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MAGNETIC TWISTED ACTIONS ON GENERAL ABELIAN C^*-ALGEBRAS

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ABSTRACT. We introduce magnetic twisted actions of $\mathscr{X} = \mathbb{R}^n$ on general abelian C^*-algebras and study the associated twisted crossed product and pseudodifferential algebras in the framework of strict deformation quantization.

KEYWORDS: *Pseudodifferential calculus, Poisson algebra, crossed product, magnetic field, dynamical system.*

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INTRODUCTION

The usual pseudodifferential calculus in phase space $\Xi := T^*\mathbb{R}^n$ is connected to crossed product C^*-algebras $\mathcal{A} \rtimes_\theta \mathscr{X}$ associated to the action by translations θ of the group $\mathscr{X} := \mathbb{R}^n$ on an abelian C^*-algebra $\mathcal{A}$ composed of functions defined on $\mathscr{X}$. Such a formalism has been used in the quantization of a physical system composed of a spin-less particle moving in $\mathscr{X}$, where the operators acting on $L^2(\mathscr{X})$ can be decomposed into the building block observables position and momentum which are associated to $\mathscr{X}$ and its dual $\mathscr{X}^*$. When dealing with Hamiltonian operators, the algebra $\mathcal{A}$ encapsulates properties of electric potentials, for instance.

During the last decade, it was shown how to incorporate correctly a variable magnetic field in the picture, cf. [4], [5], [6], [7], [12], [14], [15], [16], [17] (see also [1], [2], [3] for extensions involving nilpotent groups). This relies on twisting both the pseudodifferential calculus and the crossed product algebras by a 2-cocycle defined on the group $\mathscr{X}$ and taking values in the (Polish, non-locally compact group) $\mathcal{U}(\mathcal{A})$ of unitary elements of the algebra $\mathcal{A}$. This 2-cocycle is given by imaginary exponentials of the magnetic flux through triangles. The resulting gauge-covariant formalism has position and *kinetic* momentum as its basic observables. The latter no longer commute amongst each other due to the presence of the magnetic field. It was shown in [13] that the family of twisted crossed products indexed by $\hbar \in (0,1]$ can be understood as a strict deformation quantization (in the sense of Marc Rieffel) of a natural Poisson algebra defined by a symplectic form which is the sum of the canonical symplectic form in Ξ and a magnetic contribution.

A natural question is what happens when the algebra $\mathcal{A}$ (composed of functions defined on $\mathscr{X}$) is replaced by a general abelian C^*-algebra. By Gelfand theory this one is isomorphic to $C_0(\Omega)$, the C^*-algebra of all the complex continuous functions vanishing at infinity defined on the locally compact space Ω. To define crossed products and pseudodifferential operators we also need a continuous action θ of $\mathscr{X}$ on Ω by homeomorphisms. $C_0(\Omega)$ can be seen as a C^*-algebra of functions on $\mathscr{X}$ exactly when Ω happens to have a distinguished dense orbit. In the general case, the twisting ingredient will be "a general magnetic field", i.e. a continuous family B of magnetic fields indexed by the points of Ω and satisfying an equivariance condition with respect to the action θ.

The purpose of this article is to investigate the emerging formalism, both classical and quantal.

To the quadruplet $(\Omega, \theta, B, \mathscr{X})$ described above we first assign in Section 1 a Poisson algebra that is the setting for classical mechanics. The Poisson bracket is written with derivatives defined by the abstract action θ and it also contains the magnetic field B. Since Ω does not have the structure of a manifold, this Poisson algebra does not live on a Poisson manifold, let alone a symplectic manifold (as it is the case when a dense orbit exists). But it admits symplectic representations and, at least in the free action case, $\Omega \times \mathscr{X}^*$ is a Poisson space [8] in which symplectic manifolds (the orbits of the action raised to the phase-space Ξ) are only glued together continuously.

Twisted crossed product C^*-algebras are available in a great generality [19], [20]. We use them in Section 2 to define algebras of quantum observables with magnetic fields. By a partial Fourier transformation they can be rewritten as algebras of generalized magnetic pseudodifferential symbols. The outcome has some common points with Rieffel's pseudodifferential calculus [22], which starts from an action of $\mathbb{R}^N$ on a C^*-algebra. In our case this algebra is abelian and the action has a somehow restricted form; on the other hand the magnetic twisting cannot be covered by Rieffel's formalism. We also study Hilbert-space representations of the algebras of symbols. Their interpretation as equivariant families of usual magnetic pseudodifferential operators with anisotropic coefficients [10] is available. This will be developed in a forthcoming article and applied to spectral analysis of deterministic and random magnetic quantum Hamiltonians.

Section 4 is dedicated to a development of the magnetic composition law involving Planck's constant. The first and second terms are written using the classical Poisson algebra conterpart. We insist on reminder estimates valid in the relevant C^*-norms.

All these are used in Section 5 to show that the quantum formalism converges to the classical one when Planck's constant $\hbar$ converges to zero, in the sense of strict deformation quantization [8], [9], [22], [23]. The semiclassical limit of dynamics [8], [24] generated by generalized magnetic Hamiltonians will be studied elsewhere.

An appendix is devoted to some technical results about the behavior of the magnetic flux through triangles. These results are used in the main body of the text.

1. CLASSICAL

1.1. ACTIONS. Let $\mathcal{A}$ denote an abelian C^*-algebra. By Gelfand theory, this algebra is isomorphic to the algebra $C_0(\Omega)$ of continuous functions vanishing at infinity on some locally compact (Hausdorff) topological space Ω, and we shall treat this isomorphism as an identification. Furthermore, we shall always assume that $\mathcal{A}$ is endowed with a continuous action θ of the group $\mathscr{X} := \mathbb{R}^n$ by automorphisms: For any $x, y \in \mathscr{X}$ and $\varphi \in \mathcal{A}$ one has $\theta_0[\varphi] = \varphi$ and $\theta_x[\theta_y[\varphi]] = \theta_{x+y}[\varphi]$ and the map $\mathscr{X} \ni x \mapsto \theta_x[\varphi] \in \mathcal{A}$ is continuous for any $\varphi \in \mathcal{A}$. The triple $(\mathcal{A}, \theta, \mathscr{X})$ is usually called an (abelian) *$\mathscr{X}$-algebra*.

Equivalently, we can assume that the spectrum Ω of $\mathcal{A}$ is endowed with a continuous action of $\mathscr{X}$ by homeomorphisms, which with abuse of notation will also be denoted by θ. In other words, $(\Omega, \theta, \mathscr{X})$ is a locally compact dynamical system. We shall use all of the notations $\theta(\omega, x) = \theta_x[\omega] = \theta_\omega(x)$ for $(\omega, x) \in \Omega \times \mathscr{X}$ and choose the convention $(\theta_x[\varphi])(\omega) = \varphi(\theta_x[\omega])$ to connect the two actions.

An important, but very particular family of examples of $\mathscr{X}$-algebras is constructed using functions on $\mathscr{X}$. Denote by $BC(\mathscr{X})$ the C^*-algebra of all bounded, continuous functions $\phi : \mathscr{X} \to \mathbb{C}$. Let τ denote the action of the locally compact group $\mathscr{X} = \mathbb{R}^n$ on itself, i.e. for any $x, y \in \mathscr{X}$ we set $\tau(x, y) = \tau_x[y] := y + x$. This notation is also used for the action of $\mathscr{X}$ on $BC(\mathscr{X})$ given by $\tau_x[\varphi](y) := \varphi(y + x)$. The action is continuous only on $BC_{\mathrm{u}}(\mathscr{X})$, the C^*-subalgebra composed of bounded and uniformly continuous functions. Any C^*-subalgebra of $BC_{\mathrm{u}}(\mathscr{X})$ which is invariant under translations is an $\mathscr{X}$-algebra. Motivated by the above examples, we define $BC(\Omega) := \{\varphi : \Omega \to \mathbb{C} : f \text{ is bounded and continuous}\}$ and

$$\mathcal{B} \equiv BC_{\mathrm{u}}(\Omega) := \{\varphi \in BC(\Omega) : \mathscr{X} \ni x \mapsto \theta_x[\varphi] \in BC(\Omega) \text{ is continuous}\}.$$

By a *$\mathscr{X}$-morphism* we denote either a continuous map between the underlying spaces of two dynamical systems which intertwines the respective actions, or a morphism between two $\mathscr{X}$-algebras which also intertwines their respective actions.

Let us recall some definitions related to the dynamical system $(\Omega, \theta, \mathscr{X})$. For any $\omega \in \Omega$ we set $\mathcal{O}_\omega := \{\theta_x[\omega] : x \in \mathscr{X}\}$ for *the orbit of* ω and $\mathcal{Q}_\omega := \overline{\mathcal{O}}_\omega$ for *the quasi-orbit of* ω, which is the closure of $\mathcal{O}_\omega$ in Ω. We shall denote by $\mathbf{O}(\Omega) \equiv \mathbf{O}(\Omega, \theta, \mathscr{X})$ the set of orbits of $(\Omega, \theta, \mathscr{X})$ and by $\mathbf{Q}(\Omega) \equiv \mathbf{Q}(\Omega, \theta, \mathscr{X})$

the set of quasi-orbits of $(\Omega, \theta, \mathscr{X})$. For fixed $\omega \in \Omega$, $\varphi \in C_0(\Omega)$ and $x \in \mathscr{X}$, we set $\varphi_\omega(x) := \varphi(\theta_x[\omega]) \equiv \varphi(\theta_\omega(x))$. It is easily seen that $\varphi_\omega : \mathscr{X} \to \mathbb{C}$ belongs to $BC_{\mathrm{u}}(\mathscr{X})$. Furthermore, the C^*-algebra

$$\mathcal{A}_\omega := \{\varphi_\omega : \varphi \in C_0(\Omega)\} = \theta_\omega[C_0(\Omega)]$$

is isomorphic to the C^*-algebra $C_0(\mathcal{Q}_\omega)$ obtained by restricting the elements of $C_0(\Omega)$ to the closed invariant subset $\mathcal{Q}_\omega$. Then, one clearly obtains that

$$\theta_\omega : C_0(\Omega) \ni \varphi \mapsto \varphi_\omega = \varphi \circ \theta_\omega \in BC_{\mathrm{u}}(\mathscr{X}) \tag{1.1}$$

is a $\mathscr{X}$-morphism between $(C_0(\Omega), \theta, \mathscr{X})$ and $(BC_{\mathrm{u}}(\mathscr{X}), \tau, \mathscr{X})$ which induces a $\mathscr{X}$-isomorphism between $(C_0(\mathcal{Q}_\omega), \theta, \mathscr{X})$ and $(\mathcal{A}_\omega, \tau, \mathscr{X})$.

We recall that the dynamical system is *topologically transitive* if an orbit is dense, or equivalently if $\Omega \in \mathbf{Q}(\Omega)$. This happens exactly when the morphism (1.1) is injective for some ω. The dynamical system $(\Omega, \theta, \mathscr{X})$ is *minimal* if all the orbits are dense, i.e. $\mathbf{Q}(\Omega) = \{\Omega\}$. This property is equivalent to the fact that the only closed invariant subsets are $\emptyset$ and Ω.

DEFINITION 1.1. Let $(\mathcal{A}, \theta, \mathscr{X})$ be an $\mathscr{X}$-algebra. We define the spaces of smooth vectors

$$\mathcal{A}^\infty := \{\varphi \in \mathcal{A} : \mathscr{X} \ni x \mapsto \theta_x(\varphi) \in \mathcal{A} \text{ is } C^\infty\}.$$

For the $\mathscr{X}$-algebras $C_0(\Omega)$ and $BC_{\mathrm{u}}(\Omega)$ we will often use the notations $C_0^\infty(\Omega)$, respectively. Despite these notations, we stress that in general Ω is not a manifold; the notion of differentiability is defined only along orbits. By setting for any $\alpha \in \mathbb{N}^n$

$$\delta^\alpha : C_0^\infty(\Omega) \to C_0^\infty(\Omega), \quad \delta^\alpha \varphi := \partial_x^\alpha(\varphi \circ \theta_x)|_{x=0},$$

one defines a Fréchet structure on $C_0^\infty(\Omega)$ by the semi-norms

$$s^\alpha(\varphi) := \|\delta^\alpha \varphi\|_{C_0(\Omega)} = \sup_{\omega \in \Omega} |(\delta^\alpha \varphi)(\omega)|.$$

The spaces $C_0^\infty(\Omega)$ and $\mathcal{A}_\omega^\infty$ are dense Fréchet $*$-subalgebra of the corresponding C^*-algebra.

