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ON THE CLOSURE OF POSITIVE FLAT MOMENT MATRICES

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ABSTRACT. Let $y \equiv y^{(2d)} = \{y_i\}_{i\in\mathbb{Z}_+^n,|i|\leqslant 2d}$ denote a real n-dimensional multisequence of degree $2d$, $y_0 > 0$. Let $L_y : \mathbb{R}_{2d}[x_1,\dots,x_n] \mapsto \mathbb{R}$ denote the Riesz functional, defined by $L_y(\sum_{|i|\leqslant 2d} a_i x^i) = \sum a_i y_i$, and let $M_d(y)$ denote the corresponding moment matrix. Positivity of L_y plays a significant role in the Truncated Moment Problem and in the Polynomial Optimization Problem, but concrete conditions for positivity are unknown in general. $M_d(y)$ is *flat* if $\operatorname{rank} M_d(y) = \operatorname{rank} M_{d-1}(y)$; it is known that if $M_d(y)$ is positive semidefinite and flat, then y has a representing measure (and L_y is positive). Let $\mathcal{F}_d := \{y \equiv y^{(2d)} : M_d(y) \succeq 0 \text{ is flat}\}$. If $y \in \overline{\mathcal{F}}_d$ (the closure), then y does not necessarily have a representing measure, but L_y is positive, so $M_d(y) \succeq 0$ and, moreover, $\operatorname{rank} M_d(y) \leqslant \dim \mathbb{R}_{d-1}[x_1,\dots,x_n]$. We prove, conversely, that these positivity and rank conditions for $M_d(y)$ are sufficient for membership in $\overline{\mathcal{F}}_d$ in two basic cases: when $n = 1$, $d \geqslant 1$, and when $n = d = 2$.

KEYWORDS: *Truncated moment sequence, Riesz functional, K-positivity, flat moment matrix, representing measure, polynomial optimization.*

MSC (2010): 47A57, 44A60, 47A20.

1. INTRODUCTION

Let $y \equiv y^{(2d)} = \{y_i\}_{i\in\mathbb{Z}_+^n,|i|\leqslant 2d}$ denote a real n-dimensional multisequence of degree $2d$, $y_0 > 0$, and let $K \subseteq \mathbb{R}^n$ be a closed set. The Truncated K-Moment Problem (TKMP, cf. [7]) asks for conditions which insure that there exists a positive Borel measure μ on $\mathbb{R}^n$, with support in K, such that

$$y_j = \int x^j \mathrm{d}\mu \quad (|j| \leqslant 2d). \tag{1.1}$$

(Here, $i \equiv (i_1,\dots,i_n) \in \mathbb{Z}_+^n$, $|i| = i_1 + \cdots + i_n$, $x \equiv (x_1,\dots,x_n) \in \mathbb{R}^n$, and $x^i = x_1^{i_1}\cdots x_n^{i_n}$.) We refer to a measure μ as in (1.1) as a *K-representing measure* for y. In the case $K = \mathbb{R}^n$, we refer to (1.1) simply as the Truncated Moment Problem (TMP) and to μ as a *representing measure*. As we discuss below, TKMP is closely related to the classical Polynomial Optimization Problem (cf. [14], [15], [16], [17]) which, for $p \in \mathcal{P} \equiv \mathbb{R}[x_1, \ldots, x_n]$, seeks to compute (or estimate)

$$p_* := \inf_{x \in K} p(x).$$

Let $\mathcal{P}_m := \{p \in \mathcal{P} : \deg p \leqslant m\}$. We associate to y the *Riesz functional* $L_y : \mathcal{P}_{2d} \mapsto \mathbb{R}$ defined by $L_y(\sum_{|i| \leqslant 2d} a_i x^i) = \sum a_i y_i$. If y has a K-representing measure, then L_y is *K-positive*, i.e., $p \in \mathcal{P}_{2d}$, $p|K \geqslant 0 \Rightarrow L_y(p) \geqslant 0$; indeed, in this case, $L_y(p) = \int_K p \, d\mu \geqslant 0$. For $K = \mathbb{R}^n$, we say simply that L_y is *positive*. For K compact, a result of Tchakaloff [21] implies that if L_y is K-positive, then y has a K-representing measure, but for K noncompact, this implication fails (see (1.2) below). Nevertheless, the following result of [9] reveals the central role of K-positivity in TKMP.

THEOREM 1.1. *$y \equiv y^{(2d)}$ admits a K-representing measure if and only if y can be extended to a sequence $\widetilde{y} \equiv \widetilde{y}^{(2d+2)}$ for which $L_{\widetilde{y}}$ is K-positive.*

Further, Theorem 2.2 in [10] shows that L_y is K-positive if and only if y is the limit of multisequences having K-representing measures. In this case, y admits *approximate* K-representing measures, and, as we show in the proof of Theorem 1.5, this is significant for the Polynomial Optimization Problem. Unfortunately, in general it is quite difficult to determine whether or not a given sequence y is K-positive, even for $K = \mathbb{R}^n$. In the present note we define a class of sequences y, not necessarily admitting representing measures, for which positivity is obvious, and we study a possible concrete characterization of this class.

Following [5], we associate to y the *moment matrix* $M_d(y)$. For $p \in \mathcal{P}_d$, $p = \sum_{|i| \leqslant d} a_i x^i$, let $\widehat{p} \equiv (a_i)$ denote the vector of coefficients of p with respect to the basis for $\mathcal{P}_d$ consisting of the monomials in degree-lexicographic order. Let $\rho_d = \dim \mathcal{P}_d$. Then the moment matrix $M_d \equiv M_d(y)$ is the $\rho_d \times \rho_d$ matrix defined by

$$\langle M_d(y)\widehat{p}, \widehat{q} \rangle = L_y(pq) \quad (p, q \in \mathcal{P}_d).$$

If y has a representing measure μ, we sometimes denote $M_d(y)$ by $M_d[\mu]$; as noted above, in this case, L_y is positive. Further, if L_y is positive, then $M_d(y)$ is positive semidefinite ($M_d(y) \succeq 0$), since $\langle M_d(y)\widehat{p}, \widehat{p} \rangle = L_y(p^2) \geqslant 0$ ($p \in \mathcal{P}_d$). Neither of the preceding implications is reversible; the simplest counterexample, illustrating L_y positive, but with no representing measure, occurs with $n = 1$ and y given by

$$y^{(4)} = \{a,\ a,\ a,\ a,\ b\} \quad 0 < a < b; \tag{1.2}$$

(cf. [4], Theorem 2.2 below). The existence of y such that $M_d \succeq 0$ but L_y is not positive is established by Proposition 1.6 below. There is, however, a subclass of positive moment matrices $M_d(y)$ for which it is easy to detect the existence of representing measures (and hence positivity of L_y). Recall from [5] that $M_d(y)$ is *flat* if $\operatorname{rank} M_d(y) = \operatorname{rank} M_{d-1}(y)$. The following result of [7] describes the role of flat moment matrices in TMP.

THEOREM 1.2 (cf. [7]). *$y \equiv y^{(2d)}$ has a representing measure if and only if y can be extended to a sequence $\widetilde{y} \equiv \widetilde{y}^{(2(d+k))}$ (for some k, $0 \leqslant k \leqslant \rho_{2d} - \operatorname{rank} M_d(y) + 1$) for which $M_{d+k}(\widetilde{y})$ is positive semidefinite and flat. In particular, if $M_d(y) \succeq 0$ is flat, then y admits a unique representing measure, which is $\operatorname{rank} M_d(y)$-atomic.*

(The result in [7] is stated in terms of finitely atomic representing measures, but in [1] it is proved that the existence of a representing measure implies the existence of a finitely atomic representing measure.)

Let $\mathcal{F}_d := \{y \equiv y^{(2d)} : M_d(y) \succeq 0 \text{ is flat}\}$, which we regard as a subset of $\mathbb{R}^{\rho_{2d}}$ (equipped with the Euclidean norm). In view of the ease of detecting flatness (by simply checking the positivity and rank conditions), we are motivated to study $\overline{\mathcal{F}}_d$, the closure of $\mathcal{F}_d$ in $\mathbb{R}^{\rho_{2d}}$; equivalently, we seek to characterize in concrete terms the closure of $\{M_d(y) : y \in \mathcal{F}_d\}$ relative to any of the (equivalent) norms on the $\rho_d \times \rho_d$ matrices. Now suppose $M_d(y) = \lim_{k\to\infty} M_d(y^{[k]})$ with each $M_d(y^{[k]})$ positive and flat. Theorem 1.2 implies that for each k, $y^{[k]}$ has a representing measure, so $L_{y^{[k]}}$ is positive; since $|L_{y^{[k]}}(p) - L_y(p)| \leqslant \|y^{[k]} - y\|_\infty \|\widehat{p}\|_1$, we see that L_y is positive. It follows that $M_d(y) \succeq 0$, and lower semicontinuity of rank ([12], Proposition 1.12(i)) implies that

$$\operatorname{rank} M_d(y) \leqslant \liminf_{k\to\infty} \operatorname{rank} M_d(y^{[k]}) = \liminf_{k\to\infty} \operatorname{rank} M_{d-1}(y^{[k]}) \leqslant \rho_{d-1}. \tag{1.3}$$

These considerations lead to the following question that we study in the sequel.

