\nu + \mathrm{i}\nu h^{(\nu)}(u)u^\nu)\end{aligned}$$

$$= \sum_{\nu=1}^{k} a_{\nu,k}(h^{(\nu+1)}(u)u^{\nu+1} + \mathrm{i}\nu h^{(\nu)}(u)u^{\nu})$$
$$= \sum_{\nu=2}^{k+1} a_{\nu-1,k}h^{(\nu)}(u)u^{\nu} + \sum_{\nu=1}^{k} \mathrm{i}a_{\nu,k}\nu h^{(\nu)}(u)u^{\nu}.$$

Thus, we have

$$a_{\nu,k+1} = \sum_{\nu=2}^{k+1} a_{\nu-1,k} + \sum_{\nu=1}^{k} \mathrm{i}a_{\nu,k}\nu,$$

this ends the proof. ∎

We note that the coefficients $a_{\nu,k}$ do not depend on the choice h.

By Lemma 2.8, we have

$$\|\delta_1^k(f_n(u)) - \delta_1^k(f_m(u))\| = \Big\| \sum_{\nu=1}^{k} a_{\nu,k}(f_n^{(\nu)}(u) - f_m^{(\nu)}(u))u^{\nu} \Big\|$$
$$\leqslant \sum_{\nu=1}^{k} |a_{\nu,k}| \|f_n^{(\nu)}(u) - f_m^{(\nu)}(u)\| \to 0 \quad (n, m \to \infty),$$

which means that $\{\delta_1^k(f_n(u))\}_n$ is a Cauchy sequence. Analogously, we see that $\{\delta_1^k(g_n(u))\}_n$ is also Cauchy.

By construction, the following fact follows:

LEMMA 2.9. *Let $F^*(u_n, v_n)$ be the F^*-algebras generated by u_n and v_n. Then, they are equal to $F^*(\{e_{ij}^{(n)}\}, v_{1q_{2n}}) \oplus F^*(\{(e'_{ij})^{(n)}\}, u'_{1q_{2n-1}})$.*

Proof. Since $u_{n,j}$ and $v_{n,j}$ $(j = 1, 2)$ are all periodic unitaries, their spectra are finite. Then the projections appeared in the spectral decompositions of $u_{n,j}, v_{n,j}$ are unitarily equivalent to $e_{ij}^{(n)}$s by the properties that $F^*(u_{n,j})$ and $F^*(v_{n,j})$ are closed under the holomorphic functional calculus. ∎

LEMMA 2.10. *For any integers $k, l \geqslant 0$,*

$$\lim_{n\to\infty} \|u - u_n\|_{k,l} = \lim_{n\to\infty} \|v - v_n\|_{k,l} = 0.$$

Proof. At first, we have to verify that the sequence $\{\delta_1^k(e_{\beta_n})\}_n$ is Cauchy. By construction of e_{β_n}, we have, for $n, m \geqslant 1$,

$$\|\delta_1^k(e_{\beta_n}) - \delta_1^k(e_{\beta_m})\| \leqslant \|\delta_1^k(v^{-q_{2n-1}}g_n(u) - v^{-q_{2m-1}}g_m(u))\|$$
$$+ \|\delta_1^k(f_n(u) - f_m(u))\| + \|\delta_1^k(g_n(u)v^{q_{2n-1}} - g_m(u)v^{q_{2m-1}})\|$$
$$= \|v^{-q_{2n-1}}\delta_1^k(g_n(u)) - v^{-q_{2m-1}}\delta_1^k(g_m(u))\|$$
$$+ \|\delta_1^k(f_n(u)) - \delta_1^k(f_m(u))\|$$
$$+ \|\delta_1^k(g_n(u))v^{q_{2n-1}} - \delta_1^k(g_m(u))v^{q_{2m-1}}\|.$$

Since $p_{2n-1}/q_{2n-1} \to \theta$, the last term of the above calculation tends to 0 as $n, m \to \infty$. Therefore, $\{\delta_1^k \circ \delta_2^l(u(1-e_2^{(n)}) - u_{n,1})\}_n$ is Cauchy. Similarly, the sequence $\{\delta_1^k \circ \delta_2^l(ue_1^{(n)} - u_{n,2})\}_n$ is also a Cauchy sequence. Hence, by [8],

$$u(1-e_2^{(n)}) - u_{n,1} \to 0, \quad ue_1^{(n)} - u_{n,2} \to 0,$$

as $n \to \infty$. Using the fact that $\delta_1^k \circ \delta_2^l$ are closed, the sequences above tend to 0 as $n \to \infty$. Consequently,

$$\|u - u_n\|_{k,l} \leqslant \|u(1-e_2^{(n)}) - u_{n,1}\|_{k,l} + \|ue_1^{(n)} - u_{n,2}\|_{k,l} \to 0 \quad (n \to \infty).$$

By the similar argument, we have $\|v - v_n\|_{k,l} \to 0$ as $n \to \infty$, this ends the proof. ∎

Combining all together in this section, we conclude that our key fact follows:

