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PERTURBATION OF SEMI-FREDHOLM OPERATORS ON PRODUCTS OF BANACH SPACES

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ABSTRACT. We show new examples of Banach spaces X and Y for which the component $P\Phi_+(X,Y)$ of the perturbation class for the upper semi-Fredholm operators coincides with the strictly singular operators, or the component $P\Phi_-(X,Y)$ of the perturbation class for the lower semi-Fredholm operators coincides with the strictly cosingular operators. We also show new examples for which the equalities fail.

KEYWORDS: *Semi-Fredholm operator, strictly singular, strictly cosingular, perturbation class.*

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1. INTRODUCTION

A (continuous linear) operator $T \in \mathcal{L}(X,Y)$ is *upper semi-Fredholm* if its kernel $N(T)$ is finite dimensional and its range $R(T)$ is closed; and T is said to be *lower semi-Fredholm* if $R(T)$ is finite codimensional in Y (which entails that $R(T)$ is closed, following Lemma 3.2.4 in [11]). We denote respectively by Φ_+ and Φ_- the classes of all upper semi-Fredholm and lower semi-Fredholm operators. The class of *Fredholm operators* is $\Phi := \Phi_+ \cap \Phi_-$. Given a class $\mathcal{A}$ of operators, for each pair X, Y of Banach spaces we have a *component* given by $\mathcal{A}(X,Y) := \{T \in \mathcal{L}(X,Y) : T \in \mathcal{A}\}$. We write $\mathcal{A}(X)$ in the case $X = Y$.

Let $\mathcal{A}$ be any of the classes Φ_+, Φ_- or Φ. The *perturbation class* of $\mathcal{A}$ is defined by its components:

$$P\mathcal{A}(X,Y) := \{K \in \mathcal{L}(X,Y) \colon K + T \in \mathcal{A}(X,Y) \text{ for all } T \in \mathcal{A}(X,Y)\},$$

when $\mathcal{A}(X,Y)$ is non-empty.

The components $P\mathcal{A}(X,Y)$ have been studied by many authors, but there are no good descriptions of them in general. Kato proved that the strictly singular operators $\mathcal{SS}$ are contained in $P\Phi_+$ (Theorem 5.2 in [22]) and Vladimirskii proved that the strictly cosingular operators $\mathcal{SC}$ are contained in $P\Phi_-$ (Corollary 1 in [31]). Moreover, the classes $P\Phi_+$ and $P\Phi_-$ are both contained in the ideal $\mathcal{I}n$ of inessential operators [7], which coincides with $P\Phi$ when this perturbation class is defined.

Recall that an operator $T \in \mathcal{L}(X,Y)$ is said to be *strictly singular* if the restriction $T|_E$ to a closed subspace E is an isomorphism only if E is finite dimensional; T is said to be *strictly cosingular* if for every closed subspace F of Y the composition $Q_F T$ is surjective only if F is finite codimensional, where Q_F is the quotient operator onto Y/F; and T is said to be *inessential* if $I - ST \in \Phi(X)$ for every $S \in \mathcal{L}(Y,X)$.

The *perturbation classes problem* for semi-Fredholm operators asks whether the identities $P\Phi_+(X,Y) = \mathcal{SS}(X,Y)$ and $P\Phi_-(X,Y) = \mathcal{SC}(X,Y)$ hold whenever $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$ are non-empty respectively. Observe that a positive answer to this problem is interesting because it gives intrinsic characterizations for the perturbation classes; i.e., characterizations involving the action of an operator $T \in \mathcal{L}(X,Y)$ on subspaces or quotients, instead of considering the properties of the sums $T+U$ with all the operators in $\Phi_+(X,Y)$ or $\Phi_-(X,Y)$. It was proved in [16] that there exists a separable space Z for which $P\Phi_+(Z) \neq \mathcal{SS}(Z)$ and $P\Phi_-(Z^*) \neq \mathcal{SC}(Z^*)$. Since the space Z is a finite product of the recently constructed hereditarily indecomposable spaces [21], the perturbation classes problem remains open for many classical Banach spaces. This problem was formulated by Gohberg, Markus and Feldman (see page 74 in [14]) for the upper semi-Fredholm operators. Later, it was explicitly stated in page 101 in [11], 26.6.12 in [26], Section 3 in [30], [6] and [7].

In Section 2 we give a fairly complete list of examples of spaces X and Y for which there is a positive answer to the perturbation classes problem. We observe that the proofs of these results depend on very specific properties of the spaces considered and that most of the times these properties are not satisfied by the product of two different spaces.

In Section 3 we show that, under some conditions, the product of spaces for which there is a positive answer to the perturbation classes problem provide new examples. More precisely, given Banach spaces X_1, X_2, Y_1 and Y_2 for which $P\Phi_+(X_1,Y_1) = \mathcal{SS}(X_1,Y_1)$ and $P\Phi_+(X_2,Y_2) = \mathcal{SS}(X_2,Y_2)$, we find conditions implying

$$P\Phi_+(X_1 \times X_2, Y_1 \times Y_2) = \mathcal{SS}(X_1 \times X_2, Y_1 \times Y_2);$$

and we obtain similar results for the components of $P\Phi_-$ and $\mathcal{SC}$.

