

THE ENTIRE CYCLIC COHOMOLOGY OF NONCOMMUTATIVE 3-SPHERES

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ABSTRACT. In this paper we compute the entire cyclic cohomology of noncommutative 3-spheres. First of all, we verify the Mayer–Vietoris exact sequence of entire cyclic cohomology in the framework of Fréchet $*$ -algebras. Applying it to their noncommutative Heegaard decomposition, we deduce that their entire cyclic cohomology is isomorphic to the d’Rham homology of the ordinary 3-sphere with the complex coefficients.

KEYWORDS: *Entire cyclic cohomology, noncommutative 3-spheres, metric approximation property, Mayer–Vietoris exact sequence.*

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INTRODUCTION

Since Connes [5] constructed a generalization of periodic cyclic cohomology which is called entire cyclic cohomology, its explicit computation is executed only for few examples (cf. [3], [5], [12]). As a matter of fact, their entire cyclic cohomologies are nothing but their periodic ones. Recently, the first named author [16] computed that of smooth noncommutative 2-tori, which have the same property cited above.

In this paper, we firstly formulate the Mayer–Vietoris exact sequence for entire cyclic cohomology, then we apply it to compute for smooth noncommutative 3-spheres. The key idea is based on Meyer’s excision [14], [15] concerning the short exact sequences of Fréchet $*$ -algebras to obtain a noncommutative Mayer–Vietoris exact sequence for entire cyclic cohomology. To use his excision, we need to construct a bounded linear section for a short exact sequence of Fréchet $*$ -algebras. To ensure it, we reformulate the notion of metric approximation property in the framework of Fréchet $*$ -algebras to solve the lifting problem (see [4]). We then use Baum, Hajac, Matthes and Szymański’s method [1] for a Heegaard decomposition of smooth noncommutative 3-spheres since they pointed out an insufficient part of Matsumoto’s construction [13] in the case of C^* -algebras.

Under this circumstance, we conclude that the entire cyclic cohomology of noncommutative 3-spheres is the same as their periodic one.

Throughout this paper, θ is an irrational number in the open unit interval $(0, 1)$ and we use the notation $\mathbb{Z}_{\geq 0}$ for the set of all nonnegative integers.

1. PRELIMINARIES

We prepare some notations and basic properties used throughout the paper. Let \mathfrak{A} be a Fréchet $*$ -algebra or F^* -algebra and denote by $C^\infty([0, 1], \mathfrak{A})$ the set of all \mathfrak{A} -valued smooth functions on the closed unit interval $[0, 1]$ with respect to Fréchet topology. Given an element $f \in C^\infty([0, 1], \mathfrak{A})$ and an integer $n \geq 1$, we write by $f^{(n)}$ its n -th derivative of f at t ($0 < t < 1$) and denote by $f_+^{(n)}(0), f_-^{(n)}(1)$ the n -th derivatives at 0 or 1 as follows:

$$f_+^{(n)}(0) = \lim_{t \rightarrow 0^+} f^{(n)}(t), \quad f_-^{(n)}(1) = \lim_{t \rightarrow 1-0} f^{(n)}(t).$$

For $n = 0$, we write $f_+^{(0)}(0) = f(0), f_-^{(0)}(1) = f(1)$.

DEFINITION 1.1. For a F^* -algebra \mathfrak{A} , we define the *suspension* $S^\infty \mathfrak{A}$ of \mathfrak{A} by

$$S^\infty \mathfrak{A} = \{f \in C^\infty([0, 1], \mathfrak{A}) : f_+^{(n)}(0) = f_-^{(n)}(1) = 0 \ (n \geq 0)\}$$

and we also define the *cone* $C^\infty \mathfrak{A}$ of \mathfrak{A} by

$$C^\infty \mathfrak{A} = \{f \in C^\infty([0, 1], \mathfrak{A}) : f_-^{(n)}(1) = 0 \ (n \geq 0)\}.$$

Then we have the following short exact sequence:

$$0 \longrightarrow \mathfrak{J} \xrightarrow{i} C^\infty \mathfrak{A} \xrightarrow{q} \mathfrak{A} \longrightarrow 0,$$

where q is defined by $q(f) = f(0)$,

$$\mathfrak{J} = \{f \in C^\infty \mathfrak{A} : f(0) = 0\}$$

and i is the canonical inclusion. The map $s : \mathfrak{A} \rightarrow C^\infty \mathfrak{A}$ defined by

$$s(a)(t) = (1 - t)a \quad (a \in \mathfrak{A}, t \in [0, 1])$$

is a bounded linear section of q with respect to Fréchet topology. We need to know the entire cyclic cohomologies of $C^\infty \mathfrak{A}$ and \mathfrak{J} . We say that given two F^* -algebras \mathfrak{A} and \mathfrak{B} , the map

$$\Phi : \mathfrak{A} \rightarrow C^\infty([0, 1], \mathfrak{B})$$

is called a smooth homotopy if it is a bounded homomorphism with respect to Fréchet topology and two bounded homomorphisms $f, g : \mathfrak{A} \rightarrow \mathfrak{B}$ are smoothly homotopic if there exists a smooth homotopy Φ from \mathfrak{A} to \mathfrak{B} with $\Phi_0 = f, \Phi_1 = g$. A Fréchet algebra \mathfrak{A} is smoothly homotopic to another one \mathfrak{B} if there are two homomorphisms $f : \mathfrak{A} \rightarrow \mathfrak{B}$ and $g : \mathfrak{B} \rightarrow \mathfrak{A}$ such that $g \circ f$ (respectively $f \circ g$) is smoothly homotopic to the identity on \mathfrak{A} (respectively \mathfrak{B}). According to

Meyer [14], we know the homotopy invariance of entire cyclic cohomology in the framework of F^* -algebras:

PROPOSITION 1.2 ([14]). *If two bounded homomorphisms are smoothly homotopic, then they induce the same map on the entire cyclic cohomology.*

We also mention the following lemma:

LEMMA 1.3. *Let $S^\infty\mathfrak{A}, C^\infty\mathfrak{A}$ and \mathfrak{J} be defined above, we then have that*

$$HE^*(C^\infty\mathfrak{A}) = 0, \quad HE^*(\mathfrak{J}) \simeq HE^*(S^\infty\mathfrak{A}).$$

Proof. By Proposition 1.2, it suffices to show that $C^\infty\mathfrak{A}$ is smoothly homotopic to 0 to obtain the former isomorphism. The map

$$F : C^\infty\mathfrak{A} \rightarrow C^\infty([0, 1], C^\infty\mathfrak{A})$$

defined by

$$F_s(f)(t) = f(s + (1 - s)t) \quad (f \in C^\infty\mathfrak{A}, s, t \in [0, 1])$$

gives a smooth homotopy on $C^\infty\mathfrak{A}$. Since F_0 is the identity on $C^\infty\mathfrak{A}$ and for any $f \in C^\infty\mathfrak{A}$,

$$F_1(f)(t) = f(1) = 0.$$

We know that $C^\infty\mathfrak{A}$ is smoothly homotopic to 0. For the latter one, we introduce the map $t \mapsto f(e^{1-1/t})$ ($f \in C^\infty\mathfrak{A}, t \in [0, 1]$), which belongs to $S^\infty\mathfrak{A}$. Indeed, we note that for any $n \geq 1$, $\frac{d^n}{dt^n} f(e^{1-1/t})$ is a linear combination of some functions such as

$$f^{(k)}(e^{1-1/t}) \frac{e^{l(1-1/t)}}{t^m} \quad (k, l, m \geq 1).$$

In fact, for $n = 1$, we have that

$$\frac{d}{dt} f(e^{1-1/t}) = f^{(1)}(e^{1-1/t}) \frac{e^{1-1/t}}{t^2}.$$

Suppose that the function $\frac{d^n}{dt^n} f(e^{1-1/t})$ is a linear combination of functions

$$f^{(k)}(e^{1-1/t}) \frac{e^{l(1-1/t)}}{t^m} \quad (k, l, m \geq 1),$$

then we deduce that

$$\begin{aligned} & \frac{d}{dt} \left(f^{(k)}(e^{1-1/t}) \frac{e^{l(1-1/t)}}{t^m} \right) \\ &= f^{(k+1)}(e^{1-1/t}) \frac{e^{l(1-1/t)}}{t^{m+2}} + l f^{(k)}(e^{1-1/t}) \frac{e^{l(1-1/t)}}{t^{m+2}} - m f^{(k)}(e^{1-1/t}) \frac{e^{l(1-1/t)}}{t^{m+1}}, \end{aligned}$$

so the same is true for $\frac{d^{n+1}}{dt^{n+1}} f(e^{1-1/t})$. Because of the following equalities:

$$\lim_{t \rightarrow 0^+} f^{(k)}(e^{1-1/t}) \frac{e^{l(1-1/t)}}{t^m} = f_+^{(n)}(0) \cdot 0 = 0, \quad \lim_{t \rightarrow 1^-} f^{(k)}(e^{1-1/t}) \frac{e^{l(1-1/t)}}{t^m} = f_-^{(n)}(1) = 0,$$

for any $f \in \mathfrak{J}$, $k, l, m \geq 1$, the function $f(e^{1-1/t})$ belongs to $S^\infty \mathfrak{A}$. Let

$$r : \mathfrak{J} \rightarrow S^\infty \mathfrak{A}$$

be the map defined by

$$r(f)(t) = f(e^{1-1/t}) \quad (f \in \mathfrak{J}, t \in [0, 1])$$

and i the natural inclusion from $S^\infty \mathfrak{A}$ into \mathfrak{J} . For the proof that $r \circ i$ is smoothly homotopic to the identity on $S^\infty \mathfrak{A}$, we use the bounded homomorphism

$$G : S^\infty \mathfrak{A} \rightarrow C^\infty([0, 1], S^\infty \mathfrak{A})$$

defined by

$$G_s(f)(t) = f(se^{1-1/t} + (1-s)t) \quad (f \in S^\infty \mathfrak{A}, s, t \in [0, 1])$$

which gives a smooth homotopy connecting $r \circ i$ and the identity on $S^\infty \mathfrak{A}$. We firstly show that $G_s(f) \in S^\infty \mathfrak{A}$ for any fixed $f \in S^\infty \mathfrak{A}, s \in [0, 1]$. Since

$$\frac{d}{dt} G_s(f)(t) = f^{(1)}(se^{1-1/t} + (1-s)t) \left(\frac{s}{t^2} e^{1-1/t} + 1-s \right),$$

we know that

$$\lim_{t \rightarrow 0+} \frac{d}{dt} G_s(f)(t) = f_+^{(1)}(0) \cdot (1-s) = 0, \quad \lim_{t \rightarrow 1-0} \frac{d}{dt} G_s(f)(t) = f_-^{(1)}(1) = 0.$$

For general $n \geq 2$, we also see that

$$\lim_{t \rightarrow 0+} \frac{d^n}{dt^n} G_s(f)(t) = \lim_{t \rightarrow 1-0} \frac{d^n}{dt^n} G_s(f)(t) = 0.$$

The case for $n = 1$ has already been shown. It suffices to show that for $n \geq 2$, the function $\frac{d^n}{dt^n} G_s(f)(t)$ is a linear combination of functions like

$$f^{(k)}(se^{1-1/t} + (1-s)t) \frac{e^{l(1-1/t)}}{t^m} \quad (k, l, m \geq 1).$$

We now calculate that

$$\begin{aligned} & \frac{d}{dt} f^{(k)}(se^{1-1/t} + (1-s)t) \frac{e^{l(1-1/t)}}{t^m} \\ &= f^{(k+1)}(se^{1-1/t} + (1-s)t) \frac{e^{l(1-1/t)}}{t^m} \left(\frac{s}{t^2} + 1-s \right) \\ & \quad + f^{(k)}(se^{1-1/t} + (1-s)t) \left(\frac{le^{l(1-1/t)}}{t^{m+2}} - \frac{me^{l(1-1/t)}}{t^{m+1}} \right), \end{aligned}$$

which completes the induction process. Moreover we see that $\frac{d^n}{dt^n} G_s$ is uniformly bounded on $[0, 1]$ for each $n \geq 1$. We note that the function

$$t \mapsto \frac{e^{l(1-1/t)}}{t^m}$$

is bounded on $[0, 1]$ and that $f^{(k)}$ is also bounded since $f \in C^\infty([0, 1], \mathfrak{A})$. Hence G is a smooth homotopy connecting $r \circ i$ and the identity on $S^\infty \mathfrak{A}$ since $G_1 = r \circ i$

and G_0 is the identity on $S^\infty\mathfrak{A}$. Similarly, $i \circ r$ and the identity on \mathfrak{J} are smoothly homotopic via the smooth homotopy defined by the same way as G , which implies the following, as desired:

$$HE^*(\mathfrak{J}) \simeq HE^*(S^\infty\mathfrak{A}). \quad \blacksquare$$

2. TOEPLITZ F^* -ALGEBRAS

In this section, we construct smooth Toeplitz algebras based on 1-torus and to analyze them. They could be viewed as a quantization of 2-disc (cf. [1], [11]). Let $\{z^n\}_{n \in \mathbb{Z}}$ be the orthonormal basis of the Hilbert space $L^2(T)$ of all square integrable functions on the 1-torus T , where $z^n(t) = t^n$ ($t \in T, n \in \mathbb{Z}$), and $H^2 = H^2(T)$ the Hardy space on T which is a closed subspace of $L^2(T)$ spanned by $\{z^n\}_{n \geq 0}$. For $f \in C^\infty(T)$ the algebra of all infinitely differentiable functions on T , in which we mean that the derivation is defined by

$$\frac{d}{dt}f(t) = \lim_{r \rightarrow 0} \frac{f(e^{2\pi ir}t) - f(t)}{r},$$

we define the operator T_f for $f \in C^\infty(T)$ by

$$T_f \xi = Pf\xi \quad (\xi \in H^2),$$

where P is the projection onto H^2 . We consider the $*$ -algebra \mathcal{P} generated by T_{z^j} ($j \in \mathbb{Z}$), namely,

$$\mathcal{P} = \bigcup_{N \in \mathbb{Z}_{\geq 0}} \left\{ \sum_{i_j \in \mathbb{Z}, |i_j| \leq N} c_{i_1, \dots, i_n} T_{z^{i_1}} \cdots T_{z^{i_n}} : c_{i_1, \dots, i_n} \in \mathbb{C}, n \in \mathbb{Z}_{\geq 0} \right\}.$$

