

NUCLEAR AND TYPE I C^* -CROSSED PRODUCTS

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ABSTRACT. We prove that a C^* -crossed product $A \times_{\alpha} G$ by a locally compact group G is nuclear (respectively type I or liminal) if and only if certain hereditary C^* -subalgebras, $S_{\pi}, \mathcal{I}_{\pi} \subset A \times_{\alpha} G$ $\pi \in \widehat{K}$, are nuclear (respectively type I or liminal). Analog characterizations are proved for C^* -crossed products by compact quantum groups. These subalgebras are the analogs of the algebras of spherical functions considered by R. Godement for groups with large compact subgroups. If $K = G$ is a compact group or a compact quantum group, the algebras S_{π} are stably isomorphic with the fixed point algebras $A \otimes B(H_{\pi})^{\alpha \otimes \text{ad} \pi}$ where H_{π} is the Hilbert space of the representation π .

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1. INTRODUCTION AND PRELIMINARY RESULTS

Let G be a locally compact group and $K \subset G$ a compact subgroup. In [12] (see also [22]) the study of \widehat{G} , the set of equivalence classes of irreducible representations of G is reduced to the study of \widehat{K} and of representations of certain classes of spherical functions. In this paper we extend this approach to the case of crossed products of C^* -algebras by locally compact group and compact quantum group actions. Let (A, G, α) be a C^* -dynamical system and let $K \subset G$ be a compact subgroup.

In [17] we defined the C^* -algebras $S_{\pi}, \mathcal{I}_{\pi} \subset A \times_{\alpha} G$, $\pi \in \widehat{K}$. These are the analogs of the algebras of spherical functions. For the case $K = G$, these algebras were previously defined by Landstad in [15].

Recently, in [7], [9], we have extended the study of these algebras to the case of compact quantum group actions on C^* -algebras. If $K = G$ is a compact group or a compact quantum group, the algebras S_{π} are stably isomorphic with the fixed point algebras $A \otimes B(H_{\pi})^{\alpha \otimes \text{ad} \pi}$ where H_{π} is the Hilbert space of the representation π . In this section we will review some definitions and preliminary results.

1.1. PRELIMINARIES ON ACTIONS OF COMPACT GROUPS ON C^* -ALGEBRAS. Let K be a compact group and let $\delta : K \rightarrow \text{Aut}(A)$ be an action of K on a C^* -algebra A . Let $\pi \in \widehat{K}$. If $\pi_{ij}(g)$ are the coefficients of π_g in a fixed basis of the Hilbert space H_π of the representation $\pi, 1 \leq i, j \leq d_\pi$ we define the character of $\pi, \chi_\pi(g) = d_\pi \text{tr}(\pi_{g^{-1}}), g \in K$ where d_π is the dimension of the representation π . We consider the following mapping from B into itself :

$$(1.1) \quad P^{\pi, \delta}(a) = \int_K \chi_\pi(k) \delta_k(a) dk.$$

We define the spectral subspaces of the action δ

$$(1.2) \quad A_1^\delta(\pi) = \{a \in A : P^{\pi, \delta}(a) = a\}, \quad \pi \in \widehat{K}.$$

In particular, if $\pi = \pi_0$ is the trivial one dimensional representation, then $A_1^\delta(\pi_0) = A^\delta$ is the algebra of fixed elements under the action δ . In this case, the projection $P^{\pi_0, \delta}$ of A onto A^δ is a conditional expectation since $P^{\pi_0, \delta} = \int_K \delta_k(a) dk$ and the Haar measure is invariant. In particular, $P^{\pi_0, \delta}$ is a completely positive map.