LEMMA 1.2. (i) *For each $\omega \in \Omega$ one has $\mathcal{A}_\omega^\infty = \{\phi \in C^\infty(\mathscr{X}) : \partial^\beta \phi \in \mathcal{A}_\omega, \forall \beta \in \mathbb{N}^n\}$. In particular $\mathcal{A}_\omega^\infty \subset BC^\infty(\mathscr{X}) := \{\phi \in C^\infty(\mathscr{X}) : \partial^\beta \phi$ is bounded $\forall \beta \in \mathbb{N}^n\}$.*

(i) *Let $\varphi \in C_0(\Omega)$. Then*

$$\varphi \in C_0^\infty(\Omega) \ \leftrightarrow \ \varphi \circ \theta_\omega \in \mathcal{A}_\omega^\infty, \ \forall \omega \in \Omega.$$

Proof. The proof consists in some routine manipulations of the definitions. The only slightly non-trivial fact is to show that point-wise derivations are equivalent to the uniform ones, required by the uniform norms. This follows from the Fundamental Theorem of Calculus, using the higher order derivatives, which are assumed to be bounded. A model for such a standard argument is the proof of Lemma 2.7 in [10]. ∎

REMARK 1.3. In the following, we will use repeatedly and without further comment the identification of point-wise and uniform derivatives under the assumption that higher-order point-wise derivatives exist and are bounded.

Although in our setting the classical observables are functions defined on $\Omega \times \mathscr{X}^*$, we are going to relate them to functions on *phase space* $\Xi := \mathscr{X} \times \mathscr{X}^*$ whose points are denoted by capital letters $X = (x, \xi), Y = (y, \eta), Z = (z, \zeta)$. The dual space $\mathscr{X}^*$ also acts on itself by translations: $\tau^*_\eta(\xi) := \xi + \eta$, and this action is raised to various function spaces as above. Similarly, phase space Ξ can also be regarded as a group acting on itself by translations, $(\tau \otimes \tau^*)_{(y,\eta)}(x, \xi) := (x + y, \xi + \eta)$. Phase space Ξ acts on $\Omega \times \mathscr{X}^*$ as well, via the action $\theta \otimes \tau^*$, and this defines naturally function spaces on $\Omega \times \mathscr{X}^*$ as above; they will be used without further comment.

1.2. COCYCLES AND MAGNETIC FIELDS. Recall the definition of a 2-cocycle κ on the abelian algebra $\mathcal{A} = C_0(\Omega)$ endowed with an action θ of $\mathscr{X}$. We mention that the group $\mathcal{U}(\mathcal{A})$ of unitary elements of the unital C^*-algebra $BC(\Omega)$ coincides with $C(\Omega; \mathbb{T}) := \{\varphi \in C(\Omega) : |\varphi(\omega)| = 1, \ \forall \omega \in \Omega\}$, on which we consider the topology of uniform convergence on compact sets.

DEFINITION 1.4. *A normalized 2-cocycle on $\mathcal{A}$* is a continuous map $\kappa : \mathscr{X} \times \mathscr{X} \to \mathcal{U}(\mathcal{A})$ satisfying for all $x, y, z \in \mathscr{X}$:

$$\kappa(x + y, z)\, \kappa(x, y) = \theta_x[\kappa(y, z)]\, \kappa(x, y + z) \tag{1.2}$$

and $\kappa(x, 0) = \kappa(0, x) = 1$.

PROPOSITION 1.5. *If $\kappa : \mathscr{X} \times \mathscr{X} \to C(\Omega; \mathbb{T})$ is a 2-cocycle of $C_0(\Omega)$ then for any $\omega \in \Omega$, $\kappa_\omega(\cdot, \cdot) := \kappa(\cdot, \cdot) \circ \theta_\omega$ is a 2-cocycle of $\mathcal{A}_\omega$ with respect to the action τ.*

Proof. Everything is straightforward. To check the 2-cocycle property, one uses

$$\theta_x \circ \theta_\omega = \theta_\omega \circ \tau_x, \quad x \in \mathscr{X},\ \omega \in \Omega.$$

It is easy to show that $\kappa : \mathscr{X} \times \mathscr{X} \to C(\Omega, \mathbb{T})$ is continuous if and only if the function

$$\Omega \times \mathscr{X} \times \mathscr{X} \ni (\omega, x, y) \mapsto \kappa(\omega; x, y) := (\kappa(x, y))(\omega) \in \mathbb{T}$$

is continuous. Recalling the isomorphism $\mathcal{A}_\omega \cong C(\mathcal{Q}_\omega)$ one easily finishes the proof. ∎

We shall be interested in *magnetic 2-cocycles*.

DEFINITION 1.6. We call *magnetic field on Ω* a continuous function $B : \Omega \to \bigwedge^2 \mathscr{X}$ such that $B_\omega := B \circ \theta_\omega$ is a magnetic field (continuous closed 2-form on $\mathscr{X}$) for any ω.

Using coordinates, B can be seen as an anti-symmetric matrix $(B^{jk})_{j,k}$ where the entries are continuous functions $B^{jk} : \Omega \to \mathbb{R}$ satisfying (in the distributional sense)

$$\partial_j B_\omega^{kl} + \partial_k B_\omega^{lj} + \partial_l B_\omega^{jk} = 0, \quad \forall \omega \in \Omega,\ \forall j,k,l = 1,\ldots,n.$$

PROPOSITION 1.7. *Let B a magnetic field on Ω. Set*

$$(\kappa^B(x,y))(\omega) \equiv \kappa^B(\omega;x,y) := \exp(-\mathrm{i}\Gamma^{B_\omega}\langle 0,x,x+y\rangle),$$

where $\Gamma^{B_\omega}\langle a,b,c\rangle := \int_{\langle a,b,c\rangle} B_\omega$ is the integral (flux) of the 2-form B_ω through the triangle $\langle a,b,c\rangle$ with corners $a,b,c \in \mathscr{X}$. Then κ^B is a 2-cocycle on $C_0(\Omega)$.

Proof. The algebraic properties follow from the properties of the integration of 2-forms. For example, (1.2) is a consequence of the identity

$$\Gamma^{B_\omega}\langle 0,x,x+y\rangle + \Gamma^{B_\omega}\langle 0,x+y,x+y+z\rangle = \Gamma^{B_{\theta_x[\omega]}}\langle 0,y,y+z\rangle + \Gamma^{B_\omega}\langle 0,x,x+y+z\rangle.$$

This one follows from Stokes' Theorem, after noticing that

$$\Gamma^{B_{\theta_x[\omega]}}\langle 0,y,y+z\rangle = \Gamma^{B_\omega}\langle x,x+y,x+y+z\rangle. \tag{1.3}$$

One still has to check that $\kappa^B \in C(\Omega \times \mathscr{X} \times \mathscr{X})$. This reduces to the obvious continuity of

$$(\omega,x,y) \mapsto \Gamma^{B_\omega}\langle 0,x,x+y\rangle = \sum_{j,k=1}^{n} x_j y_k \int_0^1 \mathrm{d}t \int_0^1 \mathrm{d}s\, s\theta_{sx+sty}[B^{jk}](\omega),$$

where we have used a parametrization of the flux involving the components of the magnetic field in the canonical basis of $\mathscr{X} = \mathbb{R}^n$. ∎

By (1.3) one easily sees that $(\kappa^B)_\omega = \kappa^{B_\omega}$, where the l.h.s. was defined in Proposition 1.5, while

$$\kappa^{B_\omega}(z;x,y) := \exp(-\mathrm{i}\Gamma^{B_\omega}\langle z,z+x,z+x+y\rangle).$$

1.3. POISSON ALGEBRAS. We intend now to define a Poisson structure (cf. [8], [11]) on spaces of functions that are smooth under the action $\theta \times \tau^*$ of Ξ on $\Omega \times \mathscr{X}^*$. This Poisson algebras can be represented by families of subalgebras of $BC^\infty(\Xi)$, indexed essentially by the orbits of Ω, each one endowed with the Poisson structure induced by a magnetic symplectic form [13]. For simplicity, we shall concentrate on a Poisson subalgebra consisting of functions which have Schwartz-type behavior in the variable $\xi \in \mathscr{X}^*$. For this smaller algebra of functions, we will prove strict deformation quantization in Section 4. One can also define $C^\infty(\Omega \times \mathscr{X}^*)$ in terms of the action $\theta \otimes \tau^*$; this one is also a Poisson algebra, but we will not need it here.

We shall use $f(\xi)$ as short-hand notation for $f(\cdot,\xi)$, i.e. $f(\omega,\xi) = (f(\xi))(\omega)$ for $(\omega,\xi) \in \Omega \times \mathscr{X}^*$, and we will think of $f(\cdot,\xi)$ as an element of some algebra of functions on Ω. Note that

$$BC^\infty(\Omega \times \mathscr{X}^*) = \{f \in BC(\Omega \times \mathscr{X}^*) : f(\cdot,\xi) \in BC^\infty(\Omega) \text{ and } f(\omega,\cdot) \in BC^\infty(\mathscr{X}^*),\ \forall \omega \in \Omega, \xi \in \mathscr{X}^*\}.$$

DEFINITION 1.8. We say that $f \in BC^\infty(\Omega \times \mathscr{X}^*)$ belongs to $\mathcal{S}(\mathscr{X}^*; C_0^\infty(\Omega))$ if

(i) $\partial_\xi^\beta f(\xi) \in C_0^\infty(\Omega)$, $\forall \xi \in \mathscr{X}^*$ and

(ii) $\|f\|_{a\alpha\beta} := \sup_{\xi \in \mathscr{X}^*} \|\xi^a \delta^\alpha \partial_\xi^\beta f(\xi)\|_{C_0(\Omega)} < \infty$ for all $a, \alpha, \beta \in \mathbb{N}^n$.

PROPOSITION 1.9. *We assume from now on that $B^{jk} \in BC^\infty(\Omega)$ for any $j,k = 1,\dots,n$.*

(i) *$BC^\infty(\Omega \times \mathscr{X}^*)$ is a Poisson algebra under point-wise multiplication and the Poisson bracket*

$$\{f,g\}_B := \sum_{j=1}^n (\partial_{\xi_j} f \delta_j g - \delta_j f \partial_{\xi_j} g) - \sum_{j,k} B^{jk} \partial_{\xi_j} f \partial_{\xi_k} g. \tag{1.4}$$

(ii) *$\mathcal{S}(\mathscr{X}^*; C_0^\infty(\Omega))$ is a Poisson subalgebra of $BC^\infty(\Omega \times \mathscr{X}^*)$.*

Proof. The two vector spaces are stable under point-wise multiplication and derivations with respect to ξ and along orbits in Ω via ∂_ξ and δ, respectively. They are also stable under multiplication with elements of $BC^\infty(\Omega)$. The axioms of a Poisson algebra are verified by direct computation. ∎

To analyze the quantum calculus which is to be defined below, a change of realization is useful. Defining $\mathcal{S}(\mathscr{X}; C_0^\infty(\Omega))$ as in Definition 1.8, but with $\mathscr{X}^*$ replaced with $\mathscr{X}$, we transport by the partial Fourier transformation the Poisson structure from $\mathcal{S}(\mathscr{X}^*; C_0^\infty(\Omega))$ to $\mathcal{S}(\mathscr{X}; C_0^\infty(\Omega))$ setting

$$\begin{aligned}(\Phi \diamond_0 \Psi)(\omega; x) &:= (1 \otimes \mathcal{F})^{-1}((1 \otimes \mathcal{F})\Phi \cdot (1 \otimes \mathcal{F})\Psi)(\omega; x) \\ &= \int_{\mathscr{X}} \mathrm{d}y\ \Phi(\omega; y)\Psi(\omega; x-y), \text{ and}\end{aligned} \tag{1.5}$$

$$\begin{aligned}\{\Phi,\Psi\}^B &:= (1 \otimes \mathcal{F})^{-1}\{(1 \otimes \mathcal{F})\Phi, (1 \otimes \mathcal{F})\Psi\}_B \\ &= -\mathrm{i} \sum_{j=1}^n (Q_j \Phi \diamond_0 \delta_j \Psi - \delta_j \Phi \diamond_0 Q_j \Psi) + \sum_{j,k=1}^n B^{jk}(Q_j \Phi \diamond_0 Q_k \Psi),\end{aligned} \tag{1.6}$$

where $(Q_j\Phi)(x) = x_j \Phi(x)$ defines the multiplication operator by x_j. Obviously this also makes sense on larger spaces.

