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AFFILIATED SUBSPACES AND THE STRUCTURE OF VON NEUMANN ALGEBRAS

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ABSTRACT. The interplay between order-theoretic properties of structures of subspaces affiliated with a von Neumann algebra M and the inner structure of the algebra M is studied. The following characterization of finiteness is given: a von Neumann algebra M is finite if and only if in each representation space of M one has that closed affiliated subspaces are given precisely by strongly closed left ideals in M. Moreover, it is shown that if the modular operator of a faithful normal state φ is bounded, then all important classes of affiliated subspaces in the GNS representation space of φ coincide. Orthogonally closed affiliated subspaces are characterized in terms of the supports of normal functionals. It is proved that complete affiliated subspaces correspond to left ideals generated by finite sums of orthogonal atomic projections.

KEYWORDS: *Subspaces affiliated with a von Neuman algebra, states and weights on von Neumann algebras, modular theory.*

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INTRODUCTION AND PRELIMINARIES

The set $L(H)$ of all closed subspaces of a Hilbert space H ordered by set inclusion and endowed with orthogonal complement constitutes one of the most important examples of a complete orthomodular lattice. From the point of view of the lattice theory the lattice $L(H)$ has many specific properties. It is irreducible, atomic, and usually not modular. In order to obtain more general subspace lattices interesting from both mathematical and physical point of view one can select only distinguished subspaces of H. A natural way how to do it is to replace the lattice $L(H)$ by its sublattice consisting of those closed subspaces of H for which the corresponding projections belong to a given von Neumann algebra M acting on H. (The lattice $L(H)$ can be then recovered as a special case by taking M to be the algebra of all bounded operators on H.) In other words, one can choose only subspaces which are invariant under operators from the commutant M' of

M. This "selection procedure" leads to lattices which may be atomless, modular, and may enjoy the properties of continuous geometry; see [8]. This was the basic idea of von Neumann and Murray [9] and especially of Birkhoff and von Neumann; see the celebrated paper [2]. Their approach has led to many beautiful results in functional analysis, lattice theory, and foundations of quantum physics. Besides lattices of closed subspaces in a Hilbert space more general posets of subspaces of a (possibly incomplete) inner product space play an important role in the theory of ordered structures and quantum structures. This line of the research has attracted many authors (see e.g. [3], [4], [6], [7] and the references therein). One of the most striking classical results in this field is the Amemyia–Araki theorem [1] saying that an inner product space S is complete if and only if orthogonally closed subspaces of S form an orthomodular lattice. Related result states that orthogonally closed and splitting subspaces coincide precisely for complete spaces. This sheds some light on the role of metric completeness in ordered subspace structures. Moreover, by explaining the purely technical assumption of completeness in terms of physically more plausible "orthomodular law" the Amemyia–Araki theorem and subsequent results have contributed to better understanding of mathematical axioms of quantum mechanics. Analogously to Hilbert space lattices the structure of all closed subspaces of an inner product space enjoyes some specific properties. For example, those structures admit atoms — one dimensional subspaces. Parallel to transition from Hilbert space lattices to von Neumann projection lattices, one can replace the system of all closed subspaces of an inner product space by a smaller system of subspaces which are invariant with respect to some von Neumann algebra. This investigation was initiated by Sherstnev and Turilova, see [11], [12], [13], [14]. In [11] the research has been focused primarily on the structure of affiliated subspaces resulting from Gelfand–Naimark–Segal representation (GNS for short) generated by a normal faithful state on the algebra of all bounded operators on a Hilbert space. Among others it was shown there that in this case all major types of affiliated subspaces (such as splitting, orthogonally closed, etc.) do not coincide. The aim of this note is to generalize important results along this line from specific type I factors to general von Neumann algebras. We are going to study the position of important classes of subspaces affiliated with a von Neumann algebras and show that there is an interesting interplay between the properties of subspace posets and the structure of von Neumann algebras.

The paper is organized as follows. In the present part we introduce basic types of affiliated subspaces and show that there is a one-to-one correspondence between splitting affiliated subspaces and projections in a given von Neumann algebra that leave the basic space invariant. In the second part the subspace structure of the space of all right bounded elements corresponding to a full left Hilbert algebra is briefly studied. It is shown that the projections in the definition algebra of the semifinite normal faithful weight induce naturally splitting affiliated subspaces of this space. The core of the paper is the third section which basically deals with affiliated subspaces of the space $S = M'\Omega$, where Ω is a separating and generating vector of a von Neumann algebra M. It is shown that there is a one-to-one correspondence between affiliated subspaces of S and ideals in a von Neumann algebra M. The main result concerns the relationship between closed affiliated subspaces and affiliated subspaces for which the corresponding ideal is closed in the strong operator topology. In Theorem 2.9 we state that a von Neumann algebra M has properly infinite direct summand if and only if the above mentioned subspace classes differ for every faithful normal state on M (i.e. for every vector state given by cyclic and separating vector Ω in the standard representation of M). This generalizes considerably the work [11]. On the other hand, we show that all important classes of subspaces coincide whenever the modular operator of a given state is bounded. Therefore, unlike the situation in Amemyia–Araki theorem, we can obtain orthomodular structure of orthogonally closed affiliated subspaces even if the underlying space is far from being complete. Besides, we characterize orthogonally closed affiliated subspaces in terms of the support projections of derived functionals or in terms of the Radon–Nikodym derivatives with respect to a trace.