Since $T_f T_g - T_{fg}$ is a compact operator for any $f, g \in C^\infty(T)$ and T_f is compact if and only if $f = 0$ (cf. [8]), it is easily seen by induction that for any $T \in \mathcal{P}$, there is a unique $f \in C^\infty(T)$ and a unique compact operator S with $T = T_f + S$. Actually, if

$$T = \sum_{i_j \in \mathbb{Z}, |i_j| \leq N} c_{i_1, \dots, i_n} T_{z^{i_1}} \cdots T_{z^{i_n}} \in \mathcal{P},$$

then $T = T_f + S$, where

$$f = \sum_{i_j \in \mathbb{Z}, |i_j| \leq N} c_{i_1, \dots, i_n} z^{i_1} \cdots z^{i_n}$$

and the compact operator S is a linear combination of the operators of the form

$$T_{z^{l_1}} \cdots T_{z^{l_k}} (T_{z^n} T_{z^m} - T_{z^{n+m}}) T_{z^{l'_1}} \cdots T_{z^{l'_k}} \quad (l_1, \dots, l_k, \dots, l'_1, \dots, l'_k, n, m \in \mathbb{Z}).$$

We show that there exists a function $K_S(t, s) \in C^\infty(T^2)$ which is a polynomial of t, s and satisfies

$$(S\xi)(t) = \int_T K_S(t, s)\xi(s)ds \quad (\xi \in H^2).$$

This function K_S is called the kernel function of S . Given $n, m \in \mathbb{Z}$, it is easily verified that

$$\begin{aligned} (T_z^n T_z^m - T_{z^{n+m}})\xi(t) &= \left(\sum_{k \geq \max\{-m, -n\}} - \sum_{k \geq -m-n} \right) \langle \xi | z^k \rangle z^k(t) \\ &= \int_T \left(\sum_{k \geq \max\{-m, -n\}} - \sum_{k \geq -m-n} \right) t^k s^{-k} \xi(s) ds, \end{aligned}$$

where

$$\langle f : g \rangle = \int_T f(s) \overline{g(s)} ds \quad (f, g \in L^2(T))$$

is the usual inner product on $L^2(T)$. Then the kernel function $K_{T_z^n T_z^m - T_{z^{n+m}}}$ of $T_z^n T_z^m - T_{z^{n+m}}$ is a finite sum of the functions $t^k s^{-k}$ since there exists a finite subset $I_{n,m} \subset \mathbb{Z}$ such that

$$K_{T_z^n T_z^m - T_{z^{n+m}}}(t, s) = \left(\sum_{k \geq \max\{-m, -n\}} - \sum_{k \geq -m-n} \right) t^k s^{-k} = \pm \sum_{k \in I_{n,m}} t^k s^{-k}$$

(when $I_{n,m}$ is empty, we regard the function $K_{T_z^n T_z^m - T_{z^{n+m}}}(t, s) = 0$). Moreover, given $l \in \mathbb{Z}$, we compute that

$$\begin{aligned} K_{(T_z^n T_z^m - T_{z^{n+m}})T_{z^l}}(t, s) &= \int_T K_{T_z^n T_z^m - T_{z^{n+m}}}(t, r) K_{T_{z^l}}(r, s) dr \\ &= \pm \int_T \sum_{k \in I_{n,m}} t^k r^{-k} \sum_{k' \geq -l} r^{k'} s^{-k'} dr \\ &= \pm \sum_{k' \geq -l} \sum_{k \in I_{n,m}} t^k s^{-k'} \int_T r^{k'-k} dr = \pm \sum_{k \in I_{n,m}, k \geq -l} t^k s^{-k}, \end{aligned}$$

which implies that the kernel function $K_{(T_z^n T_z^m - T_{z^{n+m}})T_{z^l}}$ is a polynomial of t, s . By the similar computation, it follows that for $l, m, n \in \mathbb{Z}$, the kernel function $K_{T_{z^l}(T_z^n T_z^m - T_{z^{n+m}})}$ is also a polynomial. Then, by the inductive argument, we have that the kernel functions

$$K_{T_{z^l} \dots T_{z^j} (T_z^n T_z^m - T_{z^{n+m}}) T_{z^i} \dots T_{z^k}}$$

are also polynomials, which in particular belong to $C^\infty(T^2)$.

Let \mathbb{K}^∞ be the set of all compact operators S such that there exists a function $K_S \in C^\infty(T^2)$ with the property that

$$(S\xi)(t) = \int_T K_S(t, s) \xi(s) ds \quad (\xi \in H^2, t \in T).$$

By the above argument, it follows that for each operator $T \in \mathcal{P}$, there exist a function $f \in C^\infty(T)$ and an operator $S \in \mathbb{K}^\infty$ with $T = T_f + S$. Since T_g is

compact if and only if $g = 0$, the function f and the operator S are uniquely determined. We define the seminorms $\{\|\cdot\|_{k,l,m}\}$ on \mathcal{P} by

$$\|T_f + S\|_{k,l,m} = \|f^{(k)}\|_\infty + \|K_S^{(l,m)}\|_\infty \quad (k, l, m \in \mathbb{Z}_{\geq 0}),$$

where $f^{(k)}$ is the k -th derivative of f ,

$$K_S^{(l,m)} = \frac{\partial^{l+m}}{\partial t^l \partial s^m} K(t, s) \quad (K \in C^\infty(T^2)),$$

and $\|\cdot\|_\infty$ mean the supremum norms on the corresponding function spaces.

DEFINITION 2.1. The smooth Toeplitz algebra \mathcal{T}^∞ is defined by the completion of \mathcal{P} with respect to the topology induced by the seminorms $\{\|\cdot\|_{k,l,m}\}$.

Similarly as in the case of \mathcal{P} , we have that for any $T \in \mathcal{T}^\infty$, there exist a function $f \in C^\infty(T)$ and an operator $S \in \mathbb{K}^\infty$ with $T = T_f + S$. In fact, if $\{T_n\}_{n \geq 1} \subset \mathcal{P}$ converges to T with respect to the seminorms $\{\|\cdot\|_{k,l,m}\}$ with $T_n = T_{f_n} + S_n$, we compute that

$$\|f_n^{(k)} - f_{n'}^{(k)}\|_\infty = \|T_{f_n} - T_{f_{n'}}\|_{k,0,0} \leq \|T_n - T_{n'}\|_{k,0,0} \rightarrow 0 \quad (n, n' \rightarrow \infty),$$

for any $k \in \mathbb{Z}_{\geq 0}$, which ensures that there exists the function $f \in C^\infty(T)$ such that $f_n \rightarrow f$ with respect to the seminorms. Alternatively, since $\{S_n\}$ is also Cauchy, we have that for any $k, l, m \in \mathbb{Z}$,

$$\|S_n - S_{n'}\|_{k,l,m} = \|K_{S_n}^{(l,m)} - K_{S_{n'}}^{(l,m)}\|_\infty \rightarrow 0 \quad (n, n' \rightarrow \infty).$$

Hence, we find a function $K \in C^\infty(T^2)$ with $K_{S_n} \rightarrow K$ as $n \rightarrow \infty$ with respect to Fréchet topology on $C^\infty(T^2)$. Then the operator S defined by

$$S\xi(t) = \int_T K(t, s)\xi(s) ds \quad (\xi \in H^2, t \in T)$$

belongs to \mathbb{K}^∞ and $S_n - S \rightarrow 0$ as $n \rightarrow \infty$ with respect to the seminorms, which implies the conclusion. It is clear by the above argument that \mathbb{K}^∞ is a $*$ -ideal of \mathcal{T}^∞ and Fréchet closed.

We define a homomorphism $q : \mathcal{T}^\infty \rightarrow C^\infty(T)$ by $q(T_f + S) = f$, which is continuous with respect to the seminorms cited before. The following lemma is already clear:

LEMMA 2.2. We obtain the following short exact sequence as F^* -algebras:

$$0 \longrightarrow \mathbb{K}^\infty \xrightarrow{i} \mathcal{T}^\infty \xrightarrow{q} C^\infty(T) \longrightarrow 0,$$

where i is the canonical inclusion.

We next deduce the following lemma, which is a smooth version of C^* -algebra case:

LEMMA 2.3. *We have the following isomorphism:*

$$\mathbb{K}^\infty \simeq \varinjlim (M_n(\mathbb{C}), \varphi_n),$$

where the homomorphisms $\varphi_n : M_n(\mathbb{C}) \rightarrow M_{n+1}(\mathbb{C})$ are given by

$$\varphi_n(A) = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \quad (A \in M_n(\mathbb{C}), n \geq 1).$$

Proof. Let $P_n (n \geq 1)$ be the orthogonal projections on H^2 defined by

$$P_n \zeta(t) = \sum_{k=0}^{n-1} \langle \zeta : z^k \rangle z^k(t) = \sum_{k=0}^{n-1} \left(\int_T \zeta(s) s^{-k} ds \right) t^k = \int_T \sum_{k=0}^{n-1} t^k s^{-k} \zeta(s) ds \quad (\zeta \in H^2),$$

which implies that

$$K_{P_n}(t, s) = \sum_{k=0}^{n-1} t^k s^{-k}.$$

Then $P_n \mathbb{K}^\infty P_n$ is isomorphic to $M_n(\mathbb{C})$. Indeed, the kernel function $K_{P_n S P_n}$ for $S \in \mathbb{K}^\infty$ is calculated as follows: since

$$K_{S P_n}(t, s) = \int_T K_S(t, r) K_{P_n}(r, s) dr = \int_T \sum_{k=0}^{n-1} r^k s^{-k} K_S(t, r) dr,$$

we have that

$$\begin{aligned} K_{P_n S P_n}(t, s) &= \int_T K_{P_n}(t, u) K_{S P_n}(u, s) du = \int_T \sum_{k=0}^{n-1} t^k u^{-k} \left(\int_T \sum_{k'=0}^{n-1} r^{k'} s^{-k'} K_S(u, r) dr \right) du \\ &= \int_T \int_T \sum_{k, k'=0}^{n-1} t^k u^{-k} r^{k'} s^{-k'} K_S(u, r) dr du \\ &= \sum_{k, k'=0}^{n-1} t^k s^{-k'} \int_T \int_T r^{k'} u^{-k} K_S(u, r) dr du = \sum_{k, k'=0}^{n-1} c_{k, k'} t^k s^{-k'}, \end{aligned}$$

where

$$c_{k, k'} = \int_T \int_T r^{k'} u^{-k} K_S(u, r) dr du$$

are the Fourier coefficients of $K_S \in C^\infty(T^2)$.

On the other hand, we define the matrix units E_{ij} in what follows: when $i = j$, we define

$$E_{ii} = T_{z^{i-1}} T_{z^{i-1}}^* - T_{z^i} T_{z^i}^*.$$

For $i \neq j$, we define

$$E_{ij} = \begin{cases} T_{z^{j-i}} E_{ii} & (i < j), \\ E_{jj} T_{z^{j-i}} & (i > j). \end{cases}$$

It is not hard to see that $\{E_{ij}\}$ forms a family of matrix units. By taking $m = -n$ in the computation of the kernel function of $T_{z^n}T_{z^m} - T_{z^{n+m}}$, we have

$$K_{I-T_{z^n}T_{z^m}^*}(t, s) = \sum_{k=0}^{n-1} t^k s^{-k}.$$

Hence we have

$$K_{E_{ii}}(t, s) = t^{i-1} s^{-(i-1)}.$$

More generally, we obtain that

$$K_{E_{ij}}(t, s) = t^{j-1} s^{-(i-1)}.$$

Then $P_n \mathbb{K}^\infty P_n$ is generated by the matrix units $\{E_{ij}\}_{i,j=1}^n$ so that it is isomorphic to $M_n(\mathbb{C})$ with the seminorms given by

$$\|(\lambda_{kl})\|_{p,q} = \sup_{t,s \in T} \left| \sum_{k,l=0}^{n-1} \lambda_{kl} l^p k^q t^l s^{-k} \right| \quad ((\lambda_{kl}) \in M_n(\mathbb{C})).$$

For any $S \in \mathbb{K}^\infty$, $\|S - P_n S P_n\|_{l,m} \rightarrow 0$ as $n \rightarrow \infty$ for any $l, m \geq 0$ since $\{c_{k,k'}\}$ belongs to the Schwartz space on \mathbb{Z}^2 . Therefore,

$$\|S - P_n S P_n\|_{l,m} \rightarrow 0 \quad (n \rightarrow \infty)$$

for any $l, m \in \mathbb{Z}_{\geq 0}$. Hence, the conclusion follows. ■

By the above lemma, we deduce the following corollary:

COROLLARY 2.4. \mathbb{K}^∞ is a simple F^* -algebra, which is equal to the commutator F^* -ideal $[\mathcal{T}^\infty, \mathcal{T}^\infty]$ of \mathcal{T}^∞ .