PROPOSITION 2.11. *Given an irrational number $\theta \in (0,1)$, $(T_\theta^2)^\infty$ is isomorphic to the Fréchet $*$-inductive limit*

$$\varinjlim (M_{q_{2n}}(C^\infty(T)) \oplus M_{q_{2n-1}}(C^\infty(T)), \pi_n^\infty).$$

3. ENTIRE CYCLIC COHOMOLOGY OF FRÉCHET INDUCTIVE LIMITS

Let $\{\mathfrak{A}_n, i_n\}_{n\geqslant 1}$ be a family of Fréchet $*$-algebras and $i_n : \mathfrak{A}_n \to \mathfrak{A}_{n+1}$ Fréchet $*$-imbeddings. We can form the Fréchet $*$-inductive limit $\varinjlim \mathfrak{A}_n$, which is denoted by $\mathfrak{A}$. In this section, we prove that the projective limit $\varprojlim H_\varepsilon^*(\mathfrak{A}_n)$ of the entire cyclic cohomologies $\varprojlim H_\varepsilon^*(\mathfrak{A}_n)$ is isomorphic to $H_\varepsilon^*(\mathfrak{A})$. Let $[\,\cdot\,]_{\mathfrak{A}_n}$ be the entire cyclic cohomology classes on $\mathfrak{A}_n$, and the maps $\widehat{i_n^*} : H_\varepsilon^{\mathrm{ev}}(\mathfrak{A}_{n+1}) \to H_\varepsilon^{\mathrm{ev}}(\mathfrak{A}_n)$ are defined by

$$\widehat{i_n^*}([(\varphi_{2k}^{(n+1)})_k]_{\mathfrak{A}_{n+1}}) = [(i_n^{\otimes(2k+1)})^* \varphi_{2k}^{(n+1)}]_{\mathfrak{A}_n},$$

where

$$(i_n^{\otimes(2k+1)})^* \varphi_{2k}^{(n+1)}(a_0, \dots, a_{2k}) = \varphi_{2k}^{(n+1)}(i_n(a_0), \dots, i_n(a_{2k}))$$

for $a_0, \dots, a_{2k} \in \mathfrak{A}_n$. First of all, we define the notion of projective limit as follows:

DEFINITION 3.1. The projective limit $\varprojlim H_\varepsilon^{\mathrm{ev}}(\mathfrak{A}_n)$ of $H_\varepsilon^{\mathrm{ev}}(\mathfrak{A}_n)$ is the space of sequences $\{[(\varphi_{2k}^{(n)})_k]_{\mathfrak{A}_n}\}_n \in \prod\limits_{n\geqslant 1} H_\varepsilon^{\mathrm{ev}}(\mathfrak{A}_n)$ such that for any $n \geqslant 1$,

$$\widehat{i_n^*}([(\varphi_{2k}^{(n+1)})_k]_{\mathfrak{A}_{n+1}}) = [(\varphi_{2k}^{(n)})_k]_{\mathfrak{A}_n}$$

with the property that for any $k \geqslant 0, l \geqslant 1$,

$$\sup_{n\geqslant 1} \|\varphi_{2k}^{(n)}\|_l < \infty,$$

where

$$\|\varphi_{2k}^{(n)}\|_l = \sup_{a_j\in\mathfrak{A}_n,\, \|a_j\|_l\leqslant 1} |\varphi_{2k}^{(n)}(a_0,\dots,a_{2k})|.$$

We define $\varprojlim H_\varepsilon^{\mathrm{od}}(\mathfrak{A}_n)$ in the similar way as in the even case. $\{[(\varphi_{2k}^{(n)})_k]_{\mathfrak{A}_n}\}_n = \{[(\psi_{2k}^{(n)})_k]_{\mathfrak{A}_n}\}_n$ if and only if there exists $\{[(\theta_{2k+1}^{(n)})_k]_{\mathfrak{A}_n}\}_n \in \varprojlim H_\varepsilon^{\mathrm{od}}(\mathfrak{A}_n)$ such that

$$\varphi_{2k}^{(n)} - \psi_{2k}^{(n)} = b\theta_{2k-1}^{(n)} + B\theta_{2k+1}^{(n)}$$

for any $n \geqslant 1, k \geqslant 0$.

Let us construct two maps between $\varprojlim H_\varepsilon^{\mathrm{ev}}(\mathfrak{A}_n)$ and $H_\varepsilon^{\mathrm{ev}}(\mathfrak{A})$. First of all, we define $\Phi : H_\varepsilon^{\mathrm{ev}}(\mathfrak{A}) \to \varprojlim H_\varepsilon^{\mathrm{ev}}(\mathfrak{A}_n)$ by

$$\Phi([(\varphi_{2k})_k]_{\mathfrak{A}}) = \{[(\varphi_{2k}|_{\mathfrak{A}_n})_k]_{\mathfrak{A}_n}\}_n,$$