Finally, in the concluding part it is proved that a closed affiliated subspace is complete if and only if its corresponding projection is a sum of finitely many atomic projections. It means that plenty of affiliated subspaces contain no complete nontrivial affiliated subspace. As an illustration we describe all complete affiliated subspaces in case of type I factor.

Let us now introduce basic concepts and fix the notation. Throughout the paper M shall be a von Neumann algebra acting on a Hilbert space H. Denote by M' the commutant of M. By M^+ we shall denote the set of positive elements of M. The symbol $P(M)$ will be reserved for the set of all projections in M. (By a projection we mean a self-adjoint idempotent). The algebra of all bounded operators acting on a Hilbert space H is denoted by $B(H)$. The state on M is a positive normalized functional. Given a nonzero vector $\xi \in H$ we obtain so-called vector functional ω_ξ on M by putting $\omega_\xi(x) = \langle x\xi, \xi\rangle$. A vector Ω in H is called separating for M if $x\Omega \neq 0$ for every nonzero $x \in M$. The vector Ω is called generating (or cycling) for M if the space $M\Omega = \{x\Omega : x \in M\}$ is dense in H. A subspace S of H is said to be *affiliated* with M (in symbols $S\eta M$) if $x'S \subset S$ for each $x' \in M'$. Given an affiliated subspace $S\eta M$ we shall be interested in the structure of its affiliated subspaces. This structure depends heavily on the underlying von Neumann algebra M. For example, if $M = B(H)$, then every subspace S of H is affiliated with M since the commutant of M consists of multiples of the identity map. On other hand, suppose that M is a one-dimensional algebra consisting of multiples of the identity operator on H. Then $M' = B(H)$ and so only trivial subspaces of H are affiliated with M.

Let us now fix a space S affiliated with M. By the symbol $A_M(S)$ we shall denote the set of all subspaces of S which are affiliated with M. It is clear that the set $A_M(S)$ is closed under taking closures $X \to \overline{X}$. It is also stable under forming relative orthogonal complements

$$X \to X^{\perp_S} = \{y \in S : \langle x, y \rangle = 0 \text{ for all } x \in X\}\,.$$

The structure of affiliated subspaces is related to the projection structure of M in the following natural way. If $X \in A_M(S)$, then its closure $[X]$ in H is affiliated with M and so the orthogonal projection p which projects H onto $[X]$ belongs to $M'' = M$. Therefore $X \subset pH \cap S$ and if X is closed then $X = pH \cap S$. On the other hand, given a projection $p \in P(M)$, one can easily verify that $pH \cap S \in A_M(S)$. Therefore all closed affiliated subspaces of S arise from projections in M. However, in general, the ranges of different projections in $P(M)$ may have the same intersection with S and so the relationship between projections and closed affiliated subspaces may not be one-to-one.

We shall now introduce important types of affiliated subspaces studied in this paper. By the symbol $L_M(S)$ we shall denote the set of all subspaces in $A_M(S)$ which are closed in S. An affiliated subspace $X \in A_M(S)$ is called *orthogonally closed* if $X = (X^{\perp_S})^{\perp_S}$. Let the symbol $F_M(S)$ mean the set of all orthogonally closed affiliated subspaces of S. An affiliated subspace X of S is called *splitting* if $S = X \oplus X^{\perp_S}$. The set of all affiliated splitting subspaces of S shall be denoted by $E_M(S)$. Finally, the set of all subspaces in $A_M(S)$ that are complete will be denoted by $C_M(S)$. The following chain of inclusions is obvious

$$C_M(S) \subset E_M(S) \subset F_M(S) \subset L_M(S) \subset A_M(S)\,.$$

In general, all inclusions above are proper.

Among various subspace classes, the class of affiliated splitting subspaces seems to have most transparent description in terms of projections from the algebra M.

PROPOSITION 0.1. *A subspace $E \in A_M(S)$ is splitting if and only if there is a (uniquely determined) projection p in M such that*

$$E = pS\,.$$

Proof. Suppose that E is an affiliated splitting subspace and denote by p the projection projecting the underlying Hilbert space H onto the completion $[E]$ of E. (The completion will be identified with the closure taken in a complete superspace H.) As $[E]$ is invariant for M', this projection must be in M. Each $s \in S$ can be decomposed as

$$s = s_1 + s_2\,, \quad s_1 \in E, s_2 \in E^{\perp}\,.$$

As $E^{\perp_S} \subset (1-p)(H)$ we see that

$$ps = ps_1 = s_1 \in S\,.$$

It shows that $E = pS$. On the other hand, if $p \in P(M)$ with $pS \subset S$, then the space pS is obviously in $E_M(S)$. ∎

The previous proposition says that there is a one-to-one correspondence between splitting affiliated subspaces and projections in M leaving S invariant. Moreover, as a corollary of Proposition 0.1 we can also quickly see the fact, already observed in [11], that $E_M(S) = L_M(S)$ whenever M is abelian. Indeed, if M is abelian, then for each projection $p \in M \subset M'$ we have that $pS \subset S$, meaning that every closed space F in $L_M(S)$ is of the form pS, where p is the projection onto the completion of F.