In what follows, we study briefly the F^* -crossed products $\mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}$ of \mathcal{T}^∞ by the gauge action α_θ of \mathbb{Z} . Let α_θ be the action of \mathbb{Z} on \mathcal{T}^∞ defined by

$$\alpha_\theta(T_f) = T_{f_\theta} \quad (f \in C^\infty(T), n \in \mathbb{Z}),$$

where $f_\theta(z) = f(e^{2\pi i \theta} z)$, which gives a F^* -dynamical system $(\mathcal{T}^\infty, \mathbb{Z}, \alpha_\theta)$. We also consider the unitary operator U_θ on H^2 defined by

$$U_\theta \xi(t) = \xi(e^{2\pi i \theta} t) \quad (\xi \in H^2, t \in T).$$

It is easily seen that $U_\theta^* \xi(t) = U_\theta^{-1} \xi(t) = \xi(e^{-2\pi i \theta} t)$. Then we form the F^* -crossed products $\mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}$ of the F^* -dynamical system $(\mathcal{T}^\infty, \mathbb{Z}, \alpha_\theta)$, which could be viewed as the deformation quantization $(D^2 \times S^1)_\theta$ of the solid torus $D^2 \times S^1$. In fact, let $\mathcal{T}^\infty[\mathbb{Z}]$ be the $*$ -algebra of all finite sums

$$f = \sum_{n \in \mathbb{Z}, |n| \leq N} A_n U_\theta^n \quad (A_n \in \mathcal{T}^\infty, N \in \mathbb{Z}_{\geq 0}),$$

where its multiplication is determined by $U_\theta A U_\theta^{-1} = \alpha_\theta(A)$ and its $*$ -operation is given by $(A U_\theta)^* = \alpha_\theta^{-1}(A^*) U_\theta^{-1}$. For $f = \sum A_n U_\theta^n \in \mathcal{T}^\infty[\mathbb{Z}]$, we induce the

seminorms defined by

$$\|f\|_{p,q,r,s} = \sup_{n \in \mathbb{Z}} (1 + |n|^2)^p \|A_n\|_{q,r,s} \quad (p, q, r, s \in \mathbb{Z}_{\geq 0}).$$

We define the F^* -crossed product $(D^2 \times S^1)_\theta = \mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}$ by the completion of $\mathcal{T}^\infty[\mathbb{Z}]$ with respect to the seminorms cited above. For $S \in \mathbb{K}^\infty$, we calculate that

$$\alpha_\theta(S)\zeta(t) = U_\theta S U_\theta^* \zeta(t) = U_\theta \int_T K_S(t, s) \zeta(e^{-2\pi i \theta} s) ds = \int_T K_S(e^{2\pi i \theta} t, e^{2\pi i \theta} s) \zeta(s) ds$$

to obtain that

$$K_{\alpha_\theta(S)}(t, s) = K_S(e^{2\pi i \theta} t, e^{2\pi i \theta} s).$$

Therefore, we have $\alpha_\theta(\mathbb{K}^\infty) = \mathbb{K}^\infty$ so that we construct a F^* -dynamical system $(\mathbb{K}^\infty, \mathbb{Z}, \alpha_\theta)$. Since

$$K_{\alpha_\theta(P_n)}(t, s) = K_{P_n}(e^{2\pi i \theta} t, e^{2\pi i \theta} s) = \sum_{k=0}^{n-1} (e^{2\pi i \theta} t)^k (e^{2\pi i \theta} s)^{-k} = \sum_{k=0}^{n-1} t^k s^{-k} = K_{P_n}(t, s),$$

we have $\alpha_\theta(P_n \mathbb{K}^\infty P_n) = P_n \mathbb{K}^\infty P_n$. Therefore, we also construct F^* -dynamical systems $(P_n \mathbb{K}^\infty P_n, \mathbb{Z}, \alpha_\theta^{(n)})$, where $\alpha_\theta^{(n)}$ are the restrictions of α_θ on $P_n \mathbb{K}^\infty P_n$. Let i_n be the isomorphism from $P_n \mathbb{K}^\infty P_n$ onto $M_n(\mathbb{C})$ defined before and $i = \varinjlim i_n$ the isomorphism from $\varinjlim P_n \mathbb{K}^\infty P_n$ onto \mathbb{K}^∞ induced by the isomorphisms i_n . We write by $\bar{\alpha}_\theta^{(n)}$ the action $i_n \circ \alpha_\theta \circ i_n^{-1}$.

PROPOSITION 2.5. *We have the following isomorphism:*

$$\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z} \simeq \varinjlim (M_n(\mathbb{C}) \rtimes_{\bar{\alpha}_\theta^{(n)}} \mathbb{Z}, \tilde{\varphi}_n),$$

where $\tilde{\varphi}_n$ are the inclusions induced naturally by φ_n .

Proof. Since $i \circ \bar{\alpha}_\theta^{(n)} = \alpha_\theta \circ i$ for any $n \geq 1$, we have

$$\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z} \simeq \varinjlim (P_n \mathbb{K}^\infty P_n \rtimes_{\alpha_\theta^{(n)}} \mathbb{Z}, \varphi_n),$$

where $\varphi_n : P_n \mathbb{K}^\infty P_n \rtimes_{\alpha_\theta^{(n)}} \mathbb{Z} \rightarrow P_{n+1} \mathbb{K}^\infty P_{n+1} \rtimes_{\alpha_\theta^{(n+1)}} \mathbb{Z}$ are the canonical inclusions.

Moreover, since $i_n \circ \alpha_\theta^{(n)} = \bar{\alpha}_\theta^{(n)} \circ i_n$, we find isomorphisms

$$\psi_n : P_n \mathbb{K}^\infty P_n \rtimes_{\alpha_\theta^{(n)}} \mathbb{Z} \xrightarrow{\simeq} M_n(\mathbb{C}) \rtimes_{\bar{\alpha}_\theta^{(n)}} \mathbb{Z}.$$

Then since $\psi_n \circ \varphi_n = \tilde{\varphi}_n \circ \psi_n$, we conclude, as desired,

$$\varinjlim (P_n \mathbb{K}^\infty P_n \rtimes_{\alpha_\theta^{(n)}} \mathbb{Z}, \varphi_n) \simeq \varinjlim (M_n(\mathbb{C}) \rtimes_{\bar{\alpha}_\theta^{(n)}} \mathbb{Z}, \tilde{\varphi}_n). \quad \blacksquare$$

Then we construct a $*$ -homomorphism

$$\rho_n : M_n(\mathbb{C}) \rtimes_{\bar{\alpha}_\theta^{(n)}} \mathbb{Z} \rightarrow M_n(\mathbb{C}) \hat{\otimes}_\gamma \mathcal{S}(\mathbb{Z}),$$

where $\mathcal{S}(\mathbb{Z})$ is the set of all rapidly decreasing sequences $\{c_n\} \subset \mathbb{C}$ and $\widehat{\otimes}_\gamma$ means the tensor product of F^* -algebras completed by the topology induced by the seminorms defined by

$$\left\| \sum_{j=1}^N x_j \otimes y_j \right\|_{k,l} = \inf \sum_{j=1}^N \|x_j\|_k \|y_j\|_l,$$

where the infimum is taken over the all representations of $\sum_{j=1}^N x_j \otimes y_j$. Equivalently, $M_n(\mathbb{C}) \widehat{\otimes}_\gamma \mathcal{S}(\mathbb{Z})$ is regarded as $\mathcal{S}(\mathbb{Z}, M_n(\mathbb{C}))$ with the ordinary convolution as its product. For $x \in M_n(\mathbb{C}) \rtimes_{\alpha_\theta^{(n)}} \mathbb{Z}$, we define

$$\rho_n(x) = xU_\theta^n.$$

It is easily seen that it is an isomorphism. Moreover, since

$$\|\rho_n(x)\|_{p,q,r,s} = \sup_{m \in \mathbb{Z}} (1 + m^2)^p \|x_m U_\theta^n\|_{q,r,s} = \|x\|_{p,q,r,s} \quad (p, q, r, s \in \mathbb{Z}_{\geq 0}),$$

for any $x = \sum x_m U_\theta^m \in \mathcal{T}^\infty \mathbb{Z}$, it is Fréchet isometry. Therefore, we have

$$M_n(\mathbb{C}) \rtimes_{\alpha_\theta^{(n)}} \mathbb{Z} \simeq M_n(\mathbb{C}) \widehat{\otimes}_\gamma \mathcal{S}(\mathbb{Z})$$

by ρ_n . Now it is immediately known that the following fact follows:

COROLLARY 2.6. *The following isomorphism holds:*

$$\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z} \simeq \mathbb{K}^\infty \widehat{\otimes}_\gamma C^\infty(T).$$

Proof. By Proposition 2.5 and Lemma 2.3, we have that

$$\begin{aligned} \mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z} &\simeq \varinjlim (M_n(\mathbb{C}) \rtimes_{\alpha_\theta^{(n)}} \mathbb{Z}, \tilde{\varphi}_n) \simeq \varinjlim (M_n(\mathbb{C}) \widehat{\otimes}_\gamma \mathcal{S}(\mathbb{Z}), \tilde{\varphi}_n \otimes \text{id}_{\mathcal{S}(\mathbb{Z})}) \\ &\simeq \left(\varinjlim (M_n(\mathbb{C}), \varphi_n) \right) \widehat{\otimes}_\gamma \mathcal{S}(\mathbb{Z}) \simeq \mathbb{K}^\infty \widehat{\otimes}_\gamma \mathcal{S}(\mathbb{Z}). \end{aligned}$$

Since $\mathcal{S}(\mathbb{Z})$ is isomorphic to $C^\infty(T)$ sending by the Fourier transform, the conclusion follows. ■

We end this section by stating the following fact:

COROLLARY 2.7. *We have the following short exact sequence:*

$$0 \longrightarrow \mathbb{K}^\infty \otimes_\gamma \mathcal{S}(\mathbb{Z}) \xrightarrow{\tilde{i}} (D^2 \times S^1)_\theta \xrightarrow{\tilde{q}} C^\infty(T) \rtimes_{\bar{\alpha}_\theta} \mathbb{Z} \longrightarrow 0,$$

where $\bar{\alpha}_\theta : C^\infty(T) \times \mathbb{Z} \rightarrow C^\infty(T)$ is the Fréchet continuous action defined by

$$\bar{\alpha}_\theta^n(f)(z) = f(e^{2\pi i n \theta} z) \quad (f \in C^\infty(T), z \in T),$$

with a bounded linear section \tilde{s} of \tilde{q} .

Proof. Since $i \circ \alpha_\theta^n = \alpha_\theta^n \circ i$ and $q \circ \alpha_\theta^n = \bar{\alpha}_\theta^n \circ q$ for all $n \in \mathbb{Z}$, it is clear that the desired short exact sequence holds and $\tilde{s}(fU_\theta^n) = T_f U_\theta^n$ ($f \in C^\infty(T)$, $n \in \mathbb{Z}$). ■

3. METRIC APPROXIMATION PROPERTY

We introduce an analogue of the notion of metric approximation property for Banach spaces [4]. Let $\mathfrak{A}, \mathfrak{B}$ be two Banach spaces and $\mathfrak{I} \subset \mathfrak{B}$ an M -ideal. In [4], the authors prove that if \mathfrak{A} is separable and has the metric approximation property, then each contractive map $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}/\mathfrak{I}$ has a lift $\tilde{\varphi} : \mathfrak{A} \rightarrow \mathfrak{B}$ which is contractive and satisfies $q \circ \tilde{\varphi} = \varphi$, where $q : \mathfrak{B} \rightarrow \mathfrak{B}/\mathfrak{I}$ is the quotient map. Our purpose in this section is to define this property for F^* -algebras to prove lifting problem cited above. The topology on \mathfrak{A} induced by its seminorms $\{\|\cdot\|_k\}_{k \geq 0}$ is same as that induced by the metric $d_{\mathfrak{A}}$ defined by

$$d_{\mathfrak{A}}(a, b) = \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\|a - b\|_k}{1 + \|a - b\|_k} \quad (a, b \in \mathfrak{A}).$$

We say that a linear map $\varphi : \mathfrak{A} \rightarrow \mathfrak{B}$ is bounded if and only if there exists a constant $C > 0$ with

$$d_{\mathfrak{B}}(\varphi(a), 0) \leq C d_{\mathfrak{A}}(a, 0) \quad (a \in \mathfrak{A}).$$

DEFINITION 3.1. Let \mathfrak{A} be an F^* -algebra and $\{\|\cdot\|_k\}_{k \geq 0}$ its seminorms. We say that it has the *metric approximation property* if there is a family of bounded linear maps $\{\theta_n\}_{n \geq 1}$ on \mathfrak{A} with the following properties:

- (i) each θ_n has a finite rank,
- (ii) for any $a \in \mathfrak{A}$, $d_{\mathfrak{A}}(\theta_n(a), a) \rightarrow 0$ as $n \rightarrow \infty$.

We give some examples of F^* -algebras with the metric approximation property. Here we note that $d_{\mathfrak{A}}(\theta_n(a), a) \rightarrow 0$ is satisfied if and only if $\|\theta_n(a) - a\|_k \rightarrow 0$ for any $k \geq 0$.