1.2. ALGEBRAS OF SPHERICAL FUNCTIONS INSIDE THE CROSSED PRODUCT. Let now (A, G, α) be a C^* -dynamical system with G a locally compact group and $K \subset G$ a compact subgroup. Denote by $A \times_\alpha G$ the corresponding crossed product (see for instance [16]). Then the algebra $C(K)$ of all continuous functions on G can be embedded as follows in the multiplier algebra $M(A \times_\alpha G)$ of $A \times_\alpha G$: If $\varphi \in C(K)$ and $y \in C_c(G, A)$, the dense subalgebra of $A \times_\alpha G$ consisting of continuous functions with compact support from G to A , then

$$(1.3) \quad (\varphi y)(g) = \int_K \varphi(k) \alpha_k(y(k^{-1}g)) dk, \quad \text{and} \quad (y\varphi)(g) = \int_K \varphi(k) y(gk) dk.$$

In particular, if $\varphi = \chi_\pi, \varphi$ is a projection in $M(A \times_\alpha G)$ and if π_1 and π_2 are distinct elements in \widehat{K} , the projections χ_{π_1} and χ_{π_2} are orthogonal. We need the following results from Lemma 2.5. of [17]:

REMARK 1.1. The following statements hold:

- (i) If $\pi_1 \neq \pi_2$ in \widehat{K} then the projections χ_{π_1} and χ_{π_2} are orthogonal in $M(A \times_\alpha G)$.
- (ii) $\sum_\pi \chi_\pi = I$, where I is the identity of the bidual $(A \times_\alpha G)^{**}$ of $A \times_\alpha G$.

If $\pi \in \widehat{K}$, denote $S_\pi = \overline{\chi_\pi(A \times_\alpha G)\chi_\pi}$, where the closure is taken in the norm topology of $A \times_\alpha G$. Then, it is immediate that S_π is strongly Morita equivalent with the two sided ideal $J_\pi = \overline{(A \times_\alpha G)\chi_\pi(A \times_\alpha G)}$. Indeed, it can be easily verified that $X = \overline{(A \times_\alpha G)\chi_\pi}$ is an $S_\pi - J_\pi$ imprimitivity bimodule. We will consider next the action, δ of K on $A \times_\alpha G$ defined as follows: If $y \in C_c(G, A)$ set $\delta_k(y) = \alpha_k(y(k^{-1}gk))$. Then δ_k extend to automorphisms of $A \times_\alpha G$ and thus δ is

an action of K on $A \times_{\alpha} G$. The fixed point algebra $\mathcal{I} = (A \times_{\alpha} G)^{\delta}$ is called in [17] the algebra of K -central elements of the crossed product $A \times_{\alpha} G$. Denote:

$$(1.4) \quad \mathcal{I}_{\pi} = \mathcal{I} \cap S_{\pi}.$$

Then, ([17], Proposition 2.7) we have

REMARK 1.2. S_{π} is $*$ -isomorphic with $\mathcal{I}_{\pi} \otimes B(H_{\pi})$.

If $G = K$ is a compact group, then by Lemma 3 of [15] we have:

REMARK 1.3. For every $\pi \in \widehat{G}$, \mathcal{I}_{π} is $*$ -isomorphic with $(A \otimes B(H_{\pi}))^{\alpha \otimes \text{ad}\pi}$.

1.3. COMPACT QUANTUM GROUP ACTIONS ON C*-ALGEBRAS. Let $\mathcal{G} = (B, \Delta)$ be a compact quantum group [23], [24]. Here, B is a unital C*-algebra (which is the analog of the C*-algebra of continuous functions in the group case) and $\Delta : B \rightarrow B \otimes_{\min} B$ a $*$ -homomorphism such that:

- (i) $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$, where $\iota : B \rightarrow B$ is the identity map, and
- (ii) $\overline{\Delta(B)(1 \otimes B)} = \overline{\Delta(B)(B \otimes 1)} = B \otimes_{\min} B$.