1. AFFILIATED SUBSPACES GENERATED BY WEIGHTS

This section will be devoted to basic facts on affiliated subspaces arising in modular theory of von Neumann algebras. We recall basic results and fix the notation. (For details on modular theory we refer the reader to [10], [15].) Let φ be a faithful normal semifinite weight on a von Neumann algebra M. Denote by

$$\mathcal{N}_\varphi = \{x \in M : \varphi(x^* x) < \infty\}, \quad \mathfrak{A}_\varphi = \mathcal{N}_\varphi^* \cap \mathcal{N}_\varphi .$$

The set $\mathcal{N}_\varphi$ is a left ideal in M. The ideal $\mathcal{N}_\varphi$ endowed with the inner product

$$\langle x, y\rangle_\varphi = \varphi(y^* x), \quad x, y \in \mathcal{N}_\varphi ,$$

becomes an inner product space whose completion will be denoted by $\mathcal{H}_\varphi$. $\mathfrak{A}_\varphi$ is a dense subspace of $\mathcal{N}_\varphi \subset \mathcal{H}_\varphi$. If we consider $x \in N_\varphi$ as an element of the inner product space $(\mathcal{N}_\varphi, \langle\cdot,\cdot\rangle_\varphi)$, we shall emphasize this by writing $\overline{x}$. In this notation

$$\varphi(x) = \|\overline{x}^{1/2}\|_\varphi \text{ for all positive } x \text{ with } \varphi(x) < \infty .$$

The weight φ gives rise to a faithful normal representation $\pi_\varphi : M \to B(\mathcal{H}_\varphi)$ such that

$$\pi_\varphi(x)y = xy \quad x \in M, y \in \mathcal{N}_\varphi .$$

If not stated otherwise we shall identify M with $\pi_\varphi(M)$ by identifying $\pi_\varphi(x)$ with x. In other words, M will always be represented on $\mathcal{H}_\varphi$ and the commutant M' of M will always be taken in $B(\mathcal{H}_\varphi)$.

It is well known that $\mathfrak{A}_\varphi$ is a full left Hilbert algebra with the multiplication $(x, y) \to xy$ and the involution $x \to x^*$. By S_φ, J_φ, and Δ_φ we shall denote the modular data of $\mathfrak{A}_\varphi$. That is, S_φ is the closure of the conjugate linear operator $x \to x^*$ with the polar decomposition

$$S_\varphi = J_\varphi \Delta_\varphi^{1/2} ,$$

where J_φ is a conjugate linear isometry on $\mathcal{H}_\varphi$ and Δ_φ is a positive self-adjoint operator affiliated with M. A basic result of the Tomita–Takesaki modular theory says that

$$J_\varphi M J_\varphi = M' .$$

By the symbol j_φ we shall denote the canonical (conjugate linear) isomorphism of M and M'

$$j_\varphi : M \to M' : x \to J_\varphi x J_\varphi .$$

For $x \in M$ we shall write $x' = j_\varphi(x)$. An element $\eta \in \mathcal{H}_\varphi$ is called *right bounded* if

$$\sup\{\|x\eta\| : x \in \mathfrak{A}_\varphi, \|\overline{x}\| \leqslant 1\} < \infty .$$

In other words, η is right bounded if the map $\overline{x} \to x\eta$ is continuous on $\mathfrak{A}_\varphi$. By the symbol $\mathcal{B}'_\varphi$ we shall denote the set of all right bounded elements. It can be easily seen (see e.g. Lemma 1.8 on p. 5, Vol. 2 of [15]) that $\mathcal{B}'_\varphi$ is a subspace of $\mathcal{H}_\varphi$ which is affiliated with M. We shall be mainly interested in the structure of affiliated subspaces of this space. The following lemma will be useful in the sequel. It says that if the modular operator is bounded, then any projection in $\mathcal{N}_\varphi$ induces a splitting affiliated subspace.

LEMMA 1.1. *Suppose that Δ_φ is a bounded operator. For each $p \in P(M)$ such that $\varphi(p) < \infty$ we have that*

$$p\mathcal{B}'_\varphi \subset \mathcal{B}'_\varphi .$$

Proof. Let us take $p \in P(M)$ with $\varphi(p) < \infty$ and any $\xi \in \mathcal{B}'_\varphi$. We are going to prove that $p\xi \in \mathcal{B}'_\varphi$. By continuity of Δ_φ, the $*$-operation on $\mathfrak{A}_\varphi$ is continuous. There is so a constant $K \geqslant 0$ such that for all $x \in \mathfrak{A}_\varphi$

$$\|\overline{x}^*\| \leqslant K \|\overline{x}\| .$$

In other words, for each $x \in \mathfrak{A}_\varphi$ one has

$$\varphi(xx^*) \leqslant K^2 \varphi(x^*x) .$$

On the other hand, by the assumption, the map $\overline{x} \to x\xi : \mathfrak{A}_\varphi \to \mathcal{H}_\varphi$ is continuous. There is therefore a constant $L \geqslant 0$ such that

$$\|xp\xi\| \leqslant L\|\overline{xp}\| ,$$

for all $x \in \mathfrak{A}_\varphi$. (Note that $xp \in \mathfrak{A}_\varphi$ whenever $x \in \mathfrak{A}_\varphi$ because $\varphi(p)$ is finite.) Putting this together we can estimate

$$\begin{aligned}\|xp\xi\| &\leqslant L\|\overline{xp}\| \leqslant LK\|\overline{px^*}\| = LK\varphi(xpx^*)^{1/2} \\ &\leqslant LK\varphi(xx^*)^{1/2} \leqslant LK^2\varphi(x^*x)^{1/2} = LK^2\|\overline{x}\| ,\end{aligned}$$

for each $x \in \mathfrak{A}_\varphi$. This means that $p\xi \in \mathcal{B}'_\varphi$. ∎