EXAMPLE 3.2. For $n \geq 2$, let \mathbb{F}_n be the free group with n -generators. Given $g \in \mathbb{F}_n$, we denote by $|g|$ its word length, and for $f \in \mathbb{C}[\mathbb{F}_n]$ and an integer $k \in \mathbb{Z}_{\geq 0}$, we define seminorms by

$$\|f\|_k = \sup_{g \in \mathbb{F}_n} (1 + |g|)^k |f(g)|.$$

The Schwartz space $\mathcal{S}(\mathbb{F}_n)$ is defined by the completion of $\mathbb{C}[\mathbb{F}_n]$ with respect to the above seminorms. For $f \in \mathcal{S}(\mathbb{F}_n)$, we define a bounded operator $\lambda(f)$ on the Hilbert space $l^2(\mathbb{F}_n)$ by the convolution with f , that is,

$$(\lambda(f)\xi)(g) = (f * \xi)(g) = \sum_{h \in \mathbb{F}_n} f(h)\xi(h^{-1}g) \quad (g \in \mathbb{F}_n, \xi \in l^2(\mathbb{F}_n)),$$

on which the seminorms are defined by

$$\|\lambda(f)\|_k = \|f\|_k \quad (k \in \mathbb{Z}_{\geq 0}).$$

This operator is well-defined. Indeed, if $\lambda(f) = 0$, then $\lambda(f)\delta_e = 0$, where $e \in \mathbb{F}_n$ is the unit and the element $\delta_e \in l^2(\mathbb{F}_n)$ is defined by

$$\delta_e(g) = \begin{cases} 1 & (g = e), \\ 0 & (g \neq e). \end{cases}$$

Hence, for any $g \in \mathbb{F}_n$, we have that

$$0 = (\lambda(f)\delta_e)(g) = (f * \delta_e)(g) = \sum_{h \in \mathbb{F}_n} f(h)\delta_e(h^{-1}g) = f(g).$$

Therefore, $\lambda(f) = 0$ leads $f = 0$, which implies that the seminorms are well-defined. We define the F^* -algebra $C_r^*(\mathbb{F}_n)^\infty$ by the completion of the $*$ -algebra generated by the bounded operators $\lambda(f)$ ($f \in \mathcal{S}(\mathbb{F}_n)$). Here we claim that

$$C_r^*(\mathbb{F}_n)^\infty = \{\lambda(f) : f \in \mathcal{S}(\mathbb{F}_n)\}.$$

In fact, it is clear that $\lambda(f)^* = \lambda(f^*)$ for any $f \in \mathcal{S}(\mathbb{F}_n)$, where

$$f^*(g) = \overline{f(g^{-1})} \in \mathcal{S}(\mathbb{F}_n).$$

For any $T \in C_r^*(\mathbb{F}_n)^\infty$, there exists a family $\{\lambda(f_n)\}_{n \geq 1}$ ($f_n \in \mathcal{S}(\mathbb{F}_n)$) which converges to T with respect to the seminorms cited above. Then, for any $k \in \mathbb{Z}_{\geq 0}$, we have that

$$\|f_n - f_m\|_k = \|\lambda(f_n) - \lambda(f_m)\|_k \rightarrow 0 \quad (n, m \rightarrow \infty),$$

which implies that there exists a function $f \in \mathcal{S}(\mathbb{F}_n)$ which is the limit of $\{f_n\}$. Thus, for any $k \in \mathbb{Z}_{\geq 0}$, we have that

$$\|T - \lambda(f)\|_k \leq \|T - \lambda(f_n)\|_k + \|\lambda(f_n) - \lambda(f)\|_k \rightarrow 0 \quad (n \rightarrow \infty).$$

We construct a family of finite dimensional bounded linear maps $\{\theta_k\}$ on $C_r^*(\mathbb{F}_n)^\infty$ in what follows. Given an integer $k \geq 1$, let $E_k = \{g \in \mathbb{F}_n : |g| \leq k\}$ and χ_k the function on $\mathcal{S}(\mathbb{F}_n)$ defined by

$$\chi_k(g) = \begin{cases} 1 & (g \in E_k), \\ 0 & (g \notin E_k). \end{cases}$$

Since the number of elements of E_k is finite for each $k \geq 1$, the linear maps $\psi_n : \mathbb{F}_n \rightarrow \mathbb{F}_n$ defined by

$$\psi_k(\lambda(f)) = \lambda(\chi_k f)$$

have finite ranks. We define the finite linear bounded maps $\theta_k : \mathbb{F}_n \rightarrow \mathbb{F}_n$ ($k \geq 1$) by

$$\theta_k(\lambda(f)) = \lambda(e^{-|\cdot|/k} \chi_k f).$$

Then for any $l \geq 0$ and $f \in C_r^*(\mathbb{F}_n)^\infty$, we compute that

$$\begin{aligned} & \|\lambda(f) - \theta_k(\lambda(f))\|_l \\ & \leq \|\lambda(f) - \lambda(e^{-|\cdot|/k}f)\|_l + \|\lambda(e^{-|\cdot|/k}f) - \lambda(e^{-|\cdot|/k}\chi_k f)\|_l \\ & = \|f - e^{-|\cdot|/k}f\|_l + \|e^{-|\cdot|/k}f - e^{-|\cdot|/k}\chi_k f\|_l \\ & = \sup_{g \in \mathbb{F}_n} |(1+|g|)^l f(g)(1 - e^{-|g|/k})| + \sup_{g \in \mathbb{F}_n} |(1+|g|)^l f(g)e^{-|g|/k}(1 - \chi_k(g))| \\ & \leq \|f\|_l \sup_{g \in \mathbb{F}_n} |1 - e^{-|g|/k}| + \sup_{|g| \geq k+1} |(1+|g|)^l f(g)e^{-|g|/k}| \rightarrow 0 \quad (k \rightarrow \infty). \end{aligned}$$

Therefore, $C_r^*(\mathbb{F}_n)^\infty$ has the metric approximation property.

EXAMPLE 3.3. According to [16], the smooth noncommutative 2-torus T_θ^2 is isomorphic to the Fréchet inductive limit

$$\varinjlim C^\infty(T) \widehat{\otimes}_\gamma (M_{p_n}(\mathbb{C}) \oplus M_{q_n}(\mathbb{C})).$$

We show that it also has the metric approximation property. As a preparation, we verify that the Fréchet algebra $C^\infty(T) \widehat{\otimes}_\gamma M_q(\mathbb{C})$ has the metric approximation property. It suffices to show that $C^\infty(T)$ has this property since if it had this property with a family $\{\theta_n^{(q)}\}$ of bounded linear maps there, the family $\{\theta_n^{(q)} \otimes I_q\}$ would be the desired one for $C^\infty(T) \otimes M_q(\mathbb{C})$, where I_q is the identity map on $M_q(\mathbb{C})$. For $f \in C^\infty(T)$, we define the maps $\theta_n^{(q)} : C^\infty(T) \rightarrow C^\infty(T)$ by

$$\theta_n^{(q)}(f) = \sum_{|l| \leq n} \widehat{f}(l)z^l \quad (n \geq 1),$$

where $\widehat{f}(l)$ are the Fourier coefficients and $z \in C^\infty(T)$ is the canonical generator defined by $z(t) = t$ ($t \in T$). Then it is clear that they are of finite rank. For $f \in C^\infty(T)$ and $k \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} \|f - \theta_n^{(q)}(f)\|_k & = \|f^{(k)} - (\theta_n(f))^{(k)}\|_\infty = \sup_{m \in \mathbb{Z}} \left| \widehat{f^{(k)}}(m) - \sum_{|l| \leq n} \widehat{f}(l)(2\pi il)^k \delta_l(m) \right| \\ & = \sup_{m \in \mathbb{Z}} \left| \sum_{|l| \geq n+1} \widehat{f}(l)(2\pi il)^k \delta_l(m) \right| = \sup_{m \in \mathbb{Z}, |m| \geq n+1} |\widehat{f}(m)(2\pi m)^k| \rightarrow 0 \quad (n \rightarrow \infty), \end{aligned}$$

where $\delta_l(m) = 0$ ($m \neq l$), $= 1$ ($m = l$), since $\{\widehat{f}(l)\}_{l \in \mathbb{Z}}$ is a rapidly decreasing sequence by the hypothesis $f \in C^\infty(T)$. Hence $C^\infty(T)$ has the metric approximation property.

We turn to show briefly that T_θ^2 also has this property. For any $x \in T_\theta^2$, we define the sequence $\{x_n\}$ by

$$x_n = e_1^{(n)} x e_1^{(n)} + e_2^{(n)} x e_2^{(n)} \quad (n \geq 1),$$

where $e_j^{(n)}$ ($j = 1, 2$) are the projections such that

$$e_1^{(n)} x e_1^{(n)} \in C^\infty(T) \otimes M_{p_n}(\mathbb{C}), \quad e_2^{(n)} x e_2^{(n)} \in C^\infty(T) \otimes M_{q_n}(\mathbb{C})$$

for any $x \in T_\theta^2$ ([16]). We define the linear maps Φ_n on T_θ^2 by

$$\Phi_n(x) = \theta_n^{(p_n)}(e_1^{(n)} x e_1^{(n)}) + \theta_n^{(q_n)}(e_2^{(n)} x e_2^{(n)}).$$

It is easily seen that $\Phi_n(x) \rightarrow x$ with respect to the seminorms on T_θ^2 (see [16]), hence to the metric d as well. Therefore, T_θ^2 has the metric approximation property.

By the similar argument for $C^\infty(T)$, the operation of taking suspension preserves the metric approximation property.

COROLLARY 3.4. *If an F^* -algebra \mathfrak{A} has the metric approximation property, so does its suspension $S^\infty \mathfrak{A}$.*

Proof. It suffices to show that the F^* -algebra

$$C_0^\infty(0,1) = \{f \in C^\infty(0,1) : f_+^{(n)}(0) = f_-^{(n)}(1) = 0 (n \in \mathbb{Z}_{\geq 0})\}$$

has the metric approximation property. For any integer $j \geq 1$, we put

$$f_j(t) = e^{-1/jt(1-t)} \in C_0^\infty(0,1).$$

Let $\{\xi_j\}_{j=1}^\infty$ be the orthogonal family of $C_0^\infty(0,1)$ obtained by Schmidt orthogonalization of $\{f_j\}$. Then we define the linear maps $\theta_n : C_0^\infty(0,1) \rightarrow C^\infty(0,1)$ by

$$\theta_n(f)(t) = \sum_{j=1}^n \langle f | \xi_j \rangle \xi_j(t) \quad (\xi \in C_0^\infty(0,1), t \in (0,1), n \geq 1).$$

It is easily seen that the images of θ_n are included in $C_0^\infty(0,1)$. By the similar argument for $C^\infty(T)$, we obtain the conclusion. ■

For an F^* -algebra \mathfrak{A} , by \mathfrak{A}^* we denote the set of all bounded linear functionals on \mathfrak{A} , where we say a linear functional φ on \mathfrak{A} is bounded if and only if

$$\|\varphi\| = \sup_{a \in \mathfrak{A} \setminus \{0\}} \frac{|\varphi(a)|}{d_{\mathfrak{A}}(a,0)} < \infty.$$

Before we proceed to show the lifting problem, we need the following lemma:

LEMMA 3.5. *Let $\mathfrak{A}, \mathfrak{B}$ be two F^* -algebras. Suppose that \mathfrak{J} is an F^* -ideal of \mathfrak{B} and that L, N are finite dimensional subspaces of \mathfrak{A} with $L \subset N$. We consider the following diagram of bounded linear maps:*

$$\begin{array}{ccccc} L & \xrightarrow[\iota]{\subset} & N & \xrightarrow{\Psi} & \mathfrak{B} \\ & & \parallel & & \downarrow q \\ L & \xrightarrow[\iota]{\subset} & N & \xrightarrow[\varphi]{} & \mathfrak{B}/\mathfrak{J}, \end{array}$$

where q and ι are the quotient map and the natural inclusion respectively, and suppose that

$$d_{\mathfrak{B}}(q \circ \Psi(a) - \varphi(a), 0) \leq \epsilon d_{\mathfrak{A}}(a, 0) \quad (a \in L).$$

for a positive constant $\varepsilon > 0$. Then there is a bounded linear map $\varphi' : N \rightarrow \mathfrak{B}/\mathfrak{I}$ with the property that

$$\begin{cases} \varphi = q \circ \varphi', \\ d_{\mathfrak{B}}(\varphi'(a), 0) \leq d_{\mathfrak{A}}(a, 0) & (a \in N), \\ d_{\mathfrak{B}}(\varphi'(a) - \Psi(a), 0) \leq 6\varepsilon & (a \in L). \end{cases}$$

Proof. This lemma is an analogy of Lemma 2.5 in [4]. Let D' and K be the closed unit ball of $L\widehat{\otimes}_{\gamma}\mathfrak{B}$ and $N\widehat{\otimes}_{\gamma}\mathfrak{B}$ respectively, that is,

$$D' = \{\varphi : \mathfrak{B} \rightarrow L : \|\varphi\|_{L\widehat{\otimes}_{\gamma}\mathfrak{B}^*} \leq 1\} \subset L\widehat{\otimes}_{\gamma}\mathfrak{B}^*,$$

$$K = \{\varphi : \mathfrak{B} \rightarrow N : \|\varphi\|_{N\widehat{\otimes}_{\gamma}\mathfrak{B}^*} \leq 1\} \subset N\widehat{\otimes}_{\gamma}\mathfrak{B}^*,$$

where

$$\|\varphi\|_{L\widehat{\otimes}_{\gamma}\mathfrak{B}^*} = \sup_{a \in \mathfrak{B} \setminus \{0\}} \frac{d_{\mathfrak{A}}(\varphi(a), 0)}{d_{\mathfrak{B}}(a, 0)}$$

and $\|\cdot\|_{N\widehat{\otimes}_{\gamma}\mathfrak{B}^*}$ is defined by the similar way for $\|\cdot\|_{L\widehat{\otimes}_{\gamma}\mathfrak{B}^*}$, and $\text{Aff}_T(D')$ the set of all affine functions ψ on D' such that $\psi(\alpha\varphi) = \alpha\psi(\varphi)$ for all $\alpha \in T, \varphi \in D'$. It is clear that \mathfrak{I} is an M -ideal of \mathfrak{B} and the equality

$$\mathfrak{B}^* = \mathfrak{I}^{\perp} \oplus \mathfrak{I}^*$$

holds as a linear space, where \mathfrak{I}^{\perp} is the annihilator of \mathfrak{I} . Let $e : \mathfrak{B}^* \rightarrow \mathfrak{I}^{\perp}$ be the natural projection and W the image of $N\widehat{\otimes}_{\gamma}\mathfrak{B}^*$ via $1 \otimes e$, which is equal to $N\widehat{\otimes}_{\gamma}\mathfrak{I}^{\perp}$. Then D' is mapped weak* homeomorphically to $D \subset K$ through the natural embedding $\iota \otimes 1 : L\widehat{\otimes}_{\gamma}\mathfrak{B}^* \rightarrow N\widehat{\otimes}_{\gamma}\mathfrak{B}^*$. We may identify the closed unit ball of $N\widehat{\otimes}_{\gamma}\mathfrak{B}^*$ with $F = K \cap W$. We also identify the closed unit ball of $L\widehat{\otimes}_{\gamma}\mathfrak{B}^*$ with $D' \cup W'$, where $W' = (1 \otimes e)(L\widehat{\otimes}_{\gamma}\mathfrak{B}^*) = L\widehat{\otimes}_{\gamma}\mathfrak{I}^{\perp}$. It is verified by the same argument as in the proof of Lemma 2.5 in [4] that

$$(3.1) \quad (\iota \otimes 1)(1 \otimes e) = (1 \otimes e)(\iota \otimes 1),$$

$$(3.2) \quad (\iota \otimes 1)(D' \cap W') = D \cap (\iota \otimes 1)(W') = D \cap W = D \cap F.$$

Thus we have the following diagram of restrictions

$$(3.3) \quad \begin{array}{ccc} \text{Aff}_T(D \cap F) & \longleftarrow & \text{Aff}_T(F) \\ \uparrow & & \uparrow \\ \text{Aff}_T(D) & \longleftarrow & \text{Aff}_T(K). \end{array}$$

Since $1 \otimes e : L\widehat{\otimes}_{\gamma}\mathfrak{B}^* \rightarrow L\widehat{\otimes}_{\gamma}\mathfrak{I}^{\perp}$ maps D' onto $D' \cap W$, D is mapped onto $D \cap F$ by (3.1) and (3.2) and D satisfies the condition of Lemma 2.1 in [4]. Therefore, with the diagram (3.3), we obtain the conclusion by the same argument of Lemma 2.5 in [4]. ■

PROPOSITION 3.6. *Let $\mathfrak{A}, \mathfrak{B}$ be two F^* -algebras and $\mathfrak{I} \subset \mathfrak{B}$ an F^* -ideal. If \mathfrak{A} is separable and has the metric approximation property, then for any bounded linear map*

$\varphi : \mathfrak{A} \rightarrow \mathfrak{B}/\mathfrak{I}$, there exists a bounded linear map $\Phi : \mathfrak{A} \rightarrow \mathfrak{B}$ with the property that $q \circ \Phi = \varphi$, where $q : \mathfrak{B} \rightarrow \mathfrak{B}/\mathfrak{I}$ is the quotient map.