To get a better idea of the Poisson structure of $BC^\infty(\Omega \times \mathscr{X}^*)$, we will exploit the orbit structure of the dynamical system $(\Omega \times \mathscr{X}^*, \theta \otimes \tau^*, \mathscr{X} \times \mathscr{X}^*)$ and relate this big Poisson algebra to a family of smaller, symplectic-type ones. For each $\omega \in \Omega$, we can endow $\Xi = \mathscr{X} \times \mathscr{X}^*$ with a symplectic form

$$[\sigma^B_\omega]_Z(X,Y) := y \cdot \xi - x \cdot \eta + B_\omega(z)(x,y) = \sum_{j=1}^n (y_j \xi_j - x_j \eta_j) + \sum_{j,k=1}^n B^{jk}(\theta_z[\omega]) x_j y_k,$$

which makes the pair (Ξ, σ^B_ω) into a symplectic space. This canonically defines a Poisson bracket

$$\{f,g\}_{B_\omega} := \sum_{j=1}^n (\partial_{\xi_j} f \partial_{x_j} g - \partial_{x_j} f \partial_{\xi_j} g) - \sum_{j,k=1}^n B^{jk}_\omega \partial_{\xi_j} f \partial_{\xi_k} g. \tag{1.7}$$

PROPOSITION 1.10. (i) *For each $\omega \in \Omega$, the map*

$$\pi_\omega := \theta_\omega \otimes 1 : (BC^\infty(\Omega \times \mathscr{X}^*), \cdot, \{\cdot,\cdot\}_B) \to (BC^\infty(\Xi), \cdot, \{\cdot,\cdot\}_{B_\omega})$$

is a Poisson map, i.e. for all $f,g \in BC^\infty(\Omega \times \mathscr{X}^*)$

$$\pi_\omega(f \cdot g) = \pi_\omega(f) \cdot \pi_\omega(g), \quad \pi_\omega(\{f,g\}_B) = \{\pi_\omega(f), \pi_\omega(g)\}_{B_\omega}.$$

(ii) *If $\omega, \omega' \in \Omega$ belong to the same orbit, the corresponding Poisson maps are connected by a symplectomorphism (they may be called* equivalent representations of the Poisson algebra*).*

Proof. We use the notation $f_\omega := \pi_\omega(f)$ for $f \in BC^\infty(\Omega \times \mathscr{X}^*)$ and $\omega \in \Omega$.
(i) For any $\omega \in \Omega$, $f,g \in BC^\infty(\Omega \times \mathscr{X}^*)$, we have

$$(fg)_\omega(x,\xi) = ((fg) \circ (\theta_\omega \otimes 1))(x,\xi) = f(\theta_\omega(x),\xi) g(\theta_\omega(x),\xi) = (f_\omega g_\omega)(x,\xi).$$

Similarly, $(\{f,g\}_B)_\omega = \{f_\omega, g_\omega\}_{B_\omega}$ follows from direct computation, using

$$\partial_{x_j} f_\omega = \partial_{x_j}(f \circ (\theta_\omega \otimes 1)) = (\delta_j f) \circ (\theta_\omega \otimes 1) = (\delta_j f)_\omega.$$

(ii) If there exists $z \in \mathscr{X}$ such that $\theta_z[\omega] = \omega'$, then

$$\theta_{\omega'} \otimes 1 = (\theta_\omega \otimes 1) \circ (\tau_z \otimes 1),$$

where $\tau_z \otimes 1 : (\Xi, \sigma^B_\omega) \to (\Xi, \sigma^B_{\omega'})$ is a symplectomorphism. ∎

REMARK 1.11. It is easy to see that the mapping

$$\pi_\omega := \theta_\omega \otimes 1 : \mathcal{S}(\mathscr{X}^*, C^\infty_0(\Omega)) \to \mathcal{S}(\mathscr{X}^*, \mathcal{A}^\infty_\omega)$$

is a surjective morphism of Poisson algebras, for any $\omega \in \Omega$. On the second space we consider the Poisson structure defined by the magnetic field B_ω, as in [13].

For any $\omega \in \Omega$ we define the stabilizer $\mathscr{X}_\omega := \{x \in \mathscr{X} : \theta_x[\omega] = \omega\}$. This is a closed subgroup of $\mathscr{X}$, the same for all ω belonging to a given orbit. We define the subspace of Ω on which the action θ is free:

$$\Omega_0 := \{\omega \in \Omega : \mathscr{X}_\omega = \{0\}\}.$$

Obviously Ω_0 is invariant under θ and $\Omega_0 \times \mathscr{X}^*$ is invariant under the free action $\theta \otimes \tau^*$, so we can consider the Poisson algebra $BC^\infty(\Omega_0 \times \mathscr{X}^*)$ with point-wise multiplication and Poisson bracket (1.4).

For any $\mathcal{O} \in \mathbf{O}(\Omega_0)$ (the family of all the orbits of the space Ω_0) we choose a point $\omega(\mathcal{O}) \in \mathcal{O}$. Then $\theta_{\omega(\mathcal{O})} \otimes 1 : \mathscr{X} \times \mathscr{X}^* \to \Omega_0 \times \mathscr{X}^*$ is a continuous injection with range $\mathcal{O} \times \mathscr{X}^*$ (which is one of the orbits of $\Omega_0 \times \mathscr{X}^*$ under the action $\theta \times \tau^*$). Of course, one has (disjoint union)

$$\Omega_0 \times \mathscr{X}^* = \bigsqcup_{\mathcal{O} \in \mathbf{O}(\Omega_0)} \mathcal{O} \times \mathscr{X}^*.$$

In addition, $\theta_{\omega(\mathcal{O})} \otimes 1$ is a Poisson mapping on $\Xi = \mathscr{X} \times \mathscr{X}^*$ if one considers the Poisson structure induced by the symplectic form $\sigma^B_{\omega(\mathcal{O})}$.

Referring to Definition I.2.6.2 in [8], we notice that *$\Omega_0 \times \mathscr{X}^*$ is a Poisson space.*

2. QUANTUM

2.1. MAGNETIC TWISTED CROSSED PRODUCTS.

DEFINITION 2.1. We call *twisted C^*-dynamical system* a quadruplet, denoted $(\mathcal{A}, \theta, \kappa, \mathscr{X})$, where θ is an action of $\mathscr{X} = \mathbb{R}^n$ on the (abelian) C^*-algebra $\mathcal{A}$ and κ is a normalized 2-cocycle on $\mathcal{A}$ with respect to θ.

Starting from a twisted C^*-dynamical system, one can construct twisted crossed product C^*-algebras [15], [19], [20] (see also references therein). Let $L^1(\mathscr{X}; \mathcal{A})$ be the complex vector space of $\mathcal{A}$-valued Bochner integrable functions on $\mathscr{X}$ and L^1-norm

$$\|\Phi\|_{L^1} := \int_{\mathscr{X}} \mathrm{d}x \, \|\Phi(x)\|_{\mathcal{A}}.$$

For any $\Phi, \Psi \in L^1(\mathscr{X}; \mathcal{A})$ and $x \in \mathscr{X}$, we define the product

$$(\Phi \diamond^\kappa \Psi)(x) := \int_{\mathscr{X}} \mathrm{d}y \, \theta_{\frac{y-x}{2}}[\Phi(y)] \, \theta_{\frac{y}{2}}[\Psi(x-y)] \, \theta_{-\frac{x}{2}}[\kappa(y, x-y)]$$

and the involution $\Phi^{\diamond^\kappa}(x) := \overline{\Phi(-x)}$.

With these two operations, $(L^1(\mathscr{X}; \mathcal{A}), \diamond^\kappa, {}^{\diamond^\kappa})$ forms a Banach-$*$-algebra.

DEFINITION 2.2. The enveloping C^*-algebra of $L^1(\mathscr{X}; \mathcal{A})$ is called *the twisted crossed product $\mathcal{A} \rtimes^\kappa_\theta \mathscr{X}$.*

We are going to indicate now the relevant twisted crossed products, also introducing Planck's constant $\hbar$ in the formalism. We define

$$\theta^\hbar_x := \theta_{\hbar x} \quad \text{and} \quad \kappa^{B,\hbar}(x,y) = \kappa^{\frac{B}{\hbar}}(\hbar x, \hbar y),$$

which means

$$\kappa^{B,\hbar}(\omega; x, y) = \mathrm{e}^{-\frac{i}{\hbar}\Gamma^{B\omega}\langle 0, \hbar x, \hbar x + \hbar y\rangle}, \quad \forall x, y \in \mathscr{X},\ \omega \in \Omega,$$

and check easily that $(C_0(\Omega), \theta^\hbar, \kappa^{B,\hbar}, \mathscr{X})$ is a twisted C^*-dynamical system for any $\hbar \in (0,1]$. It will be useful to introduce $\Lambda^B_\hbar(x,y)$ via

$$\theta_{-\frac{\hbar}{2}x}[\kappa^{B,\hbar}(\omega; x, y)] = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Gamma^{B\omega}\langle -\frac{\hbar}{2}x, \hbar y - \frac{\hbar}{2}x, \frac{\hbar}{2}x\rangle} =: \mathrm{e}^{-\mathrm{i}\hbar\Lambda^{B\omega}_\hbar(x,y)},$$

as short-hand notation for the phase factor. This scaled magnetic flux can be parametrized explicitly as

$$\Lambda^B_\hbar(x,y) = \sum_{j,k=1}^n y_j(x_k - y_k) \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s\, \theta_{\hbar(s-\frac{1}{2})x + \hbar(t-s)y}[B^{jk}]. \tag{2.1}$$

Plugging this particular choice of 2-cocycle and $\mathscr{X}$ action into the general form of the product, one gets

$$(\Phi \diamond^B_\hbar \Psi)(x) = \int_{\mathscr{X}} \mathrm{d}y\, \theta_{\frac{\hbar}{2}(y-x)}[\Phi(y)]\theta_{\frac{\hbar}{2}y}[\Psi(x-y)]\mathrm{e}^{-\mathrm{i}\hbar\Lambda^B_\hbar(x,y)}.$$

The twisted crossed product C^*-algebra $\mathcal{A} \rtimes^{\kappa^{B,\hbar}}_{\theta^\hbar} \mathscr{X}$ will be denoted simply by $\mathfrak{C}^B_\hbar$ with self-adjoint part $\mathfrak{C}^B_{\hbar,\mathbb{R}}$ and norm $\|\cdot\|^B_\hbar$. We also call $\mathfrak{C}_0$ the enveloping C^*-algebra of $L^1(\mathscr{X};\mathcal{A})$ with the commutative product $\diamond_0$; it is isomorphic with $C_0(\mathscr{X}^*;\mathcal{A}) \cong C(\mathscr{X}^*) \otimes \mathcal{A}$.

A quick computation shows that $\pi^\hbar_\omega := \theta^\hbar_\omega \otimes 1$ intertwines the involutions associated to the C^*-algebras $\mathfrak{C}^B_\hbar$ and $\mathcal{A}_\omega \rtimes^{\kappa^{B\omega,\hbar}}_{\tau^\hbar} \mathscr{X}$, i.e. $\pi^\hbar_\omega(\Phi^{\diamond^B_\hbar}) = \pi^\hbar_\omega(\Phi)^{\diamond^{B\omega}_\hbar}$ is satisfied for every $\Phi \in \mathfrak{C}^B_\hbar$. A slightly more cumbersome task is the verification of $\pi^\hbar_\omega(\Phi \diamond^B_\hbar \Psi) = \pi^\hbar_\omega(\Phi) \diamond^{B\omega}_\hbar \pi^\hbar_\omega(\Psi)$. For any $\Phi, \Psi \in L^1(\mathscr{X};\mathcal{A})$ and $z, x \in \mathscr{X}$, we have

$$\begin{aligned}
&[\pi^\hbar_\omega(\Phi \diamond^B_\hbar \Psi)](z;x) \\
&= \int_{\mathscr{X}} \mathrm{d}y\, \big(\theta_{\frac{\hbar}{2}(y-\cdot)}[\Phi(y)]\theta_{\frac{\hbar}{2}y}[\Psi(\cdot - y)]\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Gamma^B\langle \cdot - \frac{\hbar}{2}\cdot, -\frac{\hbar}{2}\cdot + \hbar y, \frac{\hbar}{2}\cdot\rangle}\big) \circ (\theta^\hbar_\omega \otimes 1)(z;x) \\
&= \int_{\mathscr{X}} \mathrm{d}y\, \Phi(\theta_{\hbar z + \frac{\hbar}{2}(y-x)}[\omega], y)\Psi(\theta_{\hbar z + \frac{\hbar}{2}y}[\omega], x-y)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Gamma^{B_{\theta_{\hbar z}[\omega]}}\langle -\frac{\hbar}{2}x, -\frac{\hbar}{2}x + \hbar y, \frac{\hbar}{2}x\rangle} \\
&= \int_{\mathscr{X}} \mathrm{d}y\, \tau_{\frac{\hbar}{2}(y-x)}[\pi^\hbar_\omega(\Phi)(y)](z)\tau_{\frac{\hbar}{2}y}[\pi^\hbar_\omega(\Psi)(x-y)](z)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Gamma^{B\omega}\langle \hbar z - \frac{\hbar}{2}x, \hbar z - \frac{\hbar}{2}x + \hbar y, \hbar z + \frac{\hbar}{2}x\rangle} \\
&= [\pi^\hbar_\omega(\Phi) \diamond^{B\omega}_\hbar \pi^\hbar_\omega(\Psi)](z;x).
\end{aligned}$$

It follows easily that $\{\pi^\hbar_\omega\}_{\omega\in\Omega}$ defines by extension a family of epimorphisms

$$\pi^\hbar_\omega : \mathfrak{C}^B_\hbar \to \mathcal{A}_\omega \rtimes^{\kappa^{B\omega,\hbar}}_{\tau^\hbar} \mathscr{X}$$

that map a twisted crossed product defined in terms of $C_0(\Omega)$ onto more concrete C^*-algebras defined in terms of subalgebras $\mathcal{A}_\omega$ of $BC_\mathrm{u}(\mathscr{X})$.

