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SIMILARITY INVARIANTS AND COMPACT PERTURBATIONS OF HEREDITARILY INDECOMPOSABLE SPACE OPERATORS

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ABSTRACT. This paper gives a completely similarity classification of strongly irreducible operators on hereditarily indecomposable spaces by using K-theory. And we also prove that every operator with connected spectrum on complex separable hereditarily indecomposable spaces is a small compact perturbation of a strongly irreducible operator.

KEYWORDS: *Hereditarily indecomposable spaces, strongly irreducible operators, similar invariants, commutant algebras, K_0-groups.*

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INTRODUCTION

Let X be a complex Banach space and $B(X)$ be the Banach algebra of all bounded linear operators on X. When we consider the operator structure of Banach spaces, an important and fundamental problem is: how to determine the complete similarity invariants of operators. Two operators $A, B \in B(X)$ are said to be similar in $B(X)$ if there exists an invertible operator $S \in B(X)$ such that $SA = BS$. When X is finite dimensional, the Jordan Standard Theorem indicates that the eigenvalues and the generalized eigenspaces of an operator form a complete set of similarity invariants. In 1978, M.J. Cowen and R.G. Douglas pointed out that the problem has no general solution when X is a complex infinite dimensional Banach space (see [3]). Thus people can only restrict attention to special classes of operators.

In the paper, our main purpose is to characterize similarity invariants of strongly irreducible operators on hereditarily indecomposable space by using K-theory and Banach algebras. It is strongly inspired by the following two aspects: Firstly, G. Elliott, G. Gong and many other mathematicians gave successful classification of a large class of C^*-algebras by applying K-theory and tracial state space as an isomorphism invariant, which is called Elliott invariant (see [4], [5], [6], [7], [9]). Secondly, the hereditarily indecomposable space constructed by W. Gowers and B. Maurey provided very nice structure of operators, i.e., each operator T on hereditarily indecomposable space is of the form $T = \lambda + S$, where λ is a scalar operator, S is a strictly singular operator.

An infinite dimensional Banach space X is called a *hereditarily indecomposable* ($H.I.$) space if no infinite dimensional closed subspace of X is the complemented sum of two further infinite dimensional closed subspaces. The first known example of H.I. spaces was constructed by W.T. Gowers and B. Maurey [10]. Since then the class of Gowers–Maurey spaces has been extensively studied.

An operator $T \in B(X)$ is said to be strongly irreducible if there do not exist two non-trivial closed invariant subspaces M and N of T such that X is the complemented sum of M and N.

The concept of strongly irreducible operators was raised as an approximate replacement of Jordan blocks on infinite dimensional spaces. In the matrix theory of finite dimensional spaces, the Jordan Standard Theorem sufficiently reveals the internal structure of operators. Every operator on a finite dimensional space is similar to a unique Jordan standard form. A Jordan standard form is the direct sum of some fundamental elements — Jordan blocks. It is easy to prove that every strongly irreducible operator on finite dimensional spaces admits a Jordan block representation with respect to some basis. As far as we know, F. Gilfeather [8] and Z.J. Jiang gave the concept of strongly irreducible operators, respectively. Z.J. Jiang further pointed out that the strongly irreducible operators can be considered as the approximate replacement of Jordan blocks on infinite dimensional spaces and hoped that a theorem similar to the Jordan Standard Theorem can be set up with this replacement. The work of D.A. Herrero, S. Power and C.L. Jiang have answered a number of questions about operator structure of Hilbert spaces raised by D.A. Herrero and Z.J. Jiang (see [2], [11], [12], [13], [14], [15]).

The main result of this paper is that: let T_1 and T_2 be strongly irreducible operators on a H.I. space. Then T_1 and T_2 are similar if and only if the K_0-group of the commutant algebra of the direct sum of T_1 and T_2 is isomorphic to integer group $\mathbb{Z}$. This result will appear in Section 1 as Theorem 1.1.

In order to support our main result, we show that every operator on separable H.I. spaces with connected spectrum is strongly irreducible by a small compact perturbation in Section 2.

