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# CENTRALIZERS AND JORDAN DERIVATIONS FOR CSL SUBALGEBRAS OF VON NEUMANN ALGEBRAS

PENGTONG LI, DEGUANG HAN, and WAI-SHING TANG

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ABSTRACT. We investigate the centralizers and Jordan derivations for commutative subspace lattice algebras in von Neumann algebras. For any CSL subalgebra $\mathcal{A}$ of a von Neumann algebra, we prove that a (weak) Jordan centralizer $\Phi$ (i.e $\Phi : \mathcal{A} \to \mathcal{A}$ is an additive mapping satisfying $2\Phi(A^2) = \Phi(A)A + A\Phi(A)$ for all $A \in \mathcal{A}$) is automatically a centralizer. Similarly, we show that every Jordan derivation of $\mathcal{A}$ is a derivation. Additionally, we obtain concrete characterizations of centralizers for standard subalgebras of CSL algebras, and a stronger result is also obtained for standard subalgebras of nest algebras.

KEYWORDS: *Centralizers, Jordan derivations, CSL algebras, von Neumann algebras.*

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## 1. INTRODUCTION AND STATEMENTS OF THE RESULTS

The investigation for derivations and Jordan derivations of operator algebras has been an interesting topic since Kaplansky [19] started the research in this area by proving that every derivation of a type I $W^*$-algebra is inner. Kadison then continued the investigation in [18] for von Neumann algebras and he proved that every derivation on a von Neumann algebra is inner, and also proved that every derivation of a $C^*$-algebra on a Hilbert space is spatial. The complete solution for the derivation problem for $W^*$-algebra was settled by Sakai [26] who proved that every derivation of a $W^*$-algebra is inner. All these results mentioned above are concerned with self-adjoint operator algebras, and the same techniques in general do not apply to non-self-adjoint operator algebras. We refer to [7], [9], [13], [14], [15], [18], [21], [22], [25], [31] for a good resource on the results about the derivation and related problem for non-self-adjoint operator algebras. In particular we mention that Christensen [7] proved that every derivation of a nest algebra is inner. For a more general case, Gilfeather and Moore [13], [25] studied the spatiality and quasi-spatiality of derivations of certain CSL algebras. Clearly, by definition every derivation must be a Jordan derivation. The converse is not necessarily true in general. A classical result of Herstein states that every Jordan derivation on a prime ring of characteristic not two is a derivation [16]. Cusack [8] generalized Herstein's result to the case of semiprime rings (see also [6] for an alternative proof). It follows immediately that every Jordan derivation of semisimple Banach algebras is a derivation. However, most of the well studied non-self-adjoint operator algebras are not semisimple. Zhang [33] proved that every Jordan derivation of nest algebras is a derivation, and this was recently generalized by Lu [24] to CSL algebras. For several results concerning Jordan derivations of self-adjoint operator algebras, we refer to [2], [3], [17].

A related problem in this area is to characterize the centralizers for various operator algebras. For example, the theory of centralizers (also called multipliers) for $C^*$-algebras and some Banach algebras has been relatively well studied in the literature (see e.g. [1], [4] and references therein). Centralizers have also been studied in the general framework of prime rings and semiprime rings by Zalar [32] and more recently by Vukman [27], [28], [29], and Vukman and Kosi-Ulbl [20], [30], et al. Similar to Jordan derivations, there is a notion of Jordan centralizers which have been investigated recently mainly on prime or semiprime rings. However, to our knowledge nothing has been done for non-self-adjoint operator algebras which in most cases are not semiprime. In this paper we will focus on the study of centralizers and Jordan derivations on CSL algebras and more generally on CSL subalgebras of von Neumann algebras.

Let us first recall and introduce some notations and terminologies. Throughout, all algebras and vector spaces will be over $\mathbb{F}$, where $\mathbb{F}$ is either the real field or the complex field. Let $H$ be a Hilbert space. We denote by $\mathcal{B}(H)$ the algebra of all bounded linear operators on $H$ and by $I$ the identity operator on $H$. The terms "projection" and "subspace" will mean "orthogonal projection" and "norm closed linear manifold", respectively. For convenience, we will identify the subspace and the projection onto it. Let $\mathcal{L}$ be a subspace lattice on $H$, i.e., it is a strongly closed lattice of projections (or subspaces) that is closed under the usual lattice operations $\vee$ (closed linear span) and $\wedge$ (set theoretic intersection) and contains 0 and $I$. Denote Alg$\mathcal{L}$ by the algebra of all operators in $\mathcal{B}(H)$ which leave every projection in $\mathcal{L}$ invariant, that is

$$\mathrm{Alg}\mathcal{L} = \{T \in \mathcal{B}(H) : P^\perp TP = 0 \ \text{ for all } \ P \in \mathcal{L}\},$$

where $P^\perp = I - P$. Clearly, Alg$\mathcal{L}$ is a weakly closed unital algebra. We call a subspace lattice $\mathcal{L}$ a commutative subspace lattice, or CSL, if all the projections in $\mathcal{L}$ commute pairwise. In this case the associated algebra Alg$\mathcal{L}$ is called a CSL algebra. If a CSL $\mathcal{L}$ is linearly ordered by inclusion, then $\mathcal{L}$ is called a nest and Alg$\mathcal{L}$ is called a nest algebra. Putting $\mathcal{L}^\perp = \{P^\perp : P \in \mathcal{L}\}$, then $\mathcal{L}^\perp$ is a CSL if and only if $\mathcal{L}$ is a CSL. Write $\mathcal{L}'$ and $\mathcal{L}''$ for the commutant and bicommutant of $\mathcal{L}$, respectively. Then for every CSL $\mathcal{L}$, we have that $\mathcal{L} \subseteq \mathcal{L}'' \subseteq \mathcal{L}' = \mathrm{Alg}\mathcal{L} \cap$

$(\mathrm{Alg}\mathcal{L})^*$, $(\mathrm{Alg}\mathcal{L})^* = \mathrm{Alg}\mathcal{L}^\perp$ and $\mathcal{L}''$ is the von Neumann algebra generated by $\mathcal{L}$, where $(\mathrm{Alg}\mathcal{L})^* = \{A^* : A \in \mathrm{Alg}\mathcal{L}\}$ in which $A^*$ denotes the adjoint of operator $A$. For a subalgebra $\mathcal{A}$ of $\mathcal{B}(H)$, let $\mathrm{Lat}\mathcal{A}$ be the collection of all the projections which are invariant by each operator in $\mathcal{A}$. Obviously, $\mathrm{Lat}\mathcal{A}$ is a subspace lattice. We will say that an algebra $\mathcal{A}$ is reflexive if $\mathrm{AlgLat}\mathcal{A} = \mathcal{A}$ and dually, a subspace lattice $\mathcal{L}$ is reflexive if $\mathrm{LatAlg}\mathcal{L} = \mathcal{L}$. Subspace lattices need not be reflexive. However, it is well known that every CSL is reflexive ([5], Theorem 1.6.3).

Let $\mathfrak{B}$ be a von Neumann algebra on a Hilbert space $H$, $\mathcal{L}$ be a CSL in $\mathfrak{B}$, and let $\mathrm{Alg}_{\mathfrak{B}}\mathcal{L} = (\mathrm{Alg}\mathcal{L}) \cap \mathfrak{B}$. We call $\mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$ a CSL subalgebra of von Neumann algebra $\mathfrak{B}$. Then $\mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$ is a reflexive algebra since it is the intersection of two reflexive algebras. If we let $\mathcal{M}$ denote the lattice of projections in $\mathfrak{B}'$, then $\mathrm{Alg}_{\mathfrak{B}}\mathcal{L} = \mathrm{Alg}(\mathcal{L} \vee \mathcal{M})$, where $\mathcal{L} \vee \mathcal{M}$ denotes the subspace lattice generated by $\mathcal{L}$ and $\mathcal{M}$ which is usually noncommutative. Gilfeather and Larson [11], [12] introduced and studied nest subalgebra of von Neumann algebras, and they remarked that in [11], they had shown that if $\mathcal{L}$ is a nest then the subspace lattice $\mathcal{L} \vee \mathcal{M}$ is reflexive (unpublished). For the general case, it seems still unkown whether $\mathcal{L} \vee \mathcal{M}$ is reflexive. Fortunately, our research does not depend on the reflexivity of $\mathcal{L} \vee \mathcal{M}$. This is different from a general study of CSL algebras, in which the reflexivity of CSL is often used; for example, see [24].