Let $\widehat{\mathcal{G}}$ denote the set of all equivalence classes of unitary representations of \mathcal{G} or equivalently, the set of all equivalence classes of irreducible unitary co-representations of B . For each $\pi \in \widehat{\mathcal{G}}$, $\pi = [\pi_{ij}]$, $\pi_{ij} \in B$ $1 \leq i, j \leq d_{\pi}$, where d_{π} is the dimension of π , let $\chi_{\pi} = \sum_i \pi_{ii}$ be the character of π and let $F_{\pi} \in B(H_{\pi})$ be the positive, invertible matrix that intertwines π with its double contragredient representation and such that $\text{tr}(F_{\pi}) = \text{tr}(F_{\pi}^{-1}) = M_{\pi}$. Then, with the notations in [23], $F_{\pi} = [f_1(\pi_{ij})]$ where f_1 is a linear functional on the $*$ -subalgebra $\mathcal{B} \subset B$ that is linearly spanned by $\{\pi_{ij} : \pi \in \widehat{\mathcal{G}}, 1 \leq i, j \leq d_{\pi}\}$. If $a \in B$ (respectively \mathcal{B}) and ζ is a linear functional on B (respectively \mathcal{B}) we denote ([23], [24]):

$$(1.5) \quad a * \zeta = (\zeta \otimes \iota)(\Delta(a)) \in B.$$

Denote also by $\zeta \cdot a$ the following linear functional on B (respectively \mathcal{B}):

$$(1.6) \quad (\zeta \cdot a)(b) = \zeta(ab).$$

If h is the Haar state on B let $h_{\pi} = M_{\pi}h \cdot (\chi_{\pi} * f_1)$. If v_r is the right regular representation of \mathcal{G} , the Fourier transform of $a \in B$ is defined as follows (see [24]):

$$(1.7) \quad \widehat{a} = \mathcal{F}_{v_r}(a) = (\iota \otimes h \cdot a)(v_r^*).$$

Then the norm closure of the set $\widehat{B} = \{\widehat{a} : a \in B\}$ is a C*-algebra called the dual of B ([1], [24]) and \widehat{B} is a subalgebra of the algebra of compact operators, $\mathcal{C}(H_h)$ on the Hilbert space H_h of the GNS representation of B associated with the Haar state h .

Let A be a C*-algebra and $\delta : A \rightarrow M(A \otimes B)$ be a $*$ -homomorphism of A into the multiplier algebra of the minimal tensor product $A \otimes B$. Then δ is called an action of \mathcal{G} on A (or a coaction of B on A) if the following two conditions hold:

- (a) $(\iota \otimes \Delta)\delta = (\delta \otimes \iota)\delta$, and

(b) $\overline{\delta(A)(1 \otimes B)} = A \otimes B.$

Let $\pi \in \widehat{G}$. Denote $P^{\pi, \delta}(a) = (\iota \otimes h_\pi)(\delta(a)), a \in A$. Then $P^{\pi, \delta}$ is a contractive linear map from A into itself. In particular, if $\pi = \pi_0$ is the trivial one dimensional representation, then $P^{\pi_0, \delta} = (\iota \otimes h)\delta$ is the completely positive projection of norm 1 of A onto the fixed point C^* -subalgebra A^δ .

The crossed product $A \times_\delta \mathcal{G}$ is by definition, ([1], [2]), the norm closure of the set $\{(\pi_u \otimes \pi_h)(\delta(a)(1 \otimes \widehat{b})) : a \in A, b \in B\}$, where π_u is the universal representation of A and π_h is the GNS representation of B associated with the Haar state h .

Let $\pi \in \widehat{G}$. If we denote $p_\pi = (\iota \otimes h_\pi)(v_r^*)$, then $\{p_\pi\}_{\pi \in \widehat{G}}$ are mutually orthogonal projections in \widehat{B} and therefore in $A \times_\delta \mathcal{G}$ ([2], [7]). For $\pi \in \widehat{G}$ denote $\mathcal{S}_\pi = \overline{p_\pi(A \times_\delta \mathcal{G})p_\pi}$. In Lemma 3.3 of [7] it is shown that $\text{ad}(v_r)$ is an action of \mathcal{G} on the crossed product $A \times_\delta \mathcal{G}$ and the fixed point algebra $\mathcal{I} = (A \times_\delta \mathcal{G})^{\text{ad}(v_r)}$ of this action plays the role of the K -central elements in the case of groups. Let $\mathcal{I}_\pi = \mathcal{I} \cap \mathcal{S}_\pi$. Let δ_π be the following action of \mathcal{G} on $A \otimes B(H_\pi)$:

$$(1.8) \quad \delta_\pi(a \otimes m) = (\pi)_{23}(\delta(a))_{13}(1 \otimes m \otimes 1)(\pi^*)_{23}$$

where the leg-numbering notation is the usual one [1], [24]. The above δ_π equals $\delta \otimes \text{ad}(\pi)$ in the case of compact groups. Then, we have:

REMARK 1.4. The following statements hold true:

- (i) The projections $\{p_\pi\}_{\pi \in \widehat{G}}$ are mutually orthogonal and $\sum_{\pi} p_\pi = 1$ in the bidual $(A \times_\delta \mathcal{G})^{**}$.
- (ii) \mathcal{S}_π is $*$ -isomorphic with $\mathcal{I}_\pi \otimes B(H_\pi)$.
- (iii) \mathcal{I}_π is $*$ -isomorphic with $A \otimes B(H_\pi)^{\delta_\pi}$.

Part (i) is Section 2.1., Equation (2) and the discussion after that equation in [7]. Part (ii) is Remark 3.5. of [7], and part (iii) is Proposition 4.8 of [7].

2. NUCLEAR AND TYPE I CROSSED PRODUCTS

In this section we will state and prove our main results. We give necessary and sufficient conditions for a crossed product to be nuclear or type I. Our conditions are given in terms of the algebras of spherical functions inside the crossed product and in case of compact groups or compact quantum groups, in terms of the fixed point algebras of $A \otimes B(H_\pi)$ for the actions $\delta \otimes \text{ad}(\pi)$.

Recall that a C^* -algebra C is said to be of type I if for every factor representation T of C the von Neumann factor $T(C)''$ is a type I factor. C is called liminal if for every irreducible representation T of C , $T(C)$ consists of compact operators.

A C^* -algebra is called nuclear if its bidual, C^{**} , is an injective von Neumann algebra, i.e. if and only if there is a projection of norm one from $B(H_u)$ onto

C^{**} , where H_u is the Hilbert space of the universal representation of C . With the notations from Section 1, we have the following :

REMARK 2.1. Let (A, G, α) be a C^* -dynamical system with G a locally compact group and let $K \subset G$ be a compact subgroup. The following three statements hold:

- (i) S_π is nuclear if and only if \mathcal{I}_π is nuclear;
- (ii) S_π is liminal if and only if \mathcal{I}_π is liminal;
- (iii) S_π is type I if and only if \mathcal{I}_π is type I.

These statements follow from Remark 1.2.

The following is the analog of the above remark for the case of compact quantum group actions:

REMARK 2.2. Let $\mathcal{G} = (B, \Delta)$ be a compact quantum group and δ an action of \mathcal{G} on a C^* -algebra A . The following conditions are equivalent:

- (i) S_π is nuclear if and only if $(A \otimes B(H_\pi))^{\delta_\pi}$ is nuclear;
- (ii) S_π is liminal if and only if $(A \otimes B(H_\pi))^{\delta_\pi}$ is liminal;
- (iii) S_π is type I if and only if $(A \otimes B(H_\pi))^{\delta_\pi}$ is type I.

The result follows from Remark 1.4.

2.1. TYPE I CROSSED PRODUCTS. We start with the following general result:

LEMMA 2.3. *Let C be a C^* -algebra and $M(C)$ the multiplier algebra of C . Let $\{p_\lambda\} \subset M(C)$ be a family of mutually orthogonal projections of sum 1 in C^{**} . The following conditions are equivalent:*

- (i) C is type I (respectively liminal).
- (ii) *The hereditary subalgebras $S_\lambda = p_\lambda C p_\lambda \subset C$ are type I (respectively liminal) for every λ .*

Proof. Assume that C is type I (respectively liminal). Then S_λ are type I (respectively liminal) as C^* -subalgebras of a type I (liminal) C^* -algebra.

