J. OPERATOR THEORY
69:1(2013), 59–85
doi: 10.7900/jot.2010jul02.1960

KATZNELSON–TZAFRIRI TYPE THEOREM FOR INTEGRATED SEMIGROUPS

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Communicated by Florian-Horia Vasilescu

ABSTRACT. Y. Katznelson and L. Tzafriri proved that if T is a power-bounded operator and f is an analytic function, in the Wiener algebra, of spectral synthesis with respect to its peripheral spectrum then $\lim\limits_{n\to\infty} \|T^n f(T)\| = 0$. Here $f(T)$ is given by the usual functional calculus associated with T. The analogous version for bounded C_0-semigroups of operators was obtained by J. Esterle, E. Strouse and F. Zouakia and independently by Q.P. Vũ. We extend this result to α-times integrated semigroups. The proof is based on an analysis of homomorphisms from convolution Banach algebras of Sobolev type.

KEYWORDS: *Convolution, Sobolev algebra, fractional calculus, spectral synthesis, integrated semigroup, asymptotic behaviour.*

MSC (2010): Primary 43A22, 47A35; Secondary 43A45, 47D62.

INTRODUCTION

Let $\mathfrak{A}$ be a regular Banach algebra. An element a of $\mathfrak{A}$ is said to be *of spectral synthesis* with respect to a closed subset F of the character space of $\mathfrak{A}$ if there exists a sequence (b_n) in $\mathfrak{A}$ such that the Gelfand transform $\widehat{b}_n$ vanishes in a neighbourhood of F for each n, and

$$\lim_{n\to\infty} \|a - b_n\| = 0;$$

see [12], [15]. Let $A(\mathbb{T})$ be the convolution Wiener algebra formed by all continuous periodic functions $f(t) = \sum\limits_{n=-\infty}^{\infty} a_n \mathrm{e}^{\mathrm{i}nt}$, $t \in [-\pi, \pi]$, such that $\|f\|_{A(\mathbb{T})} := \sum\limits_{n=-\infty}^{\infty} |a_n| < \infty$, endowed with the norm $\|\cdot\|_{A(\mathbb{T})}$. This algebra is regular. Let $A_+(\mathbb{T})$ be the convolution closed subalgebra of $A(\mathbb{T})$ formed by the functions f with $a_n = 0$ for all $n < 0$. Let $\mathfrak{B}(X)$ denote the Banach algebra of bounded operators on a given Banach space X. Assume that $T \in \mathfrak{B}(X)$ is a power bounded operator, that is, it satisfies $\sup_{n\geqslant 0}\|T^n\| < \infty$. Clearly, the operator sum

$$f(T) := \sum_{n=0}^{\infty} a_n T^n$$

is well defined for every $f \in A_+(\mathbb{T})$. Let $\sigma(T)$ denote the spectrum of T. In [13], Katznelson and Tzafriri proved that *if $f \in A_+(\mathbb{T})$ is of spectral synthesis in $A(\mathbb{T})$ with respect to $\sigma(T)\cap\mathbb{T}$ then*

$$\lim_{n\to\infty} \|T^n f(T)\| = 0.$$

The continuous version of this theorem for bounded C_0-semigroups is as follows. Recall that the convolution algebra $L^1(\mathbb{R})$ is a regular Banach algebra.

THEOREM 0.1 ([8], Théorème 3.4 and [19], Theorem 3.2). *Let A be the infinitesimal generator of a bounded C_0-semigroup $(T(t))_{t\geqslant 0}$ on X. Let $\sigma(A)$ denote the spectrum of A. If $f \in L^1(\mathbb{R}^+)$ is of spectral synthesis in $L^1(\mathbb{R})$ with respect to $\mathrm{i}\sigma(A)\cap\mathbb{R}$ then*

$$\lim_{t\to\infty} \|T(t)\pi_0(f)\| = 0,$$

where $\pi_0\colon L^1(\mathbb{R}^+) \to \mathfrak{B}(X)$ is the bounded Banach algebra homomorphism defined by

$$\pi_0(f)x := \int_0^{\infty} f(t)\, T(t)x\,\mathrm{d}t, \quad (x \in X,\, f \in L^1(\mathbb{R}^+)).$$

We extend here Theorem 0.1 to the setting of α-times integrated semigroups. Recall that, for every $\alpha > 0$, an α-times integrated semigroup on X is a family $(T_\alpha(t))_{t\geqslant 0}$ in $\mathfrak{B}(X)$ satisfying:

(a) $T_\alpha(0) = 0$.

(b) $T_\alpha(t)T_\alpha(s) = \Big(\int_t^{t+s} - \int_0^{s}\Big)(t+s-r)^{\alpha-1}T_\alpha(r)(\mathrm{d}r/\Gamma(\alpha)),\ s,t \geqslant 0$.

(c) $t \mapsto T_\alpha(t)x \in X$ is continuous for every $x \in X$.

Such a notion arises in relation with the Abstract Cauchy Problem associated with the operator A:

$$\begin{cases} u'(t) = Au(t) & t > 0, \\ u(0) = x & x \in D(A). \end{cases}$$

Namely, in many important cases A turns out to be the *generator* of an integrated semigroup of exponential type; that is, there exists a family T_α as above and such that $\|T_\alpha(t)\| \leqslant C\mathrm{e}^{\omega t}$ ($t \geqslant 0$), for some constants $C, \omega > 0$, for which (see [3]):

$$(\lambda - A)^{-1}x = \lambda^{\alpha}\int_0^{\infty} \mathrm{e}^{-\lambda t}T_\alpha(t)x\,\mathrm{d}t \quad (\Re\lambda > \omega,\ x \in X).$$

There are certain convolution Banach algebras of Sobolev type, which are defined by fractional derivation and are well-suited for the study of integrated semigroups: Let $C_c^\infty[0,\infty)$ be the space of functions $g\chi_{[0,\infty)}$ where g is any C^∞ function of compact support in $\mathbb{R}$. For $\alpha > 0$ and a function $f \in C_c^\infty[0,\infty)$ we put $W_+^\alpha f$ to refer to the Weyl derivative of order α (whose definition will be recalled in Section 1 below). Then we define the Banach space $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ as the completion of $C_c^\infty[0,\infty)$ in the norm given by

$$\nu_\alpha(f) := \int_0^\infty |W_+^\alpha f(t)|\, t^\alpha \,\mathrm{d}t, \quad f \in C_c^\infty[0,\infty).$$

Endowed with the usual convolution product on $\mathbb{R}^+$,

$$f * g(x) := \int_0^x f(x-y)\, g(y)\,\mathrm{d}y \quad (\text{a.e. } x > 0; f, g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)),$$

and the above norm, the space $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ is a Banach algebra. Similarly, we introduce the convolution Banach algebra $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ as the completion of $C_c^\infty(\mathbb{R})$ with respect to the norm (which in the sequel will be denoted by ν_α)

$$\nu_\alpha(h) := \int_{-\infty}^\infty |W^\alpha h(t)|\, |t|^\alpha \,\mathrm{d}t, \quad h \in C_c^\infty(\mathbb{R}).$$

Here W^α is a suitable fractional derivation operator defined on all of $\mathbb{R}$ (see definition in Section 1). One has that $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ is a *regular* Banach algebra and that $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ is a closed subalgebra of $\mathcal{T}^{(\alpha)}(|t|^\alpha)$. All the above properties are proved in [9]. Some additional structure of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ as Banach algebras has been recently studied in [11], [10]. The link existing between the above algebras and integrated semigroups relies upon the following fact. Let us assume that $(T_\alpha(t))_{t\geqslant 0}$ is an α-times integrated semigroup of homogeneous growth t^α, that is, such that $\sup_{t>0} t^{-\alpha}\|T_\alpha(t)\|<\infty$. Then the mapping $\pi_\alpha : \mathcal{T}_+^{(\alpha)}(t^\alpha) \to \mathfrak{B}(X)$ defined by

$$\pi_\alpha(f)x := \int_0^\infty W_+^\alpha f(t)\, T_\alpha(t)x \,\mathrm{d}t, \quad x \in X, f \in \mathcal{T}_+^{(\alpha)}(t^\alpha), \tag{0.1}$$

is a bounded Banach algebra homomorphism [17]. (The fractional algebras $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ extend the corresponding Banach algebras introduced in [2] for integer α).

Recall that a C_0-semigroup $(T(t))_{t\geqslant 0}$ in $\mathfrak{B}(X)$ satisfies in particular

$$\lim_{t\to 0^+} T(t)x = x \quad (x \in X).$$

Analogously, we define here a C_α-*integrated semigroup* on X as an α-times integrated semigroup $(T_\alpha(t))_{t\geqslant 0}$ which, in addition to the former properties (a)–(c), satisfies

(d) $\lim\limits_{t\to 0^+} \Gamma(\alpha+1)\, t^{-\alpha} T_\alpha(t)x = x \quad (x \in X)$.

The main result of the present paper is the following extension of Theorem 0.1.

THEOREM 0.2. *For $\alpha > 0$, let $(T_\alpha(t))_{t\geqslant 0} \subseteq \mathfrak{B}(X)$ be a C_α-integrated semigroup with generator A such that*

$$\sup_{t>0} t^{-\alpha} \|T_\alpha(t)\| < \infty.$$

Suppose that $f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ is of spectral synthesis in $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ with respect to $\mathrm{i}\sigma(A) \cap \mathbb{R}$. Then

$$\lim_{t\to\infty} t^{-\alpha} \|T_\alpha(t)\, \pi_\alpha(f)\| = 0.$$

To prove the above theorem, we carry through a detailed analysis of the homomorphism π_α along the same lines as it is done for the homomorphism π_0 in the case of C_0-semigroups considered in [8]. The procedure followed in the present paper has to deal with a number of new and fairly non-trivial technicalities with respect to the proof given in [8]. The stuff needed is organized in sections as follows.

In Section 1, we present definitions, basic properties and some technical results about the algebras $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $\mathcal{T}^{(\alpha)}(|t|^\alpha)$, as well as the Riesz kernels associated with them. These kernels play a key role in the paper. In Section 2, we collect results about the relationships between duality and convolution in the above algebras of Sobolev type. Section 3 contains a couple of results involving extensions of Laplace transforms of functionals (of the algebras $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $\mathcal{T}^{(\alpha)}(|t|^\alpha)$) through open subsets of the imaginary axis. The analysis of bounded homomorphisms of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ is carried out in Section 4. First, we establish some formulae for Laplace transforms, of functionals associated to such bounded homomorphisms, which are natural extensions of those given in Section 2 of [8]. Then we give the main result of the section, Theorem 4.4, which relates homomorphisms of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ to functions of spectral synthesis, Finally, the results of previous sections are translated in terms of integrated semigroups to prove the main theorem of the paper, Theorem 0.2, in Section 5.

There is also an Appendix, written as Section 6, where some remarks about the stability of integrated semigroups are pointed out. It is worth mentioning that, in the conditions of Theorem 0.2, an integrated semigroup is ergodic provided $\sigma(A) \cap \mathrm{i}\mathbb{R} = \emptyset$ (Proposition 6.3).

1. CONVOLUTION SOBOLEV ALGEBRAS

For $\alpha > 0$ and $f \in C_c^\infty[0,\infty)$, the Weyl fractional integral of f of order α on $\mathbb{R}^+ = (0,\infty)$ is defined by

$$W_+^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_t^\infty (s-t)^{\alpha-1} f(s)\,\mathrm{d}s \quad (t \geqslant 0), \tag{1.1}$$

and then the Weyl fractional derivative of f on $\mathbb{R}^+$ is defined as

$$W_+^{\alpha} f(t) := (-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n} W_+^{-(n-\alpha)} f(t) \quad (t > 0),$$

where $n := [\alpha] + 1$ and $[\alpha]$ is the integer part of α.