2. AFFILIATED SUBSPACES GENERATED BY STATES

In this part we shall deal with the structure of affiliated subspaces of $\mathcal{B}'_\varphi$, where φ is a normal faithful state on M. We continue to use the notation of the preceding section. The situation simplifies because $\mathcal{N}_\varphi = \mathfrak{A}_\varphi = M$. Let us denote $\xi_\varphi = \overline{1} \in \mathcal{H}_\varphi$. This vector is cyclic and separating for both M and M' and

$$\varphi(x) = \langle x\xi_\varphi, \xi_\varphi\rangle_\varphi = \omega_{\xi_\varphi}(x) .$$

We shall extend φ to the whole algebra $B(\mathcal{H}_\varphi)$ by putting $\varphi = \omega_{\xi_\varphi}$. The set of right bounded vectors, $\mathcal{B}'_\varphi$, coincide with the space generated by the cyclic vector ξ_φ:

$$\mathcal{B}'_\varphi = M'\xi_\varphi = \{u\xi_\varphi : u \in M'\}$$

(see e.g. [10]). For simplicity of the notation we shall denote

$$\mathcal{S}_\varphi = \mathcal{B}_\varphi = M'\xi_\varphi\,.$$

In the present section we are going to analyze the classes of affiliated subspaces of the space $\mathcal{S}_\varphi$. First we shall describe the structure $A_M(\mathcal{S}_\varphi)$. For this, observe that given a subspace $X \in A_M(\mathcal{S}_\varphi)$, there is a linear subspace J'_X of M' such that

$$X = \{u\xi_\varphi : u \in J'_X\} = J'_X\xi_\varphi\,.$$

Moreover, since X is affiliated with M, we see that J'_X must be a left ideal in M'. Conversely, given a left ideal J' of M', the space

$$X = J'\xi_\varphi$$

lies in $A_M(\mathcal{S}_\varphi)$. The correspondence $X \to J'_X$ is a one-to-one correspondence between affiliated subspaces and left ideals which preserves the order in both directions.

In other words, the structure of affiliated subspaces $A_M(\mathcal{S}_\varphi)$, ordered by set inclusion, is isomorphic to the structure of left ideals in M'. Moreover, since M' is canonically isomorphic to M, we can say that $(A_M(\mathcal{S}_\varphi), \subset)$ is nothing but the left ideal structure in M. Having this in mind, the following natural question arises: What are the ideals in M' (and so in M) that correspond to closed affiliated subspaces? One result in this direction, obtained in [11], says that every element in $L_M(\mathcal{S}_\varphi)$ is represented by an ideal closed in the strong operator topology. We repeat this argument for the convenience of the reader.

PROPOSITION 2.1. *If $X \in L_M(\mathcal{S}_\varphi)$, then there is a unique projection $p' \in M'$ such that*

$$X = M'p'\xi_\varphi = \{x'p'\xi_\varphi : x' \in M'\}\,.$$

Proof. It is enough to show that the corresponding left ideal J'_X of M' is strongly operator closed. Take a net $(x'_\alpha) \subset J'_X$ converging strongly to $x' \in M'$. Then $(x'_\alpha\xi_\varphi)$ converges to $x'\xi_\varphi$. By closedness of X we see that $x'\xi_\varphi \in X$, meaning that $x' \in J'_X$. Since J'_X is a strongly closed left ideal, there is a (unique) projection p' in M' such that $J'_X = M'p'$. ∎

Proposition 2.1 motivates the following definition. We say that an affiliated subspace $X = J'_X\xi_\varphi \in A_M(S_\varphi)$ is *operator closed* if the corresponding left ideal J'_X in M' is closed in the strong operator topology. We shall denote by $H_M(\mathcal{S}_\varphi)$ the set of all operator closed affiliated subspaces. By the previous result the inclusion

$$L_M(\mathcal{S}_\varphi) \subset H_M(\mathcal{S}_\varphi)$$

holds universally. We shall clarify the relationship between $L_M(\mathcal{S}_\varphi)$ and $H_M(\mathcal{S}_\varphi)$ later on. In this moment, let us observe that the ordered structure $(H_M(\mathcal{S}_\varphi), \subset)$ is a complete lattice isomorphic to the projection lattices $P(M)$ and $P(M')$. Now we are going to characterize the orthogonal complement of a subspace in $L_M(S_\varphi)$, which enables us to describe the class $F_M(\mathcal{S}_\varphi)$. It turns out that this description is possible in terms of support projections of the normal functionals obtained from φ. We recall basic notation and terminology (for details see [15]). Suppose that ϱ is a normal functional on a von Neumann algebra M and $a \in M$. We denote by

$$a\varrho = \varrho(\,\cdot\, a), \quad \varrho a = \varrho(a\, \cdot\,)$$

the normal functionals obtained by substituting $x \to xa$ and $x \to ax$ into arguments of ϱ, respectively. It can be proved that there is a smallest projection $e \in M$ such that $\varrho = e\varrho$. The projection e is called the *left support* of ϱ and it is denoted by $s_1(\varrho)$.