Proof. This proof is inspired by that of Theorem 2.6 in [4]. We fix a sequence $\{a_n\}_{n \in \mathbb{N}} \subset \mathfrak{A}$ dense in \mathfrak{A} . We construct recursively the pairs $\{(L_n, \theta_n)\}_{n \in \mathbb{Z}_{\geq 0}}$ which consist of increasing finite dimensional subspaces $L_n \subset \mathfrak{A}$ with $a_n \in L_n$ for any $n \in \mathbb{Z}_{\geq 0}$ and bounded linear maps $\theta_n : \mathfrak{A} \rightarrow L_n$ with the property that for any $a \in L_{n-1}$, the inequalities

$$d_{\mathfrak{A}}(a, \theta_n(a)) \leq \frac{1}{2^n}$$

are satisfied. We put $L_0 = \{0\}$ and $\theta_0 = 0$. We suppose that for some $n \in \mathbb{Z}_{\geq 0}$ the pairs $(L_0, \theta_0), \dots, (L_n, \theta_n)$ with the above properties are given. By the approximation property of \mathfrak{A} , there exists a bounded linear map $\theta_{n+1} : \mathfrak{A} \rightarrow \mathfrak{A}$ such that for each $a \in L_n$, the inequality

$$d_{\mathfrak{A}}(a, \theta_{n+1}(a)) \leq \frac{1}{2^{n+1}}$$

holds. We define the subspace L_{n+1} of \mathfrak{A} by

$$L_{n+1} = L_n + \theta_{n+1}(\mathfrak{A}) + \mathbb{C}a_{n+1}.$$

Then we have the desired pairs $\{(L_n, \theta_n)\}_{n \in \mathbb{Z}_{\geq 0}}$. We note that $\bigcup_{n \in \mathbb{Z}_{\geq 0}} L_n$ is dense in \mathfrak{A} with respect to Fréchet topology.

Next we inductively define a family of bounded linear maps

$$\Psi_n : L_n \rightarrow \mathfrak{B} \quad (n \in \mathbb{Z}_{\geq 0})$$

such that

$$(3.4) \quad q \circ \Psi_n(a) = \varphi(a) \quad (a \in L_n).$$

Putting $\Psi_0 = 0$, suppose that for some $n \in \mathbb{Z}_{\geq 0}$, bounded linear maps Ψ_0, \dots, Ψ_n satisfying (3.4) are constructed. Then we have that for any $a \in L_{n-1}$,

$$d_{\mathfrak{B}/\mathfrak{I}}(q \circ \Psi_n \circ \theta_n(a), \varphi(a)) = d_{\mathfrak{B}/\mathfrak{I}}(\varphi \circ \theta_n(a), \varphi(a)) \leq C d_{\mathfrak{A}}(\theta_n(a), a) \leq \frac{C}{2^n}.$$

By Lemma 3.5, we find a bounded map $\Psi_{n+1} : L_{n+1} \rightarrow \mathfrak{B}$ such that $\varphi = q \circ \Psi_{n+1}$ on L_{n+1} and that for any $a \in L_{n-1}$ with

$$d_{\mathfrak{B}}(\Psi_{n+1}(a), \Psi_n \circ \theta_n(a)) \leq \frac{6C}{2^n}.$$

Therefore, we compute that

$$\begin{aligned} d_{\mathfrak{B}}(\Psi_{n+1}(a), \Psi_n(a)) &\leq \frac{6C}{2^n} + d_{\mathfrak{B}}(\Psi_n(a), \Psi_n \circ \theta_n(a)) \leq \frac{6C}{2^n} + d_{\mathfrak{A}}(a, \theta_n(a)) \\ &\leq \frac{6C+1}{2^n} \quad (a \in L_{n-1}). \end{aligned}$$

Hence for a fixed integer $n_0 \in \mathbb{Z}_{\geq 0}$, we have for all $n \geq n_0$,

$$d_{\mathfrak{B}}(\Psi_{n+1}(a), \Psi_n(a)) \leq \frac{6C + 1}{2^n}.$$

Thus, for a fixed integer $n_0 \in \mathbb{Z}_{\geq 0}$, the family of bounded linear maps $\{\Psi_n\}$ converges to some $\Psi^{(n_0)} : L_{n_0-1} \rightarrow \mathfrak{B}$. Therefore, we have the bounded linear map

$$\Psi : \bigcup_{n \in \mathbb{Z}_{\geq 0}} L_n \rightarrow \mathfrak{B}$$

such that $\Psi|_{L_n} = \Psi^{(n)}$ ($n \in \mathbb{Z}_{\geq 0}$), and we can extend it to that on the closure of $\bigcup_{n \in \mathbb{Z}_{\geq 0}} L_n$ which is equal to \mathfrak{A} . This completes the proof. ■

4. MEYER-VIEOTRIS EXACT SEQUENCE

This section is devoted to proving Mayer-Vietoris exact sequence for the entire cyclic cohomology. We firstly give a short proof of Bott periodicity for the entire cyclic cohomology by using the following Meyer’s excision for the entire cyclic theory [14]:

PROPOSITION 4.1. *Let*

$$0 \longrightarrow K \xrightarrow{i} P \xrightarrow{q} Q \longrightarrow 0$$

be a short exact sequence of F^ -algebras with a bounded linear section s of q . Then the following 6-terms exact sequence holds:*

$$\begin{CD} HE^{ev}(Q) @>{q^*}>> HE^{ev}(P) @>{i^*}>> HE^{ev}(K) \\ @. @. @. \\ @. @. @. \\ @. @. @. \\ @. @. @. \\ HE^{od}(K) @<<{i^*}<< HE^{od}(P) @<<{q^*}<< HE^{od}(Q). \end{CD}$$

This yields the following fact, which has been already shown by Brodzki and Plymen [3] using bivariant entire homology and cohomology theory:

LEMMA 4.2 (Bott periodicity for entire cyclic cohomology). *For an F^* -algebra \mathfrak{A} we have the following:*

$$HE^{ev}(S^\infty \mathfrak{A}) \simeq HE^{od}(\mathfrak{A}), \quad HE^{od}(S^\infty \mathfrak{A}) \simeq HE^{ev}(\mathfrak{A}).$$

Proof. By the exact sequence cited above, we have the following exact diagram:

$$\begin{CD} HE^{ev}(\mathfrak{A}) @>>> HE^{ev}(C^\infty \mathfrak{A}) @>>> HE^{ev}(\mathfrak{I}) \\ @. @. @. \\ @. @. @. \\ @. @. @. \\ @. @. @. \\ HE^{od}(\mathfrak{I}) @<<< HE^{od}(C^\infty \mathfrak{A}) @<<< HE^{od}(\mathfrak{A}). \end{CD}$$

By Lemma 1.3, we deduce the conclusion. ■

In what follows, we show an entire cyclic cohomology version of Mayer–Vietoris exact sequence. Before stating it, we review briefly the fibered product of F^* -algebras, which is a noncommutative analogue of the connected sum of two manifolds. Let $\mathfrak{A}_1, \mathfrak{A}_2$ and \mathfrak{B} be F^* -algebras and $f_j : \mathfrak{A}_j \rightarrow \mathfrak{B}$ ($j = 1, 2$) epimorphisms.

DEFINITION 4.3. $\{(a_1, a_2) \in \mathfrak{A}_1 \oplus \mathfrak{A}_2 : f_1(a_1) = f_2(a_2)\}$ is called the *fibered product* of $(\mathfrak{A}_1, \mathfrak{A}_2)$ along (f_1, f_2) over \mathfrak{B} , which we denote by $\mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2$. Let g_j be the projections of $\mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2$ onto \mathfrak{A}_j ($j = 1, 2$).

THEOREM 4.4 (Mayer–Vietoris exact sequence for entire cyclic cohomology). *In the situation of Definition 4.3, suppose that \mathfrak{B} has the metric approximation property and separable. Then we have that the following exact diagram holds:*

$$\begin{array}{ccccc}
 HE^{ev}(\mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2) & \longrightarrow & HE^{od}(\mathfrak{B}) & \xrightarrow{-f_1^* + f_2^*} & HE^{od}(\mathfrak{A}_1) \oplus HE^{od}(\mathfrak{A}_2) \\
 \uparrow g_1^* + g_2^* & & & & \downarrow g_1^* + g_2^* \\
 HE^{ev}(\mathfrak{A}_1) \oplus HE^{ev}(\mathfrak{A}_2) & \xleftarrow{-f_1^* + f_2^*} & HE^{ev}(\mathfrak{B}) & \longleftarrow & HE^{od}(\mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2).
 \end{array}$$

Proof. We write

$$C = \{(h_1, h_2) \in C^\infty \mathfrak{A}_1 \oplus C^\infty \mathfrak{A}_2 : f_1 \circ (h_1)_+^{(n)}(0) = (-1)^n f_2 \circ (h_2)_+^{(n)}(0) \ (n \in \mathbb{Z}_{\geq 0})\}$$

and define a map $q : C \rightarrow \mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2$ by

$$q(h_1, h_2) = (h_1(0), h_2(0)).$$

It is easily verified that the following sequence:

$$0 \longrightarrow \mathfrak{J} \xrightarrow{i} C \xrightarrow{q} \mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2 \longrightarrow 0$$

is exact, where

$$\mathfrak{J} = \{(h_1, h_2) \in C : h_j(0) = 0 \ (j = 1, 2)\}$$

and i is the canonical inclusion. Then there exists a bounded linear section s of q defined by

$$s(a_1, a_2) = ((1 - t)a_1, (1 - t)a_2), \quad ((a_1, a_2) \in \mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2, t \in [0, 1]).$$

Then by Proposition 4.1, we have the following exact diagram:

$$\begin{array}{ccccc}
 HE^{ev}(\mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2) & \longrightarrow & HE^{ev}(C) & \longrightarrow & HE^{ev}(\mathfrak{J}) \\
 \uparrow & & & & \downarrow \\
 HE^{od}(\mathfrak{J}) & \longleftarrow & HE^{od}(C) & \longleftarrow & HE^{od}(\mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2).
 \end{array}$$

Moreover, repeating the argument cited above, \mathcal{J} is smoothly homotopic to $S^\infty\mathfrak{A}_1 \oplus S^\infty\mathfrak{A}_2$. More precisely, we define the map

$$r : \mathcal{J} \rightarrow S^\infty\mathfrak{A}_1 \oplus S^\infty\mathfrak{A}_2$$

by

$$r(h_1, h_2)(t) = (h_1(e^{1-1/t}), h_2(e^{1-1/t})) \quad ((h_1, h_2) \in C)$$

and let $i : S^\infty\mathfrak{A}_1 \oplus S^\infty\mathfrak{A}_2 \rightarrow \mathcal{J}$ be the natural inclusion. It follows by the same argument discussed above that since the functions $t \mapsto h_j(e^{1-1/t})$ are in $S^\infty\mathfrak{A}_j$ ($j = 1, 2$) and using the maps

$$\begin{aligned} G_1 : \mathcal{J} &\rightarrow C^\infty([0, 1], \mathcal{J}), \\ G_2 : S^\infty\mathfrak{A}_1 \oplus S^\infty\mathfrak{A}_2 &\rightarrow C^\infty([0, 1], S^\infty\mathfrak{A}_1 \oplus S^\infty\mathfrak{A}_2), \end{aligned}$$

defined by

$$(G_j)_s(h_1, h_2)(t) = (h_1(se^{1-1/t} + (1-s)t), h_2(se^{1-1/t} + (1-s)t)) \quad (j = 1, 2),$$

\mathcal{J} is smoothly homotopic to $S^\infty\mathfrak{A}_1 \oplus S^\infty\mathfrak{A}_2$. Hence we conclude that

$$HE^*(\mathcal{J}) \simeq HE^*(S^\infty\mathfrak{A}_1) \oplus HE^*(S^\infty\mathfrak{A}_2).$$

Now we define the map $\Psi : C \rightarrow S^\infty\mathfrak{B}$ by

$$\Psi(h_1, h_2)(t) = \begin{cases} f_1 \circ h_1(1 - 2t) & (t \in [0, 1/2]), \\ f_2 \circ h_2(2t - 1) & (t \in [1/2, 1]). \end{cases}$$