The operator $W_+^{-\alpha} : C_c^\infty[0,\infty) \to C_c^\infty[0,\infty)$ defined by (1.1) is one-to-one (and continuous for the usual topology of $C_c^\infty[0,\infty)$) and then the fractional derivative W_+^α can be seen as the inverse operator of $W_+^{-\alpha}$. These operators satisfy the group law $W_+^{\alpha+\beta} = W_+^\alpha W_+^\beta$ for any $\alpha, \beta \in \mathbb{R}$ where W_+^0 is defined to be the identity operator.

We define $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ as the completion of $C_c^\infty[0,\infty)$ in the norm ν_α given by

$$\nu_\alpha(f) := \int_0^\infty |W_+^\alpha f(t)|\, t^\alpha\,\mathrm{d}t, \quad f \in C_c^\infty[0,\infty).$$

For $0 < \beta < \alpha$, we have $\mathcal{T}_+^{(\alpha)}(t^\alpha) \hookrightarrow \mathcal{T}_+^{(\beta)}(t^\beta) \hookrightarrow \mathcal{T}_+^{(0)}(t^0) \equiv L^1(\mathbb{R}^+)$, where $\hookrightarrow$ denotes continuous inclusion.

The operator $W_+^{-\alpha}$ extends as a surjective isometry $W_+^{-\alpha} \colon L^1(t^\alpha) \to \mathcal{T}_+^{(\alpha)}(t^\alpha)$ where $L^1(t^\alpha)$, endowed with the usual norm, is the Banach space of integrable functions on $\mathbb{R}^+$ with respect to the weight t^α; see Section 2 of [11]. Thus the mapping $W_+^\alpha \colon \mathcal{T}_+^{(\alpha)}(t^\alpha) \to L^1(t^\alpha)$ is defined as the inverse of $W_+^{-\alpha} \colon L^1(t^\alpha) \to \mathcal{T}_+^{(\alpha)}(t^\alpha)$, so that a function f in $L^1(\mathbb{R}^+)$ belongs to $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ if and only if there exists $F \in L^1(t^\alpha)$ such that

$$f(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (y-x)^{\alpha-1} F(y)\,\mathrm{d}y \quad \text{for a.e. } x > 0.$$

We have in fact that $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ is a commutative Banach algebra possessing bounded aproximate identities, with respect to the convolution product on $\mathbb{R}^+$. These facts can be found in Proposition 1.4 of [9] and Proposition 2.3 of [11] (the presentation of the algebra $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ done in [11] is slightly different from that one done in [9] but they are both equivalent). Also, the character space of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ coincides with $\{z \in \mathbb{C} : \Re z \geqslant 0\}$ and the Gelfand transform of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ is the Laplace transform [10].

We further consider a copy of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ on the negative real half-line. The Weyl fractional integral and derivative of a function f in $C_c^\infty(-\infty,0]$ are given by

$$W_-^{-\alpha} f(t) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad t \leqslant 0, \quad \text{and}$$

$$W_-^{\alpha} f(t) := \frac{\mathrm{d}^n}{\mathrm{d}t^n} W_-^{-(n-\alpha)} f(t), \quad t < 0,$$

respectively. Let $\mathcal{T}_-^{(\alpha)}((-t)^\alpha)$ be the convolution Banach algebra obtained as the completion of $C_c^\infty(-\infty,0]$ with respect to the norm

$$\nu_\alpha(f) := \int_{-\infty}^{0} |W_-^\alpha f(t)|\,(-t)^\alpha\,\mathrm{d}t, \quad f \in C_c^\infty(-\infty,0].$$

Of course, $\mathcal{T}_-^{(\alpha)}((-t)^\alpha)$ enjoys properties similar to those of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$.

Now we define $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ as the completion of $C_c^\infty(\mathbb{R})$ with respect to the norm

$$\nu_\alpha(f) := \int_{-\infty}^{\infty} |W^\alpha f(t)|\,|t|^\alpha\,\mathrm{d}t, \quad f \in C_c^\infty(\mathbb{R}),$$

where $W^\alpha f := W_-^\alpha f_- + W_+^\alpha f_+$, $f_- = f\chi_{(-\infty,0]}$ and $f_+ = f\chi_{[0,\infty)}$. In fact,

$$\mathcal{T}^{(\alpha)}(|t|^\alpha) = \mathcal{T}_-^{(\alpha)}((-t)^\alpha) \oplus \mathcal{T}_+^{(\alpha)}(t^\alpha),$$

and $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ is also a convolution Banach algebra, this time for the convolution on all of $\mathbb{R}$; see Theorem 1.8 of [9].

Other properties of $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ are that its character space is isomorphic to $\mathbb{R}$ and its Gelfand transform is equal to the Fourier transform. Moreover, for every $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\int_{-\infty}^{\infty} \varphi = 1$ the family $(\varepsilon^{-1}\varphi(\varepsilon^{-1}\,\cdot\,))_{0<\varepsilon\leqslant 1}$ is a bounded approximate identity in $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ (the proof of this fact is similar to the one given in Theorem 1.11(ii) of [9]). Also, $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ is *regular* on $\mathbb{R}$ since it contains the test functions.

Next we pay attention to a couple of distinguished families of functions related to the algebras $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $\mathcal{T}^{(\alpha)}(|t|^\alpha)$.

1.1. RIESZ KERNELS. For $\beta > 0$ and $t > 0$, the *Riesz kernel* $R_t^{\beta-1}$ on $[0,\infty)$ is defined by

$$R_t^{\beta-1}(s) := \begin{cases} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} & \text{if } 0 \leqslant s < t, \\ 0 & \text{if } s \geqslant t. \end{cases}$$

Then $R_t^{\beta-1} \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ whenever $\beta > \alpha$ and, though $R_t^{\alpha-1} \notin \mathcal{T}_+^{(\alpha)}(t^\alpha)$, $R_t^{\alpha-1}$ defines a multiplier of the algebra $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ by convolution. This means that

$R_t^{\alpha-1} * f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ for all $f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $t > 0$, and the inequality

$$\nu_\alpha(R_t^{\alpha-1} * f) \leqslant C_\alpha\, t^\alpha\, \nu_\alpha(f) \tag{1.2}$$

holds. This estimate follows from the formula

$$\begin{aligned} W_+^\alpha(R_t^{\alpha-1} * f)(s) = {} & \frac{\chi_{(t,\infty)}(s)}{\Gamma(\alpha)} \int_{s-t}^{s} (r+t-s)^{\alpha-1} W_+^\alpha f(r)\,\mathrm{d}r \\ & - \frac{\chi_{(0,t)}(s)}{\Gamma(\alpha)} \int_{s}^{\infty} (r+t-s)^{\alpha-1} W_+^\alpha f(r)\,\mathrm{d}r, \end{aligned} \tag{1.3}$$

valid for all $s > 0$, which is also the base to prove that the mapping

$$[0,\infty) \to \mathcal{T}_+^{(\alpha)}(t^\alpha), \quad t \mapsto R_t^{\alpha-1} * f$$

is norm continuous; see pp. 17, 34 of [9].

Moreover, the convolution product in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ can be expressed in terms of Riesz kernels ([11], Lemma 4.2):

$$f * g = \int_0^\infty W_+^\alpha f(t)\, R_t^{\alpha-1} * g\,\mathrm{d}t \quad f, g \in \mathcal{T}_+^{(\alpha)}(t^\alpha). \tag{1.4}$$

As a consequence of the above integral representation one gets that for a given closed subspace I in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$, *$I$ is an ideal of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ if and only if $R_t^{\alpha-1} * f \in I$ for all $f \in I$ and $t > 0$* ([11], Proposition 4.3).

Therefore closed ideals of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ are characterized as those closed subspaces of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ which are invariant under the action by convolution of kernels $R_t^{\alpha-1}$. Thus the family of Riesz kernels plays a similar role to the one that the translation semigroup $(\delta_t)_{t>0}$, formed by the Dirac masses on $\mathbb{R}^+$, has with respect to $L^1(\mathbb{R}^+)$. This fact seems to be of interest because the algebras $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ are not invariant under translations ([11], p. 5).

To finish this section we show that the family $\Gamma(\alpha+1)\, t^{-\alpha} R_t^{\alpha-1}$, $t > 0$, is a summability kernel for $\mathcal{T}_+^{(\alpha)}(t^\alpha)$.

PROPOSITION 1.1. *Let $\alpha > 0$. For all $f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$,*

$$\lim_{t \to 0^+} \Gamma(\alpha+1)\, t^{-\alpha} R_t^{\alpha-1} * f = f\,.$$

Proof. Let $f \in C_c^{(\infty)}[0,\infty)$ and $t > 0$. By using the formula (1.3) and the equality $\alpha t^{-\alpha} \int_{s-t}^{s} (r+t-s)^{\alpha-1} \mathrm{d}r = 1$, we obtain that

$$\begin{aligned}
&\nu_\alpha(\Gamma(\alpha+1)\, t^{-\alpha}(R_t^{\alpha-1} * f) - f) \\
&\qquad \leqslant \alpha t^{-\alpha} \int_0^t \int_s^\infty (r+t-s)^{\alpha-1} |W_+^\alpha f(r)| s^\alpha \mathrm{d}r\mathrm{d}s + \int_0^t |W_+^\alpha f(s)| s^\alpha \mathrm{d}s \\
&\qquad\quad + \alpha t^{-\alpha} \int_t^\infty \int_{s-t}^{s} (r+t-s)^{\alpha-1} |W_+^\alpha f(r) - W_+^\alpha f(s)| s^\alpha \mathrm{d}r\mathrm{d}s.
\end{aligned}$$

By Fubini's theorem and the dominated convergence theorem, it readily follows that the first integral on the right-hand member of the above inequality converges to zero as $t \to 0^+$. The second integral also tends to 0 as $t \to 0^+$ by a straightforward argument. For the third integral we apply again Fubini's theorem to get

$$\begin{aligned}
&\alpha t^{-\alpha} \int_t^\infty \int_{s-t}^{s} (r+t-s)^{\alpha-1} |W_+^\alpha f(r) - W_+^\alpha f(s)| s^\alpha \, \mathrm{d}r\mathrm{d}s \\
&\qquad = \alpha t^{-\alpha} \int_0^t \int_t^{r+t} (r+t-s)^{\alpha-1} |W_+^\alpha f(r) - W_+^\alpha f(s)| s^\alpha \, \mathrm{d}s\mathrm{d}r \\
&\qquad\quad + \alpha t^{-\alpha} \int_t^\infty \int_r^{r+t} (r+t-s)^{\alpha-1} |W_+^\alpha f(r) - W_+^\alpha f(s)| s^\alpha \, \mathrm{d}s\mathrm{d}r.
\end{aligned}$$

Now, taking into account that $W_+^\alpha f \in C_c^{(\infty)}[0,\infty)$ and the fact that $\int_t^{r+t} (r+t-s)^{\alpha-1} s^\alpha \, \mathrm{d}s \leqslant C_\alpha \, t^{2\alpha}$ for some $C_\alpha > 0$, we get that

$$t^{-\alpha} \int_0^t \int_t^{r+t} (r+t-s)^{\alpha-1} |W_+^\alpha f(r) - W_+^\alpha f(s)| s^\alpha \mathrm{d}s\mathrm{d}r \leqslant C_\alpha \|W_+^\alpha f\|_\infty t^{\alpha+1}$$

and therefore the integral tends to zero as $t \to 0^+$.