LEMMA 2.2. *For each $x \in M$ we have*

$$\varphi(x) = \varphi(x'^{*})\,.$$

Proof. By an easy computation and using basic facts of the modular theory we obtain:

$$\varphi(x) = \langle x\xi_\varphi, \xi_\varphi\rangle = \langle xJ_\varphi\xi_\varphi, J_\varphi\xi_\varphi\rangle = \langle \xi_\varphi, J_\varphi x J_\varphi \xi_\varphi\rangle = \omega_{\xi_\varphi}(x'^{*}) = \varphi(x'^{*})\,. \quad ∎$$

PROPOSITION 2.3. *Let $p, q \in P(M)$. Then the spaces $M'p'\xi_\varphi$ and $M'q'\xi_\varphi$ are orthogonal if and only if*

$$\varphi(qxp) = 0 \quad \textit{for all } x \in M\,.$$

Proof. Supposing that $M'p'\xi_\varphi \perp M'q'\xi_\varphi$, we take $x \in M$ and compute (see Lemma 2.2)

$$\varphi(qxp) = \varphi(p'x^{*\prime}q') = \langle p'x^{*\prime}q'\xi_\varphi, \xi_\varphi\rangle = \langle x'^{*}q'\xi_\varphi, p'\xi_\varphi\rangle = 0\,.$$

The reverse implication follows directly from the previous computation. ∎

THEOREM 2.4. *Suppose that $p \in P(M)$. Then*

$$(M'p'\xi_\varphi)^{\perp} = M'q'\xi_\varphi\,,$$

where $q = 1 - s_1(\varphi p)$.

Proof. First observe that the space $(M'p'\xi_\varphi)^{\perp}$ is closed and therefore operator closed. So there is a projection $q' \in M'$ such that

$$(M'p'\xi_\varphi)^{\perp} = M'q'\xi_\varphi\,.$$

In view of Proposition 2.3 the projection q is the largest projection such that

$$\varphi(pxq) = 0 \quad \text{for all } x \in M\,.$$

In other words, $1 - q$ is the smallest projection such that

$$\varphi(px(1-q)) = \varphi(px) \quad \text{for all } x \in M\,,$$

giving immediately that

$$1 - q = s_1(\varphi p)\,. \quad ∎$$

COROLLARY 2.5. *Each orthogonally closed space $X \in F_M(\mathcal{S}_\varphi)$ is of the form*

$$X = M'(1 - s_1(\varphi p))'\xi_\varphi\,,$$

for some projection $p \in M$.

Proof. It follows immediately from the previous theorem and the fact that every orthogonally closed affiliated subspace is of the form $(M'p'\xi_\varphi)^\perp$ for some projection $p \in M$. ∎

In addition to the foregoing corollary one can say that for each orthogonally closed subspace $X \in F_M(S_\varphi)$ there is a unique projection $p \in P(M)$ such that

$$X = M'p'\xi_\varphi \quad \text{and} \quad p = 1 - s_1(\varphi(1 - s_1(\varphi p)))\,.$$

An easy description of the orthogonality relation in $A_M(\mathcal{S}_\varphi)$ can be given in terms of the Radon–Nikodym derivative. Suppose that M is a semifinite von Neumann algebra and so that it admits a normal semifinite faithful trace τ. Suppose further that there is a bounded positive operator $t \in M$ such that

$$\varphi(x) = \tau(tx) \quad \text{for each } x \in M\,.$$

In this special situation there is a lucid description of orthogonality and orthogonal complement in terms of the trace class operator t.

THEOREM 2.6. *Let $p, q \in P(M)$ and $\varphi = \tau(t\cdot)$, where $t \in M^+$ and τ is a semifinite faithful normal trace on M. Then the following statements hold:*

(i) *$M'p'\xi_\varphi \perp M'q'\xi_\varphi$ if and only if $ptq = 0$.*

(ii) *$(M'p'\xi_\varphi)^\perp = M'r'\xi_\varphi$, where r is the orthogonal projection onto* Ker *pt.*

Proof. (i) As we know $M'p'\xi_\varphi \perp M'q'\xi_\varphi$ if and only if for each $x \in M$,

$$0 = \varphi(pxq) = \tau(tpxq) = \tau(qtpx)\,.$$

By substituting $x = (qtp)^*$ in the previous equality we have

$$\tau(|ptq|^2) = 0\,.$$

Therefore $|ptq| = 0$ and so $ptq = 0$.

(ii) By (i) $(M'p'\xi_\varphi)^\perp = M'r'\xi_\varphi$, where r must be the largest projection such that $ptr = 0$. This means that r is the projection onto Ker pt. ∎

Let us remark that the foregoing result generalizes result of [11], where Theorem 2.6 is proved in a special case when M is a type I factor.