where $[\,\cdot\,]_{\mathfrak{A}}$ means the same symbol as $[\,\cdot\,]_{\mathfrak{A}_n}$. Actually it is well-defined. In fact, if $[(\varphi_{2k})_k]_{\mathfrak{A}} = [(\varphi'_{2k})_k]_{\mathfrak{A}}$ then there exists an odd entire cyclic cocycle $\theta = (\theta_{2k+1})_k$ such that $(\varphi_{2k} - \varphi'_{2k})_k = (b+B)(\theta_{2k+1})_k$, where $b+B$ is the derivation on entire cyclic cocycles. It is trivial that $(\varphi_{2k}|_{\mathfrak{A}_n} - \varphi'_{2k}|_{\mathfrak{A}_n})_k = (b+B)(\theta_{2k+1}|_{\mathfrak{A}_n})_k$ for each integer $n \geqslant 1$. This means that $\{[(\varphi_{2k}|_{\mathfrak{A}_n})_k]_{\mathfrak{A}_n}\}_n = \{[(\varphi'_{2k}|_{\mathfrak{A}_n})_k]_{\mathfrak{A}_n}\}_n$. Moreover,

$$\sup_{n\geqslant 1} \|\varphi_{2k}^{(n)}|_{\mathfrak{A}_n}\|_l = \|\varphi_{2k}\|_l < \infty,$$

which implies $[(\varphi_{2k}^{(n)}|_{\mathfrak{A}_n})_k]_{\mathfrak{A}_n} \in H_\varepsilon^{\mathrm{ev}}(\mathfrak{A}_n)$.

Now we construct the inverse map Ψ of Φ. For any

$$\{[(\varphi_{2k}^{(n)})_k]_{\mathfrak{A}_n}\}_n \in \varprojlim H_\varepsilon^{\mathrm{ev}}(\mathfrak{A}_n)$$

and $a_0,\dots,a_{2k} \in \mathfrak{A}$, we can take sequences $\{b_j^{(m)}\}_m$ for $j = 0,\dots,2k$ which converge to a_j as $m \to \infty$ with respect to the seminorms $\|\cdot\|_l$ on $\varinjlim \mathfrak{A}_n$. Choose integers $N(m) \geqslant 1$ such that $b_j^{(m)} \in \mathfrak{A}_{N(m)}$ for any $0 \leqslant j \leqslant 2k$. We may assume that $N(m) = m$ by taking a larger number between $N(m)$ and m. We have that for $m > m'$, there exists an odd entire cocycle $\theta^{(m')} = (\theta_{2k+1}^{(m')})_k$ on $\mathfrak{A}_{m'}$ such that

$$\begin{aligned}\varphi_{2k}^{(m)}(b_0^{(m)},\dots,b_{2k}^{(m)}) - \varphi_{2k}^{(m')}(b_0^{(m')},\dots,b_{2k}^{(m')})& \\ = (b\theta_{2k-1}^{(m')} + B\theta_{2k+1}^{(m')})(b_0^{(m')},\dots,b_{2k}^{(m')}).&\end{aligned} \tag{3.1}$$

By Hahn–Banach theorem, we can extend $\varphi_{2k}^{(m)}$ and $\varphi_{2k}^{(m')}$ to $\widetilde{\varphi}_{2k}^{(m)}$ and $\widetilde{\varphi}_{2k}^{(m')}$ on $\mathfrak{A}$ such that

$$\|\widetilde{\varphi}_{2k}^{(m)}\|_l = \|\varphi_{2k}^{(m)}\|_l, \quad \|\widetilde{\varphi}_{2k}^{(m')}\|_l = \|\varphi_{2k}^{(m')}\|_l$$

for any $l \geqslant 1$.

LEMMA 3.2. *For any $a_0, \dots, a_{2k} \in \mathfrak{A}$, the sequence*

$$\{\widetilde{\varphi}_{2k}^{(m)}(a_0, \dots, a_{2k})\}_m$$

is bounded.

Proof. We have

$$\begin{aligned} |\widetilde{\varphi}_{2k}^{(m)}(a_0, \dots, a_{2k})| \leqslant |\widetilde{\varphi}_{2k}^{(m)}&(a_0 - b_0^{(m)}, a_1, \dots, a_{2k})| \\ &+ |\widetilde{\varphi}_{2k}^{(m)}(b_0^{(m)}, a_1 - b_1^{(m)}, a_2, \dots, a_{2k})| \\ &+ \cdots + |\widetilde{\varphi}_{2k}^{(m)}(b_0^{(m)}, \dots, b_{2k-1}^{(m)}, a_{2k} - b_{2k}^{(m)})| \\ &+ |\widetilde{\varphi}_{2k}^{(m)}(b_0^{(m)}, \dots, b_{2k}^{(m)})|. \end{aligned}$$