As regards the double integral $\int_t^\infty \int_r^{r+t}$, setting $C := \sup[\mathrm{supp}(W_+^\alpha f)]$, we have

$$\begin{aligned}
&\alpha t^{-\alpha} \int_t^\infty \int_r^{r+t} (r+t-s)^{\alpha-1} |W_+^\alpha f(r) - W_+^\alpha f(s)| s^\alpha \, \mathrm{d}s\mathrm{d}r \\
&\qquad \leqslant \alpha t^{-\alpha} \int_t^C \sup_{s \in (r, r+t)} |W_+^\alpha f(r) - W_+^\alpha f(s)| \int_r^{r+t} (r+t-s)^{\alpha-1} \mathrm{d}s \, (r+t)^\alpha \mathrm{d}r
\end{aligned}$$

$$= \int_t^C (r+t)^\alpha \sup_{s\in(r,r+t)} |W_+^\alpha f(r) - W_+^\alpha f(s)| \, \mathrm{d}r,$$

for $0 < t < C$. Since $W_+^\alpha f$ is uniformly continuous, given any $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < t < \delta$ then

$$\int_t^C (r+t)^\alpha \sup_{s\in(r,r+t)} |W_+^\alpha f(r) - W_+^\alpha f(s)| \mathrm{d}r \leqslant \varepsilon \int_t^C (r+t)^\alpha \mathrm{d}r \leqslant \frac{(2C)^{\alpha+1}}{\alpha+1}\, \varepsilon.$$

Thus putting all the above bounds together, we have shown that

$$\lim_{t\to 0^+} \Gamma(\alpha+1)\, t^{-\alpha} R_t^{\alpha-1} * f = f$$

in the norm ν_α, for every $f \in C_c^\infty[0,\infty)$. To conclude the proof, we only have to apply the density of $C_c^\infty[0,\infty)$ in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ and the estimate (1.2). ∎

As noticed before, we must deal with convolution on the whole real line. Thus a version of Riesz kernels on all of $\mathbb{R}$ is needed, which extends the family of Riesz kernels $R_t^{\alpha-1}$ defined previously for $t > 0$. We maintain the same notation for such an extension.

For $\alpha > 0$ and $t \in \mathbb{R}$, put

$$R_t^{\alpha-1}(s) := \begin{cases} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} & \text{if } 0 \leqslant s < t, \\ \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} & \text{if } t < s \leqslant 0, \\ 0 & \text{otherwise.} \end{cases}$$

The above family satisfies similar properties to those summarized prior to Proposition 1.1; see [9] and [11]. In particular we have:

(i) $R_t^{\alpha-1} * f \in \mathcal{T}^{(\alpha)}(|t|^\alpha)$ for every $t \in \mathbb{R}$ and $f \in \mathcal{T}^{(\alpha)}(|t|^\alpha)$, with

$$\nu_\alpha(R_t^{\alpha-1} * f) \leqslant C_\alpha\, |t|^\alpha\, \nu_\alpha(f).$$

(ii) For any $f, g \in \mathcal{T}^{(\alpha)}(|t|^\alpha)$,

$$f * g = \int_{-\infty}^{\infty} W^\alpha f(t)\, (R_t^{\alpha-1} * g)\, \mathrm{d}t,$$

where the integral converges in the norm topology of $\mathcal{T}^{(\alpha)}(|t|^\alpha)$.

We shall also need to consider duality in $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ and $\mathcal{T}_+^{(\alpha)}(t^\alpha)$, as well as its relationship with convolution. This matter is developed in the next section.

2. DUALITY AND CONVOLUTIONS IN THE SOBOLEV ALGEBRAS

On account of the isomorphism between $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $L^1(t^\alpha)$, the dual Banach space $\mathcal{T}_+^{(\alpha)}(t^\alpha)^*$ of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ is automatically identified with the set of almost everywhere defined functions $\phi : \mathbb{R}^+ \to \mathbb{C}$ which satisfy $t^{-\alpha}\phi \in L^\infty(\mathbb{R}^+)$. The duality is implemented by the formula

$$L_\phi(f) \equiv \langle L_\phi\, , f\rangle := \int_0^\infty W_+^\alpha f(t)\phi(t)\,\mathrm{d}t, \quad f \in \mathcal{T}_+^{(\alpha)}(t^\alpha),$$

for every $\phi \equiv L_\phi \in \mathcal{T}_+^{(\alpha)}(t^\alpha)^*$. Analogous facts hold about duality in $\mathcal{T}_-^{(\alpha)}((-t)^\alpha)$ and functions ϕ, now supported on $(-\infty, 0)$. Then we shall represent the dual $\mathcal{T}^{(\alpha)}(|t|^\alpha)^*$ as the (topological) direct sum $\mathcal{T}^{(\alpha)}(|t|^\alpha)^* = \mathcal{T}_-^{(\alpha)}((-t)^\alpha)^* \oplus \mathcal{T}_+^{(\alpha)}(t^\alpha)^*$, so that the continuous linear functionals of $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ correspond to functions φ, defined almost everywhere on $\mathbb{R}$, such that $|t|^{-\alpha}\varphi \in L^\infty(\mathbb{R})$. An important observation here is that every functional L_φ of $\mathcal{T}^{(\alpha)}(|t|^\alpha)^*$ can be considered as a tempered distribution on $\mathbb{R}$ since the Schwarz class $\mathcal{S}(\mathbb{R})$ is continuously and densely contained in $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ ([11], Proposition 2.3).

The following result translates to our setting the well-known property that $\phi * g \in C_0[0,\infty)$ provided $\phi \in C_0[0,\infty)$ and $g \in L^1(\mathbb{R}^+)$. It will be used in the proof of Theorem 4.4 below.

PROPOSITION 2.1. *If $L_\phi \in \mathcal{T}_+^{(\alpha)}(t^\alpha)^*$ is such that $t^{-\alpha}\phi \in C_0[0,\infty)$, then, for every $g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$,*

$$\lim_{t\to\infty} t^{-\alpha}\,\langle L_\phi\, , R_t^{\alpha-1} * g\rangle = 0.$$

Proof. Let ϕ be as above and let $g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$. Using the formula (1.3) and then applying Fubini's theorem, we obtain that

$$\begin{aligned}
t^{-\alpha}\langle L_\phi, R_t^{\alpha-1} * g\rangle = {} & \frac{t^{-\alpha}}{\Gamma(\alpha)}\int_0^t\int_t^{r+t}(r+t-s)^{\alpha-1}W_+^\alpha g(r)\phi(s)\mathrm{d}s\mathrm{d}r \\
& + \frac{t^{-\alpha}}{\Gamma(\alpha)}\int_t^\infty\int_r^{r+t}(r+t-s)^{\alpha-1}W_+^\alpha g(r)\phi(s)\mathrm{d}s\mathrm{d}r \\
& - \frac{t^{-\alpha}}{\Gamma(\alpha)}\int_0^t\int_0^{r}(r+t-s)^{\alpha-1}W_+^\alpha g(r)\phi(s)\mathrm{d}s\mathrm{d}r \\
& - \frac{t^{-\alpha}}{\Gamma(\alpha)}\int_t^\infty\int_0^{t}(r+t-s)^{\alpha-1}W_+^\alpha g(r)\phi(s)\mathrm{d}s\mathrm{d}r.
\end{aligned}$$

First, let us see that the first of the above four integrals tends to 0 as $t \to \infty$. Since $s^{-\alpha}\phi \in L^\infty(\mathbb{R}^+)$, we have

$$t^{-\alpha} \int_t^{r+t} (r+t-s)^{\alpha-1}|\phi(s)|\,\mathrm{d}s \leqslant \alpha^{-1}2^\alpha \|s^{-\alpha}\phi\|_\infty r^\alpha \quad (0<r<t).$$

On the other hand, the change of variables $s=(r+t)x$ gives us, for $0<r<t$ and some $C(r)>0$,

$$\begin{aligned} t^{-\alpha} \int_t^{r+t} (r+t-s)^{\alpha-1}|\phi(s)|\,\mathrm{d}s &\leqslant C(r) \sup_{(t/r+t)\leqslant x\leqslant 1} ((r+t)x)^{-\alpha}|\phi((r+t)x)| \\ &\leqslant C(r) \sup_{(1/2)\leqslant x\leqslant 1} ((r+t)x)^{-\alpha}|\phi((r+t)x)|, \end{aligned}$$

Moreover,

$$\lim_{t\to\infty} \sup_{(1/2)\leqslant x\leqslant 1} ((r+t)x)^{-\alpha}|\phi((r+t)x)| = 0$$

since $s^{-\alpha}\phi \in C_0(\mathbb{R}^+)$. The claim then follows from the dominated convergence theorem (taking sequential limits $t_n \to \infty$), since $|W_+^\alpha g(r)|r^\alpha$ is integrable on $\mathbb{R}^+$.

For the second integral, it is straightforward to check that, as in the preceding case, there exists some $C>0$ for which

$$t^{-\alpha} \int_r^{r+t} (r+t-s)^{\alpha-1}|\phi(s)|\,\mathrm{d}s \leqslant C\,r^\alpha \quad (r>t>0),$$

and from here it is clear that the second integral converges to 0 as $t\to\infty$.

Now note that the third and fourth integrals are bounded respectively by

$$\frac{C}{\Gamma(\alpha+1)}\int_0^\infty \chi_{(0,t)}(r)\Big[\Big(1+\frac{r}{t}\Big)^\alpha - 1\Big]|W_+^\alpha g(r)|r^\alpha \mathrm{d}r \ \text{ and } \ \frac{C}{\Gamma(\alpha+1)}\int_t^\infty [(r+t)^\alpha - r^\alpha]|W_+^\alpha g(r)|\mathrm{d}r,$$

which in turn go clearly to 0 as $t\to\infty$. The proof is over. ∎

In the remainder we introduce several convolutions relating functions and functionals of the algebras $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $\mathcal{T}^{(\alpha)}(|t|^\alpha)$.

The dual space $\mathcal{T}_+^{(\alpha)}(t^\alpha)^*$ becomes a (dual) Banach $\mathcal{T}_+^{(\alpha)}(t^\alpha)$-module through the action of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ on itself as a Banach algebra. We denote the dual Banach module product by $\bullet$, so that

$$(L_\varphi, f) \mapsto L_\varphi \bullet f, \quad \mathcal{T}_+^{(\alpha)}(t^\alpha)^* \times \mathcal{T}_+^{(\alpha)}(t^\alpha) \to \mathcal{T}_+^{(\alpha)}(t^\alpha)^*$$

is defined by $(L_\varphi \bullet f)(g) := L_\varphi(f * g)$ for all $g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$.

PROPOSITION 2.2. *For any $L_\varphi \in \mathcal{T}_+^{(\alpha)}(t^\alpha)^*$ and $f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$, we have*

$$L_\varphi \bullet f = L_\psi \quad \text{where } \psi(t) := L_\varphi(R_t^{\alpha-1} * f),\ t>0.$$

Proof. Let ψ be as in the statement. By properties of the Riesz kernels pointed out formerly, we obtain that $t^{-\alpha}\psi \in L^\infty(\mathbb{R}^+)$ and therefore $L_\psi \in \mathcal{T}_+^{(\alpha)}(t^\alpha)^*$. Moreover, for every $g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$,

$$L_\psi(g) = L_\varphi\Big(\int_0^\infty W_+^\alpha g(t)\, R_t^{\alpha-1} * f\,\mathrm{d}t\Big) = L_\varphi(f * g) = L_\varphi \bullet f(g),$$

where (1.4) has been used in the next-to-last equality. ∎

For a complex function F defined a.e. on $\mathbb{R}$ we put $\widetilde{F}(x) = F(-x)$. Now from the distribution theory we borrow the convolution product of tempered distributions and functions and define, for $L_\varphi \in \mathcal{T}^{(\alpha)}(|t|^\alpha)^*$ and $f \in \mathcal{T}^{(\alpha)}(|t|^\alpha)$, the functional $L_\varphi * f$ by

$$(L_\varphi * f)(g) := L_\varphi(\widetilde{f} * g), \quad g \in \mathcal{T}^{(\alpha)}(|t|^\alpha).$$

Clearly, $L_\varphi * f \in \mathcal{T}^{(\alpha)}(|t|^\alpha)^*$ and the mapping $f \mapsto L_\varphi * f$, $\mathcal{T}^{(\alpha)}(|t|^\alpha) \to \mathcal{T}^{(\alpha)}(|t|^\alpha)^*$ is linear and bounded. Also, it is readily seen that $L_\varphi * f = (L_{\widetilde{\varphi}} * \widetilde{f})\widetilde{\ }$, that is,

$$(L_\varphi * f)(g) = L_{\widetilde{\varphi}}(f * \widetilde{g}), \quad g \in \mathcal{T}^{(\alpha)}(|t|^\alpha).$$

From here and (1.4) we get that

$$L_\varphi * f(g) = \int_{-\infty}^\infty W^\alpha g(s)\, L_{\widetilde{\varphi}}(R_{-s}^{\alpha-1} * f)\,\mathrm{d}s, \tag{2.1}$$

for every $f \in \mathcal{T}^{(\alpha)}(|t|^\alpha)$, $L_\varphi \in \mathcal{T}^{(\alpha)}(|t|^\alpha)^*$ and $g \in \mathcal{T}^{(\alpha)}(|t|^\alpha)$.