Let $\mathcal{R}$ be an associative ring. Recall that $\mathcal{R}$ is prime if $a\mathcal{R}b = (0)$ implies $a = 0$ or $b = 0$, and is semiprime if $a\mathcal{R}a = (0)$ implies $a = 0$. An additive mapping $\delta : \mathcal{R} \to \mathcal{R}$ is called a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ holds for all $x, y \in \mathcal{R}$, and is called a Jordan derivation in case $\delta(x^2) = \delta(x)x + x\delta(x)$ for all $x \in \mathcal{R}$. An additive mapping $\Phi : \mathcal{R} \to \mathcal{R}$ is called a left (right) centralizer in case $\Phi(xy) = \Phi(x)y$ ($\Phi(xy) = x\Phi(y)$) holds for all pairs $x, y \in \mathcal{R}$. Call $\Phi$ a centralizer in case $\Phi$ is a both left and right centralizer. Obviously, if $a \in \mathcal{R}$ then $L_a(x) = ax$ ($R_a(x) = xa$) is a left (right) centralizer and conversely, a left (right) centralizer must be of this form in case $\mathcal{R}$ has an identity element. For $x, y \in \mathcal{R}$, the usual Jordan product is given by $x \circ y = xy + yx$. Then clearly, a Jordan derivation is an additive mapping $\delta : \mathcal{R} \to \mathcal{R}$ which satisfies $\delta(x \circ y) = \delta(x) \circ y + x \circ \delta(y)$ for all $x, y \in \mathcal{R}$. Therefore, we can define a Jordan centralizer to be an additive mapping $\Phi : \mathcal{R} \to \mathcal{R}$ which satisfies $\Phi(x \circ y) = \Phi(x) \circ y = x \circ \Phi(y)$, i.e.,

$$\Phi(xy + yx) = \Phi(x)y + y\Phi(x) = x\Phi(y) + \Phi(y)x$$

for all $x, y \in \mathcal{R}$. Since the product $\circ$ is commutative, there is no difference between the left and right Jordan centralizers. Note that, every centralizer $\Phi$ is a Jordan centralizer, and every Jordan centralizer $\Phi$ satisfies the relation

$$2\Phi(x^2) = \Phi(x)x + x\Phi(x) \tag{1.1}$$

for all $x \in \mathcal{R}$. Similar to the study of Jordan derivations, it seems natural to ask, whether the converse is true. In [32], Zalar proved that every Jordan centralizer on a semiprime ring of characteristic is not a centralizer two. More generally, Vukman ([27], Theorem 1) showed that if $\mathcal{R}$ is a semiprime ring of characteristic not two, and $\Phi : \mathcal{R} \to \mathcal{R}$ be an additive mapping satisfying the equality (1.1), then $\Phi$ is a centralizer.

Our first main result shows that Vukman's result for semiprime rings remains true for CSL subalgebras of von Neumann algebras.

THEOREM 1.1. *Let $\mathcal{L}$ be a* CSL *in a von Neumann algebra $\mathfrak{B}$ on a Hilbert space $H$, and let $\Phi : \mathrm{Alg}_{\mathfrak{B}}\mathcal{L} \to \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$ be an additive mapping such that*

$$2\Phi(A^2) = \Phi(A)A + A\Phi(A) \tag{1.2}$$

*for all $A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$. Then $\Phi$ is a centralizer.*

We remark that a CSL algebra is not necessarily prime or semiprime. Therefore, Theorem 1.1 is independent of Vukman's result and the proof relies on completely different techniques. Moreover, as Vukman pointed out in [27], the proof of his result is nontrivial even in case the considered semiprime ring has an identity element. In fact, he presented the proof for unital case, in which some classical results in prime and semiprime ring theory were used.

Using similar techniques developed in the proof of Theorem 1.1, we obtain the following result which states that every Jordan derivation on a CSL subalgebra of a von Neumann algebra is automatically a derivation. This generalizes the result of Lu ([24], Theorem 3.2) to a large class of noncommutative subspace lattice algebras.

THEOREM 1.2. *Let $\mathcal{L}$ be a* CSL *in a von Neumann algebra $\mathfrak{B}$ on a Hilbert space $H$. Then every Jordan derivation $\delta$ of $\mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$ into itself is a derivation.*

Our third result is devoted to characterizing the general form of centralizers on standard subalgebras of certain CSL algebras. For a subspace lattice $\mathcal{L}$, we adopt the notations

$$\mathcal{J}(\mathcal{L}) = \{E \in \mathcal{L} : E \neq 0 \text{ and } E_- \neq H\},$$

where $E_- = \bigvee\{F \in \mathcal{L} : E \nsubseteq F\}$. We call a subalgebra $\mathcal{A}$ of $\mathrm{Alg}\mathcal{L}$ standard, if it contains all finite rank operators in $\mathrm{Alg}\mathcal{L}$.

THEOREM 1.3. *Let $\mathcal{L}$ be a* CSL *on a Hilbert space $H$, $\mathcal{A}$ be a standard subalgebra of $\mathrm{Alg}\mathcal{L}$ and $\Phi : \mathcal{A} \to \mathcal{A}$ be a centralizer. Suppose that*

$$\bigvee\{E : E \in \mathcal{J}(\mathcal{L})\} = H \quad \textit{and} \quad \bigwedge\{E_- : E \in \mathcal{J}(\mathcal{L})\} = 0.$$

*Then there exists a densely defined, closed linear operator $T : \mathcal{D} \subseteq H \to H$ with its domain $\mathcal{D}$ invariant under every element of $\mathcal{A}$, such that $\Phi(A)x = TAx = ATx$ for all $A \in \mathcal{A}$ and $x \in \mathcal{D}$*

We will see that the proof of Theorem 1.3 depends heavily on the existence of rank one operators in the considered CSL algebras. Since a von Neumann algebra need not contain rank one operators, we are not able to extend Theorem 1.3 to all CSL subalgebras of von Neumann algebras although we conjecture that the result should hold in general.

A good deal of attention has been lavished on commutative subspace lattices which are also completely distributive. By definition, a completely distributive lattice is a subspace lattice in which distributive laws ($\wedge$ distributing over $\vee$ and vice versa) holds for infinite families of projections. However, there is a tidier description due to Longstaff [23]: A subspace lattice $\mathcal{L}$ is completely distributive if and only if $E = \bigvee\{F \in \mathcal{L} : E \nsubseteq F_-\}$ for every $E \in \mathcal{L}$, which is also equivalent to the condition $E = \bigwedge\{F_- : F \in \mathcal{L} \text{ and } F \nsubseteq E\}$ for every $E \in \mathcal{L}$. Thus if $\mathcal{L}$ is completely distributive, then $\bigvee\{E : E \in \mathcal{J}(\mathcal{L})\} = H$ and $\bigwedge\{E_- : E \in \mathcal{J}(\mathcal{L})\} = 0$. Hence Theorem 1.3 holds for completely distributive CSL algebras and in particular, holds for nest algebras. Moreover, for standard subalgebras of nest algebras, we have the following characterization for even one-sided centralizers.