We shall now return to the relationship between closed and operator closed affiliated subspaces. As we know from the previous discussion

$$L_M(\mathcal{S}_\varphi) \subset H_M(\mathcal{S}_\varphi)\,.$$

It was shown in the previous work [11] that $L_M(\mathcal{S}_\varphi) \neq H_M(\mathcal{S}_\varphi)$ provided that M is an infinite-dimensional type I factor with a separable predual. On the other hand, it was established in the same paper that $L_M(\mathcal{S}_\varphi) = H_M(\mathcal{S}_\varphi)$ on condition that φ is a tracial state. First we shall prove that the same is true if φ is close to a trace in the sense that the modular operator Δ_φ is bounded.

PROPOSITION 2.7. *If Δ_φ is bounded, then*

$$E_M(\mathcal{S}_\varphi) = H_M(\mathcal{S}_\varphi)\,.$$

Proof. Suppose that Δ_φ is bounded. Combining Lemma 1.1 and Proposition 0.1 we know that $E_M(\mathcal{S}_\varphi) = L_M(\mathcal{S}_\varphi)$. It remains to show that the space $M'p'\xi_\varphi$ is closed in $M'\xi_\varphi$ for every $p \in P(M)$. For this suppose that there is a sequence $(x_n) \subset M$ and $x \in M$ such that

$$x_n'p'\xi_\varphi \to x'\xi_\varphi \quad \text{as } n \to \infty\,.$$

Then

$$J_\varphi x_n' p' J_\varphi \xi_\varphi \to J_\varphi x' J_\varphi \xi_\varphi \quad \text{as } n \to \infty\,.$$

In other words,

$$x_n p \xi_\varphi \to x\xi_\varphi\,. \tag{2.1}$$

By Lemma 1.1 the map $\overline{x} = x\xi_\varphi \to \overline{xp} = xp\xi_\varphi$ is continuous. According to (2.1)

$$x_n pp\xi_\varphi \to xp\xi_\varphi\,, \quad \text{as } n \to \infty\,,$$

which means that $xp\xi_\varphi = x\xi_\varphi$ and so $xp = x$. Finally, $x'p' = x'$ and so $x'\xi_\varphi \in M'p'\xi_\varphi$. ∎

As a very special case of the previous proposition we can see that all types of the subspaces coincide provided that the underlying algebra is abelian. Next proposition says that this is far from being true for properly infinite algebras.

PROPOSITION 2.8. *Suppose that M is a properly infinite von Neumann algebra and φ a faithful normal state on M. Then there is a projection $p \in M$ such that the space $M'p'\xi_\varphi$ is not closed in $\mathcal{S}_\varphi$.*

Proof. By the structure theory of von Neumann algebras there is a unital von Neumann subalgebra N of M which is isomorphic to the algebra $B(K)$, where K is an infinite dimensional separable Hilbert space. Let us denote by tr the canonical trace on $B(K)$. The state φ restricts to a (normal) state on N. There is a positive trace class operator $t \in N$ such that

$$\varphi(x) = \mathrm{tr}(tx)\,, \quad x \in N\,.$$

We shall prove that there is a projection $p \in N$ with $1 - p \neq 0$ and a sequence (x_n) of operators in $N = B(K)$ such that

$$\mathrm{tr}(t(x_n p - 1)^*(x_n p - 1)) \to 0 \quad \text{as } n \to \infty\,. \tag{2.2}$$

Since φ is faithful, the operator $t^{1/2}$ is one-to-one. On the other hand, by faithfulness of φ t has a sequence of nonzero eigenvalues going to zero and so the inverse of $t^{1/2}$ cannot be continuous. By the Inverse Mapping Theorem it means that the range of $t^{1/2}$ (as it acts on K) is not the whole space K. Pick now a vector $k \in K$ such that k is not in the range of $t^{1/2}$. Let p be the projection (in $B(K) \subset M$) which projects onto the hyperplane $\{k\}^{\perp}$. Then obviously,

$$\operatorname{Ker} pt^{1/2} = 0\,. \tag{2.3}$$

Let us now consider the polar decomposition

$$pt^{1/2} = u|pt^{1/2}|\,,$$

where u is a partial isometry. By (2.3) the range projection of $(pt^{1/2})^* = t^{1/2}p$ is the unit. Therefore $u^*u = 1$. For an integer n we shall denote by g_n the characteristic function of the interval $\langle 1/n, n\rangle$ and by f_n the function $f_n(x) = (1/x)g_n(x)$. We employ the function calculus. Let p_n be a projection $p_n = g_n(|pt^{1/2}|)$. Put

$$x_n = t^{1/2} f_n(|pt^{1/2}|)u^*\,.$$

Then

$$x_n pt^{1/2} = t^{1/2} f_n(|pt^{1/2}|)u^*u|pt^{1/2}| = t^{1/2}p_n\,.$$

Observe that $p_n \nearrow 1$. Whence,

$$\begin{aligned}\operatorname{tr}(tx_n p) &= \operatorname{tr}(t^{1/2}x_n pt^{1/2}) = \operatorname{tr}(tp_n) \nearrow \operatorname{tr}(t) = 1,\\ \operatorname{tr}(tpx_n^* x_n p) &= \operatorname{tr}(t^{1/2}px_n^*x_n pt^{1/2}) = \operatorname{tr}(p_n tp_n) = \operatorname{tr}(tp_n) \nearrow 1\,.\end{aligned}$$