DEFINITION 2.3. For $f \in \mathcal{T}^{(\alpha)}(|t|^\alpha)$ and $L_\varphi \in \mathcal{T}^{(\alpha)}(|t|^\alpha)^*$, we set

$$L_\varphi \circ f := L_{\psi_-} \quad \text{and} \quad L_\varphi \diamond f := L_{\psi_+},$$

where $\psi(s) := L_{\widetilde{\varphi}}(R_{-s}^{\alpha-1} * f)$, $s \in \mathbb{R}$, $\psi_- := \psi\chi_{(-\infty,0]}$, $\psi_+ := \psi\chi_{[0,\infty)}$.

By the growth rate of ψ it follows that $L_\varphi \circ f$ and $L_\varphi \diamond f$ are elements of $\mathcal{T}_-^{(\alpha)}((-t)^\alpha)^*$ and $\mathcal{T}_+^{(\alpha)}(t^\alpha)^*$, respectively. Moreover,

$$L_\varphi * f = L_\varphi \circ f + L_\varphi \diamond f \quad \text{in } \mathcal{T}^{(\alpha)}(|t|^\alpha)^*,$$

so that $L_\varphi \circ f = (L_\varphi * f)\,|_{\mathcal{T}_-^{(\alpha)}((-t)^\alpha)}$ and $L_\varphi \diamond f = (L_\varphi * f)\,|_{\mathcal{T}_+^{(\alpha)}(t^\alpha)}$.

Thus we have that if $f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $L_{\widetilde{\varphi}} \in \mathcal{T}_+^{(\alpha)}(t^\alpha)^*$ then $L_\varphi \circ f$ and $L_{\widetilde{\varphi}} \bullet f$ are elements of $\mathcal{T}_-^{(\alpha)}((-t)^\alpha)^*$ and $\mathcal{T}_+^{(\alpha)}(t^\alpha)^*$, respectively, which are related by

$$L_\varphi \circ f = L_\psi, \quad L_{\widetilde{\varphi}} \bullet f = L_{\widetilde{\psi}}$$

where $\psi(s) := L_{\widetilde{\varphi}}(R_{-s}^{\alpha-1} * f)$ for $s < 0$.

PROPOSITION 2.4. *For any sequences* $(f_n)_{n=1}^{\infty} \subseteq \mathcal{T}_+^{(\alpha)}(t^\alpha)$ *and* $(L_{\widetilde{\varphi}_n})_{n=1}^{\infty} \subseteq \mathcal{T}_+^{(\alpha)}(t^\alpha)^*$, *we have that*

$$\lim_{n\to\infty} L_{\widetilde{\varphi}_n} \bullet f_n = 0 \text{ if and only if } \lim_{n\to\infty} L_{\varphi_n} \circ f_n = 0.$$

Proof. For $n \in \mathbb{N}$, $L_{\varphi_n} \circ f_n = L_{\psi_n}$ and $L_{\widetilde{\varphi}_n} \bullet f_n = L_{\widetilde{\psi}_n}$ where $\psi_n(s) := L_{\widetilde{\varphi}}(R_{-s}^{\alpha-1} * f_n)$ for $s < 0$, as in the remark prior to this proposition. Then, for any $g \in \mathcal{T}_-^{(\alpha)}((-t)^\alpha)$,

$$L_{\psi_n}(g) = \int_{-\infty}^{0} W_-^\alpha g(t)\psi_n(t)\,\mathrm{d}t = \int_0^\infty W_-^\alpha g(-s)\widetilde{\psi}_n(s)\,\mathrm{d}s = \int_0^\infty W_+^\alpha \widetilde{g}(s)\widetilde{\psi}_n(s)\,\mathrm{d}s = L_{\widetilde{\psi}_n}(\widetilde{g}).$$

We conclude the argument by noticing that $g \in \mathcal{T}_-^{(\alpha)}((-t)^\alpha)$ if and only if $\widetilde{g} \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$. ∎

3. LAPLACE TRANSFORM OF FUNCTIONALS

Recall that every element in $\mathcal{T}^{(\alpha)}(|t|^\alpha)^*$ is a tempered distribution. In particular, if we set $e_z := \mathrm{e}^{-z\,\cdot}$, then the Laplace transform of a functional $L_\varphi \in \mathcal{T}_+^{(\alpha)}(t^\alpha)^*$ is given by

$$\mathcal{L}(L_\varphi)(z) := L_\varphi(e_z) = \int_0^\infty W_+^\alpha(e_z)(t)\,\varphi(t)\,\mathrm{d}t = z^\alpha \mathcal{L}(\varphi)(z)\,, \quad z \in \mathbb{C}^+,$$

since $e_z \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ whenever $z \in \mathbb{C}^+$ and $W_+^\alpha(e_z) = z^\alpha e_z$ on $\mathbb{R}^+$. As a matter of fact, $\mathcal{L}(L_\varphi)$ is analytic on $\mathbb{C}^+$. On the other hand, the Laplace transform of an element $L_\varphi \in \mathcal{T}_-^{(\alpha)}((-t)^\alpha)^*$ is well defined and analytic on $\mathbb{C}^- := \{z \in \mathbb{C} : \Re z < 0\}$. Now, we have that $W_-^\alpha(e_z) = (-z)^\alpha e_z$ for $z \in \mathbb{C}^-$, so that

$$\mathcal{L}(L_\varphi)(z) = L_\varphi(e_z) = (-z)^\alpha \mathcal{L}(\varphi)(z)\,, \quad z \in \mathbb{C}^-.$$

In the sequel, we denote by $\mathcal{F}$ the Fourier transform.

PROPOSITION 3.1. *Let* $L_\rho \in \mathcal{T}_+^{(\alpha)}(t^\alpha)^*$ *such that* $\mathcal{L}(L_\rho)$ *extends continuously to* $\mathbb{C}^+ \cup U$, *where* U *is some open subset of* $\mathrm{i}\,\mathbb{R}$. *For any* $\Psi \in \mathcal{T}^{(\alpha)}(|t|^\alpha)$ *such that* $\mathcal{F}(\Psi)$ *has compact support contained in* $-\mathrm{i}\,U$, *we have that* $L_\rho * \Psi \in C^\infty(\mathbb{R}) \cap L^\infty$ *with, in fact,*

$$L_\rho * \Psi(t) = \frac{1}{2\pi} \int_{\mathrm{supp}\mathcal{F}(\Psi)} \mathcal{F}(\Psi)(y)\,\mathcal{L}(L_\rho)(\mathrm{i}y)\,\mathrm{e}^{\mathrm{i}yt}\,\mathrm{d}y \quad (t > 0).$$

Proof. For every $x > 0$, let $e_x L_\rho$ denote the functional given by

$$\langle\, e_x L_\rho\,,\, g\,\rangle := \langle\, L_\rho\,,\, e_x g\,\rangle\,, \quad g \in \mathcal{T}^{(\alpha)}(|t|^\alpha).$$

Let Ψ satisfy the assumptions of the proposition. Then $\widetilde{\Psi} * h \in \mathcal{T}^{(\alpha)}(|t|^\alpha)$ for any $h \in \mathcal{S}(\mathbb{R})$, and $\mathcal{F}(\widetilde{\Psi} * h) = \mathcal{F}(\widetilde{\Psi})\mathcal{F}(h) \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ because $\mathcal{F}(\widetilde{\Psi})$ is of compact support. We can then apply the Fourier inversion formula to get that

$$(\widetilde{\Psi} * h)(t) = \frac{1}{2\pi} \int_{\mathrm{supp}\mathcal{F}(\widetilde{\Psi})} \mathcal{F}(\widetilde{\Psi})(y)\mathcal{F}(h)(y)\mathrm{e}^{\mathrm{i}yt}\mathrm{d}y \quad (t > 0),$$

so that

$$\mathrm{e}^{-xt}\,(\widetilde{\Psi} * h)(t) = \frac{1}{2\pi} \int_{\mathrm{supp}\mathcal{F}(\Psi)} \mathcal{F}(\Psi)(y)\,\mathcal{F}(h)(-y)\,\mathrm{e}^{-(x+\mathrm{i}y)t}\,\mathrm{d}y\,.$$

The last integral converges in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ since $e_{x+\mathrm{i}y} \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$. Therefore,

$$\begin{aligned}\langle\, e_x L_\rho, \widetilde{\Psi} * h\,\rangle &= \frac{1}{2\pi} \int_{\mathrm{supp}\mathcal{F}(\Psi)} \mathcal{F}(\Psi)(y)\mathcal{F}(h)(-y)L_\rho(e_{x+\mathrm{i}y})\,\mathrm{d}y \\ &= \frac{1}{2\pi} \int_{\mathrm{supp}\mathcal{F}(\Psi)} \mathcal{F}(\Psi)(y)\mathcal{F}(h)(-y)\mathcal{L}(L_\rho)(x+\mathrm{i}y)\,\mathrm{d}y.\end{aligned}$$

Since $\mathcal{L}(L_\rho)$ is continuous on the subset $\mathrm{i}\,\mathrm{supp}\mathcal{F}(\Psi)$ of $\mathrm{i}\mathbb{R}$, the dominated convergence theorem applies and we get

$$\lim_{x\to 0}\langle e_x L_\rho, \widetilde{\Psi} * h\rangle = \frac{1}{2\pi} \int_{\mathrm{supp}\mathcal{F}(\Psi)} \mathcal{F}(\Psi)(y)\mathcal{F}(h)(-y)\mathcal{L}(L_\rho)(\mathrm{i}y)\,\mathrm{d}y.$$

On the other hand, $\lim_{x\to 0} e_x g = g$ in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ for every $g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ (see Lemma 3.6(ii) of [11]). Therefore,

$$\lim_{x\to 0}\langle e_x L_\rho, \widetilde{\Psi} * h\rangle = \langle L_\rho, \widetilde{\Psi} * h\rangle.$$

Applying Fubini's theorem, we finally obtain that

$$\langle L_\rho * \Psi, h\rangle = \int_{\mathbb{R}} \Big(\frac{1}{2\pi} \int_{\mathrm{supp}\mathcal{F}(\Psi)} \mathcal{F}(\Psi)(y)\mathcal{L}(L_\rho)(\mathrm{i}y)\mathrm{e}^{\mathrm{i}yt}\mathrm{d}y\Big)\, h(t)\,\mathrm{d}t$$

for all $h \in \mathcal{S}(\mathbb{R})$. Then the statement follows automatically. ∎

REMARK 3.2. An analogous statement to the one of the above proposition holds for elements $L_\rho \in \mathcal{T}_-^{(\alpha)}((-t)^\alpha)^*$ whose Laplace transform extends continuously to $\mathbb{C}^- \cup U$.