QUESTION 1.3. If $M_d(y) \succeq 0$ and $\operatorname{rank} M_d(y) \leqslant \rho_{d-1}$, does y belong to $\overline{\mathcal{F}}_d$?

An affirmative answer to Question 1.3 would provide a concrete sufficient condition, more general than flatness, for positivity of L_y.

Our main result provides a positive answer to Question 1.3 in two basic cases.

THEOREM 1.4. *Let $n = 1$ and $d \geqslant 1$, or let $n = d = 2$. If $M_d(y) \succeq 0$ and $\operatorname{rank} M_d(y) \leqslant \rho_{d-1}$, then $y \in \overline{\mathcal{F}}_d$. In this case, there exist moment matrices $M_d(y^{[k]})$ ($k \geqslant 1$) such that $\lim_{k\to\infty} M_d(y^{[k]}) = M_d(y)$ and for each k, $\operatorname{rank} M_d(y^{[k]}) = \operatorname{rank} M_{d-1}(y^{[k]}) = \operatorname{rank} M_d(y)$.*

We note that for y satisfying the conditions of Theorem 1.4, although L_y is positive, y does not necessarily have a representing measure; such is the case for the sequence in (1.2).

We prove the univariate case of Theorem 1.4 in Section 3 and the bivariate quartic case in Section 4. In an appendix (Section 5), we present an example with $n = 2$, $d = 3$ which also provides positive evidence for Question 1.3, but which displays behavior not present in Sections 3 and 4. Section 2 contains some background results concerning the structure of positive moment matrices. In the remainder of this section we relate Question 1.3 to the polynomial optimization problem in the case when K is semialgebraic; it is this connection which provided the original motivation for this work.

For $\mathcal{Q} \equiv \{q_0, q_1, \ldots, q_m\} \subseteq \mathcal{P}$, with $q_0 = 1$, consider the closed semialgebraic set $K \equiv K_{\mathcal{Q}} := \{x \in \mathbb{R}^n : q_j(x) \geqslant 0, 1 \leqslant j \leqslant m\}$. For $p \in \mathcal{P}$, the optimization problem entails estimating

$$p_* := \inf_{x \in K_{\mathcal{Q}}} p(x). \tag{1.4}$$

We recall the "moment relaxations" for (1.4) introduced by J.-B. Lasserre [14]. Setting $\deg q_j = 2k_j$ or $2k_j - 1$, $\deg p = 2d$ or $2d-1$, let $\alpha = \max\{k_1, \ldots, k_m, d\}$. Fix $t \geqslant \alpha$, so that $2t \geqslant \deg p, \deg q_j$ $(1 \leqslant j \leqslant m)$. For $y \equiv y^{(2t)}$ and $0 \leqslant j \leqslant m$, the *localizing matrix* $M^{(q_j)} \equiv M_t^{(q_j)}(y)$ is defined by $\langle M^{(q_j)} \widehat{f}, \widehat{g} \rangle = L_y(q_j f g)$ $(f, g \in \mathcal{P}_{t-k_j})$ (cf. [7]). Note that for $j = 0$, $M^{(1)}$ coincides with the moment matrix $M_t(y)$ associated with y. We may now define the t-th Lasserre moment relaxation for (1.4) by

$$p_t := \inf\{L_y(p) : y \equiv y^{(2t)},\ y_0 = 1,\ M_t^{(q_j)}(y) \succeq 0\ (0 \leqslant j \leqslant m)\}. \tag{1.5}$$

It is not difficult to verify that $p_t \leqslant p_*$ and that for $t' \geqslant t$, $p_{t'} \geqslant p_t$; thus, $\{p_t\}$ is convergent, and $p^{\mathrm{mom}} \equiv \lim\limits_{t \to \infty} p_t \leqslant p_*$. A result of Lasserre [14] (cf. Theorem 6.8 of [17]) shows that $p^{\mathrm{mom}} = p_*$ if the quadratic module associated to $K_{\mathcal{Q}}$ is Archimedian (so that $K_{\mathcal{Q}}$ is compact). In some cases of $K_{\mathcal{Q}}$ there is even *finite convergence* to p_*. This is the case if the algebraic variety associated to $K_{\mathcal{Q}}$ is finite [18] (cf. Theorem 6.5 of [17]), or if TKMP for $K_{\mathcal{Q}}$ can always be solved via a degree-bounded number of positive moment matrix extensions (without the requirement for a flat extension in Theorem 1.2) (cf. Proposition 3.2 of [9]).

In the general case, for fixed t, the infimum in (1.5) is not necessarily attained. Assuming that the infimum *is* attained, at some optimal sequence $y \equiv y^{\{t\}}$, we are interested in criteria which imply that $L_y(p) = p_*$, so that we have finite convergence of $\{p_s\}_{s \geqslant \alpha}$ to p_* at stage $s = t$. A basic result of [11] shows that this is the case if $\operatorname{rank} M_t(y) = \operatorname{rank} M_{t-\alpha}(y)$ (cf. Theorem 6.18 in [17]). Indeed, in this case, Corollary 1.4 in [7] implies that y has a K-representing measure, which always implies convergence at stage t; to see this last point, note that if μ is a K-representing measure for $y \equiv y^{\{t\}}$, then

$$p_* = p_* y_0 = p_* \int_K 1 \mathrm{d}\mu \leqslant \int_K p \mathrm{d}\mu = L_y(p) = p_t \leqslant p_*.$$

The following result (proved in Section 5) shows that whether or not an optimizing sequence $y \equiv y^{\{t\}}$ at stage t has a K-representing measure, if the functional L_y is merely K-positive, then we do have convergence of $\{p_s\}$ to p_* at stage t.

THEOREM 1.5. *Suppose* $p_t = L_y(p)$ *for some sequence* $y \equiv y^{(2t)}$ *for which* $y_0 = 1$ *and* $M^{(q_i)}(y) \succeq 0$ $(0 \leqslant i \leqslant m)$. *If* L_y *is K-positive, then* $p_t = p_*$.

In view of this result, Question 1.3 is motivated by the desire to have a *concrete* condition (more general than flatness) for L_y to be K-positive, particularly if y does not have (or is not known to have) a representing measure.

Note that Question 1.3 also has an affirmative answer when $n \geqslant 1$, $d = 1$; indeed, in this case, since $\rho_0 = 1$, if $M_1(y) \succeq 0$ and $\operatorname{rank} M_1(y) \leqslant \rho_0$, then clearly $M_1(y)$ is flat. This observation, and the results of Theorem 1.4, contribute positive evidence for Question 1.3, but they have no new impact on positivity for $L_{y^{[t]}}$ in the optimization problem. This is because, in these cases, positivity of $L_{y^{(2d)}}$ can always be derived from the positivity of $M_d(y)$ via sums of squares, as the following result shows.

PROPOSITION 1.6. *The following are equivalent:*

(i) $M_d(y) \succeq 0 \Rightarrow L_y$ *is positive;*

(ii) *each polynomial in* $\mathcal{P}_{2d}$ *that is nonnegative on* $\mathbb{R}^n$ *can be expressed as a sum of squares of polynomials;*

(iii) $n \geqslant 1$ *and* $d = 1$, *or* $n = 1$ *and* $d \geqslant 1$, *or* $n = d = 2$.

Proposition 1.6 (ii) $\Leftrightarrow$ (iii) is a well-known result of Hilbert (cf. [19]), and (i) $\Leftrightarrow$ (ii) may be known, but we could not find a reference, so we include a proof in Section 5. In view of Proposition 1.6, the first case where Question 1.3 could impact the optimization problem via Theorem 1.5 is $n = 2$, $d = 3$. For this case, in Example 5.2 we illustrate a sequence $y \equiv y^{(6)}$ for which positivity of L_y cannot be derived from representing measures or sums of squares, but instead is established through membership in $\overline{\mathcal{F}}_3$.