We observe that in general the perturbation classes $P\Phi_+$ and $P\Phi_-$ do not have a good behavior under products. Indeed, it was proved in [16] that:

(i) $P\Phi_+(X) = \mathcal{SS}(X)$ and $P\Phi_+(Y) = \mathcal{SS}(Y) \not\Rightarrow P\Phi_+(X \times Y) = \mathcal{SS}(X \times Y)$;
(ii) $P\Phi_-(X) = \mathcal{SC}(X)$ and $P\Phi_-(Y) = \mathcal{SC}(Y) \not\Rightarrow P\Phi_-(X \times Y) = \mathcal{SC}(X \times Y)$;

and it was proved in [7] that $P\Phi_+(X, Y \times C[0,1]) = \mathcal{SS}(X, Y \times C[0,1])$ and $P\Phi_-(X \times \ell_1, Y) = \mathcal{SC}(X \times \ell_1, Y)$ if X and Y are separable.

2. PRELIMINARIES

In the following two results we show examples of Banach spaces X and Y for which there are positive answers for the perturbation classes problem.

THEOREM 2.1. *Suppose $\Phi_+(X,Y) \neq \emptyset$. Then $P\Phi_+(X,Y) = \mathcal{SS}(X,Y)$ in the following cases:*

(i) *Y subprojective* [23], [4]*;*
(ii) *$X = Y = L_p(\mu)$, $1 \leqslant p \leqslant \infty$* [32]*;*
(iii) *X hereditarily indecomposable (Theorem* 3.14 *in* [4]*);*
(iv) *X is separable and Y contains a complemented copy of $C[0,1]$* [7]*;*
(v) *$X = L_p(0,1)$ when $1 < p < 2$ and Y satisfies the Orlicz property* [20]*;*
(vi) *$X = L_1(0,1)$ and Y is weakly sequentially complete* [20]*;*
(vii) *$X = L_p(0,1)$ with $2 \leqslant p \leqslant \infty$* [20]*;*
(viii) *X strongly subprojective* [19].

THEOREM 2.2. *Suppose $\Phi_-(X,Y) \neq \emptyset$. Then $P\Phi_-(X,Y) = \mathcal{SC}(X,Y)$ in the following cases:*

(i) *X superprojective* [23], [4]*;*
(ii) *$X = Y = L_p(\mu)$, $1 \leqslant p \leqslant \infty$* [32]*;*
(iii) *Y quotient indecomposable (Theorem* 3.14 *in* [4]*);*
(iv) *X contains a complemented copy of ℓ_1 and Y is separable* [7]*;*
(v) *$Y = L_p(0,1)$ when $2 < p < \infty$ and X^* satisfies the Orlicz property* [20]*;*
(vi) *$Y = L_p(0,1)$ with $1 \leqslant p \leqslant 2$* [20]*;*
(vii) *Y strongly superprojective* [19].

Let us recall that a Banach space X is said to be *subprojective* if each infinite dimensional closed subspace of X contains an infinite dimensional subspace M complemented in X; if M can be chosen so that its complement is isomorphic to X, then X is said to be *strongly subprojective*.

EXAMPLE 2.3 ([19]). The following spaces are strongly subprojective:

(1) The sequence spaces ℓ_p for $1 \leqslant p < \infty$ and c_0.
(2) The James space J.
(3) The Lorentz sequence spaces $d(w,p)$ for $1 \leqslant p < \infty$ and $w = (w_n)$ a non-increasing null sequence with $\sum_{n=1}^{\infty} w_n$ divergent. In particular, $\ell_{p,q}$ for $1 \leqslant p,q < \infty$.
(4) The Baernstein spaces B_p for $1 < p < \infty$.
(5) The Tsirelson space T.
(6) The function spaces $L_p(0,1)$ for $2 \leqslant p < \infty$.
(7) The function spaces $L_p(0,\infty) \cap L_2(0,\infty)$ for $1 \leqslant p \leqslant 2$.
(7) The Lorentz spaces $\Lambda_{W,p}(0,1)$, $L_{p,q}(0,\infty)$ and $L_{p,q}(0,1)$ for $2 < p < \infty$ and $1 \leqslant q < \infty$.
(9) The spaces of continuous functions $C(K)$, with K a scattered compact.

(10) Closed subspaces of the previous examples.

A Banach space X is said to be *superprojective* if each of its infinite codimensional closed subspaces is contained in some complemented infinite codimensional closed subspace of X; if moreover that subspace can be chosen isomorphic to X, then X is said to be *strongly superprojective*.