We have to verify that it is well-defined. Since $f_1 \circ h_1(0) = f_2 \circ h_2(0)$ by the definition of C , it is continuous at $t = 1/2$. For $n = 1$, we compute that

$$\begin{aligned} \lim_{t \rightarrow 1/2+0} \frac{\Psi(h_1, h_2)(t) - \Psi(h_1, h_2)(1/2)}{t - 1/2} &= \lim_{t \rightarrow 1/2+0} \frac{f_2 \circ h_2(2t - 1) - f_2 \circ h_2(0)}{t - 1/2} \\ &= f_2 \left(\lim_{t \rightarrow 1/2+0} \frac{h_2(2t - 1) - h_2(0)}{t - 1/2} \right) \\ &= 2f_2 \left(\lim_{\varepsilon \rightarrow 0+} \frac{h_2(\varepsilon) - h_2(0)}{\varepsilon} \right) = 2f_2 \circ (h_2)_+^{(1)}(0) \end{aligned}$$

and similarly, we compute that

$$\begin{aligned} \lim_{t \rightarrow 1/2-0} \frac{\Psi(h_1, h_2)(t) - \Psi(h_1, h_2)(1/2)}{t - 1/2} &= \lim_{t \rightarrow 1/2-0} \frac{f_1 \circ h_1(1 - 2t) - f_1 \circ h_1(0)}{t - 1/2} \\ &= -2f_1 \left(\lim_{\varepsilon \rightarrow 0+} \frac{h_1(\varepsilon) - h_1(0)}{\varepsilon} \right) \\ &= -2f_1 \circ (h_1)_+^{(1)}(0) = 2f_2 \circ (h_2)_+^{(1)}(0). \end{aligned}$$

Thus $\Psi(h_1, h_2)$ is differentiable once at $t = 1/2$. Suppose that it is differentiable n -times at $t = 1/2$. Here we note that

$$\Psi^{(n)}(h_1, h_2)(t) = \begin{cases} (-2)^n f_1 \circ h_1^{(n)}(1 - 2t) & t \in (0, 1/2), \\ 2^n f_2 \circ h_2^{(n)}(2t - 1) & t \in (1/2, 1), \end{cases}$$

and that

$$\Psi^{(n)}(h_1, h_2)(1/2) = (-2)^n f_1 \circ (h_1)_+^{(n)}(0) = 2^n f_2 \circ (h_2)_+^{(n)}(0)$$

by our hypothesis of induction. Then we compute that

$$\begin{aligned} & \lim_{t \rightarrow 1/2+0} \frac{\Psi(h_1, h_2)^{(n)}(t) - \Psi(h_1, h_2)^{(n)}(1/2)}{t - 1/2} \\ &= 2^n \lim_{t \rightarrow 1/2+0} \frac{f_2 \circ h_2^{(n)}(2t - 1) - f_2 \circ (h_2)_+^{(n)}(0)}{t - 1/2} \\ &= 2^n f_2 \left(\lim_{t \rightarrow 1/2+0} \frac{h_2^{(n)}(2t - 1) - (h_2)_+^{(n)}(0)}{t - 1/2} \right) \\ &= 2^n \cdot 2 f_2 \left(\lim_{\varepsilon \rightarrow 0+} \frac{h_2^{(n)}(\varepsilon) - (h_2)_+^{(n)}(0)}{\varepsilon} \right) = 2^{n+1} f_2 \circ (h_2)_+^{(n+1)}(0). \end{aligned}$$

Alternatively, we compute that

$$\begin{aligned} & \lim_{t \rightarrow 1/2-0} \frac{\Psi(h_1, h_2)^{(n)}(t) - \Psi(h_1, h_2)^{(n)}(1/2)}{t - 1/2} \\ &= (-2)^n \lim_{t \rightarrow 1/2-0} \frac{f_1 \circ h_1^{(n)}(1 - 2t) - f_1 \circ (h_1)_+^{(n)}(0)}{t - 1/2} \\ &= (-2)^n \cdot (-2) f_1 \left(\lim_{\varepsilon \rightarrow 0+} \frac{h_1^{(n)}(\varepsilon) - (h_1)_+^{(n)}(0)}{\varepsilon} \right) \\ &= (-2)^{n+1} f_1 \circ (h_1)_+^{(n+1)}(0) = 2^{n+1} f_2 \circ (h_2)_+^{(n+1)}(0). \end{aligned}$$

Therefore, $\Psi(h_1, h_2)$ is differentiable $(n + 1)$ -times for each $(h_1, h_2) \in C$, which ends the process of induction so that Ψ is well-defined. Since f_1 and f_2 are surjective, it is easily verified that Ψ is surjective. In fact, we canonically can lift them on $S^\infty \mathfrak{A}_j$, and denote them also by f_j ($j = 1, 2$). Now given an $h \in S^\infty \mathfrak{B}$, we find $\tilde{h}_j \in S^\infty \mathfrak{A}_j$ with

$$f_j(\tilde{h}_j(t)) = h(t) \quad (j = 1, 2).$$

Putting $h_1(t) = \tilde{h}_1((1-t)/2)$, $h_2(t) = \tilde{h}_2((1+t)/2)$ ($0 \leq t \leq 1$), we then check that

$$\begin{aligned} \Psi(h_1, h_2)(t) &= \begin{cases} f_1(h_1(1 - 2t)) & (0 \leq t \leq 1/2), \\ f_2(h_2(2t - 1)) & (1/2 \leq t \leq 1), \end{cases} \\ &= \begin{cases} f_1(\tilde{h}_1(1 - 2(1 - t)/2)) & (0 \leq t \leq 1/2), \\ f_2(\tilde{h}_2(2(1 + t)/2 - 1)) & (1/2 \leq t \leq 1), \end{cases} \\ &= \begin{cases} f_1(\tilde{h}_1(t)) & (0 \leq t \leq 1/2), \\ f_2(\tilde{h}_2(t)) & (1/2 \leq t \leq 1), \end{cases} \\ &= h(t), \end{aligned}$$

which implies that Ψ is surjective. As it is clear that its kernel is $C^\infty\mathfrak{J}_1 \oplus C^\infty\mathfrak{J}_2$, where

$$\mathfrak{J}_j = \text{Ker}f_j \quad (j = 1, 2),$$

we obtain the following short exact sequence:

$$(4.1) \quad 0 \longrightarrow C^\infty\mathfrak{J}_1 \oplus C^\infty\mathfrak{J}_2 \longrightarrow C \xrightarrow{\Psi} S^\infty\mathfrak{B} \longrightarrow 0.$$

Since \mathfrak{B} has the metric approximation property, so does $S^\infty\mathfrak{B}$ by Corollary 3.4. Writing $\mathfrak{J} = C^\infty\mathfrak{J}_1 \oplus C^\infty\mathfrak{J}_2$, the inverse map

$$\bar{\Psi}^{-1} : S^\infty\mathfrak{B} \rightarrow C/\mathfrak{J}$$

of the isomorphism $\bar{\Psi}$ induced by Ψ has a bounded lift

$$\tilde{\Psi}^{-1} : S^\infty\mathfrak{B} \rightarrow C$$

satisfying $\tilde{\Psi}^{-1} \circ q = \bar{\Psi}^{-1}$ by Proposition 3.6 since $\bar{\Psi}$ preserves each seminorms, where q is the quotient map from C onto C/\mathfrak{J} . Hence it is verified that $\tilde{\Psi}^{-1}$ is a bounded linear section of Ψ since we compute that

$$\Psi \circ \tilde{\Psi}^{-1} = \bar{\Psi} \circ q \circ \tilde{\Psi}^{-1} = \bar{\Psi} \circ \bar{\Psi}^{-1} = \text{id}_{S^\infty\mathfrak{B}}.$$

Therefore, we apply the above exact sequence (4.1) to Proposition 4.1 to obtain the following exact diagram:

$$\begin{array}{ccccc} HE^{\text{ev}}(S^\infty\mathfrak{B}) & \longrightarrow & HE^{\text{ev}}(C) & \longrightarrow & HE^{\text{ev}}(C^\infty\mathfrak{J}_1 \oplus C^\infty\mathfrak{J}_2) \\ \uparrow & & & & \downarrow \\ HE^{\text{od}}(C^\infty\mathfrak{J}_1 \oplus C^\infty\mathfrak{J}_2) & \longleftarrow & HE^{\text{od}}(C) & \longleftarrow & HE^{\text{od}}(S^\infty\mathfrak{B}). \end{array}$$

Since $HE^*(C^\infty\mathfrak{J}_1 \oplus C^\infty\mathfrak{J}_2) = 0$, we have that

$$HE^{\text{ev}}(C) \simeq HE^{\text{ev}}(S^\infty\mathfrak{B}) \simeq HE^{\text{od}}(\mathfrak{B}), \quad HE^{\text{od}}(C) \simeq HE^{\text{od}}(S^\infty\mathfrak{B}) \simeq HE^{\text{ev}}(\mathfrak{B}),$$

by the Bott periodicity (Lemma 4.2).

Summing up, we get the desired exact diagram in what follows:

$$(4.2) \quad \begin{array}{ccccc} HE^{\text{ev}}(\mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2) & \longrightarrow & HE^{\text{od}}(\mathfrak{B}) & \longrightarrow & HE^{\text{od}}(\mathfrak{A}_1) \oplus HE^{\text{od}}(\mathfrak{A}_2) \\ \uparrow & & & & \downarrow \\ HE^{\text{ev}}(\mathfrak{A}_1) \oplus HE^{\text{ev}}(\mathfrak{A}_2) & \longleftarrow & HE^{\text{ev}}(\mathfrak{B}) & \longleftarrow & HE^{\text{od}}(\mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2). \end{array}$$

We consider the restriction $\Phi : S^\infty\mathfrak{A}_1 \oplus S^\infty\mathfrak{A}_2 \rightarrow S^\infty\mathfrak{B}$ of Ψ . We see that it is C^∞ -homotopic to $\Pi : S^\infty\mathfrak{A}_1 \oplus S^\infty\mathfrak{A}_2 \rightarrow \mathfrak{B}$ defined by

$$\Pi(h_1, h_2)(t) = -\chi_{[0,1/2]}(t)(f_1 \circ h_1)(t) + \chi_{[1/2,1]}(t)(f_2 \circ h_2)(t)$$

for $(h_1, h_2) \in S^\infty \mathfrak{A}_1 \oplus S^\infty \mathfrak{A}_2, t \in [0, 1]$. To see this, we note that for a Fréchet continuous homomorphism $f : \mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2 \rightarrow S^\infty \mathfrak{B}$, we have

$$\bar{f}^* = -f^* : HE^*(S^\infty \mathfrak{B}) \rightarrow HE^*(\mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2)$$

by [14], where $\bar{f} : \mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2 \rightarrow S^\infty \mathfrak{B}$ is the homomorphism defined by

$$\bar{f}(a)(t) = f(a)(1 - t) \quad (a \in \mathfrak{A}, t \in [0, 1]).$$

Indeed, we prepare the map $\Theta : S^\infty \mathfrak{A}_1 \oplus S^\infty \mathfrak{A}_2 \rightarrow C^\infty([0, 1], S^\infty \mathfrak{B})$ defined by

$$\Theta_s(h_1, h_2)(t) = \begin{cases} f_1 \circ h_1(1 - 2t/(1 + s)) & (0 \leq t \leq 1/2), \\ f_2 \circ h_2(2t/(1 + s) - (1 - s)/(1 + s)) & (1/2 \leq t \leq 1), \end{cases}$$

so that it is a smooth homotopy between Ψ and the homomorphism given by

$$(h_1, h_2) \mapsto (t \mapsto \chi_{[0, 1/2]}(t)(f_1 \circ h_1)(1 - t) + \chi_{[1/2, 1]}(t)(f_2 \circ h_2)(t)).$$

Therefore, we have the homotopy equivalence of Ψ and Π . Considering the following commutative diagram:

$$\begin{array}{ccc} HE^*(C) & \longrightarrow & HE^*(S^\infty \mathfrak{A}_1 \oplus S^\infty \mathfrak{A}_2) \\ \simeq \uparrow \Psi^* & & \parallel \\ HE^*(S^\infty \mathfrak{B}) & \xrightarrow{\Gamma^* = \Pi^*} & HE^*(S^\infty \mathfrak{A}_1 \oplus S^\infty \mathfrak{A}_2) \end{array}$$

we conclude that the right upper horizontal map and the left lower horizontal map in the diagram (4.2) are both $\Pi^* = -f_1^* + f_2^*$. Finally, since the following diagram

$$\begin{array}{ccc} HE^*(S^\infty \mathfrak{A}_1) \oplus HE^*(S^\infty \mathfrak{A}_2) & \longrightarrow & HE^*(\mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2) \\ \simeq \downarrow & & \parallel \\ HE^*(\mathfrak{A}_1) \oplus HE^*(\mathfrak{A}_2) & \xrightarrow{\delta_1^* + \delta_2^*} & HE^*(\mathfrak{A}_1 \#_{\mathfrak{B}} \mathfrak{A}_2) \end{array}$$

is commutative, the vertical maps in the diagram (4.2) are both $g_1^* + g_2^*$. This completes the proof. ■

5. THE ENTIRE CYCLIC COHOMOLOGY OF NONCOMMUTATIVE 3-SPHERES

In [1], Heegaard-type quantum 3-spheres with 3-parameters are constructed as C^* -algebras. With their construction in mind, we define noncommutative 3-spheres in the framework of F^* -algebras as follows; given an irrational number θ with $0 < \theta < 1$, let T_θ^2 be the smooth noncommutative 2-torus with unitary generators u_θ, v_θ subject to $u_\theta v_\theta = e^{2\pi i \theta} v_\theta u_\theta$. There exists an isomorphism $\gamma_\theta : T_{-\theta}^2 \rightarrow T_\theta^2$ satisfying

$$\gamma_\theta(u_{-\theta}) = v_\theta, \quad \gamma_\theta(v_{-\theta}) = u_\theta,$$

by their universality. We consider the following two F^* -crossed products:

$$(D^2 \times S^1)_\theta = \mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}, \quad (D^2 \times S^1)_{-\theta} = \mathcal{T}^\infty \rtimes_{\alpha_{-\theta}} \mathbb{Z},$$

defined before. We define two epimorphisms f_j ($j = 1, 2$) such as

$$f_1 : (D^2 \times S^1)_\theta \rightarrow T_\theta^2, \quad f_2 : (D^2 \times S^1)_{-\theta} \rightarrow T_\theta^2,$$

by $f_1 = \tilde{q}_+$, $f_2 = \gamma_\theta \circ \tilde{q}_-$, where \tilde{q}_\pm are the epimorphisms from $\mathcal{T}^\infty \rtimes_{\alpha_{\pm\theta}} \mathbb{Z}$ onto $C^\infty(T) \rtimes_{\bar{\alpha}_{\pm\theta}} \mathbb{Z} = T_{\pm\theta}^2$ respectively.