On a H.I. space, an operator with connected spectrum is just a scalar operator plus a quasi-nilpotent operator. In fact, we only need to show that each quasi-nilpotent operator T is strongly irreducible by a small compact perturbation. If the kernel of T is trivial, then T is naturally strongly irreducible. If the kernel of T is of finite dimension, we could easily perturb T to kill kernel of T while keeping the operator quasi-nilpotent. The key difficulty is the following case: T is quasi-nilpotent and the dimension of kernel of T and the co-dimension of the closure of the range of T are both infinite. The following example shows that such an operator actually exists.

EXAMPLE 0.1. Let X be a Banach space with a Schauder basis $\{e_i\}$ and let $\{f_i\} \subseteq X^*$ be the sequence of coefficient functionals associated to the basis $\{e_i\}$, where X^* is the dual space of X. Let

$$T = \sum_{i=1}^{\infty} a_i f_{2i} \otimes e_{2i+2},$$

where $a_i = 1/2^i \|f_{2i}\| \|e_{2i+2}\|$. It is easy to prove that $T \in B(X)$ is a quasi-nilpotent operator, the kernel of T is $\overline{\text{span}}\{e_{2i-1}\}$ and the range of T is contained in $\overline{\text{span}}\{e_{2i}\}$.

Generally, when a Banach space X is separable, there still exists an operator $T \in B(X)$ such that T is quasi-nilpotent and the dimension of kernel of T and the co-dimension of the closure of the range of T are both infinite. It can be referred to Lemma 2.5 in this paper.

The difficulty to resolve the problem in the above case is that not all closed subspaces are complemented in H.I. spaces. Thus the skills in Hilbert spaces can not be used in H.I. spaces. So we consider the concept of quasi-complementary subspaces and use a lot of lemmas to overcome the difficulty. Up to now, we have not found a much easier way to prove it.

Now we give some definitions and notations.

In the paper, $\mathbb{C}$ denotes the complex number field, $\mathbb{Z}$ denotes the group of integers, $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_+ = \{0, 1, 2, \ldots\}$. Throughout, we assume that all Banach spaces are complex. For an operator $T \in B(X)$ denote by $\ker T$ and $\text{ran} T$ its kernel $\ker T = \{x \in X : Tx = 0\}$ and range $\text{ran} T = \{Tx : x \in X\}$, respectively. We denote by $\overline{\text{ran} T}$ the norm-closure of $\text{ran} T$. Denote the spectrum of T, the point spectrum of T and the compressed spectrum of T by $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda$ is not invertible$\}$, $\sigma_{\text{p}}(T) = \{\lambda \in \mathbb{C} : \ker(T - \lambda) \neq \{0\}\}$ and $\sigma_{\gamma}(T) = \{\lambda \in \mathbb{C} : \overline{\text{ran}(T - \lambda)} \neq X\}$, respectively.

For an operator $T \in B(X)$ denote the commutant algebra of T by $\mathcal{A}'(T) = \{S \in B(X) : ST = TS\}$. It is obvious that $\mathcal{A}'(T)$ is a Banach algebra with unit I. Denote the Jacobson radical of $\mathcal{A}'(T)$ by $\text{rad}\mathcal{A}'(T) = \{S \in \mathcal{A}'(T) : \sigma_{\mathcal{A}'(T)}(SS') = \{0\}$ for every $S' \in \mathcal{A}'(T)\}$, where $\sigma_{\mathcal{A}'(T)}(S) = \{\lambda \in \mathbb{C} : S - \lambda$ is not invertible in $\mathcal{A}'(T)\}$ is the spectrum of S in $\mathcal{A}'(T)$. It is clear that $\text{rad}\mathcal{A}'(T)$ is a closed ideal of $\mathcal{A}'(T)$.

Let $T \in B(X)$. A closed subspace M of X is said to be an invariant subspace of T if $T(M) \subseteq M$. Denote by $T|_M$ the restriction of T to M. Denote by $\text{Red}(T)$ the set $\{(M, N) : X$ is a direct sum of two closed subspaces M and N, namely $X = M \oplus N$, and M, N are invariant subspaces of $T\}$. $(M, N) \in \text{Red}(T)$ is said to be non-trivial if $M \neq \{0\}$ and $N \neq \{0\}$.

REMARK 0.2. Recall the definition of strongly irreducible operators: $T \in B(X)$ is said to be strongly irreducible if there exists no non-trivial $(M, N) \in$

$\mathrm{Red}(T)$. Denote the set of strongly irreducible operators on X by $(SI)(X)$ which is abbreviated to (SI). We can obtain the following:

(i) $T \in (SI)$ if and only if $\mathcal{A}'(T)$ has no non-trivial idempotent, namely if $P \in \mathcal{A}'(T)$ with $P^2 = P$, then $P = 0$ or $P = I$.