By the above equation (3.1),

$$\begin{aligned} \widetilde{\varphi}_{2k}^{(m)}(b_0^{(m)}, \dots, b_{2k}^{(m)}) &= \varphi_{2k}^{(m)}(b_0^{(m)}, \dots, b_{2k}^{(m)}) \\ &= \varphi_{2k}^{(m')}(b_0^{(m')}, \dots, b_{2k}^{(m')}) + (b\theta_{2k-1}^{(m')} + B\theta_{2k+1}^{(m')})(b_0^{(m')}, \dots, b_{2k}^{(m')}) \end{aligned}$$

is a constant independent of m. Using the hypothesis in Definition 3.1 and Hahn–Banach theorem, it follows that $\lim_{m\to\infty} |\widetilde{\varphi}_{2k}^{(m)}(a_0, \dots, a_{2k})|$ is dominated by the constant $|\varphi_{2k}^{(m')}(b_0^{(m')}, \dots, b_{2k}^{(m')}) + (b\theta_{2k-1}^{(m')} + B\theta_{2k+1}^{(m')})(b_0^{(m')}, \dots, b_{2k}^{(m')})|$. In particular, the sequence $\{|\widetilde{\varphi}_{2k}^{(m)}(a_0, \dots, a_{2k})|\}_m$ is bounded. ∎

Therefore, by taking the subsequence of $\{|\widetilde{\varphi}_{2k}^{(N)}(a_0, \dots, a_{2k})|\}_N$, we may assume that

$$\lim_{N\to\infty} \widetilde{\varphi}_{2k}^{(N)}(a_0, \dots, a_{2k})$$

exists, so that we define

$$\widetilde{\varphi}_{2k}(a_0, \dots, a_{2k}) = \lim_{N\to\infty} \widetilde{\varphi}_{2k}^{(N)}(a_0, \dots, a_{2k}).$$

Here we note that

$$\widetilde{\varphi}_{2k}(a_0, \dots, a_{2k}) = \lim_{m\to\infty} \widetilde{\varphi}_{2k}^{(m)}(b_0^{(m)}, \dots, b_{2k}^{(m)}).$$

In fact, by the same reason as before, we have

$$\begin{aligned} &|\widetilde{\varphi}_{2k}^{(m)}(a_0, \dots, a_{2k}) - \widetilde{\varphi}_{2k}^{(m)}(b_0^{(m)}, \dots, b_{2k}^{(m)})| \\ &\quad\leqslant |\widetilde{\varphi}_{2k}^{(m)}(a_0 - b_0^{(m)}, a_1, \dots, a_{2k})| + \cdots + |\widetilde{\varphi}_{2k}^{(m)}(b_0^{(m)}, \dots, b_{2k-1}^{(m)}, a_{2k} - b_{2k}^{(m)})| \to 0 \end{aligned}$$

as $m \to \infty$. Using the above preparation, we shall show the following fact:

LEMMA 3.3. *$(\widetilde{\varphi}_{2k})_k$ is an entire cyclic cocycle on $\mathfrak{A}$.*

Proof. Let Σ be a bounded subset of $\mathfrak{A}$ and $a_0, \ldots, a_{2k} \in \Sigma$. Then we can choose sequences $\{b_j^{(m)}\}_m \subset \bigcup \mathfrak{A}_n$ for $j = 0, \ldots, 2k$ such that $b_j^{(m)} \to a_j$ as $m \to \infty$ with respect to the topology induced by the seminorms $\|\cdot\|_l$ on $\mathfrak{A}$. In this case, the set

$$\Sigma_0 = \left\{ b_j^{(m)} \in \bigcup \mathfrak{A}_n : j = 0, \ldots, 2k, m \in \mathbb{N} \right\}$$

is bounded in $\mathfrak{A}$. So, by the equation (3.1),

$$\begin{aligned} |\widetilde{\varphi}_{2k}(a_0, \ldots, a_{2k})| &= \lim_{m \to \infty} |\widetilde{\varphi}_{2k}^{(m)}(b_0^{(m)}, \ldots, b_{2k}^{(m)})| \\ &\leqslant |\widetilde{\varphi}_{2k}^{(1)}(b_0^{(1)}, \ldots, b_{2k}^{(1)})| + |(b\theta_{2k-1}^{(1)} + B\theta_{2k+1}^{(1)})(b_0^{(1)}, \ldots, b_{2k}^{(1)})|. \end{aligned}$$

As $(\varphi_{2k}^{(1)})_k$ and $(b\theta_{2k-1}^{(1)} + B\theta_{2k+1}^{(1)})_k$ are entire on $\mathfrak{A}_1$,

$$|\widetilde{\varphi}_{2k}(a_0, \ldots, a_{2k})| \leqslant Ck!$$

for some constant $C > 0$ independent of m, which implies that $(\widetilde{\varphi}_{2k})_k$ is entire. ∎

Now we are ready to define a map $\Psi : \varprojlim H_\varepsilon^{\mathrm{ev}}(\mathfrak{A}_n) \to H_\varepsilon^{\mathrm{ev}}(\mathfrak{A})$ in the following fashion:

$$\Psi(\{[(\varphi_{2k}^{(n)})_k]_{\mathfrak{A}_n}\}_n) = [(\widetilde{\varphi}_{2k})_k]_{\mathfrak{A}}.$$

We have to verify that the definition is well-defined. Let

$$\{[(\varphi_{2k}^{(n)})_k]_{\mathfrak{A}_n}\}_n = \{[(\psi_{2k}^{(n)})_k]_{\mathfrak{A}_n}\}_n \in \varprojlim H_\varepsilon^{\mathrm{ev}}(\mathfrak{A}_n).$$