THEOREM 1.4. *Let $\mathcal{N}$ be a nest on a Hilbert space $H$, and $\mathcal{A}$ be a standard subalgebra of* $\operatorname{Alg}\mathcal{N}$. *If $\Phi$ is a norm continuous left centralizer of $\mathcal{A}$, then there exists an operator $T \in \operatorname{Alg}\mathcal{N}$ with $\|T\| = \|\Phi\|$, such that $\Phi(A) = TA$ for all $A \in \mathcal{A}$. Dually, if $\Phi$ is a norm continuous right centralizer of $\mathcal{A}$, then there exists an operator $S \in \operatorname{Alg}\mathcal{N}$ with $\|S\| = \|\Phi\|$, such that $\Phi(A) = AS$ for all $A \in \mathcal{A}$.*

## 2. PROOFS OF THEOREMS 1.1 AND 1.2

In order to prove Theorems 1.1 and 1.2, we need a decomposition result for CSL subalgebras of von Neumann algebras, which is the key to our proofs. Let $\mathcal{L}$ be a CSL in a von Neumann algebra $\mathfrak{B}$ on a Hilbert space $H$. Denote by

$$
\begin{aligned}
Q_1(\mathcal{L}) &= \overline{\operatorname{span}}\{PAP^\perp x : P \in \mathcal{L}, A \in \operatorname{Alg}_{\mathfrak{B}}\mathcal{L}, x \in H\},\\
Q_2(\mathcal{L}) &= \overline{\operatorname{span}}\{P^\perp A^* P x : P \in \mathcal{L}, A \in \operatorname{Alg}_{\mathfrak{B}}\mathcal{L}, x \in H\},\\
Q &= Q_1(\mathcal{L}) \vee Q_2(\mathcal{L}).
\end{aligned}
$$

Here, $\overline{\operatorname{span}}\{\cdot\}$ is the norm closure of linear span of a set.

LEMMA 2.1. *With the notations as above, we have:*

(i) $Q_1(\mathcal{L}), Q_2(\mathcal{L}), Q \in \mathcal{L}' \cap \mathfrak{B} \subseteq \operatorname{Alg}_{\mathfrak{B}}\mathcal{L}$, *$Q_1(\mathcal{L})$ and $Q_2(\mathcal{L})$ are commuting projections, and $Q^\perp A Q = QAQ^\perp = 0$ for all $A \in \operatorname{Alg}_{\mathfrak{B}}\mathcal{L}$;*

(ii) $Q^\perp(\operatorname{Alg}_{\mathfrak{B}}\mathcal{L})Q^\perp$ *is a von Neumann algebra on $Q^\perp H$, if $Q \neq I$;*

(iii) $\operatorname{Alg}_{\mathfrak{B}}\mathcal{L} = Q(\operatorname{Alg}_{\mathfrak{B}}\mathcal{L})Q \oplus Q^\perp(\operatorname{Alg}_{\mathfrak{B}}\mathcal{L})Q^\perp$.

*Proof.* For convenience, put $Q_1 = Q_1(\mathcal{L})$, $Q_2 = Q_2(\mathcal{L})$.

(i) Clearly, $\mathcal{L}' \cap \mathfrak{B} \subseteq \operatorname{Alg}_{\mathfrak{B}}\mathcal{L}$, $Q_1 \in \operatorname{Lat}\operatorname{Alg}_{\mathfrak{B}}\mathcal{L}$ and $Q_2 \in \operatorname{Lat}(\operatorname{Alg}_{\mathfrak{B}}\mathcal{L})^*$. Thus for every $P \in \mathcal{L}$, we have $PQ_1 = Q_1PQ_1$ and then, $Q_1P = Q_1PQ_1$ by considering adjoint. Hence $Q_1 \in \mathcal{L}'$. Let $B \in \mathfrak{B}'$. For every $A \in \operatorname{Alg}_{\mathfrak{B}}\mathcal{L}$, $P \in \mathcal{L}$ and $x \in H$, noting that $PAP^\perp \in \mathfrak{B}$, we obtain

$$Q_1BPAP^\perp x = Q_1PAP^\perp Bx = PAP^\perp Bx = BPAP^\perp x.$$

It follows from the definition of $Q_1$ that $Q_1BQ_1 = BQ_1$. Since $B^* \in \mathfrak{B}'$, we also have $Q_1B^*Q_1 = B^*Q_1$. Hence $Q_1B = BQ_1$ for all $B \in \mathfrak{B}'$, that is $Q_1 \in \mathfrak{B}$. Similarly, we can prove that $Q_2 \in \mathcal{L}' \cap \mathfrak{B}$. From $Q_1 \in \mathrm{LatAlg}_{\mathfrak{B}}\mathcal{L}$ and $Q_2 \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$ we see that $Q_2Q_1 = Q_1Q_2Q_1$. Consequently, $Q_1Q_2 = Q_2Q_1$ and $Q = Q_1 + Q_2 - Q_1Q_2 \in \mathcal{L}' \cap \mathfrak{B}$.

We now want to prove that $Q^\perp AQ = QAQ^\perp = 0$ for all $A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$. Fix an arbitrary operator $A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$. For every $P \in \mathcal{L}$, from the fact that $Q$ commutes with $P$ and the definitions of $Q_1$ and $Q_2$, we get

$$P(Q^\perp A)P^\perp = Q^\perp PAP^\perp = 0, \quad P(AQ^\perp)P^\perp = (Q^\perp P^\perp A^*P)^* = 0.$$

This means that $Q^\perp A, AQ^\perp \in (\mathrm{Alg}\mathcal{L})^*$. Also, $Q^\perp A, AQ^\perp \in \mathrm{Alg}\mathcal{L}$. Then $Q^\perp A, AQ^\perp \in \mathcal{L}'$. Hence

$$A^*Q^\perp,\ Q^\perp A^* \in \mathcal{L}' \cap \mathfrak{B}. \tag{2.1}$$

Let $T \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$, $P \in \mathcal{L}$. We have

$$Q^\perp A^*PTP^\perp = PQ^\perp A^*TP^\perp = Q_1PQ^\perp A^*TP^\perp = Q_1Q^\perp A^*PTP^\perp = 0,$$

from which it follows that $Q^\perp A^*Q_1 = 0$, that is $Q_1AQ^\perp = 0$. In a similar way, we can prove that $Q_2AQ^\perp = Q^\perp AQ_1 = Q^\perp AQ_2 = 0$. Thus $Q^\perp AQ = QAQ^\perp = 0$ for all $A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$. This proves (i).

(ii) Clearly, $Q^\perp(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})Q^\perp$ is a weakly closed operator algebra on $Q^\perp H$, containing the identity operator $Q^\perp$. Also, from (2.1) we see that $A^*Q^\perp \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$ for all $A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$, which tells us that $Q^\perp(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})Q^\perp$ is self-adjoint. Hence $Q^\perp(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})Q^\perp$ is a von Neumann algebra on $Q^\perp H$.

(iii) Obviously. ∎

Now let us prove Theorem 1.1. For clarity of exposition, we shall organize the proof in a series of lemmas, in which the notations in Theorem 1.1 will be kept.

LEMMA 2.2. *For all $A, B \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$, we have*

$$2\Phi(AB + BA) = \Phi(A)B + A\Phi(B) + \Phi(B)A + B\Phi(A).$$

*Proof.* Replacing $A$ by $A + B$ in (1.2), the desired equality can be obtained easily. ∎

In the coming three lemmas, we introduce some notations for convenience. Let $P \in \mathcal{L} \cup \{Q_1(\mathcal{L}), Q(\mathcal{L})\}$ be an arbitrary nontrivial projection. Put $P_1 = P$ and $P_2 = I - P_1$. Then $P_1, P_2 \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$ and for every $A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$, we have $A = P_1AP_1 + P_1AP_2 + P_2AP_2$, since $P_2AP_1 = 0$. Writing $\mathcal{A}_{11} = P_1(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})P_1$, $\mathcal{A}_{12} = P_1(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})P_2$ and $\mathcal{A}_{22} = P_2(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})P_2$, then

$$\mathrm{Alg}_{\mathfrak{B}}\mathcal{L} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{22}.$$

LEMMA 2.3. *We have:*
(i) $\Phi(A_{ij}) \in \mathcal{A}_{ij}$ *for* $A_{ij} \in \mathcal{A}_{ij}$*, where* $1 \leqslant i \leqslant j \leqslant 2$*;*
(ii) $\Phi(A_{12}) = \Phi(P_1)A_{12} = A_{12}\Phi(P_2)$ *for* $A_{12} \in \mathcal{A}_{12}$*.*

*Proof.* (i) We first claim that

$$\Phi(P_i) = P_i\Phi(P_i)P_i, \quad i = 1, 2. \tag{2.2}$$

In fact, since $P_1$ is idempotent, we have $2\Phi(P_1) = \Phi(P_1)P_1 + P_1\Phi(P_1)$ by the equality (1.2), which yields that $P_1\Phi(P_1)P_2 = P_2\Phi(P_1)P_2 = 0$. Hence $\Phi(P_1) = P_1\Phi(P_1)P_1$. For $i = 2$, the proof goes similarly.