As we have seen, $\mathcal{S}(\mathscr{X}^*; C_0^\infty(\Omega))$ is a Poisson subalgebra of $BC^\infty(\Omega \times \mathscr{X}^*)$. For strict deformation quantization we also need that it is a $*$-subalgebra of each of the C^*-algebras $\mathfrak{C}^B_\hbar$. Since $\mathcal{S}(\mathscr{X}^*; C_0^\infty(\Omega))$ is stable under involution, this will follow from

PROPOSITION 2.3. *If $B^{jk} \in BC^\infty(\Omega)$, then $\mathcal{S}(\mathscr{X}; C_0^\infty(\Omega))$ is a subalgebra of $(L^1(\mathscr{X}; C_0(\Omega)), \diamond^B_\hbar)$.*

Proof. Let $\Phi, \Psi \in \mathcal{S}(\mathscr{X}; C_0^\infty(\Omega))$. As $\mathcal{S}(\mathscr{X}; C_0^\infty(\Omega))$ is a subspace of $L^1(\mathscr{X}; C_0(\Omega))$, $\Phi \diamond^B_\hbar \Psi$ exists in $L^1(\mathscr{X}; C_0(\Omega))$. To prove the product $\Phi \diamond^B_\hbar \Psi$ is also in $\mathcal{S}(\mathscr{X}; C_0^\infty(\Omega))$, we need to estimate all semi-norms: let $a, \alpha, \beta \in \mathbb{N}^n$. First, we show that we can exchange differentiation with respect to x and along orbits with integration with respect to y via Dominated Convergence, i.e. that for all x and ω

$$\begin{aligned}(x^a \partial_x^\alpha \delta^\beta (\Phi \diamond^B_\hbar \Psi))(\omega; x) &= \int_{\mathscr{X}} \mathrm{d}y \; x^a \partial_x^\alpha \delta^\beta \big(\Phi(\theta_{\frac{\hbar}{2}(y-x)}[\omega], y) \Psi(\theta_{\frac{\hbar}{2}y}[\omega], x-y) \mathrm{e}^{-\mathrm{i}\hbar \Lambda^{B\omega}_\hbar(x,y)}\big) \\ &=: \int_{\mathscr{X}} \mathrm{d}y \; I^a_{\alpha\beta}(\omega; x, y)\end{aligned}$$

holds. Hence, we need to estimate the absolute value of $I^a_{\alpha\beta}$ uniformly in x and ω by an integrable function. To do that, we write out the derivatives involved in $I^a_{\alpha\beta}$,

$$\begin{aligned}&I^a_{\alpha\beta}(x,y) \\ &= x^a \partial_x^\alpha \delta^\beta \big(\theta_{\frac{\hbar}{2}(y-x)}[\Phi(y)] \theta_{\frac{\hbar}{2}y}[\Psi(x-y)] \mathrm{e}^{-\mathrm{i}\hbar \Lambda^B_\hbar(x,y)}\big) \\ &= x^a \sum_{\substack{\alpha'+\alpha''+\alpha'''=\alpha \\ \beta'+\beta''+\beta'''=\beta}} \left(\frac{-\hbar}{2}\right)^{|\alpha'|} \theta_{\frac{\hbar}{2}(y-x)}[\delta^{\alpha'+\beta'} \Phi(y)] \theta_{\frac{\hbar}{2}y}[\partial_x^{\alpha''} \delta^{\beta''} \Psi(x-y)] \partial_x^{\alpha'''} \delta^{\beta'''} \mathrm{e}^{-\mathrm{i}\hbar \Lambda^B_\hbar(x,y)}.\end{aligned}$$

Taking the $C_0(\Omega)$ norm of the above expression and using the triangle inequality, $\hbar \leqslant 1$, the fact that θ_z is an isometry as well as the estimates on the exponential of the magnetic flux from Lemma 5.1(ii), we get

$$\begin{aligned}&\|I^a_{\alpha\beta}(x,y)\|_{\mathcal{A}} \\ &\leqslant |x^a| \sum_{\substack{\alpha'+\alpha''+\alpha'''=\alpha \\ \beta'+\beta''+\beta'''=\beta}} \left(\frac{\hbar}{2}\right)^{|\alpha'|} \|\theta_{\frac{\hbar}{2}(y-x)}[\delta^{\alpha'+\beta'} \Phi(y)]\|_{\mathcal{A}} \|\theta_{\frac{\hbar}{2}y}[\partial_x^{\alpha''} \delta^{\beta''} \Psi(x-y)]\|_{\mathcal{A}} \\ &\quad \cdot \|\partial_x^{\alpha'''} \delta^{\beta'''} \mathrm{e}^{-\mathrm{i}\hbar \Lambda^B_\hbar(x,y)}\|_{\mathcal{A}} \leqslant \Big(\prod_{j=1}^n (|y_j| + |x_j - y_j|)^{a_j}\Big) \\ &\quad \cdot \sum_{\substack{\alpha'+\alpha''+\alpha'''=\alpha \\ \beta'+\beta''+\beta'''=\beta}} \|\delta^{\alpha'+\beta'} \Phi(y)\|_{\mathcal{A}} \|\partial_x^{\alpha''} \delta^{\beta''} \Psi(x-y)\|_{\mathcal{A}} \cdot \sum_{|b|+|c|=2(|\alpha'''|+|\beta'''|)} K_{bc} |y^b| |(x-y)^c|\end{aligned}$$

$$= \sum_{\substack{\alpha'+\alpha''+\alpha'''=\alpha \\ \beta'+\beta''+\beta'''=\beta}} \sum_{|b|+|c|\leqslant|a|+2|\alpha'''|+2|\beta'''|} \widetilde{K}_{bc} \|(Q^b \delta^{\alpha'+\beta'} \Phi)(y)\|_{\mathcal{A}} \|(Q^c \partial_x^{\alpha''} \delta^{\beta''} \Psi)(x-y)\|_{\mathcal{A}}.$$

The polynomial with coefficients $\widetilde{K}_{bc}$ comes from multiplying the other two polynomials in the $|y_j|$ and $|x_j - y_j|$. Taking the supremum in x only yields a function in y (independent of x and ω) which is integrable and dominates $|I^a_{\alpha\beta}(\omega;x,y)|$ since the right-hand side is a finite sum of Schwartz functions in y,

$$\begin{aligned} \sup_{x\in\mathscr{X}} \|I^a_{\alpha\beta}(x,y)\|_{\mathcal{A}} &\leqslant \sum_{\substack{\alpha'+\alpha''+\alpha'''=\alpha \\ \beta'+\beta''+\beta'''=\beta \\ |b|+|c|\leqslant|a|+2|\alpha'''|+2|\beta'''|}} \widetilde{K}_{bc} \|(Q^b \delta^{\alpha'+\beta'} \Phi)(y)\|_{\mathcal{A}} \|Q^c \partial_x^{\alpha''} \delta^{\beta''} \Psi\|_{000} \\ &= \sum_{\substack{\alpha'+\alpha''+\alpha'''=\alpha \\ \beta'+\beta''+\beta'''=\beta \\ |b|+|c|\leqslant|a|+2|\alpha'''|+2|\beta'''|}} \widetilde{K}_{bc} \|(Q^b \delta^{\alpha'+\beta'} \Phi)(y)\|_{\mathcal{A}} \|\Psi\|_{c\alpha''\beta''}. \end{aligned}$$

Hence, by Dominated Convergence, it is permissible to interchange differentiation and integration. To estimate the semi-norm of the product, we write for an integer N such that $2N \geqslant n+1$

$$\begin{aligned} \|\Phi \diamond^B_\hbar \Psi\|_{a\alpha\beta} &= \sup_{\substack{x\in\mathscr{X} \\ \omega\in\Omega}} \Big| \int_{\mathscr{X}} \mathrm{d}y\, I^a_{\alpha\beta}(\omega;x,y) \Big| \leqslant \int_{\mathscr{X}} \frac{\mathrm{d}y}{\langle y\rangle^{2N}} \langle y\rangle^{2N} \sup_{x\in\mathscr{X}} \|I^a_{\alpha\beta}(x,y)\|_{\mathcal{A}} \\ &\leqslant C_1(N) \sup_{x,y\in\mathscr{X}} \big(\langle y\rangle^{2N} \|I^a_{\alpha\beta}(x,y)\|_{\mathcal{A}}\big) \leqslant C_2(N) \max_{|b|\leqslant 2N} \sup_{x,y\in\mathscr{X}} \|y^b I^a_{\alpha\beta}(x,y)\|_{\mathcal{A}}. \end{aligned} \tag{2.2}$$

The right-hand side involves semi-norms associated to $\mathcal{S}(\mathscr{X}\times\mathscr{X}; C_0^\infty(\Omega))$ which we will estimate in terms of the semi-norms of Φ and Ψ, by arguments similar to those leading to the domination of $\|I^a_{\alpha\beta}(x,y)\|_{\mathcal{A}}$.

Thus, we have estimated $\|\Phi \diamond^B_\hbar \Psi\|_{a\alpha\beta}$ from above by a finite number of semi-norms of Φ and Ψ and $\Phi \diamond^B_\hbar \Psi \in \mathcal{S}(\mathscr{X}; C^\infty(\Omega))$. ∎

2.2. TWISTED SYMBOLIC CALCULUS. It is useful to transport the composition law $\diamond^B_\hbar$ by partial Fourier transform $1\otimes\mathcal{F} : \mathcal{S}(\mathscr{X}; C_0^\infty(\Omega)) \to \mathcal{S}(\mathscr{X}^*; C_0^\infty(\Omega))$, setting

$$f \sharp^B_\hbar g := (1\otimes\mathcal{F})\big[(1\otimes\mathcal{F})^{-1} f \diamond^B_\hbar (1\otimes\mathcal{F})^{-1} g\big]. \tag{2.3}$$

In this way one gets a multiplication on $\mathcal{S}(\mathscr{X}^*; C_0^\infty(\Omega))$ which generalizes the magnetic Weyl composition of symbols of [12], [13], [4] (and to which it reduces, actually, if Ω is just a compactification of the configuration space $\mathscr{X}$). Together with complex conjugation, they endow $\mathcal{S}(\mathscr{X}^*; C_0^\infty(\Omega))$ with the structure of a

$*$-algebra. After a short computation one gets

$$
\begin{aligned}
&(f\sharp_\hbar^B g)(\omega,\xi)\\
&= (\pi\hbar)^{-2n}\int_{\mathscr{X}} \mathrm{d}y \int_{\mathscr{X}^*} \mathrm{d}\eta \int_{\mathscr{X}} \mathrm{d}z \int_{\mathscr{X}^*} \mathrm{d}\zeta\; \mathrm{e}^{\mathrm{i}\frac{2}{\hbar}(z\cdot\eta - y\cdot\zeta)} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Gamma^{B\omega}\langle \hbar y-\hbar z, \hbar y+\hbar z, \hbar z-\hbar y\rangle}\\
&\qquad \cdot f(\theta_y[\omega],\xi+\eta)g(\theta_z[\omega],\xi+\zeta)\\
&= (\pi\hbar)^{-2n}\int_{\mathscr{X}} \mathrm{d}y \int_{\mathscr{X}^*} \mathrm{d}\eta \int_{\mathscr{X}} \mathrm{d}z \int_{\mathscr{X}^*} \mathrm{d}\zeta\; \mathrm{e}^{\mathrm{i}\frac{2}{\hbar}\sigma[(y,\eta),(z,\zeta)]} \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Gamma^{B\omega}\langle \hbar y-\hbar z, \hbar y+\hbar z, \hbar z-\hbar y\rangle}\\
&\qquad \cdot (\Theta_{(y,\eta)}[f])(\omega,x)(\Theta_{(z,\zeta)}[g])(\omega,x),
\end{aligned}
\tag{2.4}
$$

where $\sigma[(y,\eta),(z,\zeta)] := z\cdot\eta - y\cdot\zeta$ is the canonical symplectic form on $\Xi := \mathscr{X}\times\mathscr{X}^*$ and

$$(\Theta_{(y,\eta)}[f])(\omega,\xi) \equiv ((\theta_y\otimes\tau^*_\eta)[f])(\omega,\xi) = f(\theta_y[\omega],\xi+\eta).$$

This formula should be compared with the product giving Rieffel's quantization [22].

We note that $1\otimes\mathcal{F}$ can be extended to $L^1(\mathscr{X};C_0(\Omega))$ and then to $\mathfrak{C}^B_\hbar$. So we get a C^*-algebra $\mathfrak{B}^B_\hbar$, isomorphic to $\mathfrak{C}^B_\hbar$, on which the product is an extension of the twisted composition law (2.4). From the bijectivity of the partial Fourier transform and Proposition 2.3 we get the following

COROLLARY 2.4. *If the components of the magnetic field B are of class $BC^\infty(\Omega)$, then $\mathcal{S}(\mathscr{X}^*;C_0^\infty(\Omega))$ is a Fréchet $*$-subalgebra of $\mathfrak{B}^B_\hbar$.*

2.3. REPRESENTATIONS.

We first recall the definition of covariant representations of a magnetic C^*-dynamical system and the way they are used to construct representations of the corresponding C^*-algebras. We denote by $\mathcal{U}(\mathcal{H})$ the group of unitary operators in the Hilbert space $\mathcal{H}$ and by $\mathbf{B}(\mathcal{H})$ the C^*-algebra of all the linear bounded operators on $\mathcal{H}$.