Then for any $n \geqslant 1$, there exists an odd entire cyclic cocycles $\theta^{(n)} = (\theta_{2k+1}^{(n)})_k$ on $\mathfrak{A}_n$ such that

$$\varphi_{2k}^{(n)}(b_0, \ldots, b_{2k}) - \psi_{2k}^{(n)}(b_0, \ldots, b_{2k}) = (b\theta_{2k-1}^{(n)} + B\theta_{2k+1}^{(n)})(b_0, \ldots b_{2k})$$

for $b_0, \ldots, b_{2k} \in \mathfrak{A}_n$. By the above argument, there exists an odd entrie cyclic cocycle $\widetilde{\theta} = (\widetilde{\theta_{2k+1}})_k$ on $\mathfrak{A}$. Then by the definition of $b + B$, we have that

$$\begin{aligned} (b\theta_{2k-1}^{(n)} + B\theta_{2k+1}^{(n)})(a_0, \ldots, a_{2k}) &= \lim_{m \to \infty} (b\theta_{2k-1}^{(m)} + B\theta_{2k+1}^{(m)})(b_0^{(m)}, \ldots, b_{2k}^{(m)}) \\ &= \lim_{m \to \infty} (\widetilde{\varphi}_{2k}^{(m)}(b_0^{(m)}, \ldots, b_{2k}^{(m)}) - \widetilde{\psi}_{2k}^{(m)}(b_0^{(m)}, \ldots, b_{2k}^{(m)})) \\ &= \widetilde{\varphi}_{2k}(a_0, \ldots, a_{2k}) - \widetilde{\psi}_{2k}(a_0, \ldots, a_{2k}), \end{aligned}$$

which implies that $[(\widetilde{\varphi}_{2k})_k]_{\mathfrak{A}} = [(\widetilde{\psi}_{2k})_k]_{\mathfrak{A}}$.

PROPOSITION 3.4. *The following isomorphism holds as a vector space over* $\mathbb{C}$*:*

$$\varprojlim H_\varepsilon^*(\mathfrak{A}_n) \simeq H_\varepsilon^*(\mathfrak{A}).$$

Proof. We prove just in the even case. For any $[(\varphi_{2k})_k]_{\mathfrak{A}} \in H_\varepsilon^{\mathrm{ev}}(\mathfrak{A})$, we have

$$\Psi \circ \Phi([(\varphi_{2k})_k]_{\mathfrak{A}}) = \Psi(\{[(\varphi_{2k}|_{\mathfrak{A}_n})_k]_{\mathfrak{A}_n}\}_n) = [(\widetilde{\varphi_{2k}|_{\mathfrak{A}_n}})_k]_{\mathfrak{A}}.$$

For any $a_0,\dots,a_{2k}\in\mathfrak{A}$, we take sequences $\{b_j^{(m)}\}_m$ $(j=0,\dots,2k)$ which converge to a_j as $m\to\infty$ and $b_j^{(m)}\in\mathfrak{A}_m$ for $j=0,\dots,2k$. Then,

$$\widetilde{\varphi_{2k}|_{\mathfrak{A}_n}}(a_0,\dots,a_{2k})=\lim_{m\to\infty}\varphi_{2k}|_{\mathfrak{A}_m}(b_0^{(m)},\dots,b_{2k}^{(m)})=\varphi_{2k}(a_0,\dots,a_{2k}).$$

This implies that $\widetilde{\varphi_{2k}|_{\mathfrak{A}_n}}=\varphi_{2k}$, which means that $\Psi\circ\Phi$ is the identity on $H_\varepsilon^{\mathrm{ev}}(\mathfrak{A})$. On the other hand, for any $\{[(\varphi_{2k}^{(n)})_k]_{\mathfrak{A}_n}\}_n\in\varprojlim H_\varepsilon^{\mathrm{ev}}(\mathfrak{A}_n)$, we have

$$\Phi\circ\Psi(\{[(\varphi_{2k}^{(n)})_k]_{\mathfrak{A}_n}\}_n)=\Phi([(\widetilde{\varphi}_{2k})_k]_{\mathfrak{A}})=\{[(\widetilde{\varphi}_{2k}|_{\mathfrak{A}_n})_k]_{\mathfrak{A}_n}\}_n.$$

Since for $b_0,\dots,b_{2k}\in\mathfrak{A}_n$, we have

$$\widetilde{\varphi}_{2k}|_{\mathfrak{A}_n}(b_0,\dots,b_{2k})=\lim_{m\to\infty}\widetilde{\varphi}_{2k}^{(m)}(b_0,\dots,b_{2k})=\lim_{m\to\infty}\varphi_{2k}^{(m)}(b_0,\dots,b_{2k})\varphi_{2k}^{(n)}(b_0,\dots,b_{2k}).$$

Thus $\Phi\circ\Psi(\{[(\varphi_{2k}^{(n)})_k]_{\mathfrak{A}_n}\}_n)=\{[(\varphi_{2k}^{(n)})_k]_{\mathfrak{A}_n}\}_n$. Hence $\Phi\circ\Psi$ is also the identity on $\varprojlim H_\varepsilon^{\mathrm{ev}}(\mathfrak{A}_n)$. Therefore, the proof is completed. ∎

REMARK 3.5. Meyer [7] obtained the above result by means of analytic cyclic theory. We here used the original defintion by Connes [2] to conclude our result.