DEFINITION 5.1. For an irrational number θ , the *noncommutative 3-sphere* S_θ^3 is defined by the fibered product $(D^2 \times S^1)_\theta \#_{T_\theta^2} (D^2 \times S^1)_{-\theta}$ of $((D^2 \times S^1)_\theta (D^2 \times S^1)_{-\theta})$ along (f_1, f_2) over T_θ^2 .

First of all, we compute the entire cyclic cohomology of $(D^2 \times S^1)_\theta$. We note that the isomorphism $C^\infty(T) \rtimes_{\bar{\alpha}_\theta} \mathbb{Z} \simeq T_\theta^2$ holds and that by Lemma 4.3 in [16], we have

$$HE^*(C^\infty(T) \rtimes_{\bar{\alpha}_\theta} \mathbb{Z}) \simeq HE^*(T_\theta^2) = HP^*(T_\theta^2),$$

where HP^* is the functor of periodic cyclic cohomology. According to Connes [5], we know the generators of $HP^*(T_\theta^2)$ as follows:

$$HP^{\text{ev}}(T_\theta^2) = \mathbb{C}[\tau_\theta] \oplus \mathbb{C}[\tau'_\theta], \quad HP^{\text{od}}(T_\theta^2) = \mathbb{C}[\tau_\theta^{(1)}] \oplus \mathbb{C}[\tau_\theta^{(2)}],$$

where τ_θ is the unique normalized trace on T_θ^2 and

$$\begin{aligned} \tau'_\theta(a_0, a_1, a_2) &= \tau_\theta(a_0(\delta_\theta^{(1)}(a_1)\delta_\theta^{(2)}(a_2) - \delta_\theta^{(2)}(a_1)\delta_\theta^{(1)}(a_2))), \\ \tau_\theta^{(j)}(a_0, a_1) &= \tau_\theta(a_0\delta_\theta^{(j)}(a_1)) \quad (j = 1, 2), \end{aligned}$$

where $\delta_\theta^{(j)}$ are the derivations on T_θ^2 such that

$$\delta_\theta^{(1)}(u_\theta) = 2\pi i u_\theta, \quad \delta_\theta^{(1)}(v_\theta) = 0, \quad \delta_\theta^{(2)}(u_\theta) = 0, \quad \delta_\theta^{(2)}(v_\theta) = 2\pi i v_\theta.$$

PROPOSITION 5.2. *We have:*

$$HE^{\text{ev}}((D^2 \times S^1)_\theta) = \mathbb{C}[\tau'_\theta \circ \tilde{q}], \quad HE^{\text{od}}((D^2 \times S^1)_\theta) = \mathbb{C}[\tau_\theta^{(1)} \circ \tilde{q}].$$

Proof. We remember the following short exact sequence:

$$0 \longrightarrow \mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z} \xrightarrow{\tilde{i}} (D^2 \times S^1)_\theta \xrightarrow{\tilde{q}} C^\infty(T) \rtimes_{\bar{\alpha}_\theta} \mathbb{Z} \longrightarrow 0$$

appeared in Corollary 2.7. Hence we apply the above exact sequence to Proposition 4.1 to obtain the following exact diagram:

$$\begin{array}{ccccc} HE^{\text{ev}}(C^\infty(T) \rtimes_{\bar{\alpha}_\theta} \mathbb{Z}) & \xrightarrow{\tilde{q}^*} & HE^{\text{ev}}((D^2 \times S^1)_\theta) & \xrightarrow{\tilde{i}^*} & HE^{\text{ev}}(\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ HE^{\text{od}}(\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}) & \xleftarrow{\tilde{i}^*} & HE^{\text{od}}((D^2 \times S^1)_\theta) & \xleftarrow{\tilde{q}^*} & HE^{\text{od}}(C^\infty(T) \rtimes_{\bar{\alpha}_\theta} \mathbb{Z}). \end{array}$$

Alternatively, we have by Corollary 2.6 and [12] that

$$HE^*(\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}) \simeq HE^*(\mathbb{K}^\infty \widehat{\otimes}_\gamma C^\infty(T)) = H_*^{\text{DR}}(T; \mathbb{C}),$$

which implies that

$$HE^{\text{ev}}(\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}) \simeq \mathbb{C}, \quad HE^{\text{od}}(\mathbb{K}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}) \simeq \mathbb{C}.$$

Therefore, we have the following exact diagram:

$$(5.1) \quad \begin{array}{ccccc} \mathbb{C}^2 & \xrightarrow{\tilde{q}^*} & HE^{\text{ev}}((D^2 \times S^1)_\theta) & \xrightarrow{\tilde{i}^*} & \mathbb{C} \\ \uparrow & & & & \downarrow \\ \mathbb{C} & \xleftarrow{\tilde{i}^*} & HE^{\text{od}}((D^2 \times S^1)_\theta) & \xleftarrow{\tilde{q}^*} & \mathbb{C}^2. \end{array}$$

We note that there exists an element $[(\tilde{\psi}_{2k+1})] \in HE^{\text{od}}((D^2 \times S^1)_\theta)$ with the property that

$$(\tilde{\psi}_{2k+1}) = (\tilde{\psi}, 0, 0, \dots), \quad \text{and} \quad B\tilde{\psi} = \tau_\theta \circ \tilde{q}, \quad b\tilde{\psi} = 0,$$

where $b, B=AB_0$ are the operations defined by Connes [5]. Indeed, we define $\tilde{\psi}$ by

$$\tilde{\psi}(x, y) = \tau_\theta \circ \tilde{q}(x\tilde{\delta}_\theta^{(2)}(y)) \quad (x, y \in (D^2 \times S^1)_\theta),$$

where $\tilde{\delta}_\theta^{(2)}$ is the derivation on $(D^2 \times S^1)_\theta$ induced by

$$\tilde{\delta}_\theta^{(2)}\left(\sum_{n \in \mathbb{Z}} A_n U_\theta^n\right) = \sum_{n \in \mathbb{Z}} 2\pi i \theta n A_n U_\theta^n$$

for any $\sum_{n \in \mathbb{Z}} A_n U_\theta^n \in \mathcal{T}^\infty[\mathbb{Z}]$. We note that $\tilde{\delta}_\theta^{(2)}$ is Fréchet continuous since

$$\begin{aligned} \left\| \tilde{\delta}_\theta^{(2)}\left(\sum_{n \in \mathbb{Z}} A_n U_\theta^n\right) \right\|_{p,q,r,s} &= \sup_{n \in \mathbb{Z}} (1+n^2)^p \|2\pi i \theta n A_n\|_{q,r,s} \\ &\leq 2\pi\theta \sup_{n \in \mathbb{Z}} (1+n^2)^{p+1} \|A_n\|_{q,r,s} = 2\pi\theta \left\| \sum_{n \in \mathbb{Z}} A_n U_\theta^n \right\|_{p+1,q,r,s}, \end{aligned}$$

for any $p, q, r, s \in \mathbb{Z}_{\geq 0}$. In this case, let $1 \in \mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}$ be the unit. It is clear that $\tilde{\delta}_\theta^{(2)}(1) = 0$. Then by the definition of b and B , we have that

$$\begin{aligned} B\tilde{\psi}(x) &= \tilde{\psi}(1, x) + \tilde{\psi}(x, 1) = \tau_\theta \circ \tilde{q}(x\tilde{\delta}_\theta^{(2)}(1)) + \tau_\theta \circ \tilde{q}(1\tilde{\delta}_\theta^{(2)}(x)) \\ &= \tau_\theta \circ \tilde{q}(\tilde{\delta}_\theta^{(2)}(x)) \quad (x \in (D^2 \times S^1)_\theta). \end{aligned}$$

We note that for any $f \in C^\infty(T) \rtimes_{\tilde{\alpha}_\theta} \mathbb{Z}$,

$$\tau_\theta(f) = \int_T f(0)(t) dt.$$

Thus, we obtain that

$$\tau_\theta \circ \tilde{q}(\tilde{\delta}_\theta(x)) = \int_T \tilde{q}(\tilde{\delta}_\theta(x))(0)(t)dt = \int_T q(x(0))(t)dt = \tau_\theta \circ \tilde{q}(x)$$

for any $x \in (D^2 \times S^1)_\theta$, which implies that $\tilde{q}^*[\tau_\theta] = 0$. Hence, $\ker \tilde{q}^* \neq 0$ so that the left vertical map of (5.1) is not 0, therefore, injective.

Similarly, we show that the right vertical map is also injective. Since θ is an irrational number, the set $\{e^{2\pi i\theta n} \in \mathbb{C} : n \in \mathbb{Z}\}$ is dense in T . Hence, for all $r \in [0, 1]$, there exists a sequence $\{N_j\}_j \subset \mathbb{Z}$ with $|\{\theta N_j\} - r| \rightarrow 0$ as $j \rightarrow \infty$, where

$$\{x\} = x - \max_{x \geq k, k \in \mathbb{Z}} k \quad (x \in \mathbb{R}).$$

We consider the family $\{U_{\theta N_j}\}$ of unitary operators on H^2 . Since we see that for any $\zeta \in H^2$,

$$\begin{aligned} \|(U_{\theta N_j} - U_{\theta N_k})\zeta\|_{H^2}^2 &= \|(U_\theta^{N_j} - U_\theta^{N_k})\zeta\|_{H^2}^2 = \int_T |\zeta(e^{2\pi i\theta N_j}t) - \zeta(e^{2\pi i\theta N_k}t)|^2 dt \\ &= \int_T |\zeta(e^{2\pi i\theta(N_j - N_k)}t) - \zeta(t)|^2 dt \rightarrow 0 \quad (j, k \rightarrow \infty) \end{aligned}$$

by the Lebesgue dominated convergence theorem, we obtain that $\{U_{\theta N_j}\}$ has the strong limit U_r . It is easily seen that $U_r\zeta(t) = \zeta(e^{2\pi ir}t)$ ($\zeta \in H^2, t \in T$). Moreover, we define the operator h_θ on H^2 by

$$h_\theta\zeta(t) = 2\pi \sum_{j=0}^\infty \{j\theta\}c_jt^j, \quad (\zeta(t) = \sum_{j=0}^\infty c_jt^j \in H^2).$$

Since $0 \leq \{j\theta\} \leq 1$, it is easily verified that h_θ is a bounded self-adjoint positive operator on H^2 and $U_{\theta r} = e^{irh_\theta}$ for $r \in [0, 1]$ by Stone's theorem. Taking again a family $\{N_j\}_{j \in \mathbb{Z}_{\geq 0}} \subset \mathbb{Z}$ with $|e^{2\pi i\theta N_j} - e^{2\pi ir}| \rightarrow 0$ as $j \rightarrow \infty$, we have that

$$\begin{aligned} \|\alpha_{\theta N_j}(x)\zeta - \alpha_{\theta N_k}(x)\zeta\|_{H^2} &= \|U_{\theta N_j}xU_{-\theta N_j}\zeta - U_{\theta N_k}xU_{-\theta N_k}\zeta\|_{H^2} \\ &\leq \|U_{\theta N_j}x(U_{-\theta N_j} - U_{-\theta N_k})\zeta\|_{H^2} + \|(U_{\theta N_j} - U_{\theta N_k})xU_{-\theta N_k}\zeta\|_{H^2} \\ &\rightarrow 0 \quad (x \in \mathcal{T}^\infty, \zeta \in H^2) \end{aligned}$$

since the operation of product is strongly continuous. Therefore, it follows that $\alpha_r(x) = U_r x U_{-r}$ for $x \in \mathbb{B}(H^2)$. We write

$$\tilde{\delta}_\theta^{(1)}(x) = h_\theta x - x h_\theta = \text{ad}(h_\theta)(x) \quad (x \in \mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z})$$

so that

$$e^{ir\tilde{\delta}_\theta^{(1)}} = e^{ir\text{ad}(h_\theta)} = \alpha_{\theta r} \quad (r \in [0, 1]).$$

We now extend the homomorphism $\tilde{q}: \mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z} \rightarrow C^\infty(T) \rtimes_{\bar{\alpha}_\theta} \mathbb{Z}$ to that from the strong closure of $\mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}$ onto that of $C^\infty(T) \rtimes_{\bar{\alpha}_\theta} \mathbb{Z}$ faithfully acting on $L^2(T)$ because of the simplicity of $T_\theta^2 = C^\infty(T) \rtimes_{\bar{\alpha}_\theta} \mathbb{Z}$, that is, that from $\mathbb{B}(H^2)$ onto

$L^\infty(T) \rtimes_{\bar{\alpha}_\theta} \mathbb{Z}$. We also extend the trace τ_θ on T_θ^2 to that on $L^\infty(T) \rtimes_{\bar{\alpha}_\theta} \mathbb{Z}$. We use the same letters for their extensions. Then, we have that $\tilde{q} \circ \tilde{\delta}_\theta^{(1)} = \delta_\theta^{(1)} \circ \tilde{q}$ on $\mathbb{B}(H^2)$. Under the above preparation, we define the linear functional φ_0 on $\mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}$ by

$$\varphi_0(a) = -\tau_\theta \circ \tilde{q}(ah_\theta) \quad (a \in \mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}).$$

Then we compute that

$$\begin{aligned} (b\varphi_0)(a, b) &= \varphi_0(ab) - \varphi_0(ba) = -\tau_\theta \circ \tilde{q}(abh_\theta) + \tau_\theta \circ \tilde{q}(bah_\theta) \\ &= -\tau_\theta(\tilde{q}(a)\tilde{q}(b)\tilde{q}(h_\theta)) + \tau_\theta(\tilde{q}(b)\tilde{q}(a)\tilde{q}(h_\theta)) \\ &= \tau_\theta(\tilde{q}(a)\tilde{q}(h_\theta)\tilde{q}(b) - \tilde{q}(a)\tilde{q}(b)\tilde{q}(h_\theta)) \\ &= \tau_\theta(\tilde{q}(a)\tilde{q}(h_\theta b - bh_\theta)) \quad (a, b \in \mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}). \end{aligned}$$

By the definition of $\tilde{\delta}_\theta^{(1)}$ and $\tilde{q} \circ \tilde{\delta}_\theta^{(1)} = \delta_\theta^{(1)} \circ \tilde{q}$, we have that

$$(b\varphi_0)(a, b) = \tau_\theta(\tilde{q}(a)\tilde{q} \circ \tilde{\delta}_\theta^{(1)}(b)) = \tau_\theta(\tilde{q}(a)\delta_\theta^{(1)} \circ \tilde{q}(b)) = (\tau_\theta^{(1)} \circ \tilde{q})(a, b)$$

for any $a, b \in \mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z}$. Therefore, we obtain that

$$(b + B)[(\varphi_0, 0, \dots)] = [(\tau_\theta^{(1)} \circ \tilde{q}, 0, \dots)],$$

which means that $[(\tau_\theta^{(1)} \circ \tilde{q})] = 0 \in HE^{\text{od}}(\mathcal{T}^\infty \rtimes_{\alpha_\theta} \mathbb{Z})$. Hence, we have $\ker \tilde{q}^* \neq 0$ so that the right vertical map of (5.1) is also injective.