2. POSITIVE MOMENT MATRICES

In this section we recall some results concerning the structure of positive moment matrices. We begin, more generally, with a real symmetric block matrix of the form

$$M = \begin{pmatrix} A & B \\ B^{\mathrm{T}} & C \end{pmatrix}. \tag{2.1}$$

It is well known that M is positive semidefinite if and only if $A \succeq 0$ and $B = AW$ for some matrix W satisfying $C \succeq W^{\mathrm{T}}AW$ $(= B^{\mathrm{T}}W)$, or, equivalently, $C \succeq B^{\mathrm{T}}A^{+}B$, where A^{+} denotes the Moore–Penrose pseudoinverse of A (cf. [13]). In this case, let $\Delta \equiv C - W^{\mathrm{T}}AW$ denote the Schur complement, so that $\operatorname{rank} M = \operatorname{rank} A + \operatorname{rank} \Delta$. We have $\operatorname{rank} M = \operatorname{rank} A$, and we say that M is a *flat extension* of A, if and only if $C = W^{\mathrm{T}}AW$. Flat extensions are uniquely determined by A and B, for if there are matrices W and V such that $AW = B = AV$, then $W^{\mathrm{T}}AW - V^{\mathrm{T}}AV =$

$W^{\mathrm{T}}AV - V^{\mathrm{T}}AV = (AW - AV)^{\mathrm{T}}V = 0$. Given $A \succeq 0$ and $B = AW$ as above, if we set $C^{\flat} = W^{\mathrm{T}}AW$ and

$$M^{\flat} = \begin{pmatrix} A & B \\ B^{\mathrm{T}} & C^{\flat} \end{pmatrix}, \tag{2.2}$$

then clearly $M^{\flat}$ is a positive flat extension of A.

Let us denote the moment matrix $M_d \equiv M_d(y)$ as

$$M_d(y) = \begin{pmatrix} M_{d-1}(y) & B(d) \\ B(d)^{\mathrm{T}} & C(d) \end{pmatrix}. \tag{2.3}$$

Following [5] say that M_d is *flat* if $\operatorname{rank} M_d = \operatorname{rank} M_{d-1}$. If, additionally, M_d is positive, then the preceding remarks show that M_d is a flat extension of M_{d-1}. Theorem 2.19 of [7] implies that in this case, M_d has *unique* successive flat (positive) moment matrix extensions $M_{d+1}, M_{d+2}, \ldots$, and that M_d has a unique representing measure, which is $\operatorname{rank} M_d$-atomic. In the sequel, if μ is a positive Borel measure with convergent moments $y \equiv y^{(2d)}$, we sometimes denote $M_d(y)$ by $M_d[\mu]$; moreover, for a moment matrix $M_d(y)$, we sometimes refer to a representing measure for y as a representing measure for $M_d(y)$.

Let $n = 2$ and suppose $M_2(y) \succeq 0$, so that $B(2) = M_1(y)W$ for some matrix W (as above). Since $n = 2$, $C(2)$ is a Hankel matrix. In Section 4 it will be important to know that in the cases that we consider, $C^{\flat} \equiv W^{\mathrm{T}}M_1W$ is also Hankel. Here we note that in general this is not the case. Indeed, for the positive moment matrix $M_2(y)$ defined by

$$M_2(y) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix},$$

a calculation shows that

$$C^{\flat} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Assuming that $C^{\flat}$ is Hankel, in Section 4 it will also be an issue as to whether the Schur complement $\Delta \equiv C(2) - C^{\flat}$ admits a representing measure. We note here that in general this may fail. Consider

$$M_2(y) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 4 & 4 & 4 \\ 1 & 1 & 1 & 4 & 4 & 4 \\ 1 & 1 & 1 & 4 & 4 & 5 \end{pmatrix}.$$

We see that $M_2 \succeq 0$, and

$$\Delta = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$

so Theorem 2.2(iii) (just below) implies that the univariate moment sequence associated with Δ does not have a representing measure.

In the sequel we denote the successive columns of $M_d(y)$ by X^i ($|i| \leqslant d$) in degree-lexicographic order. Let $p \in \mathcal{P}_d$, $p = \sum a_i x^i$. We define a corresponding element of $\mathrm{Col} M_d(y)$, the column space of $M_d(y)$, by $p(X) = \sum a_i X^i$. Thus, each column dependence relation in $M_d(y)$ can be expressed as $p(X) = 0$ for some $p \in \mathcal{P}_d$, $p \neq 0$. Following [5], we say that $M_d(y)$ is *recursively generated* if $p, q, pq \in \mathcal{P}_d$, $p(X) = 0 \Rightarrow (pq)(X) = 0$. Positivity and recursiveness are necessary conditions for y to have a representing measure [5], and for $n = 1$ these conditions are also sufficient [4]. In the sequel we will repeatedly employ the following result, often without further reference.

THEOREM 2.1 (Structure Theorem [5]). *If $M_d(y)$ is positive semidefinite, then the following properties hold:*

(i) *$M_{d-1}(y)$ is recursively generated.*

(ii) *If $p \in \mathcal{P}_{d-1}$ and $p(X) = 0$ in $\mathrm{Col} M_{d-1}$, then $p(X) = 0$ in $\mathrm{Col} M_d$.*

(iii) *If $p \in \mathcal{P}_{d-1}$ satisfies $p(X) = 0$ in $\mathrm{Col} M_d$ and $q \in \mathcal{P}_d$ satisfies $\deg pq = d$, then $(pq)(X) = 0$ in $\mathrm{Col}\begin{pmatrix} M_{d-1}(y) & B(d) \end{pmatrix}$. Further, if M_d is flat, then $(pq)(X) = 0$ in $\mathrm{Col} M_d$.*

For $n = 1$, with $y^{(2d)} = \{y_0, \dots, y_{2d}\}$, $M_d(y)$ is a Hankel matrix, which we henceforth denote by $H_d \equiv H_d(y)$. The structure and existence of representing measures for $H_d \succeq 0$ is described by the following result.

THEOREM 2.2 ([4], Theorem 2.4, Theorem 3.9). *Suppose $H_d \equiv H_d(y)$ is positive semidefinite, with $y_0 > 0$. If H_d is positive definite, then y has a $d+1$-atomic representing measure. If H_d is singular, let $r := \min\{s : 1 \leqslant s \leqslant d : \ H_s \text{ is singular}\}$; let $v = (y_r, \dots, y_{2r-1})^{\mathrm{T}}$, and set $c \equiv (c_0, \dots, c_{r-1})^{\mathrm{T}} = M_{r-1}^{-1} v$. Then*

(a) *$y_j = c_0 y_{j-r} + \dots + c_{r-1} y_{j-1}$ ($r \leqslant j \leqslant 2d-1$); and*

(b) *$y_{2d} \geqslant c_0 y_{2d-r} + \dots + c_{r-1} y_{2d-1}$.*

Further, for H_d positive and singular, the following are equivalent:

(i) *y has a representing measure;*

(ii) *equality holds in* (b) *(above);*

(iii) *H_d is recursively generated;*

(iv) *$\mathrm{rank} H_d = r$;*

(v) *H_d is a flat extension of H_{r-1};*

(vi) *y has an r-atomic representing measure.*

For H_d positive and singular, there is strict inequality in (b) *(above) if and only if* $\mathrm{rank} H_d = r + 1$.

The conditions of Theorem 2.2 for representing measures when $n = 1$ do not extend to several variables. For the bivariate quartic case that we consider in Section 4, $y \equiv y^{(4)}$ has a representing measure if and only if $M_2(y)$ is positive semidefinite and recursively generated, and the algebraic variety $\mathcal{V}$ associated to y satisfies $\operatorname{rank} M_2(y) \leqslant \operatorname{card}\mathcal{V}$. The singular case of this result appears in [6], and generalizations to bivariate truncated K-moment problems with K a quadratic curve appear in [9]. The nonsingular case, where the conditions reduce to $M_2(y) \succ 0$, appears in [10].

3. THE CLOSURE OF THE POSITIVE FLAT HANKEL MATRICES

In this section we prove Theorem 1.4 for the case $n = 1$.