(ii) If $T \in (SI)$, then $\lambda T + \mu \in (SI)$ for every $\lambda, \mu \in \mathbb{C}, \lambda \neq 0$.

Let X be an infinite dimensional Banach space and let $T \in B(X)$. T is said to be strictly singular if there exists no infinite dimensional closed subspace M of X such that the restriction of T to M is an isomorphism.

REMARK 0.3. Suppose that T is a strictly singular operator.

(i) $\sigma(T) = \{0\} \cup \sigma_{\mathrm{p}}(T)$.

(ii) $\sigma(T)$ is an at most countable set with 0 as the only possible point of accumulation.

(iii) For every $\lambda \neq 0$, $T - \lambda$ is a Fredholm operator with index 0, namely $\dim(\ker(T-\lambda)) = \dim(X/\mathrm{ran}(T-\lambda)) < \infty$.

In the paper, the sequences $\{x_i\}_{i=1}^{\infty} \subseteq X$ and $\{f_i\}_{i=1}^{\infty} \subseteq X^*$ are abbreviated to $\{x_i\}$ and $\{f_i\}$, respectively. We denote by $\mathrm{span}\{x_i\}$ and $\mathrm{span}\{f_i\}$ the linear span of $\{x_i\}$ and $\{f_i\}$, respectively. The norm-closure of $\mathrm{span}\{x_i\}$ and the w*-closure of $\mathrm{span}\{f_i\}$ are denoted by $\overline{\mathrm{span}\{x_i\}}$ and $\overline{\mathrm{span}\{f_i\}}^{\mathrm{w}^*}$, respectively.

Let X_1 and X_2 be Banach spaces. Two operators $A_1 \in B(X_1)$ and $A_2 \in B(X_2)$ are said to be similar if there exists an invertible operator T from X_1 onto X_2 such that $T^{-1}A_2T = A_1$, denoted by $A_1 \sim A_2$. Let $A_1 \oplus A_2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ denote the direct sum of A_1 and A_2 on $X_1 \oplus X_2$. In the case where $A_1 = A_2$, we write $A_1^{(2)}$ instead of $A_1 \oplus A_1$.

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be Banach algebras and let G_1 and G_2 be groups or semigroups. We write $\mathcal{A}_1 \approx \mathcal{A}_2$ if $\mathcal{A}_1$ and $\mathcal{A}_2$ are algebraic isomorphic and $G_1 \approx G_2$ if G_1 and G_2 are group isomorphic.

1. SIMILARITY INVARIANTS

In this section we use K-theory to study the similarity invariants of operators on H.I. spaces. We show that two (SI) operators T_1 and T_2 on a H.I. space are similar if and only if the K_0-group of the commutant algebra of the direct sum $T_1 \oplus T_2$ is isomorphic to the integer group $\mathbb{Z}$.

Now we give the definition of the K_0-group of a unital Banach algebra.

Let $\mathcal{A}$ be a unital Banach algebra and let $n \in \mathbb{N}$. We denote by $M_n(\mathcal{A})$ the set of $(n \times n)$-matrices over $\mathcal{A}$. The set of idempotents in $\mathcal{A}$ is denoted by $P_1(\mathcal{A})$, namely $P_1(\mathcal{A}) = \{P \in \mathcal{A} : P^2 = P\}$. Accordingly the set of idempotents in $M_n(\mathcal{A})$ is denoted by $P_n(\mathcal{A})$, namely $P_n(\mathcal{A}) = P_1(M_n(\mathcal{A}))$. Two idempotents

$P, Q \in P_n(\mathcal{A})$ are said to be similar in $P_n(\mathcal{A})$ if there exists an invertible element $U \in M_n(\mathcal{A})$ such that $U^{-1}PU = Q$, denoted by $P \sim Q$.

Define $P_\infty(\mathcal{A}) = \bigcup_{n\in\mathbb{N}} P_n(\mathcal{A})$. We say that $P \in P_n(\mathcal{A})$ and $Q \in P_m(\mathcal{A})$ are equivalent, written $P \sim_{\mathrm{e}} Q$, if there exists $k \geqslant \max\{m, n\}$ such that $(P \oplus 0^{(k-n)}) \sim (Q \oplus 0^{(k-m)})$ in $P_k(\mathcal{A})$.