By Lemma 2.2, we have

$$\begin{aligned} 2\Phi(A_{12}) &= 2\Phi(A_{12}P_2 + P_2A_{12}) \\ &= \Phi(A_{12})P_2 + A_{12}\Phi(P_2) + \Phi(P_2)A_{12} + P_2\Phi(A_{12}) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} 2\Phi(A_{12}) &= 2\Phi(P_1A_{12} + A_{12}P_1) \\ &= \Phi(P_1)A_{12} + P_1\Phi(A_{12}) + \Phi(A_{12})P_1 + A_{12}\Phi(P_1). \end{aligned} \tag{2.4}$$

Multiplying (2.3) by $P_1$ from the right side and multiplying (2.4) by $P_2$ from the left side, we get $\Phi(A_{12})P_1 = 0$ and $P_2\Phi(A_{12}) = 0$, respectively. Hence $\Phi(A_{12}) = P_1\Phi(A_{12})P_2 \in \mathcal{A}_{12}$.

For the case of $i = j = 2$, apply Lemma 2.2 again

$$4\Phi(A_{22}) = 2\Phi(A_{22}P_2 + P_2A_{22}) = \Phi(A_{22})P_2 + A_{22}\Phi(P_2) + \Phi(P_2)A_{22} + P_2\Phi(A_{22}).$$

This together with the equality (2.2) gives that $P_1\Phi(A_{22})P_1 = P_1\Phi(A_{22})P_2 = 0$. Accordingly, $\Phi(A_{22}) = P_2\Phi(A_{22})P_2 \in \mathcal{A}_{22}$.

Similarly, we can prove that $\Phi(A_{11}) = P_1\Phi(A_{11})P_1 \in \mathcal{A}_{11}$.

(ii) It is immediate from the results of (i) and equalities (2.3)–(2.4). ∎

LEMMA 2.4. *For* $A_{11} \in \mathcal{A}_{11}, A_{12}, B_{12} \in \mathcal{A}_{12}$ *and* $A_{22} \in \mathcal{A}_{22}$*, we have:*
(i) $\Phi(A_{11}B_{12}) = A_{11}\Phi(B_{12}) = \Phi(A_{11})B_{12}$*;*
(ii) $\Phi(A_{12}B_{22}) = A_{12}\Phi(B_{22}) = \Phi(A_{12})B_{22}$*.*

*Proof.* (i) By Lemma 2.3(ii), we have

$$\Phi(A_{11}B_{12}) = A_{11}B_{12}\Phi(P_2) = A_{11}\Phi(B_{12}).$$

Combining this equality with Lemmas 2.2 and 2.3(i), we get:

$$\begin{aligned} 2\Phi(A_{11}B_{12}) &= 2\Phi(A_{11}B_{12} + B_{12}A_{11}) \\ &= \Phi(A_{11})B_{12} + A_{11}\Phi(B_{12}) + \Phi(B_{12})A_{11} + B_{12}\Phi(A_{11}) \\ &= \Phi(A_{11})B_{12} + \Phi(A_{11}B_{12}). \end{aligned}$$

Consequently, $\Phi(A_{11}B_{12}) = \Phi(A_{11})B_{12}$.

(ii) goes similarly. ∎

LEMMA 2.5. *For $A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$, $B_{12} \in \mathcal{A}_{12}$, we have:*
(i) $\Phi(AB_{12}) = A\Phi(B_{12}) = \Phi(A)B_{12}$;
(ii) $\Phi(B_{12}A) = \Phi(B_{12})A = B_{12}\Phi(A)$.

*Proof.* Since $AP_1 = P_1AP_1$, we have by Lemmas 2.3(i) and 2.4(i)

$$\Phi(AB_{12}) = \Phi(P_1AP_1B_{12}) = P_1AP_1\Phi(B_{12}) = A\Phi(B_{12}).$$

Moreover, applying the same argument as above, we have

$$\begin{aligned}\Phi(A)B_{12} &= \Phi(P_1AP_1)B_{12} + \Phi(P_1AP_2)B_{12} + \Phi(P_2AP_2)B_{12}\\ &= \Phi(P_1AP_1B_{12}) = \Phi(AB_{12}).\end{aligned}$$

This proves (i).

In a similar way, we can prove (ii). ∎

LEMMA 2.6. *For $A, B \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$, we have:*
(i) $(\Phi(AB) - A\Phi(B))Q_1(\mathcal{L}) = 0$;
(ii) $Q_2(\mathcal{L})(\Phi(AB) - A\Phi(B)) = 0$.

*Proof.* Let $T \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}, P \in \mathcal{L}$ be arbitrary elements. By Lemma 2.5(i) we get

$$\begin{aligned}\Phi(AB)PTP^{\perp} &= \Phi(ABPTP^{\perp}) = \Phi(APBPTP^{\perp})\\ &= A\Phi(PBPTP^{\perp}) = A\Phi(BPTP^{\perp}) = A\Phi(B)PTP^{\perp},\end{aligned}$$

from which it follows that (i) holds.

By Lemma 2.5(ii) we get

$$PTP^{\perp}\Phi(AB)=\Phi(PTP^{\perp}AB)=\Phi(PTP^{\perp}AP^{\perp}B)=PTP^{\perp}AP^{\perp}\Phi(B)=PTP^{\perp}A\Phi(B).$$

It follows that $(\Phi(AB) - A\Phi(B))^*Q_2(\mathcal{L}) = 0$, which means that (ii) holds. ∎

LEMMA 2.7. *Suppose that $Q_1(\mathcal{L}) \vee Q_2(\mathcal{L}) = I$. Then $\Phi$ is a right centralizer, that is, $\Phi(AB) = A\Phi(B)$ for all $A, B \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$.*

*Proof.* Denote $Q_1 = Q_1(\mathcal{L})$ and $Q_2 = Q_2(\mathcal{L})$. Define an additive mapping $\Psi : \mathrm{Alg}_{\mathfrak{B}}\mathcal{L} \to \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$ by

$$\Psi(A) = \Phi(A) - A\Phi(Q_1), \quad A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}.$$

Then clearly, $\Phi$ is a right centralizer if and only if $\Psi$ is a right centralizer.

Let $A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$. By Lemma 2.3 we have:

$$\begin{aligned}\Psi(A)Q_1 &= \Phi(A)Q_1 - A\Phi(Q_1)Q_1\\ &= \Phi(Q_1AQ_1)Q_1 + \Phi(Q_1AQ_1^{\perp})Q_1 + \Phi(Q_1^{\perp}AQ_1^{\perp})Q_1 - A\Phi(Q_1)Q_1\\ &= \Phi(AQ_1) - AQ_1\Phi(Q_1) = \Psi(AQ_1).\end{aligned} \tag{2.5}$$

Similarly

$$
\begin{aligned}
(2.6)\quad Q_1^\perp \Psi(A) &= Q_1^\perp \Phi(A) - Q_1^\perp A\Phi(Q_1)\\
&= Q_1^\perp \Phi(Q_1AQ_1) + Q_1^\perp \Phi(Q_1AQ_1^\perp) + Q_1^\perp \Phi(Q_1^\perp AQ_1^\perp) - Q_1^\perp A\Phi(Q_1)\\
&= \Phi(Q_1^\perp A) - Q_1^\perp A\Phi(Q_1) = \Psi(Q_1^\perp A).
\end{aligned}
$$

Suppose that $A, B \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$. By the assumption $Q_1 \vee Q_2 = I$, we have $Q_1^\perp Q_2 = Q_1^\perp$. It follows from Lemma 2.6(ii) that

$$
(2.7)\qquad Q_1^\perp(\Psi(AB) - A\Psi(B)) = Q_1^\perp Q_2(\Phi(AB) - A\Phi(B)) = 0.
$$