THEOREM 3.1. *If $H_d(y)$ is positive semidefinite and singular, with $y_0 > 0$, then $H_d(y) \in \overline{\mathcal{F}}_d$. If $\rho \equiv \operatorname{rank} H_d(y) \leqslant d$, then there exist positive semidefinite Hankel matrices $H_d(y^{[k]})$ ($k \geqslant 1$) such that $\operatorname{rank} H_d(y^{[k]}) = \operatorname{rank} H_{d-1}(y^{[k]}) = \rho$ and $\lim_{k\to\infty} H_d(y^{[k]}) = H_d(y)$.*

Proof. We may assume $y_0 = 1$ and write $H_d(y)$ in the block form

$$H_d(y) = \begin{bmatrix} H_{d-1}(y) & b(y) \\ b(y)^{\mathrm{T}} & y_{2d} \end{bmatrix}.$$

Of course, we have $H_{d-1}(y) \succeq 0$. If $H_{d-1}(y) \succ 0$ or $y_{2d} = b(y)^{\mathrm{T}} H_{d-1}(y)^{+} b(y)$, then $H_d(y)$ is already flat, and we are done. In view of Theorem 2.2, we need only consider the case when

$$r \equiv \operatorname{rank} H_{d-1}(y) = \rho - 1 < d, \quad y_{2d} > \widehat{y}_{2d} := b(y)^{\mathrm{T}} H_{d-1}(y)^{+} b(y).$$

Define a new moment vector $\widehat{y} = (y_0, y_1, \ldots, y_{2d-1}, \widehat{y}_{2d})$. Then

$$H_d(y) = H_d(\widehat{y}) + \eta e_{d+1} e_{d+1}^{\mathrm{T}}, \quad \eta := y_{2d} - \widehat{y}_{2d} > 0.$$

(In the above equation, e_{d+1} denotes the $(d+1)$-st unit column vector.) Since $H_d(\widehat{y})$ is a flat extension of $H_{d-1}(y)$, Theorem 2.2(vi) implies that the moment sequence $\widehat{y}$ has a rank $H_d(\widehat{y})$-atomic representing measure, i.e., there exist distinct real numbers $u_1, \ldots, u_r$ such that

$$H_d(\widehat{y}) = c_1 [u_1]_d [u_1]_d^{\mathrm{T}} + \cdots + c_r [u_r]_d [u_r]_d^{\mathrm{T}}, \quad c_1 + \cdots + c_r = 1, \quad c_1, \ldots, c_r > 0,$$

where

$$[u]_d := \begin{bmatrix} 1 & u & u^2 & \cdots & u^d \end{bmatrix}^{\mathrm{T}}.$$

For $k > 1$, define the moment vector $y^{[k]}$ by

$$y^{[k]} = \frac{1}{1 + \eta k^{-2d}} (\widehat{y} + \eta \cdot k^{-2d} (1, k, k^2, \ldots, k^{2d})).$$

Clearly, we have

$$y_0^{[k]} = 1, \quad \lim_{k\to\infty} y^{[k]} = y, \quad H_d(y^{[k]}) = \frac{1}{1+\eta k^{-2d}} H_d(\widehat{y}) + \frac{\eta k^{-2d}}{1+\eta k^{-2d}} [k]_d [k]_d^{\mathrm{T}}.$$

Note that

$$H_d(y^{[k]}) = \frac{1}{1+\eta k^{-2d}} \{c_1 [u_1]_d [u_1]_d^{\mathrm{T}} + \cdots + c_r [u_r]_d [u_r]_d^{\mathrm{T}} + \eta k^{-2d} [k]_d [k]_d^{\mathrm{T}}\},$$

so each $H_d(y^{[k]})$ has rank at most $\rho = r + 1$.

To complete the proof, we will show that $H_d(y^{[k]})$ is flat when

$$k > \max\{u_1, \ldots, u_r\}. \tag{3.1}$$

Let V be the following Vandermonde matrix,

$$V = \begin{bmatrix} [u_1]_{d-1} & \cdots & [u_r]_{d-1} & [k]_{d-1} \end{bmatrix},$$

let D denote the diagonal matrix

$$\operatorname{diag}\Big(\sqrt{c_1}, \ldots, \sqrt{c_r}, \frac{\sqrt{\eta}}{k^d}\Big),$$

and let $P \equiv P_k := VD$, so that

$$H_{d-1}(y^{[k]}) = \frac{1}{1+\eta k^{-2d}} PP^{\mathrm{T}}. \tag{3.2}$$

Now V has $r+1$ columns and d rows, and $r+1 \leqslant d$; thus, when $u_1, \ldots, u_r, k$ are distinct, V must have full column rank. From (3.2), we see that $\operatorname{rank} H_{d-1}(y^{[k]}) = \operatorname{rank} PP^{\mathrm{T}} = \operatorname{rank} P = \operatorname{rank} V = r+1 \geqslant \operatorname{rank} H_d(y^{[k]}) \geqslant \operatorname{rank} H_{d-1}(y^{[k]})$. Thus, $\operatorname{rank} H_d(y^{[k]}) = \operatorname{rank} H_{d-1}(y^{[k]}) = r+1$ when (3.1) holds, which completes the proof. ∎

Theorem 3.1 provides flat approximants $H_d^{[k]}$ for H_d of minimal rank consistent with (1.3), namely, $\operatorname{rank} H_d^{[k]} = \operatorname{rank} H_d$. It is also possible to approximate a singular positive Hankel matrix $H_d(y)$ with flat positive Hankel matrices of maximal rank d. This is the content of the following result.

THEOREM 3.2. *If $H_d(y)$ is positive semidefinite and singular, then there exist positive flat Hankel matrices $H_d(y^{[k]})$ ($k \geqslant 1$) such that* $\operatorname{rank} H_d(y^{[k]}) = \operatorname{rank} H_{d-1}(y^{[k]}) = d$ *and* $\lim_{k\to\infty} H_d(y^{[k]}) = H_d(y)$.

Proof. The proof is almost the same as for Theorem 3.1; we follow the same approach and use the same notation. Assuming $y_0 = 1$, we define $y^{[k]}$ as

$$H_d(y^{[k]}) = \frac{1}{1+\eta k^{-2d}} \Big\{ \sum_{i=1}^{r} c_i [u_i]_d [u_i]_d^{\mathrm{T}} + \frac{\eta}{(d-r)k^{2d}} \sum_{j=0}^{d-r-1} [k+j]_d [k+j]_d^{\mathrm{T}} \Big\}.$$

Clearly, $y_0^{[k]} = 1$, $\lim_{k\to\infty} H_d(y^{[k]}) = H_d(y)$, and each $H_d(y^{[k]})$ has rank at most d. As in the preceding proof, we can show that when (3.1) holds (so that $u_1, \ldots, u_r, k, k+$

$1, \ldots, k+d-r-1$ are distinct), then $H_{d-1}(y^{[k]})$ has rank d, so that $H_d(y^{[k]})$ is flat, with rank d. ∎

4. THE CLOSURE OF THE POSITIVE FLAT BIVARIATE QUARTIC MOMENT MATRICES

In this section we prove Theorem 1.4 for the case $n = d = 2$. Let J denote a real symmetric positive definite matrix. If J is at least 2×2, then J is of the form

$$J = \begin{pmatrix} A & b \\ b^{\mathrm{T}} & \gamma \end{pmatrix}, \tag{4.1}$$

where $A \succ 0$ (positive definite), b is a column vector, and $\gamma > b^{\mathrm{T}} A^{-1} b$. A calculation shows that

$$J^{-1} = \begin{pmatrix} P & v \\ v^t & \xi \end{pmatrix}, \tag{4.2}$$

where

$$P = A^{-1}(1 + \xi b b^{\mathrm{T}} A^{-1}), \quad v = -\xi A^{-1} b, \quad \xi = \frac{1}{\gamma - b^{\mathrm{T}} A^{-1} b}. \tag{4.3}$$

Let $r = \operatorname{rank}(A)$ and $[k] \equiv [k]_r = \begin{pmatrix} 1 & k & k^2 & \ldots & k^r \end{pmatrix}^{\mathrm{T}}$, and for fixed $q \geqslant 2r$ and $\tau > 0$, let

$$L \equiv L(k, r, q, \tau) = \frac{\tau}{k^q}[k][k]^{\mathrm{T}} = \tau \begin{pmatrix} \frac{1}{k^q} & \cdots & \frac{1}{k^{q-r}} \\ \vdots & \cdots & \vdots \\ \frac{1}{k^{q-r}} & \cdots & \frac{1}{k^{q-2r}} \end{pmatrix}. \tag{4.4}$$

Note that $[k][k]^{\mathrm{T}}$ is the rank-one moment matrix $H_r[\delta_k]$ for the atomic measure δ_k. The following result will be used in the sequel to establish that certain small perturbations of Hankel matrices are flat; in the Appendix we will sketch an alternate proof of Theorem 3.1 based on this result.