EXAMPLE 2.4 ([19]). The following spaces are strongly superprojective:
(1) The sequence spaces ℓ_p for $1 < p < \infty$ and c_0.
(2) The dual J^* of James' space.
(3) The dual spaces $d(w,p)^*$ of $d(w,p)$ for $1 \leqslant p < \infty$ and $w = (w_n)$ a non-increasing null sequence with $\sum_{n=1}^{\infty} w_n$ divergent. In particular, $\ell_{p,q}^*$ for $1 \leqslant p,q < \infty$.
(4) The dual spaces B_p^* of Baernstein's spaces for $1 < p < \infty$.
(5) The dual T^* of Tsirelson's space.
(6) The function spaces $L_p(0,1)$ for $1 < p \leqslant 2$.
(7) The function spaces $L_p(0,\infty) + L_2(0,\infty)$ for $2 \leqslant p < \infty$.
(8) The dual spaces $\Lambda_{W,p}(0,1)^*$, $L_{p,q}(0,\infty)^*$ and $L_{p,q}(0,1)^*$ for $2 < p < \infty$ and $1 < q < \infty$.
(9) The spaces of continuous functions $C(K)$, with K a scattered compact.
(10) Quotients of the previous examples.

Recall that a Banach space X is said to be *indecomposable* if it does not contain infinite dimensional closed subspaces M and N so that $X = M \oplus N$; equivalently, for every projection P on X, the range $R(P)$ or the kernel $N(P)$ is finite dimensional.

The space X is said to be *hereditarily indecomposable* (H.I., for short) if every closed subspace of X is indecomposable; and X is said to be *quotient indecomposable* (Q.I., for short) if every quotient of X is indecomposable.

3. PERTURBATION CLASSES ON PRODUCTS

The following result, proved in [4], is useful to determine the perturbation classes on product spaces. Observe that, although $P\Phi$ determines the operator ideal of inessential operators, the examples in [16] show that the classes $P\Phi_+$ and $P\Phi_-$ do not determine operator ideals.

LEMMA 3.1 (Lemma 3.3 in [4]). *Let $\mathcal{A}$ be one of the classes Φ_+ or Φ_-. Suppose that $\mathcal{A}(X,Y) \neq \emptyset$ and let $K \in P\mathcal{A}(X,Y)$, $A \in L(X)$ and $B \in L(Y)$. Then $BKA \in P\mathcal{A}(X,Y)$.*

We will adopt the following notation: let X_1, X_2, X, Y_1, Y_2 and Y be Banach spaces. Every operator $T \in \mathcal{L}(X_1 \times X_2, Y_1 \times Y_2)$ admits a matricial representation $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in \mathcal{L}(X_1,Y_1)$, $B \in \mathcal{L}(X_2,Y_1)$, $C \in \mathcal{L}(X_1,Y_2)$ and $D \in$

$\mathcal{L}(X_2, Y_2)$. Every operator $T \in \mathcal{L}(X_1 \times X_2, Y)$ admits a matricial representation

$$T = (A, B)$$

with $A \in \mathcal{L}(X_1, Y)$, $B \in \mathcal{L}(X_2, Y)$. Finally, every operator $T \in \mathcal{L}(X, Y_1 \times Y_2)$ admits a matricial representation

$$T = \begin{pmatrix} A \\ C \end{pmatrix}$$

or (A, C) for short, with $A \in \mathcal{L}(X, Y_1)$ and $C \in \mathcal{L}(X, Y_2)$.

PROPOSITION 3.2. *Let $\mathcal{A}$ denote Φ_+ or Φ_-.*

(i) *Suppose $\mathcal{A}(X_1 \times X_2, Y_1 \times Y_2) \neq \emptyset$ and let $T \in \mathcal{L}(X_1 \times X_2, Y_1 \times Y_2)$. Then $T \in P\mathcal{A}$ if and only if $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}$ belong to $P\mathcal{A}$.*

(ii) *Assume $\mathcal{A}(X, Y_1 \times Y_2) \neq \emptyset$ and let $T \in \mathcal{L}(X, Y_1 \times Y_2)$. Then $T \in P\mathcal{A}$ if and only if $\begin{pmatrix} A \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ C \end{pmatrix}$ belong to $P\mathcal{A}$.*

(iii) *Assume $\mathcal{A}(X_1 \times X_2, Y) \neq \emptyset$ and let $T \in \mathcal{L}(X_1 \times X_2, Y)$. Then $T \in P\mathcal{A}$ if and only if $(A, 0)$ and $(0, B)$ belong to $P\mathcal{A}$.*

Proof. (i) The "only if" implication follows from Lemma 3.1, since these matrices can be written respectively as P_1TQ_1, P_1TQ_2, P_2TQ_1 and P_2TQ_2, where P_i is the projection of $Y_1 \times Y_2$ onto Y_i, and Q_i is the projection of $X_1 \times X_2$ onto X_i.

For the converse, it is enough to observe that $P\mathcal{A}(X_1 \times X_2, Y_1 \times Y_2)$ is a subspace of $\mathcal{L}(X_1 \times X_2, Y_1 \times Y_2)$.

The proofs of statements (ii) and (iii) are similar. ∎

Let $\mathcal{A}$ denote one of the classes Φ_+ or Φ_-. Observe that an operator

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in \mathcal{L}(X_1 \times X_2, Y_1 \times Y_2)$$

belongs to $\mathcal{A}$ if and only if both A and D belong to $\mathcal{A}$. As a consequence, we obtain the following:

REMARK 3.3. Let $\mathcal{A}$ denote Φ_+ or Φ_- and suppose that both $\mathcal{A}(X_1, Y_1)$ and $\mathcal{A}(X_2, Y_2)$ are nonempty; then $\mathcal{A}(X_1 \times X_2, Y_1 \times Y_2) \neq \emptyset$.