By applying Lemma 2.6(i) we can obtain

$$
(2.8)\qquad (\Psi(AB) - A\Psi(B))Q_1 = 0.
$$

Moreover, we can write

$$
\begin{aligned}
AB &= (AQ_1 + Q_1AQ_1^\perp + Q_1^\perp AQ_1^\perp)(BQ_1 + BQ_1^\perp)\\
&= ABQ_1 + AQ_1BQ_1^\perp + Q_1AQ_1^\perp B + Q_1^\perp AB.
\end{aligned}
$$

Now, making use of Lemmas 2.3(i), 2.5 and the equalities (2.5)–(2.8) and $\Phi(Q_1) = Q_1\Phi(Q_1)Q_1$, we see that:

$$
\begin{aligned}
\Psi(AB) &= \Psi(ABQ_1) + \Psi(AQ_1BQ_1^\perp) + \Psi(Q_1AQ_1^\perp B) + \Psi(Q_1^\perp AB)\\
&= \Psi(AB)Q_1 + (\Phi(AQ_1BQ_1^\perp) - AQ_1BQ_1^\perp\Phi(Q_1))\\
&\quad + (\Phi(Q_1AQ_1^\perp B) - Q_1AQ_1^\perp B\Phi(Q_1)) + Q_1^\perp\Psi(AB)\\
&= A\Psi(B)Q_1 + \Phi(AQ_1BQ_1^\perp) + \Phi(Q_1AQ_1^\perp B) + Q_1^\perp A\Psi(B)\\
&= A\Psi(BQ_1) + A\Phi(Q_1BQ_1^\perp) + Q_1AQ_1^\perp\Phi(B) + Q_1^\perp A\Psi(B)\\
&= A\Psi(Q_1BQ_1) + A\Psi(Q_1BQ_1^\perp) + Q_1AQ_1^\perp\Psi(B) + Q_1^\perp AQ_1^\perp\Psi(B)\\
&= A\Psi(Q_1B) + A\Psi(Q_1^\perp B) = A\Psi(B).
\end{aligned}
$$

This proves that $\Psi$ is a right centralizer, and equivalently, so is $\Phi$. ∎

We are now in a position to complete the proof of Theorem 1.1.

*Proof of Theorem* 1.1. Consider the decomposition

$$
\mathrm{Alg}_{\mathfrak{B}}\mathcal{L} = Q(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})Q \oplus Q^\perp(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})Q^\perp
$$

as in Lemma 2.1. Denote by $\Phi_1$ and $\Phi_2$ the restrictions of $\Phi$ on the subalgebras $Q(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})Q$ and $Q^\perp(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})Q^\perp$, respectively. From Lemma 2.3(i) we know that $\Phi_1(QAQ) = Q\Phi_1(QAQ)Q$ and $\Phi_2(Q^\perp AQ^\perp) = Q^\perp\Phi_2(Q^\perp AQ^\perp)Q^\perp$ for all $A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$ and clearly, both $\Phi_1$ and $\Phi_2$ satisfy the equality (1.2). Note that

$$
Q(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})Q = \mathrm{Alg}_{Q\mathfrak{B}Q}(Q\mathcal{L}) = \{T \in Q\mathfrak{B}Q : (Q - QP)TQP = 0 \text{ for all } P \in \mathcal{L}\}
$$

is a CSL subalgebra of the von Neumann algebra $Q\mathfrak{B}Q$. For every $P \in \mathcal{L}, A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}, x \in H$, by Lemma 2.1 we know that $QAQ^{\perp} = 0$ and so $PAP^{\perp}x = QPAP^{\perp}x = PQA(Q - QP)x$. Recalling that

$$Q_1(\mathcal{L}) = \overline{\mathrm{span}}\{PAP^{\perp}x : P \in \mathcal{L}, A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}, x \in H\},$$
$$Q_1(Q\mathcal{L}) = \overline{\mathrm{span}}\{PQA(Q - QP)x : P \in \mathcal{L}, A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}, x \in H\},$$

we have $Q_1(\mathcal{L}) = Q_1(Q\mathcal{L})$. Similarly, $Q_2(\mathcal{L}) = Q_2(Q\mathcal{L})$. Hence

$$Q_1(Q\mathcal{L}) \vee Q_2(Q\mathcal{L}) = Q.$$

By Lemma 2.7, we know that $\Phi_1$ is a right centralizer of $Q(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})Q$. For $\Phi_2$, since any von Neumann algebra is semiprime, from Vukman's result ([27], Theorem 1), it follows that $\Phi_2$ is a centralizer of $Q^{\perp}(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})Q^{\perp}$. Consequently, $\Phi$ is a right centralizer of $\mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$.

Finally, let $A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$. The fact that $\Phi$ is a right centralizer implies that $\Phi(A) = A\Phi(I)$. Also, putting $B = I$ in Lemma 2.2, it is easy to see that $2\Phi(A) = A\Phi(I) + \Phi(I)A$. Therefore, $\Phi(A) = \Phi(I)A = A\Phi(I)$, which means that $\Phi$ is a centralizer on $\mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$. This completes the proof. ∎

Next, we will prove Theorem 1.2. The techniques are similar to those in Theorem 1.1, and the proof is an appropriate modification of that in Theorem 3.2 of [24].

*Proof of Theorem* 1.2. Let $A, B \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$. Since $\delta$ is a Jordan derivation, we have

$$\delta(A^2) = \delta(A)A + A\delta(A), \quad \delta(AB + BA) = A\delta(B) + \delta(A)B + B\delta(A) + \delta(B)A.$$

From $2ABA = A(AB + BA) + (AB + BA)A - (A^2B + BA^2)$ we see that $\delta$ also satisfies

$$\delta(ABA) = \delta(A)BA + A\delta(B)A + AB\delta(A). \tag{2.9}$$

For an arbitrary nontrivial projection $P \in \mathcal{L} \cup \{Q_1(\mathcal{L}), Q(\mathcal{L})\}$, write $P_1 = P$, $P_2 = I - P_1$, $\mathcal{A}_{ij} = P_i(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})P_j$, $1 \leqslant i \leqslant j \leqslant 2$. Then we have the decomposition $\mathrm{Alg}_{\mathfrak{B}}\mathcal{L} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{22}$. From $\delta(P_1) = P_1\delta(P_1) + \delta(P_1)P_1$ we can easily obtain that $P_1\delta(P_1)P_1 = P_2\delta(P_1)P_2 = 0$, and so

$$\delta(P_1) = P_1\delta(P_1)P_2. \tag{2.10}$$

For $A_{12} \in \mathcal{A}_{12}$, by (2.10) we have

$$\begin{aligned}\delta(A_{12}) &= \delta(P_1A_{12} + A_{12}P_1) = \delta(P_1)A_{12} + P_1\delta(A_{12}) + \delta(A_{12})P_1 + A_{12}\delta(P_1)\\ &= P_1\delta(A_{12}) + \delta(A_{12})P_1,\end{aligned}$$

from which it follows that $P_1\delta(A_{12})P_1 = P_2\delta(A_{12})P_2 = 0$. Hence

$$\delta(A_{12}) = P_1\delta(A_{12})P_2. \tag{2.11}$$

For $A_{11} \in \mathcal{A}_{11}$, from

$$2\delta(A_{11}) = \delta(P_1 A_{11} + A_{11} P_1) = \delta(P_1)A_{11} + P_1\delta(A_{11}) + \delta(A_{11})P_1 + A_{11}\delta(P_1)$$

we see that

$$P_2\delta(A_{11}) = 0. \tag{2.12}$$

Similarly, for $A_{22} \in \mathcal{A}_{22}$ we have

$$\delta(A_{22})P_1 = 0. \tag{2.13}$$

Let $A_{12} \in \mathcal{A}_{12}$ and $B \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$. Since $\delta(I) = 0$, by (2.10) we have $\delta(P_2) = -P_1\delta(P_1)P_2 \in \mathcal{A}_{12}$. Applying (2.9), (2.11) and (2.13) we get:

$$\begin{aligned} \delta(A_{12}B) &= \delta(A_{12}P_2BP_2 + P_2BP_2A_{12}) \\ &= \delta(A_{12})P_2BP_2 + A_{12}\delta(P_2BP_2) + \delta(P_2BP_2)A_{12} + P_2BP_2\delta(A_{12}) \\ &= \delta(A_{12})P_2BP_2 + A_{12}\delta(P_2BP_2) \\ &= \delta(A_{12})P_2B + A_{12}\delta(P_2)BP_2 + A_{12}P_2\delta(B)P_2 + A_{12}P_2B\delta(P_2) \\ &= \delta(A_{12})B + A_{12}\delta(B). \end{aligned} \tag{2.14}$$