LEMMA 4.1. *Suppose* $H = \begin{pmatrix} A & b \\ b^{\mathrm{T}} & c \end{pmatrix}$ *is an* $(r+1) \times (r+1)$ *positive semidefinite real matrix with* $\operatorname{rank} H = \operatorname{rank} A = r$, *i.e.,* $A \succ 0$ *and* $c = b^{\mathrm{T}} A^{-1} b$. *For fixed* $q \geqslant 2r$ *and* $\tau > 0$, *let* $L \equiv L(k, r, q, \tau)$. *Then for* k *sufficiently large,* $\operatorname{rank}(H + L) = r + 1$, *i.e.,* $H + L \succ 0$.

Proof. Let $J \equiv J_k = \begin{pmatrix} A & b \\ b^{\mathrm{T}} & c + \frac{\tau}{k^{q-2r}} \end{pmatrix}$; thus $J \succ 0$, and from (4.2)–(4.3) we can write J^{-1} in the form $J^{-1} = \begin{pmatrix} P & v \\ v^t & \xi \end{pmatrix}$, where $P = A^{-1}(1 + \xi b b^{\mathrm{T}} A^{-1})$, $v = -\xi A^{-1} b$, and $\xi = \frac{k^{q-2r}}{\tau}$. It follows that there is a constant C (independent of

k, depending only on τ, A, and (b) such that $\|J^{-1}\| \leqslant Ck^{q-2r}$. (Here $\|\cdot\|$ denotes the standard 2-norm.) Now let

$$X \equiv X_k = \tau \begin{pmatrix} \frac{1}{k^q} & \cdots & \frac{1}{k^{q-r+1}} & \frac{1}{k^{q-r}} \\ \vdots & \cdots & \vdots & \vdots \\ \frac{1}{k^{q-r+1}} & \cdots & \frac{1}{k^{q-2r+2}} & \frac{1}{k^{q-2r+1}} \\ \frac{1}{k^{q-r}} & \cdots & \frac{1}{k^{q-2r+1}} & 0 \end{pmatrix}.$$

For $A \equiv (a_{ij})_{(r+1)\times(r+1)}$, let $\|A\|_\infty = \max |a_{ij}|$, so that $\|A\| \leqslant (r+1)\|A\|_\infty$. Thus, $\|X\| \leqslant (r+1)\|X\|_\infty = \frac{(r+1)\tau}{k^{q-2r+1}}$. For $k > \tau C(r+1)$, we have $\|X\| \leqslant \frac{(r+1)\tau}{k^{q-2r+1}} < \frac{1}{Ck^{q-2r}} \leqslant \frac{1}{\|J^{-1}\|}$, whence $\|X\| < \frac{1}{\|J^{-1}\|}$. Since the operator norm is a Banach algebra norm, it now follows that $J + X$ is invertible (cf. Chapter 12 (Problem K) in [3]). Thus, $H + L$ $(= J + X)$ is invertible, i.e., $\mathrm{rank}(H + L) = r + 1$. ∎

Now we are ready to prove Theorem 1.4 for $n = d = 2$, which we restate for ease of reference.

THEOREM 4.2. *Let $n = 2$ and suppose $y \equiv y^{(4)}$ satisfies $y_{00} > 0$. Then $M \equiv M_2(y) \in \overline{\mathcal{F}}_2$ if and only if $M \succeq 0$ and $\mathrm{rank} M \leqslant 3$; in this case, there exist quartic sequences $y^{[k]}$ $(k \geqslant 1)$ such that $\lim_{k\to\infty} M_2(y^{[k]}) = M$ and $\mathrm{rank} M_2(y^{[k]}) = \mathrm{rank} M_1(y^{[k]}) = \mathrm{rank} M$.*

Proof. Since the necessity of the positivity and rank conditions is clear, we focus on sufficiency. We normalize y so that $y_{00} = 1$. If M is flat, then we are done. Since $\rho \equiv \mathrm{rank} M \leqslant 3$, we may assume that $r \equiv \mathrm{rank} M_1(y)$ satisfies $1 \leqslant r \leqslant 2$ and $r < \rho$. Write M as

$$M = \begin{pmatrix} M_1(y) & B(2) \\ B(2)^{\mathrm{T}} & C(2) \end{pmatrix}. \tag{4.5}$$

Since $M \succeq 0$, we have $\mathrm{Ran} B(2) \subseteq \mathrm{Ran} M_1$, so there is a matrix W such that

$$B(2) = M_1 W. \tag{4.6}$$

Positivity of M_1 implies that

$$C^\flat := B(2)^{\mathrm{T}} W = W^{\mathrm{T}} M_1 W \tag{4.7}$$

is independent of W satisfying (4.6), and we define $M^\flat$ by

$$M^\flat \equiv \begin{pmatrix} M_1(y) & B(2) \\ B(2)^{\mathrm{T}} & C^\flat \end{pmatrix}.$$

In the sequel we will repeatedly use the fact that

$$\mathrm{rank} M^\flat = \mathrm{rank} M_1(y), \tag{4.8}$$

often without further reference (cf. Section 2). We denote $C^\flat$ by

$$C^\flat \equiv \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}.$$

LEMMA 4.3. *The matrix $C^\flat$ is a Hankel matrix, and so $M^\flat$ is a moment matrix.*

Proof. Since $C^\flat \succeq 0$, we have $c_{21} = c_{12}$, $c_{31} = c_{13}$, and $c_{32} = c_{23}$, so it suffices to show that $c_{31} = c_{22}$.

First, consider the case when $r = 1$, so there are column dependence relations in $M_1(y)$ of the form

$$X = \alpha 1, \quad Y = \beta 1. \tag{4.9}$$

Since $M \succeq 0$, Theorem 2.1 implies that these relations extend to the columns of M and that in the columns of $[M_1 \;\; B(2)]$ we have

$$X^2 = \alpha X, \tag{4.10}$$

$$XY = \alpha Y = \beta X, \tag{4.11}$$

$$Y^2 = \beta Y. \tag{4.12}$$

By the definition of $M^\flat$, (4.9)–(4.12) also hold in $\mathrm{Col} M^\flat$, so $M^\flat$ assumes the form

$$M^\flat = \begin{pmatrix} 1 & \alpha & \beta & \alpha^2 & \alpha\beta & \beta^2 \\ \alpha & \alpha^2 & \alpha\beta & \alpha^3 & \alpha^2\beta & \alpha\beta^2 \\ \beta & \alpha\beta & \beta^2 & \alpha^2\beta & \alpha\beta^2 & \beta^3 \\ \alpha^2 & \alpha^3 & \alpha^2\beta & c_{11} & c_{12} & c_{13} \\ \alpha\beta & \alpha^2\beta & \alpha\beta^2 & c_{21} & c_{22} & c_{23} \\ \beta^2 & \alpha\beta^2 & \beta^3 & c_{31} & c_{32} & c_{33} \end{pmatrix}. \tag{4.13}$$

From (4.10), we have $c_{31} = \alpha(\alpha\beta^2)$ and from (4.11), $c_{22} = \beta(\alpha^2\beta)$, so $c_{31} = c_{22}$, as claimed.

Next, consider the case when $r = 2$, $\{1, \; X\}$ is a basis for $\mathrm{Col} M_1(y)$, and

$$Y = \alpha 1 + \beta X. \tag{4.14}$$

Positivity for M implies that the last relation holds in $\mathrm{Col} M$ and that in $\mathrm{Col}[M_1 \;\; B(2)]$,

$$XY = \alpha X + \beta X^2. \tag{4.15}$$

Further, since $\mathrm{Ran} B(2) \subseteq \mathrm{Ran} M_1$, there is a dependence relation in $\mathrm{Col}[M_1 \;\; B(2)]$ of the form

$$X^2 = r1 + sX. \tag{4.16}$$

Now, (4.14)–(4.16) must also hold in $\mathrm{Col} M^\flat$, so we have $c_{31} = ry_{02} + sy_{12}$ (from (4.16)) and, from (4.15)–(4.16), $c_{22} = \alpha y_{21} + \beta c_{21} = \alpha(ry_{01} + sy_{11}) + \beta(ry_{11} + sy_{21})$ $= r(\alpha y_{01} + \beta y_{11}) + s(\alpha y_{11} + \beta y_{21}) = ry_{02} + sy_{12} = c_{31}$.