It is not difficult to show that $T \in P\mathcal{A}(X_1 \times X_2, Y_1 \times Y_2)$ implies $A \in P\mathcal{A}(X_1, Y_1)$ and $D \in P\mathcal{A}(X_2, Y_2)$. However, the examples in [16] show that we cannot derive the same result for B and C. Thus, in order to get the result for B and C we have to impose some restrictions.

PROPOSITION 3.4. *Suppose $\Phi_+(X_1, Y) \neq \emptyset \neq \Phi_+(X_2, Y)$ and $Y \simeq Y \times Y$. Let $(A, B) \in \mathcal{L}(X_1 \times X_2, Y)$. Then*

$$(A, B) \in P\Phi_+ \Rightarrow A \in P\Phi_+(X_1, Y), B \in P\Phi_+(X_2, Y).$$

Proof. It follows from Proposition 3.2 that both $(A,0)$ and $(0,B)$ belong to $P\Phi_+$. By hypothesis, there exist closed subspaces Y_1 and Y_2 of Y such that $Y_1 \simeq Y_2 \simeq Y = Y_1 \oplus Y_2$. Thus, for each $i = 1,2$, there exists an isomorphism $U_i \in \mathcal{L}(Y)$ with range $R(U_i) = Y_i$; moreover, $\Phi_+(X_1 \times X_2, Y) \neq \emptyset$.

Let $T_i \in \Phi_+(X_i, Y)$ $(i = 1,2)$. Then $(U_1T_1, U_2T_2) \in \Phi_+$. Since $(U_1A, 0) \in P\Phi_+$, we have $(U_1(T_1 + A), U_2T_2) \in \Phi_+$. Then $U_1(T_1 + A) \in \Phi_+$, hence $T_1 + A \in \Phi_+(X_1, Y)$ (Proposition A.1.5 in [18]), and therefore, $A \in P\Phi_+$.

Similarly, it can be proved that $B \in P\Phi_+$. ∎

In order to see that the hypothesis $Y \simeq Y \times Y$ is necessary in Proposition 3.4, let X be a complex H.I. Banach space and let M be a closed subspace of X with $\dim M = \dim X/M = \infty$.

We take $X_1 = M$, $X_2 = X$ and $Y = M \times X$, we denote by $J\colon M \longrightarrow X$ the natural inclusion and we consider the operator $A \in \mathcal{L}(M, M \times X)$ defined by $A(m) := (0, Jm)$. It was proved in [16] that $(A,0) \in P\Phi_+(X_1 \times X_2, Y)$, but it is clear that $A \notin P\Phi_+(X_1, Y)$.

THEOREM 3.5. *Suppose that $\Phi_+(X_1, Y) \neq \emptyset \neq \Phi_+(X_2, Y)$, $Y \simeq Y \times Y$ and $P\Phi_+(X_i, Y) = \mathcal{SS}(X_i, Y)$ for $i = 1,2$. Then $P\Phi_+(X_1 \times X_2, Y) = \mathcal{SS}(X_1 \times X_2, Y)$.*

Proof. Let $(A,B) \in P\Phi_+(X_1 \times X_2, Y)$. Thus, by Proposition 3.4 and the hypothesis, $A \in P\Phi_+(X_1, Y) = \mathcal{SS}(X_1, Y)$ and $B \in P\Phi_+(X_2, Y) = \mathcal{SS}(X_2, Y)$. But $\mathcal{SS}$ is an operator ideal, so $(A,B) \in \mathcal{SS}(X_1 \times X_2, Y)$. ∎

Observe that if X is a complex H.I. Banach space and M is a closed subspace of X with $\dim M = \dim X/M = \infty$, then $P\Phi_+(M, M \times X) = \mathcal{SS}(M, M \times X)$ and $P\Phi_+(X, M \times X) = \mathcal{SS}(X, M \times X)$ by Theorem 2.1, but it was proved in [16] that $P\Phi_+(M \times X, M \times X) \neq \mathcal{SS}(M \times X, M \times X)$.

PROPOSITION 3.6. *Suppose $\Phi_+(X, Y_1) \neq \emptyset \neq \Phi_+(X, Y_2)$ and let $(A, C) \in \mathcal{L}(X, Y_1 \times Y_2)$. Then*

$$\begin{pmatrix} A \\ C \end{pmatrix} \in P\Phi_+ \Rightarrow A \in P\Phi_+(X, Y_1), C \in P\Phi_+(X, Y_2).$$

Proof. Let $T \in \Phi_+(X, Y_1)$. Then $(T, 0) \in \Phi_+(X, Y_1 \times Y_2)$, and by Proposition 3.2, $(A, 0) \in P\Phi_+$, so we have $(T + A, 0) \in \Phi_+$; hence $T + A \in \Phi_+(X, Y_1)$, and we have proved that $A \in P\Phi_+$.