Assume now that all S_λ are type I (liminal). Let T be a nondegenerate factor representation (respectively an irreducible representation) of C . Since, by assumption, $\sum p_\lambda = 1$ it follows that $\sum p_\lambda C$ is norm dense in C . Therefore, there is a λ such that the restriction of T to $p_\lambda C$, $T|_{p_\lambda C} \neq 0$. Then $T|_{J_\lambda} \neq 0$, where $J_\lambda = \overline{C p_\lambda C}$ is the two sided ideal of C generated by p_λ . Since T is a factor representation of C (respectively an irreducible representation of C) and the bicommutant $T(J_\lambda)''$ is a nonzero weakly closed ideal of $T(C)''$ it follows that $T(J_\lambda)'' = T(C)''$. Therefore T has the same type with $T|_{J_\lambda}$. On the other hand, it can be checked that J_λ is strongly Morita equivalent with S_λ in the sense of Rieffel, [20], with imprimitivity bimodule $C p_\lambda$. Therefore, since S_λ is assumed to be type I (respectively liminal), it follows from the discussion in [20] (respectively [11]) that J_λ is type I (respectively liminal). It then follows that the representation

T is a type I representation (respectively $T(C)$ consists of compact operators). Since T was arbitrary, we are done. ■

We will state next some consequences of the above lemma.

THEOREM 2.4. *Let (A, G, α) be a C^* -dynamical system with G a locally compact group and let $K \subset G$ be a compact subgroup. Then the following conditions are equivalent:*

- (i) $A \rtimes_{\alpha} G$ is type I (respectively liminal).
- (ii) All hereditary C^* -subalgebras $S_{\pi} \subset A \rtimes_{\alpha} G$, $\pi \in \widehat{K}$ are type I (respectively liminal).
- (iii) All C^* -subalgebras of K -central elements, $\mathcal{I}_{\pi} \subset S_{\pi}$, $\pi \in \widehat{K}$ are type I (respectively liminal).

Proof. The equivalence of the conditions (i)–(iii) follows from Remarks 1.1 and 2.1 and Lemma 2.3. ■

If $G = K$ is a compact group, then the conditions (i)–(iii) in the above theorem are equivalent with:

- (iv) *The fixed point algebra A^{α} is type I (respectively liminal) ([13], Theorem 3.2).*

We will prove next an analogous result for compact quantum group actions. In Theorem 19 of [2] it is shown that the crossed product of a C^* -algebra by an ergodic action of a compact quantum group is a direct sum of full algebras of compact operators, hence a liminal C^* -algebra. Since, in the ergodic case, S_{π} are finite dimensional, the next result is an extension of Boca's result to the case of general compact quantum group actions.

For compact quantum groups we have the following result:

THEOREM 2.5. *Let $\mathcal{G} = (B, \Delta)$ be a compact quantum group and δ an action of \mathcal{G} on a C^* -algebra A . The following conditions are equivalent:*

- (i) $A \rtimes_{\delta} \mathcal{G}$ is type I (respectively liminal).
- (ii) All hereditary C^* -subalgebras $S_{\pi} \subset A \rtimes_{\delta} \mathcal{G}$, $\pi \in \widehat{\mathcal{G}}$, are type I (respectively liminal).
- (iii) All C^* -subalgebras $\mathcal{I}_{\pi} \subset S_{\pi}$, $\pi \in \widehat{\mathcal{G}}$, are type I (respectively liminal).
- (iv) All C^* -algebras $A \otimes B(H_{\pi})^{\delta_{\pi}}$, $\pi \in \widehat{\mathcal{G}}$ are type I (respectively liminal).

The result follows from Remark 1.4 and Lemma 2.3.

2.2. NUCLEAR CROSSED PRODUCTS. We start with the following lemma which is certainly known but we could not find a reference for it:

LEMMA 2.6. *A C^* -algebra C is nuclear if and only if for every state φ of C , $T_{\varphi}(C)''$ is an injective von Neumann algebra, where T_{φ} is the GNS representation of C associated with φ .*

Proof. If C is nuclear then C^{**} is an injective von Neumann algebra ([10], Theorem 6.4). Therefore, so is $T_\varphi(C)''$ which is isomorphic with an algebra of the form eC^{**} for a certain projection, $e \in (C^{**})'$.