As a consequence,

$$\operatorname{tr}(t(x_n p - 1)^*(x_n p - 1)) = 1 + \operatorname{tr}(tpx_n^* x_n p) - 2\operatorname{Re}(\operatorname{tr}(tx_n p)) \to 0, \quad n \to \infty\,.$$

Therefore (2.2) has been established. The equality (2.2) implies that

$$x'_n p' \xi_\varphi \to \xi_\varphi\,, \quad \text{as } n \to \infty\,.$$

Indeed, using Lemma 2.2 we see

$$\begin{aligned}\|x'_n p'\xi_\varphi - \xi_\varphi\|^2 &= \varphi((x'_n p' - 1)^*(x'_n p' - 1)) = \varphi((x_n p - 1)^*(x_n p - 1))\\ &= \operatorname{tr}(t(x_n p - 1)^*(x_n p - 1)) \to 0 \quad \text{as } n \to 0\,.\end{aligned}$$

We have shown that ξ_φ is in the closure of $M'p'\xi_\varphi$. This means that $M'p'\xi_\varphi$ is dense in $\mathcal{S}_\varphi = M'\xi_\varphi$. However, as $p' \neq 1$, $M'p'\xi_\varphi$ is a proper subspace of $M'\xi_\varphi$. In other words,

$$M'p'\xi_\varphi \in H_M(\mathcal{S}_\varphi) \setminus L_M(\mathcal{S}_\varphi)\,. \quad \blacksquare$$

THEOREM 2.9. *Let M be a σ-finite von Neumann algebra. The following conditions are equivalent:*

(i) *M is not finite.*

(ii) *$L_M(\mathcal{S}_\varphi) \neq H_M(\mathcal{S}_\varphi)$ for each faithful normal state φ on M.*

Proof. (i) $\Rightarrow$ (ii) If M is not finite, then there is a nonzero properly infinite direct summand of M. So M decomposes as

$$M = zM \oplus (1-z)M\,,$$

where z is a central projection, not equal to 1, such that zM is finite and $(1-z)M$ is properly infinite. Let φ be a faithful normal state on M. We shall continue to identify M and $\pi_\varphi(M)$. Since the modular data for (M,φ) are given by direct sums of the corresponding modular data of the restriction of φ to zM and $(1-z)M$, respectively, we see that

$$\xi_\varphi = z\xi_\varphi + (1-z)\xi_\varphi\,,$$

where $z\xi_\varphi$ and $(1-z)\xi_\varphi$ are separating and generating vectors for the algebra zM (living on $z(\mathcal{H}_\varphi)$) and $(1-z)M$ (living on $(1-z)(\mathcal{H}_\varphi)$), respectively. By Proposition 2.8 and its proof there is a proper subprojection $p \leqslant 1-z$ such that

$$(1-z)\xi_\varphi \in \overline{(1-z')p'M'} = \overline{p'M'}\,.$$

So for $q = z+p$ we have

$$q' \neq 1 \quad \text{and} \quad \xi_\varphi \in \overline{M'q'\xi_\varphi}\,.$$

Hence, $M'q'\xi_\varphi \in H_M(\mathcal{S}_\varphi) \setminus L_M(\mathcal{S}_\varphi)$.

(ii) $\Rightarrow$ (i) If M is finite, then it admits a normal faithful tracial state τ on M. But in this case $H_M(\mathcal{S}_\tau) = L_M(\mathcal{S}_\tau)$, which is a special case of Proposition 2.7. ∎

Let us remark that by combining the previous results we obtain the following fact. If a von Neumann algebra M admits a faithful normal state which has a bounded modular operator, then M has to finite.

3. COMPLETE AFFILIATED SUBSPACES

In the concluding part we shall characterize complete affiliated spaces. We restrict our discussion to the case $\mathcal{S}_\varphi$, where φ is a normal faithful state on M.

THEOREM 3.1. *Let $p \in P(M)$ be nonzero. The space $M'p'\xi_\varphi$ is complete if and only if p is a sum of finitely many atomic projections.*

Proof. Suppose that the space $M'p'\xi_\varphi$ is complete. It means that the left ideal $I = M'p'$ endowed with the norm

$$\|x\|_\varphi = \sqrt{\varphi(x^*x)}\,, \quad x \in I\,,$$

is a Hilbert space. There is an obvious inequality between the von Neumann algebra norm and the Hilbert space norm $\|\cdot\|_\varphi$:

$$\|x\|_\varphi^2 = \varphi(x^*x) \leqslant \|x^*x\| = \|x\|^2\,, \quad x \in I\,.$$

Thanks to this inequality both norms have to be equivalent by the Inverse Mapping Theorem. In other words, I is a reflexive space and its closed subspace

$I \cap I^* = p'M'p'$ is reflexive as well. However, it means that the hereditary subalgebra $p'M'p'$ is finite dimensional. Therefore p' is a finite sum of orthogonal atomic projections. By the isomorphism between M and M' we conclude that the same must hold for p.