Finally, consider the case when $r = 2$, $X = \alpha 1$, and $\{1, Y\}$ is a basis for $\mathrm{Col} M_1$. Then, as above, in $M^\flat$ we must have column relations $X = \alpha 1$, $X^2 = \alpha X$, and $XY = \alpha Y$, and these relations imply $c_{31} = \alpha y_{12} = c_{22}$. ∎

Now $C^\flat$ is a positive Hankel matrix and $M^\flat$ is a positive flat moment matrix, whose moment sequence we denote by $y^\flat$. Further, $\Delta \equiv C(2) - C^\flat$ is also a positive Hankel matrix and if $\sigma := \mathrm{rank}\Delta$, then we have $\rho \equiv \mathrm{rank} M = r + \sigma$. Since $\rho \leqslant 3$ and we may assume that M is not flat, then we have $1 \leqslant r \leqslant 2$ and $1 \leqslant \sigma \leqslant 2$. We denote Δ by

$$\Delta \equiv \begin{pmatrix} u & v & w \\ v & w & f \\ w & f & g \end{pmatrix}$$

and we define a block matrix (compatible in block sizes with (4.5)) by

$$M^\Delta \equiv \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix},$$

so that $M_2(y) = M^\flat + M^\Delta$.

Case I. $u = 0$. Since $\Delta \succeq 0$, it then follows that $v = w = 0$, so then $f = 0$ as well, with $g > 0$. In this case, let $\widetilde{y}^{[k]}$ denote the moment sequence of the atomic measure

$$\mu^{[k]} := \frac{g}{k^8} \delta_{(k,k^2)},$$

and let

$$y^{[k]} := y^\flat + \widetilde{y}^{[k]}, \tag{4.17}$$

so that

$$M_2^{[k]} \equiv M_2(y^{[k]}) = M^\flat + M_2[\mu^{[k]}]. \tag{4.18}$$

Clearly, $\lim\limits_{k\to\infty} \widetilde{y}_{ij}^{[k]} = 0$ for $(i,j) \neq (0,4)$, and $\widetilde{y}_{04}^{[k]} = g$, so $\lim\limits_{k\to\infty} M_2^{[k]} = M$. To show that $M_2^{[k]}$ is flat, we consider several cases.

Subcase I(a). *In* $\mathrm{Col} M_1(y)$ *there is a relation* $X = \alpha 1$. So $M_1(y)$ is of the form

$$M_1(y) = \begin{pmatrix} 1 & \alpha & \beta \\ \alpha & \alpha^2 & \alpha\beta \\ \beta & \alpha\beta & \gamma \end{pmatrix}, \tag{4.19}$$

with $\gamma \geqslant \beta^2$. Calculations show that

$$\det[M_1^{[k]}]_{2\times 2} = \frac{g(k-\alpha)^2}{k^8}, \text{ and} \tag{4.20}$$

$$\det M_1^{[k]} = \frac{g(k-\alpha)^2(\gamma - \beta^2)}{k^8}. \tag{4.21}$$

If $\operatorname{rank} M_1(y) = 1$ (equivalently, $\gamma = \beta^2$), then (4.20)–(4.21) imply $\operatorname{rank} M_1^{[k]} = 2$ (for $k > \alpha$), so, using (4.8) and (4.18),

$$2 = \operatorname{rank} M_1^{[k]} \leqslant \operatorname{rank} M_2^{[k]} \leqslant \operatorname{rank} M^\flat + \operatorname{rank} M_2[\delta_{(k,k^2)}] = 2,$$

whence $M_2^{[k]}$ is flat.

Next, if $\operatorname{rank} M_1(y) = 2$, with column basis $\{1,\ Y\}$ (equivalently, $\gamma > \beta^2$), then (4.21) implies that $\operatorname{rank} M_1^{[k]} = 3$ for $k > \alpha$, whence

$$3 = \operatorname{rank} M_1^{[k]} \leqslant \operatorname{rank} M_2^{[k]} \leqslant \operatorname{rank} M^\flat + \operatorname{rank} M_2[\delta_{(k,k^2)}] = \operatorname{rank} M_1(y) + 1 = 3, \tag{4.22}$$

so again $M_2^{[k]}$ is flat.

Subcase I(b). $\operatorname{rank} M_1(y) = 2$ *and* $\{1,\ X\}$ *is a column basis for* $\operatorname{Col} M_1(y)$. So $H \equiv M_1(y)$ is a flat extension of $[M_1(y)]_{2\times 2}$. Since

$$M_1[\mu^{[k]}] = g \begin{pmatrix} \frac{1}{k^8} & \frac{1}{k^7} & \frac{1}{k^6} \\ \frac{1}{k^7} & \frac{1}{k^6} & \frac{1}{k^5} \\ \frac{1}{k^6} & \frac{1}{k^5} & \frac{1}{k^4} \end{pmatrix},$$

it is of the form $L \equiv L(k, 2, 8, g)$ (cf. (4.4)). Lemma 4.1 thus implies that $M_1^{[k]} \equiv H + L$ has rank 3, so it follows exactly as in (4.22) that $M_2^{[k]}$ is flat.

Case II. $u > 0$ *and* Δ *admits a representing measure.* In this case, from Theorem 2.2(vi), Δ admits a σ-atomic representing measure, either of the form

$$u\delta_{v/u} \tag{4.23}$$

or of the form

$$\xi\delta_{x_1} + \tau\delta_{x_2} \quad (\xi + \tau = u). \tag{4.24}$$

Corresponding to (4.23), define a bivariate measure $\mu^{[k]}$ by

$$\mu^{[k]} := \frac{u}{k^4}\delta_{(k,(v/u)k)}, \tag{4.25}$$

and corresponding to (4.24) define

$$\mu^{[k]} := \frac{\xi}{k^4}\delta_{(k,kx_1)} + \frac{\tau}{k^4}\delta_{(k,kx_2)}. \tag{4.26}$$

It is straightforward to check that in both cases, the moments for $\mu^{[k]}$ of degree $\leqslant 3$ converge to 0 as $k \to \infty$ and that the degree 4 moments coincide with those of M^Δ. Note also that in both cases $M_1[\mu^{[k]}]$ is of the form

$$M_1[\mu^{[k]}] = \begin{pmatrix} \frac{u}{k^4} & \frac{u}{k^3} & \frac{v}{k^3} \\ \frac{u}{k^3} & \frac{u}{k^2} & \frac{v}{k^2} \\ \frac{v}{k^3} & \frac{v}{k^2} & \frac{w}{k^2} \end{pmatrix}. \tag{4.27}$$

In the sequel, we denote $M_1(y)$ by

$$M_1(y) \equiv \begin{pmatrix} 1 & a & b \\ a & c & d \\ b & d & e \end{pmatrix}. \tag{4.28}$$

A calculation now shows that $\mathcal{D}(k) := -k^6 \det(M_1(y) + M_1[\mu^{[k]}])$ is a polynomial in k of degree 6, for which the coefficients of k^2 and k^4 are equal, respectively, to

$$\kappa_2 \equiv (d^2 - ce)u + v^2 - uw, \text{ and} \tag{4.29}$$

$$\kappa_4 \equiv -((c-a^2)w + (e-b^2)u + 2v(ab-d)). \tag{4.30}$$

We will refer to (4.29) and (4.30) in the sequel. We now consider four subcases based on the values of $r \equiv \operatorname{rank} M_1(y)$ and $\sigma \equiv \operatorname{rank} \Delta$.

Subcase II(1). $r = 1, \sigma = 1$. Since $r = 1$, we have

$$c = a^2, \quad d = ab, \quad ce = d^2, \quad e = b^2. \tag{4.31}$$

Since $\sigma = 1$, we have

$$w = \frac{v^2}{u}, \quad f = \frac{v^3}{u^2}, \quad g = \frac{v^4}{u^3}. \tag{4.32}$$

Let $\widetilde{y}^{[k]}$ denote the moment sequence of the atomic measure $\mu^{[k]} := \frac{u}{k^4}\delta_{(k,(v/u)k)}$, and define $y^{[k]}$ and $M^{[k]}$ using (4.17) and (4.18), i.e., $y^{[k]} := y^\flat + \widetilde{y}^{[k]}$ and $M_2^{[k]} \equiv M_2(y^{[k]}) = M^\flat + M_2(\widetilde{y}^{[k]})$. As noted above, the moments of degree $\leqslant 3$ for $\mu^{[k]}$ converge to 0 as $k \to \infty$ and the moments of degree 4 coincide with the degree 4 moments in M^Δ. It follows that $\lim_{x\to\infty} M_2^{[k]} = M$. Further, a calculation using (4.27)–(4.28) now shows that

$$\det[M_1^{[k]}]_{2\times 2} = \frac{u(k-a)^2}{k^4}.$$

Thus, for $k > \alpha$,

$$2 \leqslant \operatorname{rank} M_1^{[k]} \leqslant \operatorname{rank} M_2^{[k]} \leqslant \operatorname{rank} M^\flat + \operatorname{rank} M_2(\widetilde{y}^{[k]}) = r + 1 = 2,$$

so $M_2^{[k]}$ is flat.