In a similar way, for $A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$ and $B_{12} \in \mathcal{A}_{12}$, from (2.9), (2.11) and (2.12) one has

$$\delta(AB_{12}) = \delta(A)B_{12} + A\delta(B_{12}). \tag{2.15}$$

We write $Q_1 = Q_1(\mathcal{L}), Q_2 = Q_2(\mathcal{L})$, and let $A, B \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$. For every $T \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$ and $P \in \mathcal{L}$, we have by (2.15)

$$\begin{aligned} \delta(ABPTP^{\perp}) &= \delta(AB)PTP^{\perp} + AB\delta(PTP^{\perp}), \text{ and} \\ \delta(ABPTP^{\perp}) &= \delta(A(PBPTP^{\perp})) = \delta(A)PBPTP^{\perp} + A\delta(PBPTP^{\perp}) \\ &= \delta(A)BPTP^{\perp} + A\delta(BPTP^{\perp}) \\ &= \delta(A)BPTP^{\perp} + A\delta(B)PTP^{\perp} + AB\delta(PTP^{\perp}). \end{aligned}$$

It follows that $(\delta(AB) - \delta(A)B - A\delta(B))PTP^{\perp} = 0$. Consequently

$$(\delta(AB) - \delta(A)B - A\delta(B))Q_1 = 0. \tag{2.16}$$

Similarly, we can prove that

$$Q_2(\delta(AB) - \delta(A)B - A\delta(B)) = 0. \tag{2.17}$$

Define a new Jordan derivation $\widetilde{\delta} : \mathrm{Alg}_{\mathfrak{B}}\mathcal{L} \to \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$ by

$$\widetilde{\delta}(A) = \delta(A) - (A\delta(Q_1) - \delta(Q_1)A), \quad A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}.$$

Then $\widetilde{\delta}(Q_1^{\perp}) = \widetilde{\delta}(Q_1) = 0$ by (2.10). Applying (2.9) for $\widetilde{\delta}$, it is easy to see that for any $A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$, we have

$$\widetilde{\delta}(AQ_1) = \widetilde{\delta}(A)Q_1 \quad \text{and} \quad \widetilde{\delta}(Q_1^{\perp}A) = Q_1^{\perp}\widetilde{\delta}(A). \tag{2.18}$$

We claim that if $Q_1(\mathcal{L}) \vee Q_2(\mathcal{L}) = I$, then $\delta$ is a derivation.

For this, it suffices obviously to prove that $\widetilde{\delta}$ is a derivation. In fact, let $A, B \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$. Since $\widetilde{\delta}$ is also a Jordan derivation, the equality (2.17) holds if replacing $\delta$ by $\widetilde{\delta}$. Thus $Q_2(\widetilde{\delta}(Q_1^{\perp}AB) - \widetilde{\delta}(Q_1^{\perp}A)B - Q_1^{\perp}A\widetilde{\delta}(B)) = 0$. Because of $Q_1(\mathcal{L}) \vee Q_2(\mathcal{L}) = I$, we get that $Q_1^{\perp}(\widetilde{\delta}(Q_1^{\perp}AB) - \widetilde{\delta}(Q_1^{\perp}A)B - Q_1^{\perp}A\widetilde{\delta}(B)) = 0$. Also, by (2.18) we have $Q_1(\widetilde{\delta}(Q_1^{\perp}AB) - \widetilde{\delta}(Q_1^{\perp}A)B - Q_1^{\perp}A\widetilde{\delta}(B)) = 0$. Hence

$$\widetilde{\delta}(Q_1^{\perp}AB) = \widetilde{\delta}(Q_1^{\perp}A)B + Q_1^{\perp}A\widetilde{\delta}(B). \tag{2.19}$$

By using (2.16) and (2.18), a similar argument gives that

$$\widetilde{\delta}(ABQ_1) = \widetilde{\delta}(A)BQ_1 + A\widetilde{\delta}(BQ_1). \tag{2.20}$$

Moreover, we have by (2.14)–(2.15) and (2.18)–(2.20)

$$\begin{aligned}
\widetilde{\delta}(AB) &= \widetilde{\delta}(ABQ_1) + \widetilde{\delta}(AQ_1BQ_1^{\perp}) + \widetilde{\delta}(Q_1AQ_1^{\perp}B) + \widetilde{\delta}(Q_1^{\perp}AB) \\
&= \widetilde{\delta}(A)BQ_1 + A\widetilde{\delta}(BQ_1) + \widetilde{\delta}(A)Q_1BQ_1^{\perp} + A\widetilde{\delta}(Q_1BQ_1^{\perp}) \\
&\quad + \widetilde{\delta}(Q_1AQ_1^{\perp})B + Q_1AQ_1^{\perp}\widetilde{\delta}(B) + \widetilde{\delta}(Q_1^{\perp}A)B + Q_1^{\perp}A\widetilde{\delta}(B) \\
&= \widetilde{\delta}(A)(BQ_1 + Q_1BQ_1^{\perp}) + A\widetilde{\delta}(BQ_1 + Q_1BQ_1^{\perp}) \\
&\quad + \widetilde{\delta}(Q_1AQ_1^{\perp} + Q_1^{\perp}A)B + (Q_1AQ_1^{\perp} + Q_1^{\perp}A)\widetilde{\delta}(B) \\
&= \widetilde{\delta}(AQ_1)B + A\widetilde{\delta}(Q_1B) + \widetilde{\delta}(AQ_1^{\perp})B + A\widetilde{\delta}(Q_1^{\perp}B) = \widetilde{\delta}(A)B + A\widetilde{\delta}(B).
\end{aligned}$$

This proves that $\widetilde{\delta}$ is a derivation. Hence the claim holds.

For the general case, put $Q = Q_1(\mathcal{L}) \vee Q_2(\mathcal{L})$. By Lemma 2.1 we have the decomposition

$$\mathrm{Alg}_{\mathfrak{B}}\mathcal{L} = Q(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})Q \oplus Q^{\perp}(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})Q^{\perp}.$$

Define a new Jordan derivation $\widehat{\delta} : \mathrm{Alg}_{\mathfrak{B}}\mathcal{L} \to \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$ by

$$\widehat{\delta}(A) = \delta(A) - (A\delta(Q) - \delta(Q)A), \quad A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}.$$

Then $\widehat{\delta}(Q) = \widehat{\delta}(Q^{\perp}) = 0$. For every $A \in \mathrm{Alg}_{\mathfrak{B}}\mathcal{L}$, from (2.9) it follows that $\widehat{\delta}(QAQ) = Q\widehat{\delta}(A)Q$ and $\widehat{\delta}(Q^{\perp}AQ^{\perp}) = Q^{\perp}\widehat{\delta}(A)Q^{\perp}$. Denote by $\widehat{\delta}_1$ and $\widehat{\delta}_2$ the restrictions of $\widehat{\delta}$ on the subalgebras $Q(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})Q$ and $Q^{\perp}(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})Q^{\perp}$, respectively. Then $\widehat{\delta}_1$ and $\widehat{\delta}_2$ are both Jordan derivations. Since $Q_1(Q\mathcal{L}) \vee Q_2(Q\mathcal{L}) = Q$, by the claim we know that $\widehat{\delta}_1$ is a derivation. Also, recalling that $Q^{\perp}(\mathrm{Alg}_{\mathfrak{B}}\mathcal{L})Q^{\perp}$ is a von Neumann algebra, then the fact that any von Neumann algebra is semiprime implies that $\widehat{\delta}_2$ is also a derivation. Therefore, $\widehat{\delta}$ is a derivation, and so is $\delta$. The proof is complete. ∎

## 3. PROOFS OF THEOREMS 1.3 AND 1.4

In this section, rank one operators are a useful tool. Given vectors $x$ and $y$ in a Hilbert space $H$, we define the operator $x \otimes y$ by $z \mapsto (z, y)x$ for $z \in H$.