PROPOSITION 3.3. *Let U be an open subset of* $\mathrm{i}\mathbb{R}$. *Let $L_{\rho_1} \in \mathcal{T}_-^{(\alpha)}((-t)^\alpha)^*$ and $L_{\rho_2} \in \mathcal{T}_+^{(\alpha)}(t^\alpha)^*$ be such that $\mathcal{L}(L_{\rho_1})$ and $\mathcal{L}(L_{\rho_2})$ have continuous extensions to $\mathbb{C}^- \cup U$ and $\mathbb{C}^+ \cup U$, respectively, and verify $\mathcal{L}(L_{\rho_1})\,|_U = -\mathcal{L}(L_{\rho_2})\,|_U$. Then,*

$$(L_{\rho_1} + L_{\rho_2}) * \Psi = 0\,,$$

for every $\Psi \in \mathcal{T}^{(\alpha)}(|t|^\alpha)$ such that $\mathrm{supp}\mathcal{F}(\Psi) \subseteq -\mathrm{i}\,U$.

Proof. Take $\varphi \in \mathcal{S}(\mathbb{R})$ such that its Fourier transform has compact support in $\mathbb{R}$ and $\int_{-\infty}^{\infty} \varphi = 1$. Then the family $\varphi_n := n\varphi(n\,\cdot\,)$, $n \in \mathbb{N}$, is a bounded approximate identity in $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ (see Section 2, prior to the definition of Riesz kernels). Hence $\Psi = \lim_{n\to\infty} \Psi * \varphi_n$ in $\mathcal{T}^{(\alpha)}(|t|^\alpha)$ in particular.

Now, $\mathcal{F}(\Psi * \varphi_n)$ is a compact subset of $\mathcal{F}(\Psi)$, so it is a compact subset of $-\mathrm{i}U$. By Proposition 3.1, $(L_{\rho_1} + L_{\rho_2}) * (\Psi * \varphi_n) = 0$ for all n. Then, as $n \to \infty$, the weak* continuity in $(\mathcal{T}^{(\alpha)}(|t|^\alpha))^*$ implies that $(L_{\rho_1} + L_{\rho_2}) * \Psi = 0$. ∎

4. BOUNDED HOMOMORPHISMS FROM $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ AND FUNCTIONS OF SPECTRAL SYNTHESIS

For any element $f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$, let $\sigma(f)$ denote its spectrum in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$. Let u be the function e_1 in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$. The spectrum of u is $\sigma(u) = \{(1+z)^{-1} : z \in \overline{\mathbb{C}^+}\} \cup \{0\}$, and its n-th convolution product is $u^{*n}(t) = (1/(n-1)!)t^{n-1}\mathrm{e}^{-t}$ $(t > 0)$. It is shown in Proposition 1.1 of [10] that $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ is polynomially generated by u; that is, $\mathcal{T}_+^{(\alpha)}(t^\alpha) = \overline{\operatorname{span}}\{u^{*n} : n \in \mathbb{N}\}$.

Let $\mathfrak{B}$ be a Banach algebra with a unit e and let π be a bounded algebra homomorphism $\pi\colon \mathcal{T}_+^{(\alpha)}(t^\alpha) \oplus \mathbb{C}\delta_0 \to \mathfrak{B}$ such that $\pi(\delta_0) = e$, where δ_0 is the Dirac mass at the origin. Let $\mathfrak{A}$ be the closure of the image $\pi(\mathcal{T}_+^{(\alpha)}(t^\alpha))$ in $\mathfrak{B}$. We denote by π^* the adjoint mapping of $\pi\colon \mathcal{T}_+^{(\alpha)}(t^\alpha) \to \mathfrak{A}$, so that $\pi^* : \mathfrak{A}^* \to \mathcal{T}_+^{(\alpha)}(t^\alpha)^*$, and by $\sigma(\pi(u)) \equiv \sigma_{\mathfrak{B}}(\pi(u))$ the spectrum of $\pi(u)$ in $\mathfrak{B}$.

Set $Z_1(\pi) := \{z \in \mathbb{C} : (z+1)^{-1} \in \sigma(\pi(u))\}$ and denote by $Z(\pi)$ the complement of the connected component of $\mathbb{C}\backslash Z_1(\pi)$ which contains $\mathbb{C}^-$. Notice that $Z(\pi)$ is well-defined since $Z_1(\pi) \subseteq \overline{\mathbb{C}^+}$.

PROPOSITION 4.1. *Under the above assumptions and notations, the following hold:*

(i) $(e - (z+1)\pi(u))^{-1} \in \mathfrak{A} \oplus \mathbb{C}e$ *for all* $z \in \mathbb{C}\backslash Z(\pi)$.

(ii) *Let* $L_{\widetilde{\varphi}} := \pi^*(T)$ *where* $T \in \mathfrak{A}^*$. *Then the mapping*

$$z \mapsto \langle T, \pi(u)(e - (z+1)\pi(u))^{-1}\rangle, \quad \mathbb{C} \setminus Z(\pi) \to \mathbb{C} \tag{4.1}$$

is an analytic extension of $\mathcal{L}(L_\varphi)$ *to* $\mathbb{C} \setminus Z(\pi)$, *which we continue denoting by* $\mathcal{L}(L_\varphi)$.

Proof. (i) Put $\lambda \equiv \lambda(z) := (z+1)^{-1}$. Clearly, if $z \in \mathbb{C} \setminus Z(\pi)$ then $\lambda \in \mathbb{C} \setminus \sigma(\pi(u))$. Moreover, $\{-1\} \subseteq \mathbb{C}^- \subseteq \mathbb{C} \setminus Z(\pi)$ and therefore $\lambda(\mathbb{C} \setminus Z(\pi))$ is contained in the unbounded connected component of $\mathbb{C} \setminus \sigma(\pi(u))$. Since the boundary of $\sigma_{\mathfrak{A}\oplus\mathbb{C}e}(\pi(u))$ is included in the boundary of $\sigma(\pi(u))$ we obtain that $\lambda(\mathbb{C} \setminus Z(\pi)) \subseteq \mathbb{C} \setminus \sigma_{\mathfrak{A}\oplus\mathbb{C}e}(\pi(u))$, as claimed.

(ii) Let $z \in \mathbb{C}^- := \{w \in \mathbb{C} : \Re w < 0\}$. Since $(z+1)^{-1} \notin \sigma(u)$, we have that $\delta_0 - (z+1)u$ is invertible in $\mathcal{T}_+^{(\alpha)}(t^\alpha) \oplus \mathbb{C}\delta_0$. Indeed, if $\nu_\alpha((z+1)u) < 1$ then

$$(\delta_0 - (z+1)u)^{-1} = \delta_0 + \sum_{n=1}^{\infty} (z+1)^n \, u^{*n} \,,$$

so that

$$u * (\delta_0 - (z+1)u)^{-1}(x) = e^{-x} + \sum_{n=1}^{\infty} (z+1)^n \, \frac{x^n \, e^{-x}}{n!} = e^{zx}.$$

Now, let $T \in \mathfrak{A}^*$ and set $L_{\widetilde{\varphi}} = \pi^*(T)$. Since $L_\varphi \in \mathcal{T}_-^{(\alpha)}((-t)^\alpha)^*$, the Laplace transform of L_φ is well-defined and it is analytic on $\mathbb{C}^-$. Moreover,

$$\mathcal{L}(L_\varphi)(z) = \int_{-\infty}^{0} W_-^\alpha(e_z)(t) \, \varphi(t) \, \mathrm{d}t = \int_{0}^{\infty} W_+^\alpha(e_{-z})(s) \, \widetilde{\varphi}(s) \, \mathrm{d}s = L_{\widetilde{\varphi}}(e_{-z})$$

for any $z \in \mathbb{C}^-$. In particular, if $|z+1| < \nu_\alpha(u)^{-1}$ we have

$$\begin{aligned}\mathcal{L}(L_\varphi)(z) &= \langle \pi^*(T), u * (\delta_0 - (z+1)u)^{-1} \rangle \\ &= \langle T, \pi(u * (\delta_0 - (z+1)u)^{-1}) \rangle = \langle T, \pi(u)(e - (z+1)\pi(u))^{-1} \rangle.\end{aligned}$$

By the identity principle for analytic functions, the above equality holds on the whole left half-plane $\mathbb{C}^-$. Now, observe that the mapping

$$z \mapsto \langle T, \, \pi(u)(e - (z+1)\pi(u))^{-1} \rangle, \quad \mathbb{C} \setminus Z(\pi) \to \mathbb{C}$$

is analytic, so it is an analytic extension of $\mathcal{L}(L_\varphi)$ to $\mathbb{C} \setminus Z(\pi)$. ∎

A first consequence of the proposition is an analytic extension result for the Laplace transform linked to π and the product $\circ$.

COROLLARY 4.2. *If $L_{\widetilde{\varphi}} = \pi^*(T) \in \pi^*(\mathfrak{A}^*)$ then $\mathcal{L}(L_\varphi \circ f)$ extends to an analytic function on $\mathbb{C} \setminus Z(\pi)$ for every $f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$. In particular, for $z \in \mathbb{C}^+ \setminus Z(\pi)$ we have*

$$\mathcal{L}(L_\varphi \circ f)(z) = \langle T, \pi(f)\pi(u)(e - (z+1)\pi(u))^{-1} \rangle.$$

Proof. Let $L_{\widetilde{\varphi}} = \pi^*(T)$ for some $T \in \mathfrak{A}^*$. Recall that $L_\varphi \circ f = L_{\psi_-}$ with $\psi(s) := L_{\widetilde{\varphi}}(R_{-s}^{\alpha-1} * f)$, $s \in \mathbb{R}$ (see Definition 2.3). Next, we show that $L_{\widetilde{\psi_-}} \in \pi^*(\mathfrak{A}^*)$:

For every $g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$,

$$\begin{aligned}L_{\widetilde{\psi_-}}(g) &= \int_0^\infty W_+^\alpha g(t) L_{\widetilde{\varphi}}(R_t^{\alpha-1} * f) \mathrm{d}t = L_{\widetilde{\varphi}}\Big(\int_0^\infty W_+^\alpha g(t) R_t^{\alpha-1} * f \, \mathrm{d}t \Big) \\ &= L_{\widetilde{\varphi}}(g * f) = \langle \pi^*(T), g * f \rangle = \langle T, \pi(g)\pi(f) \rangle.\end{aligned}$$

Let $T\pi(f)$ denote the element of $\mathfrak{A}^*$ defined by $\langle T\pi(f), a\rangle = \langle T, a\pi(f)\rangle$ for all $a \in \mathfrak{A}$. Then one has that

$$L_{\widetilde{\psi}_-}(g) = \langle T, \pi(g)\pi(f)\rangle = \langle T\pi(f), \pi(g)\rangle = \langle \pi^*(T\pi(f)), g\rangle$$

and therefore $L_{\widetilde{\psi}^-} = \pi^*(T\pi(f)) \in \pi^*(\mathfrak{A}^*)$ as claimed.

The statement follows now from Proposition 4.1. In addition, we have $\mathcal{L}(L_\varphi \circ f)(z) = \langle T, \pi(f)\pi(u)(e-(z+1)\pi(u))^{-1}\rangle$ if z is taken in $\mathbb{C}^+ \setminus Z(\pi)$. ∎

Now we relate the products $\circ$ and $\diamond$ for functionals associated with the homomorphism π. The result extends Proposition 2.5 of [8].