4. ENTIRE CYCLIC COHOMOLOGY OF $(T_\theta^2)^\infty$

Summing up the argument discussed in the previous sections, we are ready to obtain the next main result:

THEOREM 4.1. *The entire cyclic cohomology $H_\varepsilon^*((T_\theta^2)^\infty)$ of the noncommutative 2-torus $(T_\theta^2)^\infty$ is isomorphic to $\mathbb{C}^4$ as linear spaces, especially*

$$\begin{cases} H_\varepsilon^{\mathrm{ev}}((T_\theta^2)^\infty)=HP^{\mathrm{ev}}((T_\theta^2)^\infty)\simeq\mathbb{C}^2,\\ H_\varepsilon^{\mathrm{od}}((T_\theta^2)^\infty)=HP^{\mathrm{od}}((T_\theta^2)^\infty)\simeq\mathbb{C}^2,\end{cases}$$

where $HP^*((T_\theta^2)^\infty)$ is the periodic cyclic cohomology of $(T_\theta^2)^\infty$.

Proof. By Proposition 3.4, we have

$$\begin{aligned} H_\varepsilon^*((T_\theta^2)^\infty)&\simeq H_\varepsilon^*(\varinjlim(C^\infty(T)\otimes(M_{q_{2n}}(\mathbb{C})\oplus M_{q_{2n-1}}(\mathbb{C})),\ \pi_n^\infty))\\ &\simeq\varprojlim H_\varepsilon^*((C^\infty(T)\otimes(M_{q_{2n}}(\mathbb{C})\oplus M_{q_{2n-1}}(\mathbb{C})),(\pi_n^\infty)^*))\end{aligned}$$

We have the following decomposition by applying Khalkhali ([4], Proposition 7) in the case of F^*-algebras:

$$\begin{aligned} &H_\varepsilon^*(C^\infty(T)\otimes(M_{q_{2n}}(\mathbb{C})\oplus M_{q_{2n-1}}(\mathbb{C})))\\ &\qquad\simeq H_\varepsilon^*(C^\infty(T)\otimes M_{q_{2n}}(\mathbb{C}))\oplus H_\varepsilon^*(C^\infty(T)\otimes M_{q_{2n-1}}(\mathbb{C})).\end{aligned}$$

We also deduce applying Khalkhali ([4], Theorem 6) in the case of F^*-algebras that

$$H^*_\varepsilon(C^\infty(T) \otimes M_q(\mathbb{C})) \simeq H^*_\varepsilon(C^\infty(T)) \quad (q \geqslant 1).$$

Since the above two phenomena are shown for $HP^*((T^2_\theta)^\infty)$ as well and we can see that

$$H^*_\varepsilon(C^\infty(T)) = HP^*(C^\infty(T)) \simeq \mathbb{C} \quad (* = \mathrm{ev}, \mathrm{od})$$

([2], Theorem 2 (page 208) and Theorem 25 (page 382)), then we obtain that

$$H^*_\varepsilon(C^\infty(T) \otimes (M_q(\mathbb{C}))) = HP^*(C^\infty(T) \otimes (M_q(\mathbb{C}))) \quad (* = \mathrm{ev}, \mathrm{od}).$$

We then have the following commutative diagram:

$$\begin{array}{ccc} HP^{\mathrm{ev}}(\mathfrak{A}_{n+1}) & \xrightarrow[i^*]{\simeq} & H^{\mathrm{ev}}_\varepsilon(\mathfrak{A}_{n+1}) \\ {\scriptstyle (\pi_n^\infty)^*}\downarrow & & \downarrow{\scriptstyle (\pi_n^\infty)^*} \\ HP^{\mathrm{ev}}(\mathfrak{A}_n) & \xrightarrow[\simeq]{i^*} & H^{\mathrm{ev}}_\varepsilon(\mathfrak{A}_n), \end{array}$$

where i^* is the canonical inclusion map. Then we work on the periodic cyclic cohomology in what follows: we consider homomorphisms

$$\begin{aligned}(\pi_n^\infty)^* : HP^{\mathrm{ev}}&(C^\infty(T) \otimes (M_{q_{2n+2}}(\mathbb{C}) \oplus M_{q_{2n+1}}(\mathbb{C}))) \\ &\to HP^{\mathrm{ev}}(C^\infty(T) \otimes (M_{q_{2n}}(\mathbb{C}) \oplus M_{q_{2n-1}}(\mathbb{C}))).\end{aligned}$$