Similarly, we get $C \in P\Phi_+$. ∎

THEOREM 3.7. *Suppose $\Phi_+(X, Y_1) \neq \emptyset \neq \Phi_+(X, Y_2)$ and $P\Phi_+(X, Y_i) = \mathcal{SS}(X, Y_i)$ for $i = 1,2$. Then $P\Phi_+(X, Y_1 \times Y_2) = \mathcal{SS}(X, Y_1 \times Y_2)$.*

Proof. Let $T \in P\Phi_+(X, Y_1 \times Y_2)$ be an operator with matricial representation

$$\begin{pmatrix} A \\ C \end{pmatrix}.$$

By Proposition 3.6 and the hypothesis, A belongs to $P\Phi_+(X,Y_1) = \mathcal{SS}(X,Y_1)$ and C belongs to $P\Phi_+(X,Y_2) = \mathcal{SS}(X,Y_2)$. Since $\mathcal{SS}$ is an operator ideal, $(A,C) \in \mathcal{SS}(X,Y_1 \times Y_2)$. ∎

Let us consider now the perturbation classes for lower semi-Fredholm operators.

PROPOSITION 3.8. *Suppose $\Phi_-(X,Y_1) \neq \emptyset \neq \Phi_-(X,Y_2)$ and $X \simeq X \times X$. Let $(A,C) \in \mathcal{L}(X,Y_1 \times Y_2)$. Then*

$$(A,C) \in P\Phi_-(X,Y_1 \times Y_2) \Rightarrow A \in P\Phi_-(X,Y_1), C \in P\Phi_-(X,Y_2).$$

Proof. Assume $(A,C) \in P\Phi_-(X,Y_1 \times Y_2)$. Then, by Proposition 3.2,

$$\begin{pmatrix} A \\ 0 \end{pmatrix} \in P\Phi_-(X,Y_1 \times Y_2) \quad \text{and} \quad \begin{pmatrix} 0 \\ C \end{pmatrix} \in P\Phi_-(X,Y_1 \times Y_2).$$

Take any pair of operators $T_1 \in \Phi_-(X,Y_1)$ and $T_2 \in \Phi_-(X,Y_2)$. By hypothesis, X contains two subspaces X_1 and X_2 isomorphic to X and such that $X = X_1 \oplus X_2$. Let $U_1 \in \mathcal{L}(X)$ be an operator which is bijective from X_1 onto X and has kernel $N(U_1) = X_2$, and let $U_2 \in \mathcal{L}(X)$ be an operator which is bijective from X_2 onto X and has kernel $N(U_2) = X_1$. Note that

$$\begin{pmatrix} T_1U_1 \\ T_2U_2 \end{pmatrix} \in \Phi_-(X,Y_1 \times Y_2).$$

Moreover, by Lemma 3.1,

$$\begin{pmatrix} AU_1 \\ 0 \end{pmatrix} \in P\Phi_-(X,Y_1 \times Y_2).$$

Therefore,

$$\begin{pmatrix} (T_1 + A)U_1 \\ T_2U_2 \end{pmatrix} \in \Phi_-(X,Y_1 \times Y_2),$$

so $(T_1 + A)U_1 \in \Phi_-(X,Y_1)$; hence $T_1 + A \in \Phi_-(X,Y_1)$ (see Proposition A.1.5 in [18]), which proves that $A \in P\Phi_-(X,Y_1)$.

With a similar procedure, we prove that $C \in P\Phi_-(X,Y_2)$. ∎

In order to see that the hypothesis $X \simeq X \times X$ is necessary in Proposition 3.8, let Z be a complex reflexive H.I. Banach space and let M be a closed subspace of Z with $\dim M = \dim Z/M = \infty$.

We take $X = M^* \times Z^*$, $Y_1 = M^*$ and $Y_2 = Z^*$, we denote by $J\colon M \longrightarrow Z$ the natural inclusion and we consider the operator $B \in \mathcal{L}(M^* \times Z^*, M^*)$ defined by $B(m^*, x^*) := J^*x^*$. It was proved in [16] that $(B,0) \in P\Phi_-(X,Y_1 \times Y_2)$, but it is clear that $B \notin P\Phi_-(X,Y_1)$.

THEOREM 3.9. *Suppose that $\Phi_-(X,Y_1) \neq \emptyset \neq \Phi_-(X,Y_2)$, $P\Phi_-(X,Y_i) = \mathcal{SC}(X,Y_i)$ for $i = 1,2$ and $X \simeq X \times X$. Then $P\Phi_-(X,Y_1 \times Y_2) = \mathcal{SC}(X,Y_1 \times Y_2)$.*

Proof. Take an operator $(A,C) \in P\Phi_-(X, Y_1 \times Y_2)$. By Proposition 3.8,

$$A \in P\Phi_-(X, Y_1) = \mathcal{SC}(X, Y_1), \quad C \in P\Phi_-(X, Y_2) = \mathcal{SC}(X, Y_2).$$

Thus, since $\mathcal{SC}$ is an operator ideal, $(A,C) \in \mathcal{SC}(X, Y_1 \times Y_2)$. ∎

Note that if X is a complex reflexive H.I. space and M is a closed subspace of X with $\dim M = \dim X/M = \infty$, then $P\Phi_-(M^* \times X^*, M^*) = \mathcal{SC}(M^* \times X^*, M^*)$ and $P\Phi_-(M^* \times X^*, X^*) = \mathcal{SC}(M^* \times X^*, X^*)$ by Theorem 2.2, but it was proved in [16] that $P\Phi_-(M^* \times X^*, M^* \times X^*) \neq \mathcal{SC}(M^* \times X^*, M^* \times X^*)$.