Subcase II(2). $r = 1$, $\sigma = 2$. As in the previous case, $ce = d^2$, but in this case, since $\sigma = 2$ and Δ has a representing measure, we have $uw - v^2 > 0$ (cf. Theorem 2.2). Since $\sigma = 2$, we have a representing measure for Δ of the form $\xi\delta_{x_1} + \tau\delta_{x_2}$ (where $x_1 \neq x_2$, $\xi, \tau > 0$, $\xi + \tau = u$), and, following as in (4.26), we define

$$\mu^{[k]} := \frac{\xi}{k^4}\delta_{(k,kx_1)} + \frac{\tau}{k^4}\delta_{(k,kx_2)}.$$

As in the previous case, the low order moments of $\mu^{[k]}$ converge to 0 and the degree 4 moments coincide with those of M^Δ. Thus, defining $y^{[k]}$ and $M^{[k]}$ as above (using (4.17) and (4.18)), we have $\lim_{x\to\infty} M_2^{[k]} = M$. Further, $M_1[\mu^{[k]}]$ has the same form as in (4.27), and since $ce = d^2$, (4.29) shows that $\kappa_2 = v^2 - uw > 0$. It follows that for all sufficiently large k, $\mathcal{D}(k) \neq 0$, so

$$3 = \operatorname{rank} M_1^{[k]} \leqslant \operatorname{rank} M_2^{[k]} \leqslant \operatorname{rank} M^\flat + \operatorname{rank} M_2[\mu^{[k]}] \leqslant 3, \tag{4.33}$$

whence $M_2(y^{[k]})$ is flat.

Subcase II(3). *$r = 2$, $\sigma = 1$, $\{1, X\}$ is dependent in* $\operatorname{Col} M_1(y)$*, and $\{1, Y\}$ is independent.* Since $\sigma = 1$, we have $wu = v^2$. We define $\mu^{[k]}$, with moment sequence that we denote by $\widetilde{y}^{[k]}$, as in (4.25), and we define $y^{[k]}$ and $M_2^{[k]}$ using (4.17) and (4.18). Exactly as above, $\lim_{x\to\infty} M_2^{[k]} = M$, so it remains to show that $M_2^{[k]}$ is flat. If $ce - d^2 > 0$, then since $wu = v^2$, using (4.29) we see that $\kappa_2 = -(ce - d^2)u < 0$, and it follows as in (4.33) that $\operatorname{rank} M_1^{[k]} = \operatorname{rank} M_2^{[k]} = \operatorname{rank} M = 3$. Next, suppose $ce - d^2 = 0$. Since $\{1, X\}$ is dependent and $r = 2$, it follows that $X = 0$, whence $a = c = d = 0$ and $e - b^2 > 0$. Now (4.30) shows that $\kappa_4 = -(e - b^2)u^2 < 0$, so it follows as above (from (4.33)) that $\operatorname{rank} M_1^{[k]} = \operatorname{rank} M_2^{[k]} = \operatorname{rank} M = 3$.

Subcase II(4). *$r = 2$, $\sigma = 1$, and $\{1, X\}$ is a basis for* $\operatorname{Col} M_1(y)$*.* If $ce > d^2$ (equivalently, $\{X, Y\}$ is independent), we may proceed as in Subcase II(3) (using $wu = v^2$ and (4.29)) to see that $\kappa_2 < 0$ and to thereby conclude from (4.33) that $M_2^{[k]}$ is flat, with $\operatorname{rank} M_2^{[k]} = \operatorname{rank} M = 3$. Thus, we may assume that $ce = d^2$. Since $\sigma = 1$, the relations of (4.32) hold, particularly $wu = v^2$, whence $\kappa_2 \equiv (d^2 - ce)u + v^2 - uw = 0$. Let us turn our attention to κ_4 (cf. (4.30)). Of course, if $\kappa_4 \neq 0$, then, as above, for k sufficiently large, we have $\mathcal{D}(k) \neq 0$, whence (4.33) implies that $M_2^{[k]}$ is flat (with rank 3). Since $ce = d^2$ and $X \neq 0$, positivity implies that in $\operatorname{Col} M$ we have a dependence relation of the form $Y = \alpha X$, whence

$$b = \alpha a, \quad d = \alpha c, \quad e = \alpha^2 c. \tag{4.34}$$

A calculation now shows that $\kappa_4 = 0$ if and only if $v = \alpha u$. In this case, since $\sigma = 1$, it then follows that $w = \alpha^2 u$, $f = \alpha^3 u$, $g = \alpha^4 u$. A further calculation shows that in this case, $\mathcal{D}(k) \equiv 0$, so $\operatorname{rank} M_1^{[k]} < 3$; since $\lim_{x\to\infty} M_2^{[k]} = M$, we have $\operatorname{rank} M_2^{[k]} \geqslant \operatorname{rank} M = 3$ (for large k), so we conclude that $M_2^{[k]}$ is *not* flat. To rectify this, if $\alpha \neq 0$ we redefine $\mu^{[k]}$ as

$$\mu^{[k]} := \frac{u}{k^4}\delta_{(k,\alpha(k+(1/k)))}$$

and set $M_2^{[k]} := M^\flat + M_2[\mu^{[k]}]$. It follows readily that $\lim_{x\to\infty} M_2[\mu^{[k]}] = M^\Delta$, so $\lim_{x\to\infty} M_2^{[k]} = M$. Since $\{1, X\}$ is a basis, $c > a^2$; further, $\det M_1^{[k]} = \frac{(c-a^2)\alpha^2 u}{k^6} > 0$, so we conclude as in (4.33) that $3 = \operatorname{rank} M_1^{[k]} = \operatorname{rank} M_2^{[k]} = \operatorname{rank} M$. In the case when $\alpha = 0$, we redefine $\mu^{[k]}$ as

$$\mu^{[k]} := \frac{u}{k^4}\delta_{(k,(1/k))}.$$

Setting $M_2^{[k]} := M^\flat + M_2[\mu^{[k]}]$, it follows that $\lim_{x\to\infty} M_2^{[k]} = M$, and since $\det M_1^{[k]} = \frac{(c-a)^2 u}{k^6} > 0$, we see that $\operatorname{rank} M_1^{[k]} = \operatorname{rank} M_2^{[k]} = \operatorname{rank} M = 3$.

Case III. *$u > 0$ and Δ admits no representing measure.* This occurs precisely when Δ is of the form

$$\Delta \equiv \begin{pmatrix} u & v & \frac{v^2}{u} \\ v & \frac{v^2}{u} & \frac{v^3}{u^2} \\ \frac{v^2}{u} & \frac{v^3}{u^2} & g \end{pmatrix},$$

where $g > \frac{v^4}{u^3}$ (cf. Theorem 2.2). Since $\operatorname{rank}\Delta = 2$, we have $r = 1$, so $c = a^2$, $d = ab$, $e = b^2$. We define

$$M_2^{[k]} := M^\flat + M_2\left[\frac{u}{k^4}\delta_{(k,(v/u)k)}\right] + M_2\left[\frac{s}{k^4}\delta_{(0,k)}\right],$$

where $s = g - \frac{v^4}{u^3}$. It follows that $\lim_{x\to\infty} M_2^{[k]} = M$. Moreover, a calculation shows that

$$\det M_1^{[k]} = \frac{s(au + bu - ku - av)^2}{k^6 u},$$

and since $u > 0$, it follows that for large k we have

$$3 = \operatorname{rank} M_1^{[k]} \leqslant \operatorname{rank} M_2^{[k]} \leqslant \operatorname{rank} M^\flat + \operatorname{rank} M_2\left[\frac{u}{k^4}\delta_{(k,\frac{v}{u}k)}\right] + \operatorname{rank} M_2\left[\frac{s}{k^4}\delta_{(0,k)}\right] = 3,$$

whence $M_2^{[k]}$ is flat. ∎

5. APPENDIX

In this section we present the proofs of Theorem 1.5 and Proposition 1.6, an alternate proof of Theorem 3.1 based on Lemma 4.1, and two examples concerning Question 1.3 in the case $n = 2$, $d = 3$.