Now we note that

$$\begin{aligned}HP^{\mathrm{ev}}&(C^\infty(T) \otimes (M_{q_{2n+2}}(\mathbb{C}) \oplus M_{q_{2n+1}}(\mathbb{C}))) \\ &\simeq HP^{\mathrm{ev}}(C^\infty(T) \otimes M_{q_{2n+2}}(\mathbb{C})) \oplus HP^{\mathrm{ev}}(C^\infty(T) \otimes M_{q_{2n+1}}(\mathbb{C}))\end{aligned}$$

and moreover, since $HP^{\mathrm{od}}(M_q(\mathbb{C})) = 0$,

$$\begin{aligned}HP^{\mathrm{ev}}&(C^\infty(T) \otimes M_q(\mathbb{C})) \\ &\simeq (HP^{\mathrm{ev}}(C^\infty(T)) \otimes HP^{\mathrm{ev}}(M_q(\mathbb{C}))) \oplus (HP^{\mathrm{od}}(C^\infty(T)) \otimes HP^{\mathrm{od}}(M_q(\mathbb{C}))) \\ &\simeq \mathbb{C}[\textstyle\int_T] \otimes \mathbb{C}[\mathrm{Tr}_q] \simeq \mathbb{C}[\textstyle\int_T \otimes \mathrm{Tr}_q],\end{aligned}$$

where $\int_T$ and Tr_q are the usual integral on $C^\infty(T)$ and the trace on $M_q(\mathbb{C})$ respectively. Here, we consider the following diagram:

$$\begin{array}{ccc} HP^{\mathrm{ev}}(\mathfrak{A}_{n+1}) & \xrightarrow{\simeq} & \mathbb{C}[\int_T \otimes \mathrm{Tr}_{q_{2n+2}}] \oplus \mathbb{C}[\int_T \otimes \mathrm{Tr}_{q_{2n+1}}] \\ {\scriptstyle (\pi_n^\infty)^*}\downarrow & & \downarrow{\scriptstyle (\pi_n^\infty)^*} \\ HP^{\mathrm{ev}}(\mathfrak{A}_n) & \xrightarrow[\simeq]{} & \mathbb{C}[\int_T \otimes \mathrm{Tr}_{q_{2n}}] \oplus \mathbb{C}[\int_T \otimes \mathrm{Tr}_{q_{2n-1}}], \end{array}$$

where the horizonal isomorphisms are defined by

$$\begin{aligned}&HP^{\mathrm{ev}}(\mathfrak{A}_n) \to \mathbb{C}[\textstyle\int_T \otimes \mathrm{Tr}_{q_{2n}}] \oplus [\textstyle\int_T \otimes \mathrm{Tr}_{q_{2n-1}}], \\ &\varphi \mapsto \varphi|_{(C^\infty(T)\otimes M_{q_{2n}}(\mathbb{C}))\oplus 0} \oplus \varphi|_{0\oplus(C^\infty(T)\otimes M_{q_{2n-1}}(\mathbb{C}))}.\end{aligned}$$

We check that the diagram above is also commutative.

So, we regard $(\pi_n^\infty)^*$ as the linear map from $\mathbb{C}[\int_T \otimes \mathrm{Tr}_{q_{2n+2}}] \oplus \mathbb{C}[\int_T \otimes \mathrm{Tr}_{q_{2n+1}}]$ into $\mathbb{C}[\int_T \otimes \mathrm{Tr}_{q_{2n}}] \oplus \mathbb{C}[\int_T \otimes \mathrm{Tr}_{q_{2n-1}}]$. Let us recall that we write the matrix P_{n+1} by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ used in the definition of π_n^∞. Then we have

$$\Big(\Big(\int_T \otimes \mathrm{Tr}_{q_{2n+2}}\Big) \oplus 0\Big)(\pi_n^\infty(\xi)) = a\Big(\int_T \otimes \mathrm{Tr}_{q_{2n}}\Big)(\mathbf{1} \otimes (x_{ij})) + b\Big(\int_T \otimes \mathrm{Tr}_{q_{2n-1}}\Big)(\mathbf{1} \otimes (y_{ij})) \tag{4.1}$$

for each

$$\xi = (\mathbf{1} \otimes (x_{ij})) \oplus (\mathbf{1} \otimes (y_{ij})) \in (C^\infty(T) \otimes M_{q_{2n}}(\mathbb{C})) \oplus (C^\infty(T) \otimes M_{q_{2n-1}}(\mathbb{C})),$$

where $\mathbf{1}$ is the function which evaluates 1 at each point of T. In fact, by the definition of π_n^∞, we have

$$\pi_n^\infty((\mathbf{1} \otimes (x_{ij}) \oplus (\mathbf{1} \otimes (y_{ij})) = \begin{pmatrix} x_{11}I_a & \dots & x_{1q'}I_a & & & \\ \vdots & & \vdots & & & \\ x_{q'1}I_a & \dots & x_{q'q'}I_a & & & \\ & & & y_{11}I_b & \dots & y_{1q}I_b \\ & & & \vdots & & \vdots \\ & & & y_{q1}I_b & \dots & y_{qq}I_b \end{pmatrix} \oplus \begin{pmatrix} x_{11}I_c & \dots & x_{1q'}I_c & & & \\ \vdots & & \vdots & & & \\ x_{q'1}I_c & \dots & x_{q'q'}I_c & & & \\ & & & y_{11}I_d & \dots & y_{1q}I_d \\ & & & \vdots & & \vdots \\ & & & y_{q1}I_d & \dots & y_{qq}I_d \end{pmatrix},$$