DEFINITION 2.5. Given a magnetic C^*-dynamical system $(\mathcal{A},\theta^\hbar,\kappa^{B,\hbar},\mathscr{X})$, we call *covariant representation* $(\mathcal{H},r,T)$ a Hilbert space $\mathcal{H}$ together with two maps $r:\mathcal{A}\to\mathbf{B}(\mathcal{H})$ and $T:\mathscr{X}\to\mathcal{U}(\mathcal{H})$ satisfying:

(i) r is a non-degenerate representation,

(ii) T is strongly continuous and $T(x)T(y) = r[\kappa^{B,\hbar}(x,y)]T(x+y)$, $\forall x,y\in\mathscr{X}$,

(iii) $T(x)\; r(\varphi)\; T(x)^* = r[\theta^\hbar_x(\varphi)]$, $\forall x\in\mathscr{X}$, $\varphi\in\mathcal{A}$.

LEMMA 2.6. *If $(\mathcal{H},r,T)$ is a covariant representation of $(\mathcal{A},\theta^\hbar,\kappa^{B,\hbar},\mathscr{X})$, then $\mathfrak{Rep}^T_r$ defined on $L^1(\mathscr{X};\mathcal{A})$ by*

$$\mathfrak{Rep}^T_r(\Phi) := \int_{\mathscr{X}} \mathrm{d}y\, r[\theta^\hbar_{y/2}(\Phi(y))]T(y)$$

extends to a representation of $\mathfrak{C}^B_\hbar$.

By composing with the partial Fourier transformation, one gets representations of the pseudodifferential C^*-algebra $\mathfrak{B}^B_\hbar$, denoted by

$$\mathfrak{Op}^T_r : \mathfrak{B}^B_\hbar \to \mathbf{B}(\mathcal{H}), \quad \mathfrak{Op}^T_r(f) := \mathfrak{Rep}^T_r[(1 \otimes \mathcal{F})^{-1}(f)]. \tag{2.5}$$

Given any $\omega \in \Omega$, we shall now construct a representation of $\mathfrak{C}^B_\hbar$ in $\mathcal{H} = L^2(\mathscr{X})$. Let r_ω be the representation of $\mathcal{A}$ in $\mathbf{B}(\mathcal{H})$ given for $\varphi \in \mathcal{A}$, $u \in \mathcal{H}$ and $x \in \mathscr{X}$ by

$$[r_\omega(\varphi)u](x) = [\theta_x(\varphi)](\omega)\, u(x) \equiv \varphi(\theta_x[\omega])\, u(x).$$

Let also $T^\hbar_\omega$ be the map from $\mathscr{X}$ into the set of unitary operators on $\mathcal{H}$ given by

$$[T^\hbar_\omega(y)u](x) := \kappa^{B,\hbar}(\omega; \frac{x}{\hbar}, y)\, u(x + \hbar y) = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Gamma^{B_\omega}\langle 0, x, x+\hbar y\rangle} u(x + \hbar y).$$

PROPOSITION 2.7. *$(\mathcal{H}, r_\omega, T^\hbar_\omega)$ is a covariant representation of the magnetic twisted C^*-dynamical system.*

Proof. Use the definitions, Stokes theorem for the magnetic field B_ω and the identities

$$\Gamma^{B_\omega}\langle x, x+\hbar y, x+\hbar y+\hbar z\rangle = \Gamma^{B_{\theta_x[\omega]}}\langle 0, \hbar y, \hbar y+\hbar z\rangle, \text{ and}$$
$$\Gamma^{B_\omega}\langle 0, x+\hbar y, x\rangle = -\Gamma^{B_\omega}\langle 0, x, x+\hbar y\rangle,$$

valid for all $x, y, z \in \mathscr{X}$ and $\omega \in \Omega$. ∎

The integrated form $\mathfrak{Rep}^\hbar_\omega := \mathfrak{Rep}^{T^\hbar_\omega}_{r_\omega}$ has the following action on $L^1(\mathscr{X}; \mathcal{A})$:

$$\begin{aligned}[\mathfrak{Rep}^\hbar_\omega(\Phi)u](x) &= \int_{\mathscr{X}} \mathrm{d}z\, \Phi(\theta_{x+\frac{\hbar z}{2}}[\omega]; z)\, \kappa^{B,\hbar}(\omega; \frac{x}{\hbar}, z)\, u(x+\hbar z) \\ &= \hbar^{-n} \int_{\mathscr{X}} \mathrm{d}y\, \Phi\Big(\theta_{\frac{x+y}{2}}[\omega]; \frac{1}{\hbar}(y-x)\Big)\, \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Gamma^{B_\omega}\langle 0,x,y\rangle}\, u(y),\end{aligned} \tag{2.6}$$

and the corresponding representation $\mathfrak{Op}^\hbar_\omega$ of the C^*-algebra $\mathfrak{B}^B_\hbar$ has the following form on suitable $f \in \mathfrak{B}^B_\hbar$:

$$[\mathfrak{Op}^\hbar_\omega(f)u](x) = (2\pi\hbar)^{-n} \int_{\mathscr{X}} \mathrm{d}y \int_{\mathscr{X}^*} \mathrm{d}\xi\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(x-y)\cdot\xi} f(\theta_{\frac{x+y}{2}}[\omega]; \xi) \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Gamma^{B_\omega}\langle 0,x,y\rangle} u(y). \tag{2.7}$$

It is clear that $\mathfrak{Op}^\hbar_\omega$ is *not* a faithful representation, since (2.7) only involves the values taken by f on $\mathcal{O}_\omega \times \mathscr{X}^*$, where $\mathcal{O}_\omega$ is the orbit passing through ω. It is rather easy to show that the kernel of $\mathfrak{Op}^\hbar_\omega$ can be identified with the twisted crossed product $C_0(\mathcal{Q}_\omega) \rtimes^{\kappa^B,\hbar}_{\theta^\hbar} \mathscr{X}$, constructed as explained above, with Ω replaced by $\mathcal{Q}_\omega := \overline{\mathcal{O}}_\omega$, the quasi-orbit generated by the point ω.

REMARK 2.8. The expert in the theory of quantum magnetic fields might recognize in (2.7) the expression of a magnetic pseudodifferential operator with symbol $f \circ (\theta_\omega \otimes 1)$, written in the transverse gauge for the magnetic field B_ω.

Then it will be a simple exercise to write down analogous representations associated to continuous (fields of) vector potentials $A : \Omega \to \wedge^1 \mathscr{X}$ generating the magnetic field (i.e. $B_\omega = \mathrm{d}A_\omega$, $\forall \omega \in \Omega$) and to check an obvious principle of gauge-covariance.

We show now that the family of representations $\{\mathfrak{Op}^\hbar_\omega : \omega \in \Omega\}$ actually has as a natural index set the orbit space of the dynamical system, up to unitary equivalence.

PROPOSITION 2.9. *Let ω, ω' be two elements of Ω, belonging to the same orbit under the action θ. Then, for any $\hbar \in (0,1]$, one has $\mathfrak{Rep}^\hbar_\omega \cong \mathfrak{Rep}^\hbar_{\omega'}$ and $\mathfrak{Op}^\hbar_\omega \cong \mathfrak{Op}^\hbar_{\omega'}$ (unitary equivalence of representations).*

Proof. By assumption, there exists an element x_0 of $\mathscr{X}$ such that $\theta_{x_0}[\omega'] = \omega$. For $u \in \mathcal{H}$ and $x \in \mathscr{X}$ we define the unitary operator

$$(U^\hbar_{\omega,\omega'}\, u)(x) := \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Gamma^{B_{\omega'}}\langle 0, x_0, x_0 + x\rangle}\; u(x + x_0)\,.$$

To show unitary equivalence of the two representations, it is enough to show that for all $\varphi \in \mathcal{A}$ and $y \in \mathscr{X}$

$$U^\hbar_{\omega,\omega'} r_{\omega'}(\varphi) = r_\omega(\varphi) U^\hbar_{\omega,\omega'} \quad \text{and} \quad U^\hbar_{\omega,\omega'} T^\hbar_{\omega'}(y) = T^\hbar_\omega(y) U^\hbar_{\omega,\omega'}.$$

The first one is obvious. The second one reduces to

$$\begin{aligned}\Gamma^{B_{\omega'}}\langle 0, x_0, x_0 + x\rangle &+ \Gamma^{B_{\omega'}}\langle 0, x_0 + x, x_0 + x + \hbar y\rangle \\ &= \Gamma^{B_{\omega'}}\langle x_0, x_0 + x, x_0 + x + \hbar y\rangle + \Gamma^{B_{\omega'}}\langle 0, x_0, x_0 + x + \hbar y\rangle,\end{aligned}$$

which is true by Stokes theorem. ∎

REMARK 2.10. The proposition reveals what we consider to be the main practical interest of the formalism we develop in the present article. To a fixed real symbol f and to a fixed value $\hbar$ of Planck's constant one associates a family $\{H^\hbar_\omega := \mathfrak{Op}^\hbar_\omega(f) : \omega \in \Omega\}$ of self-adjoint magnetic pseudodifferential operators on the Hilbert space $\mathcal{H} := L^2(\mathscr{X})$, indexed by the points of a dynamical system $(\Omega, \theta, \mathscr{X})$ and satisfying the equivariance condition

$$H^\hbar_{\theta_x[\omega]} = (U^\hbar_{\omega,\theta_x[\omega]})^{-1} H^\hbar_\omega U^\hbar_{\omega,\theta_x[\omega]}, \quad \forall (\omega, x) \in \Omega \times \mathscr{X}. \tag{2.8}$$

In concrete situations, such equivariance conditions usually carry some physical meaning. In a future publication we are going to extend the formalism to unbounded symbols f, getting realistic magnetic Quantum Hamiltonians organized in equivariant families, which will be studied in the framework of spectral theory.

To define other types of representations, we consider now Ω endowed with a θ-invariant measure μ. Such measures always exist, since $\mathscr{X}$ is abelian hence amenable. We set $\mathcal{H}'$ for the Hilbert space $L^2(\Omega, \mu)$ and consider first the faithful representation: $\widetilde{r} : \mathcal{A} \to \mathbf{B}(\mathcal{H}')$ with $[\widetilde{r}(\varphi)v](\omega) := \varphi(\omega)\; v(\omega)$ for all $v \in \mathcal{H}'$ and

$\omega \in \Omega$. Then, (by a standard construction in the theory of twisted crossed products) the *regular representation of the magnetic* C^**-dynamical system* $(\mathcal{A}, \theta^\hbar, \kappa^{B,\hbar}, \mathscr{X})$ *induced by* $\widetilde{r}$ is the covariant representation $(L^2(\mathscr{X};\mathcal{H}'), r, T^\hbar)$:

$$r : \mathcal{A} \to \mathbf{B}[L^2(\mathscr{X};\mathcal{H}')], [r(\varphi)w](\omega;x) := (\widetilde{r}(\theta_x(\varphi))[w(x)])(\omega) = \varphi(\theta_x(\omega))w(\omega;x),$$

$$T^\hbar : \mathscr{X} \to \mathcal{U}[L^2(\mathscr{X};\mathcal{H}')], \quad [T^\hbar(y)w](\omega;x) := \kappa^{B,\hbar}(\omega;\frac{x}{\hbar},y)\; w(\omega;x+\hbar y).$$

We identify freely $L^2(\mathscr{X};\mathcal{H}')$ with $L^2(\Omega \times \mathscr{X})$ with the obvious product measure, so $r(\varphi)$ is the operator of multiplication by $\varphi \circ \theta$ in $L^2(\Omega \times \mathscr{X})$. Due to Stokes' Theorem, this is again a covariant representation. The integrated form $\mathfrak{REP}^\hbar := \mathfrak{Rep}_r^{T^\hbar}$ associated to $(r, T^\hbar)$ is given on $L^1(\mathscr{X};\mathcal{A})$ by

$$[\mathfrak{REP}^\hbar(\Phi)w](\omega;x) = \hbar^{-n} \int_{\mathscr{X}} \mathrm{d}y\; \Phi(\theta_{\frac{x+y}{2}}[\omega];\frac{y-x}{\hbar})\; \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Gamma^{B\omega}\langle 0,x,y\rangle}\; w(\omega;y)$$

and it admits the direct integral decomposition

$$\mathfrak{REP}^\hbar(\Phi) = \int_\Omega^\oplus \mathrm{d}\mu\,(\omega)\; \mathfrak{Rep}^\hbar_\omega(\Phi). \tag{2.9}$$

The group $\mathscr{X}$, being abelian, is amenable, and thus the regular representation $\mathfrak{REP}^\hbar$ is faithful. The corresponding representation $\mathfrak{OP}^\hbar : \mathfrak{B}^B_\hbar \to \mathbf{B}[L^2(\mathscr{X};\mathcal{H}')]$ is given for f with partial Fourier transform in $L^1(\mathscr{X};\mathcal{A})$ by

$$[\mathfrak{OP}^\hbar(f)w](\omega;x) = (2\pi\hbar)^{-n}\int_{\mathscr{X}}\int_{\mathscr{X}^*} \mathrm{d}y\,\mathrm{d}\eta\; \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(x-y)\cdot\eta} f(\theta_{\frac{x+y}{2}}[\omega],\eta)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\Gamma^{B\omega}\langle 0,x,y\rangle} w(\omega;y).$$

3. ASYMPTOTIC EXPANSION OF THE PRODUCT

The proof of strict deformation quantization hinges on the following theorem:

THEOREM 3.1 (Asymptotic expansion of the product). *Assume the components of B are in $BC^\infty(\Omega)$. Let $\Phi, \Psi \in \mathcal{S}(\mathscr{X};C_0^\infty(\Omega))$ and $\hbar \in (0,1]$. Then the product $\Phi \diamond^B_\hbar \Psi$ can be expanded in powers of $\hbar$,*

$$\Phi \diamond^B_\hbar \Psi = \Phi \diamond_0 \Psi - \hbar \frac{\mathrm{i}}{2}\{\Phi,\Psi\}^B + \hbar^2 R^{\diamond,2}_\hbar(\Phi,\Psi), \tag{3.1}$$

where $\{\Phi,\Psi\}^B$ is defined as in equation (1.6). *All terms are in $\mathcal{S}(\mathscr{X};C_0^\infty(\Omega))$ and $R^{\diamond,2}_\hbar(\Phi,\Psi)$ is bounded uniformly in $\hbar$, $\parallel R^{\diamond,2}_\hbar(\Phi,\Psi) \parallel^B_\hbar \leqslant C$.*

Proof. We are going to use Einstein's summation convention, i.e. repeated indices in a product are summed over. Two types of terms in the product formula need to be expanded in $\hbar$, the group action of $\mathcal{X}$ on Ω,

$$\begin{aligned}
(\theta_{\frac{\hbar}{2}y}[\Phi(x)])(\omega) = \Phi(\theta_{\frac{\hbar}{2}y}[\omega];x) &= \Phi(\omega;x) + \hbar \int_0^1 \mathrm{d}\tau \, \frac{1}{2} y_j \theta_{\tau\frac{\hbar}{2}y}[(\delta_j\Phi)(\omega;x)] \\
&=: \Phi(\omega;x) + \hbar (R^{\theta,1}_{\hbar,y}(\Phi))(\omega;x) \\
&= \Phi(\omega;x) + \frac{\hbar}{2} y_j (\delta_j\Phi)(\omega;x) + \hbar^2 \int_0^1 \mathrm{d}\tau \, \frac{1}{4}(1-\tau) y_j y_k \theta_{\tau\frac{\hbar}{2}y}[(\delta_j\delta_k\Phi)(\omega;x)] \\
&=: \Phi(\omega;x) + \frac{\hbar}{2} y_j (\delta_j\Phi)(\omega;x) + \hbar^2 (R^{\theta,2}_{\hbar,y}(\Phi))(\omega;x),
\end{aligned}$$

and the exponential of the magnetic flux,

$$\begin{aligned}
\mathrm{e}^{-\mathrm{i}\hbar\Lambda^B_\hbar(x,y)} &= 1 + \hbar \int_0^1 \mathrm{d}\tau \, \frac{\mathrm{d}}{\mathrm{d}\varepsilon} (\mathrm{e}^{-\mathrm{i}\varepsilon\Lambda^B_\varepsilon(x,y)})|_{\varepsilon=\tau\hbar} =: 1 + \hbar R^{\kappa,1}_\hbar(x,y) \\
&= 1 - \hbar \mathrm{i} \Lambda^B_0(x,y) + \hbar^2 \int_0^1 \mathrm{d}\tau \, (1-\tau) \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} (\mathrm{e}^{-\mathrm{i}\varepsilon\Lambda^B_\varepsilon(x,y)})|_{\varepsilon=\tau\hbar} \\
&=: 1 - \hbar \frac{\mathrm{i}}{2} B^{jk} y_j (x_k - y_k) + \hbar^2 R^{\kappa,2}_\hbar(x,y).
\end{aligned}$$

We will successively plug these expansions into the product formula, keeping only terms of $\mathcal{O}(\hbar^2)$:

$$\begin{aligned}
&(\Phi \diamond^B_\hbar \Psi)(x) \\
&= \int_{\mathcal{X}} \mathrm{d}y \, \Big(\Phi(y) + \frac{\hbar}{2}(y_j - x_j)(\delta_j\Phi)(y) + \hbar^2 (R^{\theta,2}_{\hbar,y-x}(\Phi))(y) \Big) \theta_{\frac{\hbar}{2}y}[\Psi(x-y)] \mathrm{e}^{-\mathrm{i}\hbar\Lambda^B_\hbar(x,y)} \\
&= \int_{\mathcal{X}} \mathrm{d}y \, \Phi(y) \Big(\Psi(x-y) + \frac{\hbar}{2} y_j (\delta_j\Psi)(x-y) + \hbar^2 (R^{\theta,2}_{\hbar,y}(\Psi))(x-y) \Big) \mathrm{e}^{-\mathrm{i}\hbar\Lambda^B_\hbar(x,y)} \\
&\quad + \frac{\hbar}{2} \int_{\mathcal{X}} \mathrm{d}y \, (y_j - x_j)(\delta_j\Phi)(y)(\Psi(x-y) + \hbar (R^{\theta,1}_{\hbar,y}(\Psi))(x-y)) \mathrm{e}^{-\mathrm{i}\hbar\Lambda^B_\hbar(x,y)} \\
&\quad + \hbar^2 \int_{\mathcal{X}} \mathrm{d}y \, (R^{\theta,2}_{\hbar,y-x}(\Phi))(y) \theta_{\frac{\hbar}{2}y}[\Psi(x-y)] \mathrm{e}^{-\mathrm{i}\hbar\Lambda^B_\hbar(x,y)} \\
&= \int_{\mathcal{X}} \mathrm{d}y \, \Phi(y)\Psi(x-y)(1 - \hbar\mathrm{i}\Lambda^B_0(x,y) + \hbar^2 R^{\kappa,2}_\hbar(x,y)) \\
&\quad + \frac{\hbar}{2} \int_{\mathcal{X}} \mathrm{d}y \, (-(\delta_j\Phi)(y)(Q_j\Psi)(x-y) + (Q_j\Phi)(y)(\delta_j\Psi)(x-y))(1 + \hbar R^{\kappa,1}_\hbar(x,y))
\end{aligned}$$

$$
\begin{aligned}
&\quad + \hbar^2 \int_{\mathcal{X}} \mathrm{d}y \, \Big[(R^{\theta,2}_{\hbar,y-x}(\Phi))(y) \theta_{\frac{\hbar}{2}y}[\Psi(x-y)] \mathrm{e}^{-\mathrm{i}\hbar \Lambda^B_\hbar(x,y)} \\
&\quad + \Phi(y)(R^{\theta,2}_{\hbar,y}(\Psi))(x-y) \mathrm{e}^{-\mathrm{i}\hbar\Lambda^B_\hbar(x,y)} - \frac{1}{2}(\delta_j\Phi)(y)(Q_j R^{\theta,1}_{\hbar,y}(\Psi))(x-y)\mathrm{e}^{-\mathrm{i}\hbar\Lambda^B_\hbar(x,y)} \Big] \\
&= \int_{\mathcal{X}} \mathrm{d}y \, \Phi(y)\Psi(x-y) + \frac{\hbar}{2} \int_{\mathcal{X}} \mathrm{d}y \, \big((Q_j\Phi)(y)(\delta_j\Psi)(x-y) \\
&\quad - (\delta_j\Phi)(y)(Q_j\Psi)(x-y) - \mathrm{i}B^{jk}(Q_j\Phi)(y)(Q_k\Psi)(x-y)\big) \\
&\quad + \hbar^2 \int_{\mathcal{X}} \mathrm{d}y \, \Big[\big((R^{\theta,2}_{\hbar,y-x}(\Phi))(y)\theta_{\frac{\hbar}{2}y}[\Psi(x-y)] + \Phi(y)(R^{\theta,2}_{\hbar,y}(\Psi))(x-y)\big)\mathrm{e}^{-\mathrm{i}\hbar\Lambda^B_\hbar(x,y)} \\
&\quad - \frac{1}{2}(\delta_j\Phi)(y)(Q_jR^{\theta,1}_{\hbar,y}(\Psi))(x-y)\mathrm{e}^{-\mathrm{i}\hbar\Lambda^B_\hbar(x,y)} + \frac{1}{2}\big((Q_j\Phi)(y)(\delta_j\Psi)(x-y) \\
&\quad - (\delta_j\Phi)(y)(Q_j\Psi)(x-y)\big)R^{\kappa,1}_\hbar(x,y) + \Phi(y)\Psi(x-y)R^{\kappa,2}_\hbar(x,y) \Big] \\
&=: (\Phi \diamond_0 \Psi)(x) - \hbar \frac{\mathrm{i}}{2}\{\Phi,\Psi\}^B(x) + \hbar^2 (R^{\diamond,2}_\hbar(\Phi,\Psi))(x).
\end{aligned}
$$

In the above, we have used $(y_j - x_j)\Psi(x-y) = -(Q_j\Psi)(x-y)$, $y_j\Phi(y) = (Q_j\Phi)(y)$ and the explicit expression for $\Lambda^B_0(x,y)$. Clearly, the leading-order and sub-leading-order terms are again in $\mathcal{S}(\mathscr{X}; C^\infty_0(\Omega))$. Thus also $R^{\diamond,2}_\hbar(\Phi,\Psi) = \hbar^{-2}(\Phi \diamond^B_\hbar \Psi - \Phi \diamond_0 \Psi + \hbar\frac{\mathrm{i}}{2}\{\Phi,\Psi\}^B)$ is an element of $\mathcal{S}(\mathscr{X}; C^\infty_0(\Omega))$ for all $\hbar \in (0,1]$.

The most difficult part of the proof is to show that the $\hbar$-dependent C^*-norm of the remainder $R^{\diamond,2}_\hbar(\Phi,\Psi)$ can be uniformly bounded in $\hbar$. The first ingredient is the fact that the $\hbar$-dependent C^*-norm of the twisted crossed product is dominated by the $L^1(\mathcal{X};\mathcal{A})$-norm for all values of $\hbar \in (0,1]$,

$$
\|\Phi\|^B_\hbar \leqslant \|\Phi\|_{L^1}, \quad \forall \Phi \in \mathcal{S}(\mathscr{X}; C^\infty_0(\Omega)) \subset L^1(\mathscr{X}; C_0(\Omega)) \subset \mathfrak{C}^B_\hbar.
$$

Hence, if we can find $\hbar$-independent L^1 bounds on each term of the remainder, we have also estimated the $\hbar$-dependent C^*-norm uniformly in $\hbar$.

There are four distinct types of terms in the remainder. Let us start with the first: we define

$$
(R^{\diamond,2}_{\hbar,1}(\Phi,\Psi))(x) := \int_{\mathcal{X}} \mathrm{d}y \, (R^{\theta,2}_{\hbar,y-x}(\Phi))(y)\theta_{\frac{\hbar}{2}y}[\Psi(x-y)]\mathrm{e}^{-\mathrm{i}\hbar\Lambda^B_\hbar(x,y)}.
$$

Then we have

$$
\begin{aligned}
\|R^{\diamond,2}_{\hbar,1}(\Phi,\Psi)\|^B_\hbar &\leqslant \|R^{\diamond,2}_{\hbar,1}(\Phi,\Psi)\|_{L^1(\mathcal{X};\mathcal{A})} \\
&\leqslant \int_{\mathcal{X}} \mathrm{d}x \int_{\mathcal{X}} \mathrm{d}y \, \|(R^{\theta,2}_{\hbar,y-x}(\Phi))(y)\|_{\mathcal{A}} \|\theta_{\frac{\hbar}{2}y}[\Psi(x-y)]\|_{\mathcal{A}} \|\mathrm{e}^{-\mathrm{i}\hbar\Lambda^B_\hbar(x,y)}\|_{\mathcal{A}} \\
&= \int_{\mathcal{X}} \mathrm{d}x \int_{\mathcal{X}} \mathrm{d}y \, \|(R^{\theta,2}_{\hbar,y-x}(\Phi))(y)\|_{\mathcal{A}} \|\Psi(x-y)\|_{\mathcal{A}}
\end{aligned}
$$

$$= \int_{\mathcal{X}} \mathrm{d}x \int_{\mathcal{X}} \mathrm{d}y \; \|(R^{\theta,2}_{\hbar,-x}(\Phi))(y)\|_{\mathcal{A}} \|\Psi(x)\|_{\mathcal{A}}.$$

We inspect $\|(R^{\theta,2}_{\hbar,-x}(\Phi))(y)\|_{\mathcal{A}}$ more closely:

$$\begin{aligned}\|(R^{\theta,2}_{\hbar,-x}(\Phi))(y)\|_{\mathcal{A}} &\leqslant \frac{1}{4} \int_0^1 \mathrm{d}\tau \; |(-x_j)(-x_k)| \|\theta_{-\tau\frac{\hbar}{2}x}[(\delta_j \delta_k \Phi)(y)]\|_{\mathcal{A}} \\ &= \frac{1}{8} |x_j x_k| \|(\delta_j \delta_k \Phi)(y)\|_{\mathcal{A}}.\end{aligned}$$

If we plug that back into the estimate of the L^1 norm, we get

$$\begin{aligned}\|R^{\diamond,2}_{\hbar,1}(\Phi,\Psi)\|^B_{\hbar} &\leqslant \frac{1}{8} \int_{\mathcal{X}} \mathrm{d}x \int_{\mathcal{X}} \mathrm{d}y \; \|(\delta_j \delta_k \Phi)(y)\|_{\mathcal{A}} \|(Q_j Q_k \Psi)(x)\|_{\mathcal{A}} \\ &= \frac{1}{8} \|\delta_j \delta_k \Phi\|_{L^1} \|Q_j Q_k \Psi\|_{L^1}.\end{aligned}$$

The right-hand side is finite by the definition of $\mathcal{S}(\mathscr{X}; C^\infty_0(\Omega))$. Similarly, the second term can be estimated, just the roles of Φ and Ψ are reversed.