COROLLARY 4.3. *For every $L_{\widetilde{\varphi}} \in \pi^*(\mathfrak{A}^*)$ and $f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$,*

$$\mathcal{L}(L_\varphi \circ f) + \mathcal{L}(L_\varphi \diamond f) = \mathcal{L}(L_\varphi)\,\mathcal{L}(f) \quad \textit{in } \mathbb{C}^+ \setminus Z(\pi). \tag{4.2}$$

Proof. Let $L_{\widetilde{\varphi}} \in \pi^*(\mathfrak{A}^*)$ and $z \in \mathbb{C}^+ \setminus Z(\pi)$ be fixed. By Proposition 4.1 and Corollary 4.2, the mapping $\Phi : \mathcal{T}_+^{(\alpha)}(t^\alpha) \to \mathbb{C}$ given for each $f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ by

$$\Phi(f) := \mathcal{L}(L_\varphi \circ f)(z) + \mathcal{L}(L_\varphi \diamond f)(z) - \mathcal{L}(L_\varphi)(z)\mathcal{L}(f)(z)$$

is well-defined and linear. Furthermore, it is continuous. To prove that we note firstly that $f \to \mathcal{L}(f)(z)$ is clearly continuous. Secondly, recall that $L_\varphi \diamond f = L_{\psi_+} \in \mathcal{T}_+^{(\alpha)}(t^\alpha)^*$ with $\psi(s) = L_{\widetilde{\varphi}}(R_{-s}^{\alpha-1} * f)$, $s \in \mathbb{R}$. Hence,

$$\begin{aligned}|\mathcal{L}(L_\varphi \diamond f)(z)| = |L_{\psi_+}(e_z)| &\leqslant |z|^\alpha \int_0^\infty \mathrm{e}^{-\Re z t}\, |L_{\widetilde{\varphi}}(R_{-t}^{\alpha-1} * f)|\,\mathrm{d}t \\ &\leqslant C\,\nu_\alpha(f)\,|z|^\alpha \int_0^\infty t^\alpha \mathrm{e}^{-t\Re z}\,\mathrm{d}t = C(z)\nu_\alpha(f).\end{aligned}$$

Finally, by Corollary 4.2 we have that

$$\mathcal{L}(L_\varphi \circ f)(z) = \langle T, \pi(f)\pi(u)(e-(z+1)\pi(u))^{-1}\rangle$$

for all $z \in \mathbb{C}^+ \setminus Z(\pi)$. Thus $|\mathcal{L}(L_\varphi \circ f)(z)| \leqslant C'(z)\nu_\alpha(f)$. In conclusion, Φ is bounded on $\mathcal{T}_+^{(\alpha)}(t^\alpha)$.

Now we want to prove that $\Phi = 0$. Clearly, it will be enough to show that Φ vanishes on $C_c^\infty(\mathbb{R}^+)$. So take $f \in C_c^\infty(\mathbb{R}^+)$. Then we have that $L_\varphi * f \in C^{(\infty)}(\mathbb{R})$ and $\operatorname{supp}(L_\varphi * f) \subseteq \operatorname{supp}(L_\varphi) + \operatorname{supp}(f) \subseteq (-\infty, a]$ where $a := \sup(\operatorname{supp}(f)) \geqslant 0$ ([18], Theorem 6.30 b) and Theorem 6.37 b)). Hence, $\mathcal{L}(L_\varphi * f)$ exists and it is analytic at least on $\mathbb{C}^-$. Moreover, $\mathcal{L}(L_\varphi * f) = \mathcal{L}(L_\varphi)\mathcal{L}(f)$ on $\mathbb{C}^-$, where all factors have sense simultaneously. On the other hand, we have by definition that $L_\varphi * f = L_\varphi \circ f + L_\varphi \diamond f$, where $L_\varphi \circ f$ and $L_\varphi \diamond f$ are supported on $(-\infty, 0]$ and $[0, \infty)$, respectively. Since $\operatorname{supp}(L_\varphi * f) \subseteq (-\infty, a]$, it follows that $\operatorname{supp}(L_\varphi \diamond f) \subseteq [0, a]$, so that its Laplace transform is an entire function. Therefore, $\mathcal{L}(L_\varphi * f) = \mathcal{L}(L_\varphi \circ f) + \mathcal{L}(L_\varphi \diamond f)$ on $\mathbb{C}^-$. Then, $\mathcal{L}(L_\varphi \circ f) + \mathcal{L}(L_\varphi \diamond f) = \mathcal{L}(L_\varphi)\mathcal{L}(f)$ on

$\mathbb{C}^-$. Indeed, this equality holds on $\mathbb{C} \setminus Z(\pi)$ by the identity theorem for analytic functions. This concludes the proof. ∎

Next, we give the key result on functions in $\mathcal{T}_+^{(\alpha)}(t^\alpha) \subseteq \mathcal{T}^{(\alpha)}(|t|^\alpha)$ which are of spectral synthesis with respect to $(-\mathrm{i}\, Z(\pi)) \cap \mathbb{R}$. Such a result is in the spirit of Théorème 2.7 of [8].

THEOREM 4.4. *Let $(T_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathfrak{A}^*$ and denote $L_{\widetilde{\varphi}_n} := \pi^*(T_n)$, $n \in \mathbb{N}$. If $f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ is a function of spectral synthesis with respect to $(-\mathrm{i}\, Z(\pi)) \cap \mathbb{R}$ then*

$$\lim_{n\to\infty} t_n^{-\alpha} L_{\varphi_n} \circ (R_{t_n}^{\alpha-1} * g) \circ f = 0,$$

in the weak topology of $\mathcal{T}_-^{(\alpha)}((-t)^\alpha)^*$, for every $g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ such that $\lim_{n\to\infty} t_n = \infty$.*

Proof. For $g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ and (t_n) as above, set $g_n := t_n^{-\alpha} R_{t_n}^{\alpha-1} * g$ $(n \in \mathbb{N})$. Note that $\nu_\alpha(g_n) \leqslant C\nu_\alpha(g)$ for all n. We want to prove that $\lim_{n\to\infty} L_{\varphi_n} \circ g_n \circ f = 0$ weakly* in $\mathcal{T}_-^{(\alpha)}((-t)^\alpha)^*$.

The fact that $(T_n)_{n\in\mathbb{N}}$ is uniformly bounded implies that both sequences $(L_{\varphi_n} \circ g_n)$ and $(L_{\varphi_n} \diamond g_n)$ are uniformly bounded in $\mathcal{T}_-^{(\alpha)}((-t)^\alpha)^*$ and $\mathcal{T}_+^{(\alpha)}(t^\alpha)^*$, respectively. Indeed, for all n,

$$\|L_{\varphi_n} \circ g_n\|,\ \|L_{\varphi_n} \diamond g_n\| \leqslant C \sup_n \|\pi^*(T_n)\| \nu_\alpha(g).$$

Also, since $\mathcal{T}_-^{(\alpha)}((-t)^\alpha)$ and $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ are separable Banach spaces their bounded subsets are weak* metrizable and then by the Banach–Alaouglu theorem there exist (taking subsequences if necessary) the two weak* limits

$$\lim_{n\to\infty} L_{\varphi_n} \circ g_n =: L_{\rho_1} \in \mathcal{T}_-^{(\alpha)}((-t)^\alpha)^* \quad \text{and} \tag{4.3}$$

$$\lim_{n\to\infty} L_{\varphi_n} \diamond g_n =: L_{\rho_2} \in \mathcal{T}_+^{(\alpha)}(t^\alpha)^*. \tag{4.4}$$

On the other hand, the family $(\mathcal{L}(L_{\varphi_n} \circ g_n))$ is a normal family of analytic functions in $\mathbb{C}\setminus Z(\pi)$: By Proposition 4.2, $\mathcal{L}(L_{\varphi_n} \circ g_n)(z)$ extends holomorphically to $z \in \mathbb{C} \setminus Z(\pi)$ as

$$\mathcal{L}(L_{\varphi_n} \circ g_n)(z) := \langle T_n, \pi(g_n)\pi(u)(e - (z+1)\pi(u))^{-1} \rangle.$$

Since the sequences (T_n) and $(\nu_\alpha(g_n))$ are uniformly bounded and the mapping $z \in \mathbb{C}^+\setminus Z(\pi) \mapsto \pi(u)(e - (z+1)\pi(u))^{-1} \in \mathfrak{A}$ is analytic in $\mathbb{C}\setminus Z(\pi)$, we have that $(\mathcal{L}(L_{\varphi_n} \circ g_n))$ is uniformly bounded on compact subsets of $\mathbb{C}\setminus Z(\pi)$ as well. In other words, $(\mathcal{L}(L_{\varphi_n} \circ g_n))$ is a normal family on $\mathbb{C}\setminus Z(\pi)$.

Therefore, by the Montel theorem we can assume (by passing to a subsequence if necessary) that there exists an analytic function H in $\mathbb{C}\setminus Z(\pi)$ such that

$$\lim_{n\to\infty} \mathcal{L}(L_{\varphi_n} \circ g_n) = H \tag{4.5}$$

uniformly on compact subsets of sets of $\mathbb{C}\backslash Z(\pi)$. Also, it is readily seen that $(\mathcal{L}(L_{\varphi_n} \diamond g_n))_{n\in\mathbb{N}}$ converges to an analytic function on $\mathbb{C}^+$ uniformly on compact subsets.

Now, we are going to prove that $\lim_{n\to\infty} \mathcal{L}(g_n)(z) = 0$ for any $z \in \mathbb{C}^+$. Note that

$$\mathcal{L}(g_n)(z) = \int_0^\infty g_n(t) e^{-zt}\, dt = \int_0^\infty W_+^\alpha g_n(t) D^{-\alpha}(e_z)(t)\, dt$$

for every $z \in \mathbb{C}^+$, where

$$D^{-\alpha}(e_z)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e_z(s)\, ds\,, \quad t > 0.$$

Then $t^{-\alpha} D^{-\alpha}(e_z) \in C_0([0,\infty)$ and so $\lim_{n\to\infty} \mathcal{L}(g_n)(z) = 0$ $(z \in \mathbb{C}^+)$, by Proposition 2.1.

Applying formula (4.2) to φ_n and g_n, and then using (4.5) and (4.4), we obtain

$$H = \lim_{n\to\infty} \mathcal{L}(L_{\varphi_n} \circ g_n) = -\lim_{n\to\infty} \mathcal{L}(L_{\varphi_n} \diamond g_n) = -\mathcal{L}(L_{\rho_2})$$

in $\mathbb{C}^+\backslash Z(\pi)$. Hence, the function given by

$$F(z) := \begin{cases} H(z) & \text{if } z \in \mathbb{C}\backslash Z(\pi), \\ -\mathcal{L}(L_{\rho_2})(z) & \text{if } z \in \mathbb{C}^+, \end{cases}$$

is well-defined and analytic on $\mathbb{C}\backslash(Z(\pi) \cap i\mathbb{R})$. In addition, $F\mid_{\mathbb{C}^-} = \mathcal{L}(L_{\rho_1})$ by (4.3) and (4.5), and also $F\mid_{\mathbb{C}^+} = -\mathcal{L}(L_{\rho_2})$.

Denote $U := i\mathbb{R}\backslash(Z(\pi) \cap i\mathbb{R})$. Recall that f is assumed to be a function of spectral synthesis with respect to $(-iZ(\pi)) \cap \mathbb{R}$. Thus there exists a sequence $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{T}^{(\alpha)}(|t|^\alpha)$ such that $\operatorname{supp}\mathcal{F} f_n \subseteq -iU$ and $\lim_{n\to\infty} \nu_\alpha(f - f_n) = 0$. Clearly, $\lim_{n\to\infty} (L_{\rho_1} + L_{\rho_2}) * f_n = (L_{\rho_1} + L_{\rho_2}) * f$ in the weak* topology of $\mathcal{T}^{(\alpha)}(|t|^\alpha)^*$. On the other hand, $\mathcal{L}(L_{\rho_2})$ can be continuously extended to U as H, and so we get that $\mathcal{L}(L_{\rho_2})\mid_U := -\mathcal{L}(L_{\rho_1})\mid_U$. Then, by Proposition 3.3 we have that $(L_{\rho_1} + L_{\rho_2}) * f_n = 0$ for every $n \in \mathbb{N}$. Therefore, $(L_{\rho_1} + L_{\rho_2}) * f = 0$ in $\mathcal{T}^{(\alpha)}(|t|^\alpha)^*$. In particular, if $h \in \mathcal{T}_-^{(\alpha)}((-t)^\alpha)$ then

$$(L_{\rho_1} * f)(h) = -(L_{\rho_2} * f)(h) = -L_{\rho_2}(\widetilde{f} * h) = 0$$

since $\operatorname{supp}(L_{\rho_2}) \subseteq [0,\infty)$ and $\operatorname{supp}(\widetilde{f} * h) \subseteq (-\infty, 0]$. In other words, $L_{\rho_1} \circ f = 0$.