Suppose now that p is a finite sum of atomic projections. The same holds for p'. The central cover of the atomic projection is an atomic projection in the center. Hence

$$p' = p'_1 + \cdots + p'_n\,,$$

such that the following holds: each p'_i is a sum of orthogonal atomic projections and the central covers $c(p'_1), c(p'_2), \ldots, c(p'_n)$ are mutually orthogonal atomic projections in the center of M'. Therefore $M'c(p'_i)$ is a type I factor for all i. We shall prove that each space $X'_i = M'p'_i\xi_\varphi$ is complete. Assume that this space is not zero. As p'_i is a sum of finitely many atomic projections we have that the vector functional ϱ_i corresponding to the vector $p'_i\,\xi_\varphi$ restricts to a finite combination of pure states on $M'c(p'_i)$. Each space X'_i is unitarily isomorphic to the GNS inner product space corresponding to this restriction of ϱ_i. But such a space is complete by e.g. [5]. As $M'p'\xi_\varphi$ is a direct sum of complete spaces it must be complete. ∎

It has been proved in [11] that for $M = B(H)$ (where H is separable and infinite dimensional) the affiliated subspace $M'p'\xi_\varphi$ is splitting if the projection p (acting on H) is finite dimensional. The previous result says that the space $M'p'\xi_\varphi$ is not only splitting. In fact, it is even complete, which is a much stronger statement. Moreover, in the light of the previous result, only finite dimensional projections in $B(H)$ generate complete affiliated subspaces in $\mathcal{S}_\varphi$. In order to illustrate this situation, we shall now give a concrete picture of complete subspaces affiliated with the type I factor. Let H be a separable infinite dimensional Hilbert space and φ a normal faithful state on $B(H)$. Then there is an orthonormal basis (e_n) and a sequence of strictly positive numbers (λ_k) such that $\sum\limits_{k=1}^{\infty} \lambda_k = 1$ and

$$\varphi = \sum_{k=1}^{\infty} \lambda_k \omega_{e_k}\,.$$

We shall use the following realization of the modular data for $(B(H), \varphi)$:

$$\mathcal{H}_\varphi = H \otimes H\,, \quad \xi_\varphi = \sum_{k=1}^{\infty} \lambda_k^{1/2}(e_k \otimes e_k)\,.$$

$M = B(H)$ will be identified with $B(H) \otimes 1$ and so $M' = 1 \otimes B(H)$. Each operator $x \in B(H)$ can be represented by a matrix with respect to the basis (e_n). We shall denote by $\widetilde{x}$ the operator whose matrix results from entrywise complex conjugation of the matrix of x. The modular conjugation acts as

$$j(x) = J_\varphi(x \otimes 1)J_\varphi = 1 \otimes \widetilde{x}\,.$$

The space $\mathcal{S}_\varphi$ becomes

$$\mathcal{S}_\varphi = M'\xi_\varphi = \Big\{ \sum_{k=1}^{\infty} \lambda_k^{1/2}(e_k \otimes xe_k) : x \in B(H) \Big\}.$$

PROPOSITION 3.2. *Let $p \in B(H)$ be a finite dimensional nonzero projection. Then there is an orthogonal sequence $h_1, \dots, h_n$ in H such that*

$$M'p'\xi_\varphi = (h_1 \otimes H) \oplus (h_2 \otimes H) \oplus \cdots \oplus (h_n \otimes H).$$

Proof. Write

$$p = p_{\xi_1} + p_{\xi_2} + \cdots + p_{\xi_s},$$

where $\xi_1, \dots, \xi_s$ is an orthonormal sequence and p_{ξ_i} the projection with range $sp(\xi_i)$. By $\widetilde{\xi_i}$ we shall denote the element of H whose coordinates with respect to the basis (e_n) are term by term complex conjugate of that of ξ_i. We can compute

$$M'p'\xi_\varphi = \{(1 \otimes x)(1 \otimes \widetilde{p})\xi_\varphi : x \in B(H)\} = \Big\{ \sum_{k=1}^{\infty} \lambda_k^{1/2}(e_k \otimes x\widetilde{p}e_k) : x \in B(H) \Big\}$$
$$= \Big\{ \sum_{k=1}^{\infty} \lambda_k^{1/2}\Big(e_k \otimes \sum_{i=1}^{s} \langle e_k, \widetilde{\xi_i}\rangle x\widetilde{\xi_i}\Big) : x \in B(H) \Big\}.$$

Since the collection $\widetilde{x}\widetilde{\xi}_1, \widetilde{x}\widetilde{\xi}_2, \dots, \widetilde{x}\widetilde{\xi}_s$, where x runs through all $B(H)$ may be an arbitrary collection of the vectors in H, we have

$$M'p'\xi_\varphi = \Big\{ \sum_{k=1}^{\infty} \lambda_k^{1/2}\Big(e_k \otimes \sum_{i=1}^{s} \langle e_k, \widetilde{\xi_i}\rangle\, v_i\Big) : v_1, \dots, v_s \in H \Big\}$$
$$= (u_1 \otimes H) + (u_2 \otimes H) + \cdots + (u_s \otimes H),$$

where

$$u_i = \sum_{k=1}^{\infty} \lambda_k^{1/2} \langle e_k, \widetilde{\xi_i}\rangle e_k.$$

Employing the Gramm–Schmidt orthogonalization process we can find an orthonormal sequence $h_1, \dots, h_n$ giving the same linear span as the sequence $u_1, \dots, u_s$. Then

$$M'p'\xi_\varphi = (h_1 \otimes H) \oplus (h_2 \otimes H) \oplus \cdots \oplus (h_n \otimes H). \quad \blacksquare$$

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