PROPOSITION 3.10. *Suppose $\Phi_-(X_1, Y) \neq \emptyset \neq \Phi_-(X_2, Y)$. If $(A,B) \in P\Phi_-(X_1 \times X_2, Y)$ then $A \in P\Phi_-(X_1, Y)$ and $B \in P\Phi_-(X_2, Y)$.*

Proof. Take $T \in \Phi_-(X_1, Y)$. Thus $(T,0) \in \Phi_-(X_1 \times X_2, Y)$. Besides, as $(A,B) \in P\Phi_-(X_1 \times X_2, Y)$, Proposition 3.2 yields $(A,0) \in P\Phi_-(X_1 \times X_2, Y)$, and therefore, $(T + A, 0) \in \Phi_-(X_1 \times X_2, Y)$, which leads to $T + A \in \Phi_-(X_1, Y)$. We have just proved that $A \in P\Phi_-(X_1, Y)$. Analogously, it can be proved that $B \in P\Phi_-(X_2, Y)$. ∎

THEOREM 3.11. *Suppose that $\Phi_-(X_1, Y)$ and $\Phi_-(X_2, Y)$ are non-empty and that $P\Phi_-(X_i, Y) = \mathcal{SC}(X_i, Y)$ for $i = 1,2$. Then $P\Phi_-(X_1 \times X_2, Y) = \mathcal{SC}(X_1 \times X_2, Y)$.*

Proof. Let $(A,B) \in P\Phi_-(X_1 \times X_2, Y)$. Then, by Proposition 3.10,

$$A \in P\Phi_-(X_1, Y) = \mathcal{SC}(X_1, Y), \quad B \in P\Phi_-(X_2, Y) = \mathcal{SC}(X_2, Y),$$

and as $\mathcal{SC}$ is an operator ideal, it follows $(A,B) \in \mathcal{SC}(X_1 \times X_2, Y)$. ∎

4. EXAMPLES AND COUNTEREXAMPLES

In this section we describe a few concrete examples of products of Banach spaces for which a positive answer to the perturbation classes problem is derived from the results of the previous section, showing the power and the limitations of these results. We observe that, using Theorems 2.1 and 2.2, we could easily show plenty of additional examples. Note that the condition $Y \simeq Y \times Y$ in these theorems is not very restrictive when we consider classical Banach spaces.

We also describe new counterexamples to the perturbation classes problem which are different from those obtained in [16]. In particular, in these new counterexamples the semi-Fredholm operators do not coincide with the Fredholm operators.

Let us write $L_p = L_p(0,1)$ for $1 \leqslant p \leqslant \infty$. Recall that $\Phi_+(L_p, L_q) \neq \emptyset$ for $1 \leqslant q \leqslant p \leqslant 2$ (see Corollary 2.f.5 in [25]).

Let $1 \leqslant q \leqslant p_1 < \cdots < p_n \leqslant 2$. It follows from Theorem 3.5 that

$$P\Phi_+(L_{p_1} \times \cdots \times L_{p_n}, L_q) = \mathcal{SS}(L_{p_1} \times \cdots \times L_{p_n}, L_q).$$

Moreover, by Theorem 3.7, for $1 \leqslant q_1 < \cdots < q_m \leqslant p_1 < \cdots < p_n \leqslant 2$ we get

$$\begin{aligned} P\Phi_+(L_{p_1} \times \cdots \times L_{p_n}, L_{q_1} \times \cdots \times L_{q_m}) \\ = \mathcal{SS}(L_{p_1} \times \cdots \times L_{p_n}, L_{q_1} \times \cdots \times L_{q_m}). \end{aligned} \tag{4.1}$$

REMARK 4.1. In the case $m = n$, if we only assume $1 < q_i \leqslant p_i < 2$ for $i = 1, \ldots, n$, we cannot derive directly the equality (4.1), although the result is valid. Indeed, it follows from Theorem 6 in [20] that

$$P\Phi_+(L_{p_i}, L_{q_1} \times \cdots \times L_{q_m}) = \mathcal{SS}(L_{p_i}, L_{q_1} \times \cdots \times L_{q_m})$$

for each $i = 1, \ldots, n$, and we can apply Theorem 3.7 to derive (4.1).

Using duality, we can derive similar results for lower semi-Fredholm operators. Indeed, $\Phi_-(L_p, L_q) \neq \emptyset$ for $2 \leqslant q \leqslant p \leqslant \infty$. Thus, the arguments given in the case of Φ_+ allow us to show that for $2 \leqslant q_1 < \cdots < q_m \leqslant p_1 < \cdots < p_n \leqslant \infty$ we have

$$\begin{aligned} P\Phi_-(L_{p_1} \times \cdots \times L_{p_n}, L_{q_1} \times \cdots \times L_{q_m}) \\ = \mathcal{SC}(L_{p_1} \times \cdots \times L_{p_n}, L_{q_1} \times \cdots \times L_{q_m}), \end{aligned} \tag{4.2}$$

and we have the same limitation described in Remark 4.1.