Conversely, if $T_\varphi(C)''$ is injective for every state φ , let $\{\varphi_i\}$ be a maximal family of states for which the corresponding cyclic representations T_{φ_i} are disjoint. Then T_{φ_i} and $T = \bigoplus T_{\varphi_i}$ can be extended to normal representations \bar{T}_{φ_i} and \bar{T} of C^{**} with \bar{T} a normal isomorphism. Therefore, C^{**} is isomorphic with $\bigoplus T_{\varphi_i}(C)''$. Since all $T_{\varphi_i}(C)''$ are injective (by assumption), from Proposition 3.1 of [10] it follows that C^{**} is injective and thus C is nuclear. ■

Throughout the rest of this section all algebras, groups and quantum groups are assumed to be separable. The following lemma is the analog of Lemma 2.3 for the case of nuclear crossed products.

LEMMA 2.7. *Let C be a separable C^* -algebra and $\{q_\lambda\} \subset M(C)$ be a family of mutually orthogonal projections such that $\sum_\lambda q_\lambda = 1$ in C^{**} . The following statements are equivalent:*

- (i) C is nuclear.
- (ii) The hereditary C^* -subalgebras S_λ are nuclear for all λ .

Proof. Assume first that C is nuclear. Then, by Corollary 3.3 (4) of [4], every hereditary subalgebra of C is nuclear. Hence S_λ is nuclear for every λ .

Assume now that (ii) holds that is: all S_λ are nuclear C^* -algebras. We will show that for every cyclic representation T_φ of C , $T_\varphi(C)''$ is injective and the result will follow from the previous lemma. Let φ be a state of C . Then, by reduction theory, $\bar{T}_\varphi(C^{**}) = T_\varphi(C)''$ is the direct integral of factors $\bar{T}_\psi(C^{**}) = T_\psi(C)''$ where ψ are factor states of C , $\bar{T}_\varphi(C^{**}) = \int \bar{T}_\psi(C^{**}) d\mu(\psi)$ where μ is the central measure associated with the state φ and the integral is taken over the state space of C ([21], Theorem 3.5.2). Applying Proposition 6.5 of [5] it follows that $T_\varphi(C)''$ is injective if and only if almost all of the factors $T_\psi(C)''$ are injective. We have, therefore, reduced our problem to the following:

Assuming that all hereditary subalgebras S_λ are nuclear, show that for every cyclic factor representation T of C we have that $T(C)''$ is an injective von Neumann algebra.

Let T be a non degenerate cyclic factor representation of C . Since $\sum_\lambda q_\lambda = 1$ in C^{**} there is a λ such that the restriction $T|_{q_\lambda C} \neq 0$. Hence the restriction of T to the closed two sided ideal $J_\lambda = \overline{Cq_\lambda C}$ is non zero. Since T is a factor representation and J_λ is a two sided ideal it follows that $T(C)'' = T(J_\lambda)''$. We show next that under our assumptions $T(J_\lambda)''$ is injective and thus $T(C)''$ is injective. We noticed above that S_λ is strongly Morita equivalent with J_λ . Since C is separable, so are S_λ and J_λ . By Theorem 1.2 of [3] S_λ and J_λ are stably isomorphic. Since S_λ is nuclear it follows that J_λ is nuclear. By Lemma 2.6 we have that $T(J_\lambda)''$ is injective and the proof is complete. ■

From the proof of the previous lemma it follows:

COROLLARY 2.8. *A separable C^* -algebra C is nuclear if and only if for every factor state ψ of C , $T_\psi(C)''$ is an injective von Neumann algebra, where T_ψ is the GNS representation of C associated with ψ .*

We can now state our main results of this section.