This operator has rank one if and only if both $x$ and $y$ are nonzero. It is easy to check that, if $A$ and $B$ are linear operators on $H$ such that $B$ is bounded, then $A(x\otimes y)B=(Ax)\otimes(B^*y)$.

The following lemma will get repeated use.

LEMMA 3.1 ([23]). *Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $H$. Then the rank one operator $x\otimes y\in \mathrm{Alg}\mathcal{L}$ if and only if there exists $E\in\mathcal{J}(\mathcal{L})$ such that $x\in E$ and $y\in E_-^{\perp}$, where $E_-^{\perp}$ means $(E_-)^{\perp}$.*

*Proof of Theorem* 1.3. If $\mathcal{A}$ contains the identity operator, then the conclusion is clear, because $\Phi(A)=\Phi(I)A=A\Phi(I)$ holds for all $A\in\mathcal{A}$.

In the following we do not assume that $\mathcal{A}$ contains the identity operator. Let $A\in\mathcal{A}$ and $\lambda\in\mathbb{F}$. For every $E\in\mathcal{J}(\mathcal{L})$, choosing a nonzero vector $y\in E_-^{\perp}$, then $x\otimes y\in\mathcal{A}$ for every $x\in E$ by Lemma 3.1. From $\Phi(\lambda A)x\otimes y=\Phi(\lambda Ax\otimes y)=\lambda\Phi(A)x\otimes y$ we see that $\Phi(\lambda A)x=\lambda\Phi(A)x$ for all $x\in E$. Since $\bigvee\{E:E\in\mathcal{J}(\mathcal{L})\}=H$, we have $\Phi(\lambda A)=\lambda\Phi(A)$. This shows that $\Phi$ is linear.

Let $E\in\mathcal{J}(\mathcal{L})$. Fix a unit vector $y_E\in E_-^{\perp}$. Then $x\otimes y_E\in\mathcal{A}$ for every $x\in E$. Define a linear mapping $T_E:E\to H$ by

$$T_Ex=\Phi(x\otimes y_E)y_E,\quad x\in E.$$

If $A\in\mathcal{A}$ and $x\in E$, then $Ax\in E$ and $\Phi(Ax\otimes y_E)=\Phi(A)x\otimes y_E=A\Phi(x\otimes y_E)$. It follows that

$$T_EAx=\Phi(A)x=AT_Ex. \tag{3.1}$$

*Claim* 1. Let $E,F\in\mathcal{J}(\mathcal{L})$ such that $EF\neq 0$ and let $x\in E\wedge F$. Then $T_Ex=T_Fx$.

In fact, taking a nonzero vector $y\in E_-^{\perp}$ and noting that $E_-^{\perp}\subseteq(E\wedge F)_-^{\perp}$, it follows from (3.1) that $T_Ex\otimes y=\Phi(x\otimes y)=T_{E\wedge F}x\otimes y$. Hence $T_Ex=T_{E\wedge F}x$. Similarly, we have $T_Fx=T_{E\wedge F}x$. So $T_Ex=T_Fx$.

*Claim* 2. If $x_1+x_2+\cdots+x_n=0$, where $x_i\in E_i$, $E_i\in\mathcal{J}(\mathcal{L})$ then $T_{E_1}x_1+T_{E_2}x_2+\cdots+T_{E_n}x_n=0$.

Let us proceed its proof by induction. For simplicity, denote by $T_i=T_{E_i}$. If $n=2$, then $x_1+x_2=0$ implies $x_1,x_2\in E_1\wedge E_2$. So $T_1x_1+T_2x_2=T_1x_1-T_2x_1=0$ by Claim 1, as desired. Suppose that the claim holds for natural number $n$. Let $x_1+x_2+\cdots+x_{n+1}=0$, where $x_i\in E_i$ and $E_i\in\mathcal{J}(\mathcal{L})$, $i=1,2,\cdots,n+1$. Then $T_1x_1+T_1E_1x_2+\cdots+T_1E_1x_{n+1}=0$. Since $E_1\in\mathrm{Alg}\mathcal{L}$, we have $E_1x_i\in E_1\wedge E_i$ and by Claim 1, $T_1E_1x_i=T_iE_1x_i$ for each $i\geqslant 2$. Thus

$$T_1x_1+T_2E_1x_2+T_3E_1x_3+\cdots+T_{n+1}E_1x_{n+1}=0. \tag{3.2}$$

On the other hand, taking $E_1^{\perp}x_1=0$ into account, we obtain $E_1^{\perp}x_2+E_1^{\perp}x_3+\cdots+E_1^{\perp}x_{n+1}=0$, where $E_1^{\perp}x_i\in E_i$, $i=2,3,\ldots,n+1$. By the inductive hypothesis, one has

$$T_2E_1^{\perp}x_2+T_3E_1^{\perp}x_3+\cdots+T_{n+1}E_1^{\perp}x_{n+1}=0. \tag{3.3}$$

It follows from (3.2) plus (3.3) that $T_1x_1+T_2x_2+\cdots+T_{n+1}x_{n+1}=0$, completing the induction step.

Denote by $\mathcal{D}_0$ the linear span of $\{E : E \in \mathcal{J}(\mathcal{L})\}$. Define a linear mapping $T_0 : \mathcal{D}_0 \to H$ such that $T_0|_E = T_E$ for every $E \in \mathcal{J}(\mathcal{L})$, where $T_0|_E$ denotes the restriction of $T_0$ to $E$. By Claim 2, it is easy to see that $T_0$ is well defined. From (3.1), we have clearly

$$\Phi(A)x = T_0Ax = AT_0x \tag{3.4}$$

for all $A \in \mathcal{A}$ and $x \in \mathcal{D}_0$. Put

$$\mathcal{D} = \{x \in H : (x,y) \in \overline{G(T_0)} \text{ for some } y \in H\},$$

where $G(T_0) = \{(x, T_0x) : x \in \mathcal{D}_0\}$ is the graph of $T_0$, and $\overline{G(T_0)}$ denotes the norm closure. Then $\mathcal{D}$ is a linear manifold and $\mathcal{D}_0 \subseteq \mathcal{D}$. Clearly, $\mathcal{D}$ is dense in $H$. For every $x \in \mathcal{D}$, we want to prove that there exists a unique $y \in H$ such that $(x,y) \in \overline{G(T_0)}$. In fact, assume that $(0,y) \in \overline{G(T_0)}$. Let a sequence $\{x_n\}_1^\infty$ in $\mathcal{D}_0$ such that $x_n \to 0$ and $T_0x_n \to y$. For any $E \in \mathcal{J}(\mathcal{L})$, fix a nonzero vector $u \in E$. Then $u \otimes v \in \mathcal{A}$ for all $v \in E_-^\perp$. Applying (3.4) we obtain

$$(y,v)u = \lim_{n\to\infty}(u \otimes v)T_0x_n = \lim_{n\to\infty}\Phi(u \otimes v)x_n = 0.$$

Hence $(y,v) = 0$. So $y \in E_-$ for all $E \in \mathcal{J}(\mathcal{L})$. From the assumption $\bigwedge\{E_- : E \in \mathcal{J}(\mathcal{L})\} = 0$, it follows that $y = 0$, as required.

Thus, we can define a mapping $T : \mathcal{D} \subseteq H \to H$ in an obvious way, such that $G(T) = \overline{G(T_0)}$. Clearly, $T$ is a densely defined, closed linear operator and extends $T_0$.