Proof of Theorem 1.5. Theorem 2.2 of [10] shows that L_y is K-positive if and only if y is in the closure of the multisequences having K-representing measures. Thus, for $\varepsilon > 0$, there exists a multisequence $y_\varepsilon \equiv y_\varepsilon^{(2t)}$, having a K-representing measure μ_ε, such that

$$1 - \varepsilon < (y_\varepsilon)_0 < 1 + \varepsilon.$$

Since $|L_y(p) - L_{y_\varepsilon}(p)| \leqslant \|y - y_\varepsilon\|_1 \|\widehat{p}\|_\infty$, we may further assume that

$$L_y(p) - \varepsilon < L_{y_\varepsilon}(p) < L_y(p) + \varepsilon.$$

Consider first the case when $p_* \geqslant 0$, and let $\varepsilon > 0$. Now $p_t = L_y(p) > L_{y_\varepsilon}(p) - \varepsilon = \int p\,d\mu_\varepsilon - \varepsilon \geqslant p_*(y_\varepsilon)_0 - \varepsilon \geqslant p_*(1-\varepsilon) - \varepsilon = p_* - \varepsilon(p_* + 1)$. Thus, it follows that $p_t \geqslant p_*$, as desired. In case $p_* < 0$, we have $p_t = L_y(p) > L_{y_\varepsilon}(p) - \varepsilon =$

$\int p\,\mathrm{d}\mu_\varepsilon - \varepsilon \geqslant p_*(y_\varepsilon)_0 - \varepsilon > p_*(1+\varepsilon) - \varepsilon = p_* + \varepsilon(p_* - 1)$, whence $p_t \geqslant p_*$, and the result again follows. ∎

Proof of Proposition 1.6. The equivalence of (ii) and (iii) is Hilbert's theorem on sums of squares (cf. [19], [20]). Suppose (ii) holds and that $M_d(y) \succeq 0$. To show that L_y is positive, suppose $p \in \mathcal{P}_{2d}$ satisfies $p|\mathbb{R}^n \geqslant 0$. Then $p = \sum p_i^2$ for certain $p_i \in \mathcal{P}_d$, whence $L_y(p) = \sum \langle M_d(y)\widehat{p}_i, \widehat{p}_i\rangle \geqslant 0$; thus (ii) implies (i). Conversely, suppose there is a polynomial $p \in \mathcal{P}_{2d}$ such that $p|\mathbb{R}^n \geqslant 0$, but p is not in the convex cone Σ^2 in $\mathcal{P}_{2d}$ consisting of sums of squares. Corollary 3.50 in [17] shows that Σ^2 is closed in $\mathcal{P}_{2d}$, so it follows from the Separation Theorem (cf. Corollary 34.2 in [2]) that there is a linear functional $L : \mathcal{P}_{2d} \mapsto \mathbb{R}$ such that $L|\Sigma^2 \geqslant 0$ and $L(p) < 0$. If $y \equiv y^{(2d)}$ is the moment sequence of L, i.e., $y_i = L(x^i)$ for $|i| \leqslant 2d$, then clearly $M_d(y) \succeq 0$ and $L_y = L$, so L_y is not positive. ∎

Alternate proof of Theorem 3.1. In the notation of the proof of Theorem 3.1, it suffices to show that $\mathrm{rank} H_{d-1}(y^{[k]}) = r + 1$. We apply Lemma 4.1 with $H \equiv H_r(y)$ and $M \equiv H_{r-1}(y)$. From Theorem 2.2, $M \succ 0$ and $\mathrm{rank} H = \mathrm{rank} M = r$. Let $q = 2d$ and set $L \equiv L(k,r,q,\eta) = \frac{\eta}{k^q}[k]_r[k]_r^{\mathrm{T}}$, so that $H_r(y^{[k]}) = H + L$. Lemma 4.1 implies that for k sufficiently large, $\mathrm{rank}(H+L) = r+1$. Thus, $r+1 = \mathrm{rank} H_r(y^{[k]}) \leqslant \mathrm{rank} H_{d-1}(y^{[k]}) \leqslant \mathrm{rank} H_d(y^{[k]}) \leqslant r+1$, so the result follows. ∎

Recall from Section 4 that if $M_2(y)$ satisfies the hypothesis of Theorem 4.1, then the Schur complement Δ is Hankel, and this forms the basis for the proof. We next present an example concerning the case $n = 2$, $d = 3$, in which the positivity and rank conditions are satisfied, but Δ is not Hankel.

EXAMPLE 5.1. Consider $M \equiv M_3(y)$ of the form

$$M = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 4 \end{pmatrix}.$$

M is positive semidefinite, with column relations $X^2 = 1$, $X^3 = X$, $X^2Y = Y$, and $Y^3 = 2Y$. Thus, $\mathrm{rank} M = 6$, so M satisfies the positivity and rank conditions of Question 1.3. It is easy to check that by propagating the column relations forward, i.e., by defining $X^4 := X^2$, $X^3Y := XY$, $X^2Y^2 := Y^2$, $XY^3 := 2XY$, $Y^4 := 2Y^2$, we thereby construct a moment matrix $M_4(\widetilde{y})$ that is a flat extension of M, so y has a 6-atomic representing measure by Theorem 1.2. Is $M \in \overline{\mathcal{F}}_3$? If we try to use the method of Section 4, we must verify that the Schur complement Δ is Hankel. We see that $B(3) = M_2(y)W$ where

$$W = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, we see

$$C^\flat \equiv W^{\mathrm{T}} M_2 W = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 4 \end{pmatrix}, \quad \Delta \equiv C(3) - C^\flat = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Despite the fact that the Schur complement is not Hankel, we can nevertheless show that $M \in \overline{\mathcal{F}}_3$. Indeed, let

$$M_3^{[k]} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1+\frac{1}{k} & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1+\frac{1}{k} & 0 & 1 & 0 & 0 & \frac{1}{\sqrt{k}} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{1}{\sqrt{k}} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 2 & \frac{1}{\sqrt{k}} & 0 & 0 & 0 \\ 0 & 1+\frac{1}{k} & 0 & 0 & 0 & \frac{1}{\sqrt{k}} & (1+\frac{1}{k})^2+\frac{1}{k} & 0 & 1+\frac{1}{k} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{\sqrt{k}} & 0 & 0 & 1+\frac{1}{k} & 0 & 2 \\ 0 & 1 & 0 & \frac{1}{\sqrt{k}} & 0 & 0 & 1+\frac{1}{k} & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 2 & 0 & 4 \end{pmatrix}.$$

Clearly, $\lim_{k\to\infty} M_3^{[k]} = M$, and it is straightforward to check that $M_3^{[k]} \succeq 0$ and that $\mathrm{rank} M_3^{[k]} = \mathrm{rank} M_2^{[k]} = \mathrm{rank} M = 6$. Thus we see that $M \in \overline{\mathcal{F}}_3$.

In cases where y has no representing measure and where sums of squares are not available (cf. Proposition 1.6), membership in $\overline{\mathcal{F}}_d$ provides an alternate criterion for positivity of L_y. We illustrate this approach to positivity in the following example.

EXAMPLE 5.2. Let $n = 2$, $d = 3$, so sums of squares are not available. For $c \in \mathbb{R}$, $\kappa > 0$, consider $y^{(6)}$ defined by

$$M \equiv M_3(y) = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & c \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & c \\ 0 & 0 & 1 & 0 & 1 & c & 0 & 1 & c & 1+c^2 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & c \\ 0 & 0 & 1 & 0 & 1 & c & 0 & 1 & c & 1+c^2 \\ 1 & 1 & c & 1 & c & 1+c^2 & 1 & c & 1+c^2 & 2c+c^3 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & c \\ 0 & 0 & 1 & 0 & 1 & c & 0 & 1 & c & 1+c^2 \\ 1 & 1 & c & 1 & c & 1+c^2 & 1 & c & 1+c^2 & 2c+c^3 \\ c & c & 1+c^2 & c & 1+c^2 & 2c+c^3 & c & 1+c^2 & 2c+c^3 & \gamma \end{pmatrix},$$

where $\gamma = 1 + 3c^2 + c^4 + \kappa$. Then $M_3 \succeq 0$, and there are column relations $1 = X = X^2 = X^3$, $Y = XY = X^2Y$, $X + cY = Y^2 = Y^2X$, whence $\mathrm{rank} M = 3$. Since $\kappa > 0$, $Y^3 \neq XY + cY^2$, so M is not recursively generated, whence y has no representing measure. Let $M^\flat$ be the flat positive moment matrix obtained from M by replacing γ by $1 + 3c^2 + c^4$. For $k > 0$, let $M_3^{[k]} := M^\flat + M_3[\frac{\kappa}{k^6}\delta_{(0,k)}]$. It is easy to check that $\lim_{k\to\infty} M_3^{[k]} = M$, and that $M_3^{[k]}$ is positive and flat, with $\mathrm{rank} M = \mathrm{rank} M_3^{[k]} = \mathrm{rank} M_2^{[k]} = 3$. Thus, $y \in \overline{\mathcal{F}}_3$, so L_y is positive, but positivity is not a consequence of representing measures or sums of squares.

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