THEOREM 2.9. *Let (A, G, α) be a C^* -dynamical system with G a locally compact group and let $K \subset G$ be a compact subgroup. Then the following conditions are equivalent:*

- (i) $A \rtimes_\alpha G$ is a nuclear C^* -algebra.
 - (ii) All hereditary C^* -subalgebras $S_\pi \subset A \rtimes_\alpha G$, $\pi \in \widehat{K}$ are nuclear.
 - (iii) All C^* -subalgebras of K -central elements, $\mathcal{I}_\pi \subset S_\pi$, $\pi \in \widehat{K}$ are nuclear.
- Furthermore, any of the previous three equivalent conditions implies:

- (iv) A is nuclear.

In addition, if G is amenable, i.e. if the group C^* -algebra $C^*(G)$ is nuclear the conditions (i)–(iv) are equivalent.

Proof. The equivalence of the conditions (i)–(iii) follows from Remarks 1.1 and 2.1 and Lemma 2.7. On the other hand, if the crossed product, $A \rtimes_\alpha G$ is nuclear, then, applying Theorem 4.6 of [19] it follows that $A \rtimes_\alpha G \rtimes_{\widehat{\alpha}} \widehat{G}$ is nuclear, where $\widehat{\alpha}$ is the dual coaction. Since by biduality ([18], Theorem 3.1) this latter crossed product is isomorphic with $A \otimes \mathcal{C}(\mathcal{H})$ where $\mathcal{C}(\mathcal{H})$ is the C^* -algebra of compact operators on a certain Hilbert space \mathcal{H} it follows that A is a nuclear C^* -algebra. Finally, if G is amenable and A is nuclear, then by Proposition 14 of [14] the crossed product $A \rtimes_\alpha G$ is nuclear and therefore in this case (iv) \Rightarrow (i). ■

In the proof of the implication (i) \Rightarrow (iv) of the above theorem we have used the fact that every locally compact group is co-amenable and Raeburn’s result. The next result is the analog of the previous one for the case of compact quantum groups. A compact quantum group $\mathcal{G} = (B, \Delta)$ is automatically amenable since \widehat{B} is a subalgebra of compact operators, but not co-amenable, in general, since B is not necessarily nuclear.

We will state next the corresponding result for compact quantum group actions.

THEOREM 2.10. *Let (A, \mathcal{G}, δ) be a quantum C^* -dynamical system, where $\mathcal{G} = (B, \Delta)$ is a compact quantum group. The following three conditions are equivalent:*

- (i) $A \rtimes_\alpha \mathcal{G}$ is nuclear.
- (ii) All hereditary C^* -subalgebras $S_\pi \subset A \rtimes_\alpha \mathcal{G}$ are nuclear.
- (iii) All C^* -algebras $(A \otimes B(H_\pi))^{\delta_\pi}$ are nuclear.

Furthermore, each of the above condition is implied by

- (iv) A is a nuclear C^* -algebra.

In addition, if the quantum group \mathcal{G} is co-amenable, i.e. if B is a nuclear C^* -algebra, then the conditions (i)–(iv) are equivalent with the following:

(v) A^δ is nuclear.

Proof. The equivalence of (i)–(iii) follows from Lemma 1.4 and Lemma 2.7. We now prove that (iv) implies (iii). Let $\pi \in \widehat{\mathcal{G}}$. If A is nuclear, then $A \otimes B(H_\pi)$ is a nuclear C^* -algebra. The projection of $A \otimes B(H_\pi)$ onto the fixed point algebra $(A \otimes B(H_\pi))^{\delta\pi}$ is obviously a completely positive map. Therefore, by Corollary 3.4. (4) of [4] it follows that $(A \otimes B(H_\pi))^{\delta\pi}$ is nuclear. Assume now that \mathcal{G} is co-amenable. Therefore, B is nuclear. Then, by applying Corollary 7 of [6] it follows that A^δ is nuclear if and only if A is nuclear and thus (v) \Leftrightarrow (iv). Since S_{π_0} is isomorphic with A^δ , we have that (iii) \Rightarrow (iv) and the proof is completed. ■

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