In conclusion, we have proved that any weak* cluster point of the sequence $(L_{\varphi_n} \circ g_n \circ f)_{n\in\mathbb{N}}$ is 0. This implies (recall again the Banach–Alaouglu theorem and the metrizability of bounded weak* subsets of $\mathcal{T}_-^{(\alpha)}((-t)^\alpha)^*$) that $\lim_{n\to\infty} t_n^{-\alpha} L_{\varphi_n} \circ (R_{t_n}^{\alpha-1} * g) \circ f = 0$, as we wanted to show. ∎

5. KATZNELSON–TZAFRIRI THEOREM FOR INTEGRATED SEMIGROUPS

On the base of what has been established in previous sections, we go on to prove the main result of the paper. This is the extension to the setting of integrated semigroups of the Esterle–Strouse–Zouakia and Vũ's theorem referred to in the Introduction (Theorem 0.2).

From now on, we will denote by X an arbitrary Banach space. Given an α-times integrated semigroup $(T_\alpha(t))_{t\geqslant 0}$ on X, we will say for simplicity that it is *sub-homogeneous of growth* α if

$$\sup_{t>0} t^{-\alpha}\|T_\alpha(t)\| < \infty.$$

Recall that we have defined in the Introduction a C_α-semigroup as an α-times integrated semigroup on X such that

$$\lim_{t\to 0} \Gamma(\alpha+1)\, t^{-\alpha} T_\alpha(t)x = x \quad (x \in X).$$

The following result tells us that the sub-homogeneous C_α-semigroups on X are in a one-to-one correspondence with bounded Banach algebra homomorphisms $\pi_\alpha : \mathcal{T}_+^{(\alpha)}(t^\alpha) \to \mathfrak{B}(X)$ for which $\pi_\alpha(\mathcal{T}_+^{(\alpha)}(t^\alpha))X$ is dense in X. The corresponding result for $\alpha = 0$ can be seen in [5], [8], [14] for instance.

Proposition 5.1. *Let X be a Banach space and let $\alpha > 0$. If $(T_\alpha(t))_{t\geqslant 0}$ is a sub-homogeneous C_α-semigroup on X then the bounded homomorphism $\pi_\alpha : \mathcal{T}_+^{(\alpha)}(t^\alpha) \to \mathfrak{B}(X)$ given by* (0.1),

$$\pi_\alpha(f)x = \int_0^\infty W_+^\alpha f(t) T_\alpha(t)x \, \mathrm{d}t \quad (x \in X),$$

is such that $\pi_\alpha(\mathcal{T}_+^{(\alpha)}(t^\alpha))X$ is dense in X.

Conversely, for every bounded homomorphism $\pi_\alpha : \mathcal{T}_+^{(\alpha)}(t^\alpha) \to \mathfrak{B}(X)$ such that $\pi_\alpha(\mathcal{T}_+^{(\alpha)}(t^\alpha))X$ is dense in X, the family defined by

$$T_\alpha(t)x := \pi_\alpha(R_t^{\alpha-1} * g)y, \quad (x = \pi_\alpha(g)y \in \mathcal{T}_+^{(\alpha)}(t^\alpha);\ t \geqslant 0),$$

is a sub-homogeneous C_α-semigroup on X whose associated homomorphism defined by formula (0.1) *is π_α.*

Proof. Let $(T_\alpha(t))_{t\geqslant 0} \subseteq \mathfrak{B}(X)$ be a sub-homogeneous C_α-semigroup. Since the mapping $f \mapsto W_+^\alpha(f)\, t^\alpha$, $\mathcal{T}_+^{(\alpha)}(t^\alpha) \to L^1(\mathbb{R}^+)$ is an isometric isomorphism of Banach spaces, we can take a sequence $(e_n)_{n\in\mathbb{N}} \subseteq \mathcal{T}_+^{(\alpha)}(t^\alpha)$ such that $\Gamma(\alpha+1)^{-1} W_+^\alpha(e_n)(t)\, t^\alpha = n\chi_{[0,\frac{1}{n}]}(t)$, $t > 0$, for each $n \in \mathbb{N}$. Then, for any $x \in X$ and

$n \in \mathbb{N}$,

$$x - \pi_\alpha(e_n)x = x - n \int_0^{1/n} \Gamma(\alpha+1)\, t^{-\alpha} T_\alpha(t)x\mathrm{d}t = n \int_0^{1/n} (x - \Gamma(\alpha+1)\, t^{-\alpha} T_\alpha(t)x)\mathrm{d}t\,.$$

From this and the fact that $(T_\alpha(t))_{t>0}$ is a C_α-semigroup, we conclude that

$$\|x - \pi_\alpha(e_n)x\| \leqslant \sup_{t\in(0,\frac{1}{n})} \|x - \Gamma(\alpha+1)\, t^{-\alpha} T_\alpha(t)x\| \to 0, \quad \text{as } n \to \infty.$$

This shows that $\pi_\alpha(\mathcal{T}_+^{(\alpha)}(t^\alpha))X$ is dense in X.

Conversely, let $\pi_\alpha : \mathcal{T}_+^{(\alpha)}(t^\alpha) \to \mathfrak{B}(X)$ be an arbitrary bounded homomorphism such that $\pi_\alpha(\mathcal{T}_+^{(\alpha)}(t^\alpha))X$ is dense in X. Then X is a Banach (bi-)module on $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ through the action π_α such that $\mathcal{T}_+^{(\alpha)}(t^\alpha)X = X$, by the Cohen's factorization theorem, since $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ has a bounded approximate identity, (see [9]). It means that for every $x \in X$ there exist $g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $y \in X$ such that $x = \pi_\alpha(g)y$. For a given multiplier μ of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ define

$$\pi_\alpha(\mu)x = \pi_\alpha(\mu * g)y.$$

The expression above does not depend on the decomposition $x = \pi_\alpha(g)y$, it gives rise to a bounded linear operator on X, and makes π_α a bounded algebra homomorphism from the Banach algebra of multipliers of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $\mathfrak{B}(X)$ ([7], Proposition 5.2). In particular one can define $T_\alpha(t) := \pi_\alpha(R_t^{\alpha-1})$, $t \geqslant 0$, since $R_t^{\alpha-1}$ is a multiplier of $\mathcal{T}_+^{(\alpha)}(t^\alpha)$.

The fact that π_α is a bounded homomorphism and that $R_t^{\alpha-1}$ is moreover an α-times integrated semigroup implies that $(T_\alpha(t))$ satisfies properties (a), (b), (c) of the Introduction, so it is an α-times integrated semigroup. Also, $(T_\alpha(t))$ is subhomogeneous because $\|R_t^{\alpha-1}\| \leqslant Ct^\alpha$ ($t > 0$); see Section 1. Now, for $f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $x = \pi_\alpha(g)y \in X$,

$$\begin{aligned}\pi_\alpha(f)\pi_\alpha(g)y = \pi_\alpha(f * g)y \ &= \pi_\alpha\Big(\int_0^\infty W_+^\alpha f(t) R_t^{\alpha-1} * g\, \mathrm{d}t\Big)y \\ &= \int_0^\infty W_+^\alpha f(t)\pi_\alpha(R_t^{\alpha-1} * g)y\, \mathrm{d}t = \int_0^\infty W_+^\alpha f(t) T_\alpha(t)\pi_\alpha(g)y\, \mathrm{d}t,\end{aligned}$$

where we have used the representation (1.4) in the second equality and the continuity of π_α in the third one. Hence

$$\pi_\alpha(f)x = \int_0^\infty W_+^\alpha f(t) T_\alpha(t)x\, \mathrm{d}t \quad \forall f \in \mathcal{T}_+^{(\alpha)}(t^\alpha), x \in X.$$

Finally notice that

$$\Gamma(\alpha+1)t^{-\alpha}T_\alpha(t)\pi_\alpha(f) = \pi_\alpha(\Gamma(\alpha+1)t^{-\alpha}R_t^{\alpha-1} * f)$$

for any $f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $t > 0$. Then, by the factorization $X = \pi_\alpha(\mathcal{T}_+^{(\alpha)}(t^\alpha))X$, Proposition 1.1 and the continuity of π_α we obtain that

$$\lim_{t\to 0^+} \Gamma(\alpha+1)t^{-\alpha}T_\alpha(t)x = x \quad (x \in X).$$

Thus we have shown that $(T_\alpha(t))_{t\geqslant 0}$ is a C_α-semigroup, and the proof is over. ∎

REMARK 5.2. Given any sub-homogeneous α-times integrated semigroup $(T_\alpha(t))$ on X, there always exists a closed subspace X_0 of X such that the family $T_\alpha(t)\mid_{X_0}$ belongs to $\mathfrak{B}(X_0)$ and satisfies properties (a)–(d) of the Introduction. In fact, X_0 can be taken as the closure of $\pi_\alpha(\mathcal{T}_+^{(\alpha)}(t^\alpha))X$ in X, where $\pi_\alpha : \mathcal{T}_+^{(\alpha)}(t^\alpha) \to \mathfrak{B}(X)$ is as in (0.1). Then, as a consequence of above proposition, $(T_\alpha(t)\mid_{X_0})$ is a C_α-integrated semigroup on X_0.

Next, we give the proof of Theorem 0.2. Let us point out first the following lemma:

LEMMA 5.3. *Let A be the generator of a sub-homogeneous α-times integrated semigroup $T_\alpha(t)$. Let π_α be defined as in* (0.1). *Then, $-\mathrm{i}Z(\pi_\alpha)\cap\mathbb{R} = \mathrm{i}\sigma(A)\cap\mathbb{R}$.*

Proof. Set $B := -A$ and $D(B)$ for the domain of B. Note that $(B+I)^{-1} = \int_0^\infty \mathrm{e}^{-t}T_\alpha(t)\,\mathrm{d}t = \pi_\alpha(u) \in \mathfrak{B}(X)$. In particular, $X = (B+I)D(B)$. These facts imply that $[I-(z+1)\pi_\alpha(u)]X = (B-zI)D(B)$ for every $z \in \mathbb{C}$. As a result, $Z_1(\pi_\alpha) = \sigma(B)$. Therefore, $Z(\pi_\alpha)\cap \mathrm{i}\mathbb{R} = Z_1(\pi_\alpha)\cap\mathrm{i}\mathbb{R} = -\sigma(A)\cap\mathrm{i}\mathbb{R}$. ∎

Proof of Theorem 0.2. Let $f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ be a function of spectral synthesis with respect to the subset $\mathrm{i}\sigma(A)\cap\mathbb{R}$. Since $T_\alpha(t) = \pi_\alpha(R_t^{\alpha-1})$, we have to prove that $\lim_{t\to\infty}\pi_\alpha(t^{-\alpha}R_t^{\alpha-1} * f) = 0$ in norm. Take $g \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ and $(t_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^+$ be such that $\lim_{n\to\infty} t_n = \infty$. Put $g_n := t_n^{-\alpha}R_{t_n}^{\alpha-1} * g$, $n \in \mathbb{N}$. For $h \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$, by the Hahn–Banach theorem there exists a sequence $(T_n)_{n\in\mathbb{N}} \subseteq \mathfrak{A}^*$ such that $\|T_n\| = 1$, and

$$\|\pi_\alpha(h * f * g_n)\| = \langle T_n, \pi_\alpha(h * f * g_n)\rangle, \quad \text{for all } n.$$

Hence, if $L_{\widetilde{\varphi}_n} := \pi_\alpha^*(T_n)$, we have that

$$\|\pi_\alpha(h * f * g_n)\| = \langle L_{\widetilde{\varphi}_n} \bullet g_n \bullet f, h\rangle, \quad \text{for all } n.$$

By Proposition 2.4 one has that $\lim_{n\to\infty} L_{\widetilde{\varphi}_n} \bullet \vartheta_n \bullet f = 0$ weakly* in $\mathcal{T}_+^{(\alpha)}(t^\alpha)^*$ if and only if $\lim_{n\to\infty} L_{\varphi_n} \circ g_n \circ f = 0$ weakly* in $\mathcal{T}_-^{(\alpha)}((-t)^\alpha)^*$. The latter limit holds as a consequence of Theorem 4.4, since f is of spectral synthesis with respect to $\mathrm{i}\,\sigma(A) \cap \mathbb{R} = -\mathrm{i}Z(\pi_\alpha) \cap \mathbb{R}$. Therefore

$$\lim_{n\to\infty} t_n^{-\alpha} \|\pi_\alpha(R_{t_n}^{\alpha-1} * f * g * h)\| = \lim_{n\to\infty} \|\pi_\alpha(h * f * g_n)\| = 0,$$

that is,

$$\lim_{n\to\infty} t_n^{-\alpha} \|T_\alpha(t_n)\pi_\alpha(f * F)\| = 0 \quad \text{for every } F \in \mathcal{T}_+^{(\alpha)}(t^\alpha),$$

since $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ factorizes and $T_\alpha(t_n) = \pi_\alpha(R_{t_n}^{\alpha-1})$. Finally, letting F running over a bounded approximate identity family in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$, we get the desired result:

$$\lim_{n\to\infty} t_n^{-\alpha} \|T_\alpha(t_n)\pi_\alpha(f)\| = 0. \quad \blacksquare$$

REMARK 5.4. There are proofs of Theorem 0.1 on the basis of the Ingham's tauberian theorem or using the notion of complete trajectories; see pp. 84, 85 of [4]. We wonder if such arguments admit analogues, in the setting of integrated semigroups, which could be fruitfully employed to give alternative proofs of Theorem 0.2.