Next we describe the counterexamples. Let us denote the *semi-Fredholm operators* by $\Phi_\pm(X, Y) := \Phi_+(X, Y) \cup \Phi_-(X, Y)$. For an operator $T \in \Phi_\pm(X, Y)$, the *index of* T is defined by

$$\operatorname{ind}(T) := \dim N(T) - \dim Y/R(T) \in \mathbb{Z} \cup \{\pm\infty\}.$$

For the convenience of the reader, we include a proof for the following result.

PROPOSITION 4.2. *Let X be an* H.I. *Banach space. Then every operator $T \in \Phi_\pm(X)$ is Fredholm with* $\operatorname{ind}(T) = 0$.

Proof. If X is a complex space, it is not difficult to show that every $T \in \mathcal{L}(X)$ can be written as $T = \lambda I + K$, where λ is a complex number and K is a strictly singular operator. When T is semi-Fredholm, we have $\lambda \neq 0$. Thus the result follows from the stability of the index of a Fredholm operator under strictly singular perturbations.

Now assume that X is a real space. Then $\mathcal{L}(X)/\mathcal{SS}(X)$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or the quaternions $\mathbb{H}$ (Theorem 2 in [12]); hence $\dim \mathcal{L}(X)/\mathcal{SS}(X) \leqslant 4$. We consider the complexification $\widehat{X}$ of X. It is not difficult to show (see Proposition 2.8 in [17]) that $\dim \mathcal{L}(\widehat{X})/\mathcal{SS}(\widehat{X}) \leqslant \dim \mathcal{L}(X)/\mathcal{SS}(X)$.

Suppose that there exists $T \in \Phi_\pm(X)$ with $\operatorname{ind}(T) \neq 0$. Then it is easy to check that $\widehat{T}(x + \mathrm{i}y) := Tx + \mathrm{i}Ty$ defines an operator $\widehat{T} \in \Phi_\pm(\widehat{X})$ with $\operatorname{ind}(\widehat{T}) = \operatorname{ind}(T)$. Since $\operatorname{ind}(\widehat{T}) \neq 0$,

$$\{\lambda \in \mathbb{C} : \widehat{T} - \lambda I \notin \Phi(\widehat{X})\}$$

is an infinite set. Indeed, it includes the boundary of the open set

$$\{\lambda \in \mathbb{C} : \operatorname{ind}(\widehat{T} - \lambda I) = \operatorname{ind}(\widehat{T})\}.$$

Hence the spectrum of the image of $\widehat{T}$ in the Banach algebra $\mathcal{L}(\widehat{X})/\mathcal{SS}(\widehat{X})$ would be an infinite set, which is not possible since the algebra is finite dimensional. ∎

Proposition 4.2 can be restated by saying that an H.I. Banach space is isomorphic to no proper subspace or quotient. Moreover, we can find in [13] examples of real H.I. spaces for which $\mathcal{L}(X)/\mathcal{SS}(X)$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

The following result of Argyros and Felouzis will be necessary in our construction.

THEOREM 4.3 (Corollary 3 in [8]). *For each $1 < p < \infty$ there exists a reflexive* H.I. *Banach space X_p with a quotient isomorphic to ℓ_p.*

Obviously, X_p does not contain copies of ℓ_p, and therefore every operator in $\mathcal{L}(\ell_p, X_p)$ or $\mathcal{L}(X_p, \ell_p)$ is inessential [15]. We need an improvement of this result.

LEMMA 4.4. *Every operator $B \in \mathcal{L}(\ell_p, X_p)$ is strictly cosingular.*

Proof. Suppose that B is not strictly cosingular. Then there exists an infinite codimensional closed subspace N of X_p such that $Q_N B$ is surjective. Therefore, the kernel $N(Q_N B) = B^{-1}(N)$ is closed and infinite codimensional.

In the case $p = 2$, we can write $\ell_2 = M \oplus B^{-1}(N)$, where M is an infinite dimensional closed subspace of ℓ_2. Hence $X_2 = B(M) \oplus N$ with $B(M)$ infinite dimensional and closed (see Theorem 5.10 in [29]). Since X_2 is H.I., we would conclude that N is finite dimensional; hence X_2 isomorphic to ℓ_2, which is not possible.

In the general case, we can do something similar with a more involved argument. Note that $Q_N B$ is surjective; hence it induces an isomorphism from $\ell_p/B^{-1}(N)$ onto X_p/N. As ℓ_p is strongly superprojective (Example 2.4), $B^{-1}(N)$ is contained in an infinite codimensional complemented subspace M_0 isomorphic to ℓ_p. Therefore, we can write $\ell_p = M \oplus M_0$, where M is an infinite dimensional closed subspace of ℓ_p. Moreover, since $Q_N B$ is surjective, $B(M_0)$ is an infinite codimensional closed subspace of X_p that contains N and $X_p = B(M) \oplus B(M_0)$. Thus, proceeding like in the case $p = 2$, we get a contradiction. ∎

Given an operator $T \in \mathcal{L}(X_p \times \ell_p)$ we consider the matricial representation