Now to the second type of term: we define

$$(R^{\diamond,2}_{\hbar,3}(\Phi,\Psi))(x) := -\frac{1}{2} \int_{\mathcal{X}} \mathrm{d}y \; (\delta_j \Phi)(y) (Q_j R^{\theta,1}_{\hbar,y}(\Psi))(x-y) \mathrm{e}^{-\mathrm{i}\hbar \Lambda^B_{\hbar}(x,y)}$$

and estimate

$$2\|R^{\diamond,2}_{\hbar,3}(\Phi,\Psi)\|^B_{\hbar} \leqslant 2\|R^{\diamond,2}_{\hbar,3}(\Phi,\Psi)\|_{L^1} \leqslant \int_{\mathcal{X}} \mathrm{d}x \int_{\mathcal{X}} \mathrm{d}y \; \|(\delta_j \Phi)(y)\|_{\mathcal{A}} \|(Q_j R^{\theta,1}_{\hbar,y}(\Psi))(x)\|_{\mathcal{A}}.$$

The last factor needs to be estimated by hand:

$$\|(Q_j R^{\theta,1}_{\hbar,y}(\Psi))(x)\|_{\mathcal{A}} \leqslant \frac{1}{2} \int_0^1 \mathrm{d}\tau \; |x_j y_k| \|\theta_{\tau\frac{\hbar}{2}y}[(\delta_k \Psi)(x)]\|_{\mathcal{A}} = \frac{1}{2} |x_j y_k| \|(\delta_k \Psi)(x)\|_{\mathcal{A}}.$$

This leads to the bound

$$\|R^{\diamond,2}_{\hbar,3}(\Phi,\Psi)\|^B_{\hbar} \leqslant \frac{1}{4} \int_{\mathcal{X}} \mathrm{d}x \int_{\mathcal{X}} \mathrm{d}y \; \|(\delta_j \Phi)(y)\|_{\mathcal{A}} |x_j y_k| \|(\delta_k \Psi)(x)\|_{\mathcal{A}} = \frac{1}{4} \|Q_k \delta_j \Phi\|_{L^1} \|Q_j \delta_k \Psi\|_{L^1}.$$

The right-hand side is again finite since $\Phi, \Psi \in \mathcal{S}(\mathscr{X}; C^\infty_0(\Omega))$ does not depend on $\hbar$.

Estimating the two magnetic terms is indeed a bit more involved: we define

$$(R^{\diamond,2}_{\hbar,4}(\Phi,\Psi))(x) := \frac{1}{2} \int_{\mathcal{X}} \mathrm{d}y \; (Q_j \Phi)(y) (\delta_j \Psi)(x-y) R^{\kappa,1}_{\hbar}(x,y).$$

The usual arguments show the C^*-norm can be estimated by

$$\|R^{\diamond,2}_{\hbar,4}(\Phi,\Psi)\|^B_{\hbar} \leqslant \int_{\mathcal{X}} \mathrm{d}x \int_{\mathcal{X}} \mathrm{d}y \; \|(Q_j \Phi)(y)\|_{\mathcal{A}} \|(\delta_j \Psi)(x-y)\|_{\mathcal{A}} \|R^{\kappa,1}_{\hbar}(x,y)\|_{\mathcal{A}}$$

which warrants a closer inspection of the last term: first of all, we note that

$$\begin{aligned}
R^{\kappa,1}_\hbar(x,y) &= \int_0^1 \mathrm{d}\tau \; \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big(\mathrm{e}^{-\mathrm{i}\varepsilon \Lambda^B_\varepsilon(x,y)}\big)\big|_{\varepsilon=\tau\hbar} \\
&= \int_0^1 \mathrm{d}\tau \; \Big(- \mathrm{i}\Lambda^B_\varepsilon(x,y) - \mathrm{i}\varepsilon \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Lambda^B_\varepsilon(x,y)\Big) \mathrm{e}^{-\mathrm{i}\varepsilon \Lambda^B_\varepsilon(x,y)}\big|_{\varepsilon=\tau\hbar} .
\end{aligned}$$

If we use Lemma 5.1 and $\hbar \leqslant 1$, this leads to the following norm estimate of $R^{\kappa,1}_\hbar(x,y)$:

$$\begin{aligned}
\|R^{\kappa,1}_\hbar(x,y)\|_{\mathcal{A}} &\leqslant \int_0^1 \mathrm{d}\tau \; \Big(\|\Lambda^B_{\tau\hbar}(x,y)\|_{\mathcal{A}} + \hbar\tau \Big\{ \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Lambda^B_\varepsilon(x,y)|_{\varepsilon=\tau\hbar}\Big\}_{\mathcal{A}}\Big) \|\mathrm{e}^{-\mathrm{i}\varepsilon\Lambda^B_{\tau\hbar}(x,y)}\|_{\mathcal{A}} \\
&\leqslant \|B^{jk}\|_{\mathcal{A}} |y_j| |x_k - y_k| + \frac{1}{2} \|\delta_l B^{jk}\|_{\mathcal{A}} |y_j| |x_k - y_k| \big(|x_l - y_l| + |y_l|\big).
\end{aligned}$$

Put together, this allows us to estimate the norm of $R^{\diamond,2}_{\hbar,4}$ by

$$\begin{aligned}
\|R^{\diamond,2}_{\hbar,4}(\Phi,\Psi)\|^B_\hbar \leqslant \|B^{mk}\|_{\mathcal{A}} \|Q_j Q_m \Phi\|_{L^1} \|Q_k \delta_j \Psi\|_{L^1} \\
+ \frac{1}{2} \|\delta_l B^{mk}\|_{\mathcal{A}} \big(\|Q_j Q_m Q_l \Phi\|_{L^1} \|Q_k \delta_j \Psi\|_{L^1} \\
+ \|Q_j Q_m \Phi\|_{L^1} \|Q_k Q_l \delta_j \Psi\|_{L^1} \big).
\end{aligned}$$

Now on to the last term,

$$\big(R^{\diamond,2}_{\hbar,6}(\Phi,\Psi)\big)(x) := \int_{\mathcal{X}} \mathrm{d}y \; \Phi(y) \Psi(x-y) R^{\kappa,2}_\hbar(x,y).$$

Using the explicit form of $R^{\kappa,2}_\hbar(x,y)$,

$$\begin{aligned}
R^{\kappa,2}_\hbar(x,y) &= \int_0^1 \mathrm{d}\tau \; (1-\tau) \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\big(\mathrm{e}^{-\mathrm{i}\varepsilon\Lambda^B_\varepsilon(x,y)}\big)\big|_{\varepsilon=\tau\hbar} \\
&= \int_0^1 \mathrm{d}\tau \; (1-\tau) \Big[- \mathrm{i}2 \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Lambda^B_\varepsilon(x,y) - \mathrm{i}\varepsilon \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\Lambda^B_\varepsilon(x,y) \\
&\qquad - \Big(\Lambda^B_\varepsilon(x,y) + \varepsilon \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Lambda^B_\varepsilon(x,y)\Big)^2 \Big] \Big|_{\varepsilon=\tau\hbar} \mathrm{e}^{-\mathrm{i}\tau\hbar\Lambda^B_{\tau\hbar}(x,y)} ,
\end{aligned}$$

in conjunction with the estimates of Lemma 5.1 (which are uniform in τ), we get

$$\begin{aligned}
\|R^{\kappa,2}_\hbar(x,y)\|_{\mathcal{A}} &\leqslant \int_0^1 \mathrm{d}\tau \; (1-\tau) \Big[2 \Big\| \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Lambda^B_\varepsilon(x,y)|_{\varepsilon=\tau\hbar}\Big\|_{\mathcal{A}} + \tau \Big\| \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\Lambda^B_\varepsilon(x,y)|_{\varepsilon=\tau\hbar}\Big\|_{\mathcal{A}} \\
&\qquad + \Big(\|\Lambda^B_{\hbar\tau}(x,y)\|_{\mathcal{A}} + \tau \Big\| \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Lambda^B_\varepsilon(x,y)|_{\varepsilon=\tau\hbar}\Big\| \Big)^2 \Big] \|\mathrm{e}^{-\mathrm{i}\tau\hbar\Lambda^B_{\tau\hbar}(x,y)}\|_{\mathcal{A}}
\end{aligned}$$

$$= \int_0^1 \mathrm{d}\tau\ (1-\tau)\Big[2\Big\|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Lambda^B_\varepsilon(x,y)|_{\varepsilon=\tau\hbar}\Big\|_{\mathcal{A}} + \tau\Big\|\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\Lambda^B_\varepsilon(x,y)|_{\varepsilon=\tau\hbar}\Big\|_{\mathcal{A}}$$
$$+ \|\Lambda^B_{\hbar\tau}(x,y)\|^2_{\mathcal{A}} + 2\tau\|\Lambda^B_{\hbar\tau}(x,y)\|_{\mathcal{A}}\Big\|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Lambda^B_\varepsilon(x,y)|_{\varepsilon=\tau\hbar}\Big\|_{\mathcal{A}}$$
$$+ \tau^2\Big\|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Lambda^B_\varepsilon(x,y)|_{\varepsilon=\tau\hbar}\Big\|^2_{\mathcal{A}}\Big].$$

Hence, we can bound the $\hbar$-dependent C^*-norm of $R^{\diamond,2}_{\hbar,6}$ by

$$\begin{aligned}
&\|R^{\diamond,2}_{\hbar,6}\|^B_\hbar \\
&\leqslant \|\delta_l B^{jk}\|_{\mathcal{A}}(\|Q_j\Phi\|_{L^1}\|Q_kQ_l\Psi\|_{L^1} + \|Q_jQ_l\Phi\|_{L^1}\|Q_k\Psi\|_{L^1}) \\
&\quad + \frac{1}{6}\|\delta_l\delta_m B^{jk}\|_{\mathcal{A}}(\|Q_j\Phi\|_{L^1}\|Q_kQ_lQ_m\Psi\|_{L^1} + \|Q_jQ_m\Phi\|_{L^1}\|Q_kQ_l\Psi\|_{L^1} \\
&\quad + \|Q_jQ_lQ_m\Phi\|_{L^1}\|Q_k\Psi\|_{L^1}) + \frac{1}{2}\|B^{jk}\|_{\mathcal{A}}\|B^{j'k'}\|_{\mathcal{A}}\|Q_jQ_{j'}\Phi\|_{L^1}\|Q_kQ_{k'}\Psi\|_{L^1} \\
&\quad + \frac{1}{3}\|B^{jk}\|_{\mathcal{A}}\|\delta_{l'}B^{j'k'}\|_{\mathcal{A}}(\|Q_jQ_{j'}\Phi\|_{L^1}\|Q_kQ_{k'}Q_{l'}\Psi\|_{L^1} \\
&\quad + \|Q_jQ_{j'}Q_{l'}\Phi\|_{L^1}\|Q_kQ_{k'}\Psi\|_{L^1}) \\
&\quad + \frac{1}{12}\|\delta_l B^{jk}\|_{\mathcal{A}}\|\delta_{l'}B^{j'k'}\|_{\mathcal{A}}(\|Q_jQ_{j'}\Phi\|_{L^1}\|Q_kQ_{k'}Q_lQ_{l'}\Psi\|_{L^1} \\
&\quad + 2\|Q_jQ_{j'}Q_l\Phi\|_{L^1}\|Q_kQ_{k'}Q_{l'}\Psi\|_{L^1} + \|Q_jQ_{j'}Q_lQ_{l'}\Phi\|_{L^1}\|Q_kQ_{k'}\Psi\|_{L^1}).
\end{aligned}$$

Putting all these individual estimates together, the result yields a bound on $\|R^{\diamond,2}_\hbar(\Phi,\Psi)\|^B_\hbar$ which is uniform in $\hbar$ and the proof of the theorem is finished. ∎

COROLLARY 3.2. *Assume the components of B are in $BC^\infty(\Omega)$. Let $f, g \in \mathcal{S}(\mathscr{X}^*; C^\infty_0(\Omega))$ and $\hbar \in (0,1]$. Then the product $f \sharp^B_\hbar g$ can be expanded in powers of $\hbar$,*

$$f\sharp^B_\hbar g = fg - \hbar\frac{\mathrm{i}}{2}\{f,g\}_B + \hbar^2 R^{\sharp,2}_\hbar(f,g), \tag{3.2}$$

where fg is the pointwise product and $\{f,g\}_B$ is the magnetic Poisson bracket defined as in equation (1.4)*. All terms are in $\mathcal{S}(\mathscr{X}^*; C^\infty_0(\Omega))$ and the remainder satisfies*

$$\| R^{\sharp,2}_\hbar(f,g) \|_{\mathfrak{B}^B_\hbar} \leqslant C$$

uniformly in $\hbar$.