Finally, we show that $\mathcal{D}$ is invariant under every element of $\mathcal{A}$, and $\Phi(A)x = TAx = ATx$ for all $A \in \mathcal{A}$ and $x \in \mathcal{D}$. For this, let $A \in \mathcal{A}$ and $x \in \mathcal{D}$. Then $(x, Tx) \in \overline{G(T_0)}$. Hence there exists a sequence $\{x_n\}_1^\infty$ in $\mathcal{D}_0$ such that $x_n \to x$, $T_0x_n \to Tx$. So $Ax_n \to Ax$, $AT_0x_n \to ATx$ and $\Phi(A)x_n \to \Phi(A)x$. By (3.4) we have $\Phi(A)x_n = AT_0x_n = T_0Ax_n$. Consequently, $\Phi(A)x = ATx$ and $(Ax_n, T_0Ax_n) = (Ax_n, AT_0x_n) \to (Ax, ATx)$. Then $(Ax, ATx) \in G(T)$. Hence $Ax \in \mathcal{D}$ and $TAx = ATx$. This completes the proof. ∎

To prove Theorem 1.4, we need to cite the following lemma.

LEMMA 3.2 ([10]). *Let $\mathcal{N}$ be a nest on a Hilbert space $H$. Then the finite rank operators of* Alg $\mathcal{N}$ *are strongly dense in* Alg $\mathcal{N}$*; in particular, there is a net $F_\alpha$ of finite rank operators in* Alg $\mathcal{N}$*, each of which has norm at most one, such that $F_\alpha$ converges to $I$ strongly.*

*Proof of Theorem* 1.4. We first consider the "left" case. If $\mathcal{A}$ contains the identity operator, then $\Phi(A) = \Phi(I)A$ for all $A \in \mathcal{A}$.

Assume that $\mathcal{A}$ does not contain the identity operator. Similar to the proof of Theorem 1.3, we can see that $\Phi$ is linear. We now want to prove that there exists an operator $T \in \mathcal{B}(H)$ such that $\Phi(A) = TA$ for all $A \in \mathcal{A}$. For this, we distinguish two cases.

*Case* 1. $H_- \neq H$. Choose a unit vector $x_0 \in H_-^\perp$. Then $x \otimes x_0 \in \mathcal{A}$ for any $x \in H$. Define a linear mapping $T : H \to H$ by

$$Tx = \Phi(x \otimes x_0)x_0, \quad x \in H.$$

Clearly, $T$ is continuous and for all $A \in \mathcal{A}$, $x \in H$, we have $\Phi(A)x = \Phi(A)(x \otimes x_0)x_0 = \Phi(Ax \otimes x_0)x_0 = TAx$. Hence $\Phi(A) = TA$.

*Case* 2. $H_- = H$. Then $\mathcal{N} \neq \{0, H\}$ and $H = \bigvee\{E \in \mathcal{N} : E \neq H\}$ by the definition of $H_-$. Let $E \in \mathcal{N} \setminus \{0, H\}$ and pick a unit vector $x_E \in E^\perp$. Since $E_- \subseteq E$, we have $x \otimes x_E \in \mathcal{A}$ for each $x \in E$. Define a continuous linear mapping $T_E : E \to H$ by

$$T_E x = \Phi(x \otimes x_E)x_E, \quad x \in E.$$

Then $\Phi(A)x = T_E Ax$ for all $A \in \mathcal{A}$ and $x \in E$. Further, suppose that $E_1, E_2 \in \mathcal{N} \setminus \{0, H\}$ such that $E_1 \subseteq E_2$. Then for $x \in E_1$, it follows that $T_{E_1}Ax = \Phi(A)x = T_{E_2}Ax$ holds for all $A \in \mathcal{A}$. Applying Lemma 3.2 and the continuity of $T_{E_1}$ and $T_{E_2}$, we obtain $T_{E_1}x = T_{E_2}x$. Denote $K$ by the subspace $\bigcup\{E \in \mathcal{N} : E \neq H\}$ which is not necessarily closed. Then we can define a linear mapping $T' : K \to H$ such that $T'|_E = T_E$ for every $E \in \mathcal{N} \setminus \{0, H\}$. Obviously,

$$\Phi(A)x = T'Ax \tag{3.5}$$

for all $A \in \mathcal{A}$, $x \in K$. We now want to prove that $T'$ is continuous. Let $x \in K$. Then $x \in E$ for some $E \in \mathcal{N}$ with $E \neq H$. Picking a unit vector $y \in E^\perp$, then $x \otimes y \in \mathcal{A}$ and for any $u \in K$, we have $\Phi(x \otimes y)u = (T'x \otimes y)u$ by (3.5). Noting that both $\Phi(x \otimes y)$ and $T'x \otimes y$ are continuous on $H$ and $K$ is dense in $H$, we get that $\Phi(x \otimes y) = T'x \otimes y$. Hence, $\|T'x\| = \|T'x \otimes y\| \leqslant \|\Phi\|\|x\|$. So $T'$ is continuous. Therefore $T'$ can be extended to be a continuous linear operator $T$ on $H$. By (3.5), we have clearly $\Phi(A) = TA$ for all $A \in \mathcal{A}$, as desired.

Now we will show that $\|\Phi\| = \|T\|$. Obviously, $\|\Phi\| \leqslant \|T\|$. To prove the reverse, let $x \in H$ be arbitrary. By Lemma 3.2, there is a net $F_\alpha$ of finite rank operators in $\mathcal{A}$ with $\|F_\alpha\| \leqslant 1$, which converges to $I$ strongly. Thus for each $F_\alpha$,

$$\|TF_\alpha x\| = \|\Phi(F_\alpha)x\| \leqslant \|\Phi\|\|F_\alpha\|\|x\| \leqslant \|\Phi\|\|x\|.$$

Taking the limit, we get $\|Tx\| \leqslant \|\Phi\|\|x\|$ and so $\|T\| \leqslant \|\Phi\|$.

Let $F_\alpha$ be as above. Then $TF_\alpha$ converges to $T$ strongly. Since $\Phi(F_\alpha) \in \mathcal{A}$, we have $TF_\alpha \in \mathcal{A}$ for each $F_\alpha$. So $T \in \operatorname{Alg}\mathcal{N}$. This completes the proof in case $\Phi$ is a left centralizer.

Finally, assume that $\Phi$ is a right centralizer. Obviously, $\mathcal{N}^\perp$ is still a nest on $H$ and $\mathcal{A}^*$ is a standard subalgebra of $\operatorname{Alg}\mathcal{N}^\perp$. Define a mapping $\Psi : \mathcal{A}^* \to \mathcal{A}^*$ by $\Psi(A^*) = \Phi(A)^*$ for $A \in \mathcal{A}$. Then for $A, B \in \mathcal{A}$, we have

$$\Psi(A^*B^*) = \Psi((BA)^*) = \Phi(BA)^* = (B\Phi(A))^* = \Phi(A)^*B^* = \Psi(A^*)B^*.$$

Hence $\Psi$ is a left centralizer of $\mathcal{A}^*$. Also, $\Psi$ is clearly linear, norm continuous and $\|\Psi\| = \|\Phi\|$. It follows that there is an operator $T \in \operatorname{Alg}\mathcal{N}^\perp$ with $\|T\| = \|\Psi\|$, such that $\Psi(A^*) = TA^*$ for every $A \in \mathcal{A}$. Take $S = T^*$. Then $S \in \operatorname{Alg}\mathcal{N}$, $\|S\| = \|\Phi\|$ and $\Phi(A) = AS$ for all $A \in \mathcal{A}$. The proof is complete. ∎

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PENGTONG LI, DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY OF AERONAUTICS AND ASTRONAUTICS, NANJING, 210016, P.R. CHINA

*E-mail address*: pengtongli@nuaa.edu.cn

DEGUANG HAN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CENTRAL FLORIDA, ORLANDO, 32816, U.S.A.

*E-mail address*: Deguang.Han@ucf.edu

WAI-SHING TANG, DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, SINGAPORE, 119076, SINGAPORE

*E-mail address*: mattws@nus.edu.sg

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