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

THEOREM 4.5. *The operator $T \in \mathcal{L}(X_p \times \ell_p)$ is lower semi-Fredholm if and only if the operators $A \in \mathcal{L}(X_p)$ and $D \in \mathcal{L}(\ell_p)$ are lower semi-Fredholm. Consequently,*

$$P\Phi_-(X_p \times \ell_p) \neq \mathcal{SC}(X_p \times \ell_p) \quad \text{and} \quad P\Phi_+(X_p^* \times \ell_p^*) \neq \mathcal{SS}(X_p^* \times \ell_p^*).$$

Proof. Suppose $T \in \Phi_-(X_p \times \ell_p)$. By Lemma 4.4, the entry $B \in \mathcal{L}(\ell_p, X_p)$ is strictly cosingular. Hence the operator

$$T_0 := \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$$

is lower semi-Fredholm. This fact implies $A \in \Phi_-(X_p)$, and as X_p is H.I., Proposition 4.2 yields $A \in \Phi$.

Observe that $R(T_0) \cap (\{0\} \times \ell_p)$ is finite codimensional in $\{0\} \times \ell_p$. Let $(0, z) \in R(T_0)$. Then

$$(0, z) = (Ax, Cx) + (0, Dy) \quad \text{with } x \in N(A) \text{ and } y \in \ell_p.$$

Since $N(A)$ is finite dimensional, we conclude that $R(D)$ is finite codimensional; hence $D \in \Phi_-(\ell_p)$ (Theorem 5.10 in [29]).

Conversely, suppose that $A \in \Phi_-(X_p)$ and $D \in \Phi_-(\ell_p)$; hence A is a Fredholm operator.

In the case $p = 2$, D is a right Atkinson operator (lower semi-Fredholm with complemented kernel); hence

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

is a right Atkinson operator. Since $B \in \mathcal{L}(\ell_p, X_p)$ and $C \in \mathcal{L}(X_p, \ell_p)$ are inessential, T is right Atkinson (Theorem 7.23 in [1]); in particular, $T \in \Phi_-$.

In the general case, we consider the operators $S_1 \in \mathcal{L}(X_p, X_p \times \ell_p)$ and $S_2 \in \mathcal{L}(\ell_p, X_p \times \ell_p)$ given by $S_1 x := (Ax, Cx)$ and $S_2 y := (0, Dy)$.

Since A is Fredholm and C is inessential, the operators $(A, 0)$ and $(0, C)$ are respectively left Atkinson (upper semi-Fredholm with complemented range) and inessential, so S_1 is left Atkinson (Theorem 7.23 in [1]). In particular, $R(S_1)$ is isomorphic to a subspace of X_p. Also, $R(S_2)$ is isomorphic to a subspace of ℓ_p. Since X_p and ℓ_p are totally incomparable, $R(T_0) = R(S_1) + R(S_2)$ is a closed subspace of $X_p \times \ell_p$ [27].

It remains to show that $R(T_0)$ is finite codimensional. Since $M := C^{-1}R(D)$ is finite codimensional in X_p, it is enough to show that $R(T_0)$ contains the subspaces $\{0\} \times R(D)$ and $A(M) \times \{0\}$. Obviously, it contains the first one. Moreover, given $m \in M$ we can write $Cm = -Dz$ for some $z \in \ell_p$. Hence $(Am, 0) = (Am, Cm) + (0, Dz) = T_0(m, z)$; thus it also contains the second one.

For the second part, let $Q \in \mathcal{L}(X_p, \ell_p)$ be a surjective operator, whose existence is guaranteed by Theorem 4.3. The first part shows that

$$K := \begin{pmatrix} 0 & 0 \\ Q & 0 \end{pmatrix}$$

defines an operator $K \in P\Phi_-(X_p \times \ell_p)$ which is not strictly cosingular.

Since the space $X_p \times \ell_p$ is reflexive, the conjugate operator K^* belongs to $P\Phi_+(X_p^* \times \ell_p^*)$ but $K^* \notin \mathcal{SS}(X_p^* \times \ell_p^*)$. ∎

REMARK 4.6. One interesting difference between the counterexamples in [16] and those in Theorem 4.5 is that in the former ones all semi-Fredholm operators are Fredholm. However, in these new examples $\Phi_-(X_p \times \ell_p)$ and $\Phi_+(X_p^* \times \ell_p^*)$ are both different from the corresponding sets of Fredholm operators.

In the case $1 < p < \infty$, $p \neq 2$, there are operators in $\Phi_-(\ell_p)$ which are not right Atkinson. This can be derived from the fact that, taken q so that $1/p + 1/q = 1$, ℓ_q contains non-complemented subspaces isomorphic to ℓ_q (see Corollary 3.2 in [10] for $1 < q < 2$ and Theorem 6 in [28] for $2 < q < \infty$); hence ℓ_p contains closed non-complemented subspaces M with ℓ_p/M isomorphic to ℓ_p. As a consequence, there are operators in $\Phi_-(X_p \times \ell_p)$ which are not right Atkinson.

In the case $p = 2$, the operators in $\Phi_-(X_2 \times \ell_2)$ are right Atkinson because every closed subspace in ℓ_2 is complemented.

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