6. APPENDIX: ASYMPTOTIC BEHAVIOUR OF SEMIGROUPS

Let $T(t)$ be a C_0-semigroup on a Banach space X. An orbit $T(\cdot)x$, with $x \in X$, is said to be stable if $\lim\limits_{t\to\infty} \|T(t)x\| = 0$. The semigroup is called stable if every orbit is stable [4].

The above notion is important in connection with the asymptotic stability of well-posed abstract Cauchy problems. Obviously, the Esterle–Strouse–Zouakia and Vũ's theorem can be regarded as a result on stability of orbits $T(\cdot)x$, for $x \in \mathrm{Im}\ \pi_0(f)$ and appropriate f in $L^1(\mathbb{R}^+)$.

Further, it is proven in [8] that the aforementioned theorem implies (in a certainly non-trivial way) the following celebrated result, due originally to W. Arendt and C.J.K. Batty [1], and Y. Lyubich and Q.P. Vũ [16].

THEOREM 6.1. *If A generates a uniformly bounded C_0-semigroup $(T(t))_{t\geqslant 0}$ such that $\sigma(A)\cap \mathrm{i}\mathbb{R}$ is countable and the adjoint operator A^* has no eigenvalues then $(T(t))_{t\geqslant 0}$ is stable.*

A notion of stability of once integrated semigroups has been considered in p. 363 of [6]. This is as follows: A once integrated semigroup $T_1(t)$ is said to be stable if $\lim\limits_{t\to\infty} T_1(t)x$ exists for every $x \in X$. Such a definition entails that the generator A_1 of $T_1(t)$ must be invertible ([6], p. 363). As a partial converse of this property, the following version of Theorem 6.1 is given in Theorem 5.6 of [6].

For A_1 invertible, if $\sigma(A_1) \cap \mathrm{i}\mathbb{R}$ is countable and the adjoint operator A^* has no eigenvalues then $T_1(t)$ is stable in the sense just indicated above.

Here, we do not deal with stability of integrated semigroups. Instead, we consider another type of asymptotic behaviour (which is of clear ergodic nature when the integrated semigroup $T_\alpha(t)$ comes from a C_0-semigroup $T(t)$ as given by $T_\alpha(t) = \Gamma(\alpha)^{-1}\int_0^t (t-s)^{\alpha-1}T(s)\,\mathrm{d}s$): An α-times integrated semigroup is *ergodic* if $\lim_{t\to\infty} t^{-\alpha}T_\alpha(t)x$ exists for all $x \in X$ ([6], p. 353). This property is important because it reflects the asymptotic behaviour of solutions of (some) ill-posed abstract Cauchy problems ([6], p. 352). The ergodicity of an α-times integrated semigroup $T_\alpha(t)$ such that $\sup_{t\geqslant 1} t^{-\alpha}\|T_\alpha(t)\| \leqslant \infty$ is characterized in [6] in terms of Abel-ergodicity or/and ergodic decompositions of the Banach space X.

DEFINITION 6.2. Given an α-times integrated semigroup $(T_\alpha(t))_{t\geqslant 0}$ on X, we call an orbit $T_\alpha(\cdot)x$ $(x \in X)$ *$o(t^\alpha)$-ergodic* if

$$\lim_{t\to\infty} t^{-\alpha}\|T_\alpha(t)x\| = 0\,.$$

We say that $(T_\alpha(t))_{t\geqslant 0}$ is $o(t^\alpha)$-ergodic if $T_\alpha(\cdot)x$ is $o(t^\alpha)$-ergodic for every $x \in X$.

PROPOSITION 6.3. *Let $T_\alpha(t)$, A and π_α be under the assumptions of Theorem* 0.2. *Let $\rho(A)$ be the resolvent set of A. If $\sigma(A)\cap i\mathbb{R} = \emptyset$ then*

$$\lim_{t\to\infty} t^{-\alpha}\|T_\alpha(t)(\lambda - A)^{-1}\| = 0 \quad \forall\lambda \in \rho(A).$$

In consequence, $(T_\alpha(t))_{t\geqslant 0}$ is $o(t^\alpha)$-ergodic; that is,

$$\lim_{t\to\infty} t^{-\alpha}\|T_\alpha(t)x\| = 0 \quad \forall x \in X.$$

Proof. Take λ with $\Re\lambda > 0$, so that $\lambda \in \rho(A)$ in particular. Recall that the function $e_\lambda(t) = \mathrm{e}^{-\lambda t}$ $(t > 0)$ is in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ with $W_+^\alpha e_\lambda = \lambda^\alpha e_\lambda$ (see Section 4, prior to Proposition 3.1). Hence, for $x \in X$,

$$(\lambda - A)^{-1}x = \int_0^\infty (W_+^\alpha e_\lambda)(t)\,T_\alpha(t)x\,\mathrm{d}t = \pi_\alpha(e_\lambda)x.$$

Notice now that every function in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$, so e_λ, is of spectral synthesis for the empty set. Hence, by Theorem 0.2, one gets

$$\lim_{t\to\infty} t^{-\alpha}\|T_\alpha(t)(\lambda - A)^{-1}\| = \lim_{t\to\infty} t^{-\alpha}\|T_\alpha(t)\pi_\alpha(e_\lambda)\| = 0.$$

For arbitrary μ in $\rho(A)$ it is enough to apply the resolvent identity for μ and λ with $\Re\lambda > 0$ to obtain $\lim_{t\to\infty} t^{-\alpha}\|T_\alpha(t)(\mu - A)^{-1}\| = 0$.

Finally, the $o(t^\alpha)$-ergodicity of $T_\alpha(t)$ follows by factorising $x = (\mu - A)^{-1}(\mu - A)x$ if x belongs to the domain $D(A)$ of A, and then for arbitrary $x \in X$ by the density of $D(A)$ in X and the uniform boundedness of $t^{-\alpha}T_\alpha(t)$. ∎

REMARK 6.4. (i) The assumption $\sigma(A) \cap \mathrm{i}\mathbb{R} = \emptyset$ implies that $0 \in \rho(A)$ and therefore we obtain from the above proposition that

$$\lim_{t\to\infty} t^{-\alpha}\|T_\alpha(t)A^{-1}\| = 0.$$

This extends Corollary 3.3 of [19].

(ii) For a bounded C_0-semigroup $T(t)$ the necessary property given in the first part of Proposition 6.3 is in fact an equivalence:

$$\lim_{t\to\infty} t^{-\alpha}\|T_\alpha(t)(\lambda - A)^{-1}\| = 0 \quad \forall \lambda \in \rho(A) \Leftrightarrow \sigma(A) \cap \mathrm{i}\mathbb{R} = \emptyset;$$

see Corollary 5.2.6 of [20]. Thus we wonder if this equivalence also holds for integrated semigroups. One of the ingredients to prove the above result is that if $f \in L^1(\mathbb{R}^+)$ is such that $\lim_{t\to\infty} \|T(t)\pi_0(f)\| = 0$ then its Fourier transform $\mathcal{F}(f)$ vanishes on $\sigma(A) \cap \mathrm{i}\mathbb{R}$. Unfortunately, the argument used in Theorem 5.2.6 of [20] to show this property of $\mathcal{F}(f)$ does not work for integrated semigroups. The reason is that in the latter case one cannot appeal to translations $\delta_s * f$ in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$. The natural substitute in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ for δ_s is the family of Riesz kernels $R_s^{\alpha-1}$, but then by taking convolutions $R_s^{\alpha-1} * f$ one gets that the lower bound obtained for semigroups in Theorem 5.2.6 of [20] fails for integrated semigroups.

(iii) An alternative way to show the $o(t^\alpha)$-ergodicity of $T_\alpha(t)$ in Proposition 6.3 relies on an idea employed succesfully in [8] to prove Theorem 6.1 from Théorème 3.4 of [8].

Let $\mathfrak{S}$ be the set formed by all functions $f \in \mathcal{T}_+^{(\alpha)}(t^\alpha)$ which are of spectral synthesis with respect to $\mathrm{i}\sigma(A) \cap \mathbb{R}$. By Theorem 0.2 it follows that the subset Y of X given by

$$Y := \{\pi_\alpha(f)x : f \in \mathfrak{S}, x \in X\},$$

defines a family of $o(t^\alpha)$-ergodic orbits of a C_α-semigroup $(T_\alpha(t))_{t\geqslant 0}$. Trivially, if the set Y is dense in X then the integrated semigroup is $o(t^\alpha)$-ergodic. From this observation, and since the empty set is of spectral synthesis, using Proposition 5.1 and the Cohen's factorization theorem we obtain that $\lim_{t\to\infty} t^{-\alpha}\|T_\alpha(t)x\| = 0$ $\forall x \in X$, whenever $\sigma(A) \cap \mathrm{i}\mathbb{R} = \emptyset$.

(Note that if one assumes additionally that the integrated semigroup $T_\alpha(t)$ is Lipschitz continuous then that conclusion follows automatically by Theorem 2.4 of [6].)

Since Definition 6.2 is formally equal to stability of C_0-semigroups (for $\alpha = 0$) we would like to extend the second part of Proposition 6.3 for integrated semigroups, in the form of a result like Theorem 6.1. One of the main points in the proof of that theorem in [8] is that the subset Y given in Remark 6.4(iii) is dense in X (for $\alpha = 0$), when $\sigma(A) \cap \mathrm{i}\mathbb{R}$ is countable, because *the countable subsets of* $\mathbb{R}$ *are of spectral synthesis for* $L^1(\mathbb{R})$. In contrast, and even for *integer* α, points in $\mathbb{R}$ *are not of spectral synthesis for* $\mathcal{T}_+^{(\alpha)}(t^\alpha)$ (with the only exception of the point $t = 0$).

Thus, if we want to establish the analogous of Theorem 6.1 for integrated semigroups the first task to do is to study primary ideals and weak versions of spectral synthesis in $\mathcal{T}_+^{(\alpha)}(t^\alpha)$. This programme is in progress currently.

Acknowledgements. Authors are grateful to an anonymous referee for a patient reading and valuable suggestions which led to improve the presentation of this paper.

This research has been partially supported by Projects MTM2010-16679 of the M.C.Y.T., Spain, and E-64 of the D.G. Aragón. The second author was partially supported by a FPU Grant from the Ministry of Education of Spain

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Received July 2, 2010; revised June 29, 2012; posted on February 8, 2013.