where $q = q_{2n-1}$, $q' = q_{2n}$ and so on. Then, it follows that

$$\begin{aligned} \Big(\Big(\int_T \otimes \mathrm{Tr}_{q_{2n+2}}\Big) \oplus 0\Big)(\pi_n^\infty(\xi)) &= a\sum_{i=1}^{q_{2n}} x_{ii} + b\sum_{i=1}^{q_{2n-1}} y_{ii} \\ &= a\Big(\int_T \otimes \mathrm{Tr}_{q_{2n}}\Big)(\mathbf{1} \otimes (x_{ij})) + b\Big(\int_T \otimes \mathrm{Tr}_{q_{2n-1}}\Big)(\mathbf{1} \otimes (y_{ij})). \end{aligned}$$

Similarly, we have

$$\begin{aligned} &\Big(0 \oplus \Big(\int_T \otimes \mathrm{Tr}_{q_{2n+1}}\Big)\Big)(\pi_n^\infty(\xi)) \\ &\qquad = c\Big(\int_T \otimes \mathrm{Tr}_{q_{2n}}\Big)(\mathbf{1} \otimes (x_{ij})) + d\Big(\int_T \otimes \mathrm{Tr}_{q_{2n-1}}\Big)(\mathbf{1} \otimes (y_{ij})). \end{aligned} \tag{4.2}$$

On the other hand, we check that

$$\Big(\Big(\int_T \otimes \mathrm{Tr}_{q_{2n+2}}\Big) \oplus 0\Big)(\pi_n^\infty((z^k \otimes I_{q_{2n}}) \oplus 0)) = 0,$$

$$\Big(0 \oplus \Big(\int_T \otimes \mathrm{Tr}_{q_{2n+1}}\Big)\Big)(\pi_n^\infty((z^k \otimes I_{q_{2n}}) \oplus 0)) = 0,$$

$$\Big(\Big(\int_T \otimes \mathrm{Tr}_{q_{2n+2}}\Big) \oplus 0\Big)(\pi_n^\infty(0 \oplus (z^k \otimes I_{q_{2n-1}}))) = 0, \quad \text{and}$$

$$\Big(0 \oplus \Big(\int_T \otimes \mathrm{Tr}_{q_{2n+1}}\Big)\Big)(\pi_n^\infty(0 \oplus (z^k \otimes I_{q_{2n-1}}))) = 0,$$

for each integer $k \geqslant 1$. Indeed, for example, it is easily verified that if

$$\begin{pmatrix} 0 & & & z \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \in M_q(C^\infty(T)),$$

$$\begin{pmatrix} 0 & & & z \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}^k = \begin{cases} z^\nu \otimes I_q & (k \equiv 0 \mod q), \\ \begin{pmatrix} 0 & & * \\ & \ddots & \\ * & & 0 \end{pmatrix} & (k \not\equiv 0 \mod q), \end{cases}$$

for some integer $\nu \geqslant 1$. Thus, we have that

$$\Big(\int_T \otimes \mathrm{Tr}_q\Big)\left(\begin{pmatrix} 0 & & & z \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}^k\right) = \begin{cases} \int_T z^\nu \mathrm{d}z & (k \equiv 0 \mod q), \\ 0 & (k \not\equiv 0 \mod q), \end{cases} = 0.$$

Since the space of Laurent polynomials is dense in $C^\infty(T)$ with respect to Fréchet topology, we then conclude that (4.1) and (4.2) hold for every $\xi \in \mathfrak{A}_n$. Hence, it is verified that $(\pi_n^\infty)^*$ is an isomorphism by the fact that

$$\begin{aligned} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \det P_{n+1} \\ &= \det \begin{pmatrix} a_{4n+4} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{4n+3} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{4n+2} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{4n+1} & 1 \\ 1 & 0 \end{pmatrix} = 1 \neq 0. \end{aligned}$$

Finally, we conclude that

$$H_\varepsilon^{\mathrm{ev}}((T_\theta^2)^\infty) \simeq \varprojlim(\mathbb{C} \oplus \mathbb{C}, (\pi_n^\infty)^*) \simeq \mathbb{C}^2.$$

Analogously, the same consequence is obtained in the odd case. We note that

$$\begin{aligned} &HP^{\mathrm{od}}(C^\infty(T)\otimes M_q(\mathbb{C})) \\ &\quad\simeq (HP^{\mathrm{ev}}(C^\infty(T))\otimes HP^{\mathrm{od}}(M_q(\mathbb{C})))\oplus(HP^{\mathrm{od}}(C^\infty(T))\otimes HP^{\mathrm{ev}}(M_q(\mathbb{C}))) \\ &\quad\simeq \mathbb{C}[\psi\otimes \mathrm{Tr}_q], \end{aligned}$$

where $\psi(f,g)=\int_T f(t)g'(t)\mathrm{d}t$ for $f,g\in C^\infty(T)$. This ends the proof. ∎

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