Summing up, we obtain the following exact diagram:

$$(5.2) \quad \begin{array}{ccccc} \mathbb{C}^2 & \xrightarrow{\tilde{q}^*} & HE^{\text{ev}}((D^2 \times S^1)_\theta) & \xrightarrow{0} & \mathbb{C} \\ \uparrow & & & & \downarrow \\ \mathbb{C} & \xleftarrow{0} & HE^{\text{od}}((D^2 \times S^1)_\theta) & \xleftarrow{\tilde{q}^*} & \mathbb{C}^2, \end{array}$$

to conclude that

$$HE^*((D^2 \times S^1)_\theta) \simeq \mathbb{C}^2/\mathbb{C} \simeq \mathbb{C}$$

as required. Moreover, we easily see that $\tilde{q}^* \neq 0$. Hence $\tilde{q}^*[\tau'_\theta] = [\tau'_\theta \circ \tilde{q}]$ and $\tilde{q}^*[\tau_\theta^{(2)}] = [\tau_\theta^{(2)} \circ \tilde{q}]$ are the generators of the corresponding entire cyclic cohomology. ■

We need the following lemma to end up the main result:

LEMMA 5.3. *We have the following equalities:*

- (i) $\tau_\theta \circ \gamma_\theta = \tau_{-\theta}$ and $\tau'_\theta \circ \gamma_\theta = -\tau'_{-\theta}$,
- (ii) $\tau_\theta^{(1)} \circ \gamma_\theta = \tau_{-\theta}^{(2)}$ and $\tau_\theta^{(2)} \circ \gamma_\theta = \tau_{-\theta}^{(1)}$.

Proof. Since $\tau_\theta \circ \gamma_\theta$ is a normalized trace on $T_{-\theta}^2$, it follows by uniqueness that $\tau_\theta \circ \gamma_\theta = \tau_{-\theta}$. We firstly verify that

$$\delta_\theta^{(1)} \circ \gamma_\theta = \delta_{-\theta}^{(2)}, \quad \delta_\theta^{(2)} \circ \gamma_\theta = \delta_{-\theta}^{(1)}.$$

In fact, it is sufficient to verify these equalities for generators. We compute that

$$\begin{aligned} \delta_\theta^{(j)} \circ \gamma_\theta(u_{-\theta}) &= \delta_\theta^{(j)}(v_\theta) = \begin{cases} 0 & (j = 1), \\ 2\pi i v_\theta & (j = 2); \end{cases} \\ \delta_\theta^{(j)} \circ \gamma_\theta(v_{-\theta}) &= \delta_\theta^{(j)}(u_\theta) = \begin{cases} 2\pi i u_\theta & (j = 1), \\ 0 & (j = 2). \end{cases} \end{aligned}$$

We then deduce that

$$\begin{aligned} &\tau'_\theta(\gamma_\theta(b_0), \gamma_\theta(b_1), \gamma_\theta(b_2)) \\ &= \tau_\theta(\gamma_\theta(b_0)((\delta_\theta^{(1)} \circ \gamma_\theta(b_1))(\delta_\theta^{(2)} \circ \gamma_\theta(b_2)) - (\delta_\theta^{(2)} \circ \gamma_\theta(b_1))(\delta_\theta^{(1)} \circ \gamma_\theta(b_2))) \\ &= \tau_\theta(\gamma_\theta(b_0(\delta_{-\theta}^{(2)}(b_1)\delta_{-\theta}^{(1)}(b_2) - \delta_{-\theta}^{(1)}(b_1)\delta_{-\theta}^{(2)}(b_2))) \\ &= -\tau_\theta \circ \gamma_\theta(b_0(\delta_{-\theta}^{(1)}(b_1)\delta_{-\theta}^{(2)}(b_2) - \delta_{-\theta}^{(2)}(b_1)\delta_{-\theta}^{(1)}(b_2))) \\ &= -\tau_{-\theta}(b_0(\delta_{-\theta}^{(1)}(b_1)\delta_{-\theta}^{(2)}(b_2) - \delta_{-\theta}^{(2)}(b_1)\delta_{-\theta}^{(1)}(b_2))) \quad (b_0, b_1, b_2 \in T_{-\theta}^2). \end{aligned}$$

Moreover, for $b_0, b_1 \in T_{-\theta}^2$, we calculate that

$$\tau_\theta^{(1)} \circ \gamma_\theta(b_0, b_1) = \tau_\theta^{(1)}(\gamma_\theta(b_0)\delta_\theta^{(1)}(\gamma_\theta(b_1))) = \tau_\theta^{(1)}(\gamma_\theta(b_0\delta_{-\theta}^{(2)}(b_1))) = \tau_{-\theta}^{(2)}(b_0, b_1).$$

Similarly we have that $\tau_\theta^{(2)} \circ \gamma_\theta = \tau_{-\theta}^{(1)}$. ■

Under the above preparation, we determine the entire cyclic cohomology of noncommutative 3-spheres S_θ^3 . By Theorem 4.4, we have the following exact diagram:

$$\begin{array}{ccccc} HE^{ev}(S_\theta^3) & \longrightarrow & HE^{od}(T_\theta^2) & \xrightarrow{-f_1^*+f_2^*} & G_\theta^1 \oplus G_{-\theta}^1 \\ s_1^*+s_2^* \uparrow & & & & \downarrow s_1^*+s_2^* \\ G_\theta^0 \oplus G_{-\theta}^0 & \xleftarrow{-f_1^*+f_2^*} & HE^{ev}(T_\theta^2) & \longleftarrow & HE^{od}(S_\theta^3), \end{array}$$

where $G_{\pm\theta}^0 = HE^{ev}((D^2 \times S^1)_{\pm\theta})$, $G_{\pm\theta}^1 = HE^{od}((D^2 \times S^1)_{\pm\theta})$ respectively. By Proposition 5.2 and the description in its proof, the above diagram becomes the following one:

$$\begin{array}{ccccc} HE^{ev}(S_\theta^3) & \longrightarrow & \mathbb{C}^2 & \xrightarrow{-f_1^*+f_2^*} & \mathbb{C}^2 \\ s_1^*+s_2^* \uparrow & & & & \downarrow s_1^*+s_2^* \\ \mathbb{C}^2 & \xleftarrow{-f_1^*+f_2^*} & \mathbb{C}^2 & \longleftarrow & HE^{od}(S_\theta^3). \end{array}$$

We describe precisely the maps $-f_1^* + f_2^*$ to compute $HE^*(S_\theta^3)$. For the even case, we check the map

$$-f_1^* + f_2^* : HPE^{ev}(T_\theta^2) = \mathbb{C}[\tau_\theta] \oplus \mathbb{C}[\tau'_\theta] \rightarrow \mathbb{C}[\tau'_\theta \circ \tilde{q}] \oplus \mathbb{C}[\tau'_{-\theta} \circ \tilde{q}] = G_\theta^0 \oplus G_{-\theta}^0.$$

We have $f_1^*[\tau_\theta] = [\tau_\theta \circ \tilde{q}] = 0$ by the calculation in Proposition 5.2 and $f_1^*[\tau'_\theta] = [\tau'_\theta \circ \tilde{q}]$. Alternatively, it follows from Lemma 5.3 that $f_2^*[\tau_\theta] = [\tau_{-\theta} \circ \tilde{q}] = 0$ by the same reason for the case of f_1^* and that $f_2^*[\tau'_\theta] = [\tau'_\theta \circ \tilde{q}] = -[\tau'_{-\theta} \circ \tilde{q}]$ by Lemma 5.3. On the other hand, for the odd case, we consider the map

$$-f_1^* + f_2^* : HP^{\text{od}}(T_\theta^2) = \mathbb{C}[\tau_\theta^{(1)}] \oplus \mathbb{C}[\tau_\theta^{(2)}] \rightarrow \mathbb{C}[\tau_\theta^{(2)} \circ \tilde{q}] \oplus \mathbb{C}[\tau_{-\theta}^{(2)} \circ \tilde{q}] = G_\theta^1 \oplus G_{-\theta}^1.$$

Similarly, we compute, by Lemma 5.3, that:

$$\begin{aligned} f_1^*[\tau_\theta^{(2)}] &= [\tau_\theta^{(2)} \circ \tilde{q}], & f_1^*[\tau_\theta^{(1)}] &= [\tau_\theta^{(1)} \circ \tilde{q}] = 0, & f_2^*[\tau_\theta^{(1)}] &= [\tau_\theta^{(1)} \circ \gamma_\theta \circ \tilde{q}] = [\tau_{-\theta}^{(2)} \circ \tilde{q}], \\ f_2^*[\tau_\theta^{(2)}] &= [\tau_\theta^{(2)} \circ \gamma_\theta \circ \tilde{q}] = [\tau_{-\theta}^{(1)} \circ \tilde{q}] = 0. \end{aligned}$$

Therefore, we have the following exact diagram:

$$\begin{array}{ccccc} HE^{\text{ev}}(S_\theta^3) & \xrightarrow{0} & \mathbb{C}^2 & \xrightarrow{(\lambda, \mu) \mapsto (-\mu, \lambda)} & \mathbb{C}^2 \\ \uparrow & & & & \downarrow 0 \\ \mathbb{C}^2 & \xleftarrow{(\lambda, \mu) \mapsto (-\mu, -\mu)} & \mathbb{C}^2 & \xleftarrow{\quad} & HE^{\text{od}}(S_\theta^3), \end{array}$$

by which we conclude that

$$\begin{aligned} HE^{\text{ev}}(S_\theta^3) &\simeq \text{coker}\{\mathbb{C} \oplus \mathbb{C} \ni (\lambda, \mu) \mapsto (-\mu, -\mu) \in \mathbb{C} \oplus \mathbb{C}\} \simeq \mathbb{C}, \\ HE^{\text{od}}(S_\theta^3) &\simeq \text{ker}\{\mathbb{C} \oplus \mathbb{C} \ni (\lambda, \mu) \mapsto (-\mu, -\mu) \in \mathbb{C} \oplus \mathbb{C}\} \simeq \mathbb{C}. \end{aligned}$$

This completes our computation of the entire cyclic cohomology of noncommutative 3-spheres.

THEOREM 5.4. *The entire cyclic cohomology of noncommutative 3-spheres is isomorphic to the d' Rham homology of the ordinary 3-spheres with complex coefficients.*

REFERENCES

- [1] P.F. BAUM, P.M. HAJAC, R. MATTHES, W. SZYMAŃSKI, The K -theory of Heegaard-type quantum 3-spheres, *K-theory* **35**(2005), 159–186.
- [2] B. BLACKADAR, *K-Theory for Operator Algebras*, 2nd edn., Math. Res. Inst. Publ., vol. 5, Cambridge Univ. Press, Cambridge, MA 1998.
- [3] J. BRODZKI, R. PLYMEN, Entire cyclic homology of Schatten ideals, *Homology Homotopy Appl.* **7**(2005), 37–52.
- [4] M. CHOI, E.G. EFFROS, Lifting problems and the cohomology of C^* -algebras, *Canad. J. Math.* **24**(1977), 1092–1111.
- [5] A. CONNES, *Noncommutative Geometry*, Academic Press, San Diego, CA 1994.
- [6] J. CUNTZ, Excision in periodic cyclic theory for topological algebras, cyclic cohomology and noncommutative geometry, in *Cyclic Cohomology and Noncommutative Geometry (Waterloo, ON, 1995)*, Fields Inst. Commun., vol. 17, Amer. Math. Soc., Providence, RI 1997, pp. 43–53.

- [7] J. CUNTZ, D. QUILLEN, Excision in bivariant periodic cyclic cohomology, *Invent. Math.* **127**(1997), 67–98.
- [8] K.R. DAVIDSON, *C*-Algebras by Example*, Fields Inst. Monographs, vol. 6, Amer. Math. Soc., Providence, RI 1996.
- [9] U. HAAGERUP, An example of a non nuclear C*-algebra, which has the metric approximation property, *Invent. Math.* **50**(1979), 279–293.
- [10] M. KHALKHALI, On the entire cyclic cohomology of Banach algebras, *Comm. Algebra* **22**(1994), 5861–5874.
- [11] S. KLIMEK, A. LESNIEWSKI, Quantum Riemann surfaces. I. The unit disc, *Comm. Math. Phys.* **146**(1992), 103–122.
- [12] V. MATHAI, D. STEVENSON, Entire cyclic cohomology of stable continuous trace algebras, *Bull. London Math. Soc.* **39**(2007), 71–75.
- [13] K. MATSUMOTO, Non-commutative three dimensional spheres, *Japan. J. Math.* **17**(1991), 333–356.
- [14] R. MEYER, Analytic cyclic cohomology, Ph.D. Dissertation, Münster, 1999, arXiv. math. KT/9906205.
- [15] R. MEYER, Excision in entire cyclic cohomology, *J. Eur. Math. Soc.* **3**(2001), 269–286.
- [16] K. KAWASHIMA, The entire cyclic cohomology of noncommutative 2-tori, arXiv. math. KT/0709.1253.
- [17] F. TREVES, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York-London 1967.

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