Proof. The proof follows from equations (2.3), (1.5), (1.6) and Theorem 3.1, keeping in mind that the partial Fourier transforms are isomorphisms

$$\mathcal{S}(\mathscr{X}^*; C^\infty(\Omega)) \overset{\rightarrow}{\longleftarrow} \mathcal{S}(\mathscr{X}; C^\infty(\Omega))$$

that extend to automorphisms between the C^*-algebras $\mathfrak{B}^B_\hbar$ and $\mathfrak{C}^B_\hbar$. ∎

4. STRICT DEFORMATION QUANTIZATION

To make this precise, we repeat an already standard concept. For more details and motivation, the reader could see [8], [22], [23] and references therein.

DEFINITION 4.1. Let $(\mathcal{S}, \circ, \{\cdot,\cdot\})$ be a real Poisson algebra which is densely contained on the selfadjoint part $\mathfrak{C}_{0,\mathbb{R}}$ of an abelian C^*-algebra $\mathfrak{C}_0$. *A strict deformation quantization of the Poisson algebra* $\mathcal{S}$ is a family of $\mathbb{R}$-linear injections $(\mathfrak{Q}_\hbar : \mathcal{S} \to \mathfrak{C}_{\hbar,\mathbb{R}})_{\hbar\in I}$, where $I \subset \mathbb{R}$ contains 0 as an accumulation point, $\mathfrak{C}_{\hbar,\mathbb{R}}$ is the selfadjoint part of the C^*-algebra $\mathfrak{C}_\hbar$, with products and norms denoted by $\diamond_\hbar$ and $\|\cdot\|_\hbar$, $\mathfrak{Q}_0$ is just the inclusion map and $\mathfrak{Q}_\hbar(\mathcal{S})$ is a subalgebra of $\mathfrak{C}_{\hbar,\mathbb{R}}$.

The following conditions are required for each $\Phi, \Psi \in \mathcal{S}$:

(i) Rieffel axiom: the mapping $I \ni \hbar \mapsto \|\mathfrak{Q}_\hbar(\Phi)\|_\hbar$ is continuous.

(ii) Von Neumann axiom:

$$\lim_{\hbar\to 0}\left\| \frac{1}{2}[\mathfrak{Q}_\hbar(\Phi) \diamond_\hbar \mathfrak{Q}_\hbar(\Psi) + \mathfrak{Q}_\hbar(\Psi) \diamond_\hbar \mathfrak{Q}_\hbar(\Phi)] - \mathfrak{Q}_\hbar(\Phi \circ \Psi)\right\|_\hbar = 0.$$

(iii) Dirac axiom:

$$\lim_{\hbar\to 0}\left\| \frac{i}{\hbar}[\mathfrak{Q}_\hbar(\Phi) \diamond_\hbar \mathfrak{Q}_\hbar(\Psi) - \mathfrak{Q}_\hbar(\Psi) \diamond_\hbar \mathcal{Q}_\hbar(\Phi)] - \mathfrak{Q}_\hbar(\{\Phi, \Psi\})\right\|_\hbar = 0.$$

Putting this into the present context, we have

THEOREM 4.2. *Assume that* $B^{jk} \in BC^\infty(\Omega)$ *and* $I = [0,1]$. *Then the family of injections*

$$(\mathcal{S}(\mathscr{X}, C_0^\infty(\Omega))_\mathbb{R} \hookrightarrow \mathfrak{C}^B_{\hbar,\mathbb{R}})_{\hbar\in I}$$

defines a strict deformation quantization.

Proof. By Proposition 1.9 and Proposition 2.3, $\mathcal{S}(\mathscr{X}, C_0^\infty(\Omega))_\mathbb{R}$ can be seen a Poisson algebra with respect to $\diamond_0$ and $\{\cdot,\cdot\}^B$ as well as a subalgebra of the real part of each of the twisted crossed product $\mathfrak{C}^B_\hbar$.

Von Neumann and Dirac axioms are direct consequences of Theorem 3.1.

The Rieffel axiom can be checked exactly as in [13], which builds on results from [18], [21]. The fact that the algebra $\mathcal{A}$ in [13] consisted of continuous functions defined on the group $\mathscr{X}$ itself does not play any role here. ∎

A partial Fourier transform transfers these results directly to $\mathcal{S}(\mathscr{X}^*, C_0^\infty(\Omega))$ and $\mathfrak{B}^B_\hbar$, objects which are natural in the context of Weyl calculus. In this way we extend the main result of [13] to magnetic twisted actions on *general* abelian C^*-algebras.

COROLLARY 4.3. *Assume that* $B^{jk} \in C^\infty(\Omega)$. *Let* $I = [0,1]$. *Then the family of injections*

$$(\mathcal{S}(\mathscr{X}^*, C_0^\infty(\Omega))_\mathbb{R} \hookrightarrow \mathfrak{B}^B_{\hbar,\mathbb{R}})_{\hbar\in I}$$

defines a strict deformation quantization, where the Poisson structure in $\mathcal{S}(\mathscr{X}^*, C_0^\infty(\Omega))_\mathbb{R}$ *is given by point-wise multiplication and the Poisson bracket* $\{\cdot,\cdot\}_B$.

Proof. The proof is straightforward from the Corollary 3.2 and the above theorem, after noticing that the partial Fourier transform is an isomorphism between the Poisson algebras $\mathcal{S}(\mathscr{X}^*; C_0^\infty(\Omega))$ and $\mathcal{S}(\mathscr{X}; C_0^\infty(\Omega))$, and it extends to an isomorphisms between the C^*-algebras $\mathfrak{B}_\hbar^B$ and $\mathfrak{C}_\hbar^B$. ∎

5. APPENDIX: ESTIMATES ON THE MAGNETIC FLUX

In the next lemma we gather some useful estimates on the scaled magnetic flux and its exponential, that are used in the proofs of Propositions 2.3 and 3.1.

LEMMA 5.1. *Assume the components of B are in $BC^\infty(\Omega)$ and $\hbar \in (0,1]$.*

(i) *For all $a, \alpha \in \mathbb{N}^n$ there exist constants $C^j > 0$, $C^{jk} > 0$, $j,k \in \{1,\ldots,n\}$, depending on B^{jk} and its δ-derivatives up to $(|a|+|\alpha|)$th order, such that*

$$\|\partial_x^a \delta^\alpha \Lambda_\hbar^B(x,y)\|_{\mathcal{A}} \leqslant \sum_{j=1}^n C_1^j |y_j| + \sum_{j,k=1}^n C_2^{jk} |y_j||x_k - y_k|.$$

(ii) *For all $a, \alpha \in \mathbb{N}^n$ there exists a polynomial $p_{a\alpha}$ in $2n$ variables, with coefficients $K_{bc} \geqslant 0$, such that*

$$\begin{aligned}\|\partial_x^a \delta^\alpha \mathrm{e}^{-\mathrm{i}\hbar \Lambda_\hbar^B(x,y)}\|_{\mathcal{A}} &\leqslant p_{a\alpha}(|y_1|,\ldots,|y_n|,|x_1-y_1|,\ldots,|x_n-y_n|) \\ &= \sum_{|b|+|c|\leqslant 2(|a|+|\alpha|)} K_{bc}|y^b||(x-y)^c|.\end{aligned}$$

(iii) *The following estimates which are uniform in $\hbar$ and τ hold :*

$$\|\Lambda_{\hbar\tau}^B(x,y)\|_{\mathcal{A}} \leqslant \sum_{jk} \|B^{jk}\|_{\mathcal{A}} |y_j||x_k - y_k|,$$

$$\left\| \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \Lambda_\varepsilon^B(x,y)|_{\varepsilon=\tau\hbar} \right\|_{\mathcal{A}} \leqslant \sum_{jkl} \|\delta_l B^{jk}\|_{\mathcal{A}} |y_j||x_k-y_k|(|x_l-y_l|+|y_l|),$$

$$\begin{aligned}&\left\| \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} \Lambda_\varepsilon^B(x,y)|_{\varepsilon=\tau\hbar} \right\|_{\mathcal{A}} \\ &\quad \leqslant \sum_{jklm} \|\delta_l \delta_m B^{jk}\|_{\mathcal{A}} |y_j||x_k-y_k|(|x_l-y_l||x_m-y_m| + |y_l||x_m-y_m| + |y_l||y_m|).\end{aligned}$$

Proof. (i) and (ii) follow directly from the explicit parametrization of the magnetic flux.

(iii) Throughout the proof we are going to use Einstein's summation convention, i.e. repeated indices in a product are summed over from 1 to $\dim(\mathscr{X})$. From the explicit parametrization (2.1)

$$\Lambda_\varepsilon^B(x,y) = y_j(x_k - y_k) \int_0^1 \mathrm{d}t \int_0^t \mathrm{d}s\, \theta_{\varepsilon(s-\frac{1}{2})x+\varepsilon(t-s)y}[B^{jk}],$$

we compute first and second derivative of $\Lambda^B_\varepsilon(x,y)$ with respect to ε, using dominated convergence to interchange differentiation with respect to the parameter ε and integration with respect to t and s,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Lambda^B_\varepsilon(x,y)=y_j(x_k-y_k)\int_0^1\mathrm{d}t\int_0^t\mathrm{d}s\,\Big(s(x_l-y_l)+ty_l-\frac{1}{2}x_l\Big)\theta_{\varepsilon(s-\frac{1}{2})x+\varepsilon(t-s)y}[\delta_l B^{jk}],$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\Lambda^B_\varepsilon(x,y)=y_j(x_k-y_k)\int_0^1\mathrm{d}t\int_0^t\mathrm{d}s\,\Big(s(x_l-y_l)+ty_l-\frac{1}{2}x_l\Big)\Big(s(x_m-y_m)+ty_m-\frac{1}{2}x_m\Big)$$
$$\cdot\,\theta_{\varepsilon(s-\frac{1}{2})x+\varepsilon(t-s)y}[\delta_l\delta_m B^{jk}].$$

The estimate on the flux itself follows from the fact that all the automorphisms θ_z are isometric in $\mathcal{A}$:

$$\begin{aligned}\|\Lambda^B_{\tau\hbar}(x,y)\|_{\mathcal{A}} &\leqslant |y_j||x_k-y_k|\int_0^1\mathrm{d}t\int_0^t\mathrm{d}s\,\|\theta_{\varepsilon(s-\frac{1}{2})x+\varepsilon(t-s)y}[B^{jk}]\|_{\mathcal{A}}\\ &\leqslant \|B^{jk}\|_{\mathcal{A}}|y_j||x_k-y_k|.\end{aligned}$$

Using the triangle inequality to estimate $|x_l|$ from above by $|x_l-y_l|+|y_l|$, we get

$$\begin{aligned}&\Big\|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Lambda^B_\varepsilon(x,y)|_{\varepsilon=\tau\hbar}\Big\|_{\mathcal{A}}\\ &\leqslant |y_j||x_k-y_k|\int_0^1\mathrm{d}t\int_0^t\mathrm{d}s\,\Big(s|x_l-y_l|+t|y_l|+\frac{1}{2}|x_l|\Big)\|\theta_{\tau\hbar(s-\frac{1}{2})x+\tau\hbar(t-s)y}[\delta_l B^{jk}]\|_{\mathcal{A}}\\ &=\|\delta_l B^{jk}\|_{\mathcal{A}}|y_j||x_k-y_k|\int_0^1\mathrm{d}t\int_0^t\mathrm{d}s\,\Big(s|x_l-y_l|+t|y_l|+\frac{1}{2}|x_l|\Big)\\ &\leqslant \|\delta_l B^{jk}\|_{\mathcal{A}}|y_j||x_k-y_k|(|x_l-y_l|+|y_l|).\end{aligned}$$

In a similar fashion, we obtain the estimate for the second-order derivative,

$$\begin{aligned}&\Big\|\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}\Lambda^B_\varepsilon(x,y)|_{\varepsilon=\tau\hbar}\Big\|_{\mathcal{A}}\\ &\leqslant |y_j||x_k-y_k|\int_0^1\mathrm{d}t\int_0^t\mathrm{d}s\,\Big|\Big(s(x_l-y_l)+ty_l-\frac{1}{2}x_l\Big)\Big(s(x_m-y_m)+ty_m-\frac{1}{2}x_m\Big)\Big|\\ &\qquad\cdot\|\theta_{\tau\hbar(s-\frac{1}{2})x+\tau\hbar(t-s)y}[\delta_l\delta_m B^{jk}]\|_{\mathcal{A}}\\ &\leqslant \|\delta_l\delta_m B^{jk}\|_{\mathcal{A}}|y_j||x_k-y_k|\int_0^1\mathrm{d}t\int_0^t\mathrm{d}s\,\Big(s^2|x_l-y_l||x_m-y_m|+2st|y_l||x_m-y_m|\end{aligned}$$

$$+ s|x_l - y_l||x_m| + t|y_l||x_m| + t^2|y_l||y_m| + \frac{1}{4}|x_l||x_m|\Big)$$
$$\leqslant \|\delta_l \delta_m B^{jk}\|_{\mathcal{A}} |y_j||x_k - y_k|(|x_l - y_l||x_m - y_m| + |y_l||x_m - y_m| + |y_l||y_m|).$$

This finishes the proof. ∎

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