Clearly, $\sim_{\mathrm{e}}$ is an equivalence relation on $P_\infty(\mathcal{A})$. Therefore we may form the quotient $\bigvee(\mathcal{A}) = P_\infty(\mathcal{A})/\sim_{\mathrm{e}}$. Let $[P]$ denote the equivalence class of $P \in P_\infty(\mathcal{A})$ in $\bigvee(\mathcal{A})$. One easily checks that the operation

$$\begin{array}{rcl}\bigvee(\mathcal{A}) \times \bigvee(\mathcal{A}) & \longrightarrow & \bigvee(\mathcal{A}), \\ ([P],[Q]) & \longmapsto & [P \oplus Q].\end{array}$$

is well-defined and turns $\bigvee(\mathcal{A})$ into a commutative semigroup.

We define $K_0(\mathcal{A})$ to be the Grothendieck group of $\bigvee(\mathcal{A})$ ([1]).

The following is the main theorem in this section.

THEOREM 1.1. *Let X be a* H.I. *space, let $A_1, A_2 \in (SI)(X)$ and let*

$$A = A_1 \oplus A_2.$$

Then $A_1 \sim A_2$ if and only if $K_0(\mathcal{A}'(A)) \approx \mathbb{Z}$.

In order to prove the theorem, we need the following lemmas.

LEMMA 1.2. *Let X_1 and X_2 be Banach spaces. Let $A_i \in B(X_i)$ such that*

$$\mathcal{A}'(A_i)/\mathrm{rad}\mathcal{A}'(A_i) \approx \mathbb{C}$$

for $i = 1, 2$ and A_1 is not similar to A_2. If there exist $T \in B(X_1, X_2)$ and $S \in B(X_2, X_1)$ such that $A_2T = TA_1$ and $A_1S = SA_2$, then $ST \in \mathrm{rad}\mathcal{A}'(A_1)$ and $TS \in \mathrm{rad}\mathcal{A}'(A_2)$.

Proof. Since $A_2T = TA_1$ and $A_1S = SA_2$, then $A_1ST = SA_2T = STA_1$. Thus $ST \in \mathcal{A}'(A_1)$. In the same way, $TS \in \mathcal{A}'(A_2)$.

Suppose that $ST \notin \mathrm{rad}\mathcal{A}'(A_1)$ and $TS \notin \mathrm{rad}\mathcal{A}'(A_2)$. Since

$$\mathcal{A}'(A_i)/\mathrm{rad}\mathcal{A}'(A_i) \approx \mathbb{C}$$

for $i = 1, 2$, then $ST = \lambda_1 + R_1$ and $TS = \lambda_2 + R_2$, where $0 \neq \lambda_1, \lambda_2 \in \mathbb{C}$, $R_1 \in \mathrm{rad}\mathcal{A}'(A_1)$, $R_2 \in \mathrm{rad}\mathcal{A}'(A_2)$. By the definition of Jacobson radical, $\sigma_{\mathcal{A}'(A_1)}(R_1) = \sigma_{\mathcal{A}'(A_2)}(R_2) = \{0\}$. Then $\sigma(R_1) = \sigma(R_2) = \{0\}$, so $\sigma(ST) = \{\lambda_1\}$ and $\sigma(TS) = \{\lambda_2\}$. Since $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$, then ST and TS are both invertible. So S and T are both invertible and thus $A_1 \sim A_2$, which is a contradiction.

From the above, $ST \in \mathrm{rad}\mathcal{A}'(A_1)$ or $TS \in \mathrm{rad}\mathcal{A}'(A_2)$. Without loss of generality, we may assume that $ST \in \mathrm{rad}\mathcal{A}'(A_1)$. Then $\sigma(ST) = \sigma_{\mathcal{A}'(A_1)}(ST) = \{0\}$, so $\sigma(TS) = \{0\}$ by $\sigma(ST)\backslash\{0\} = \sigma(TS)\backslash\{0\}$. Since $\mathcal{A}'(A_2)/\mathrm{rad}\mathcal{A}'(A_2) \approx \mathbb{C}$, then $TS = \lambda + R$, where $\lambda \in \mathbb{C}$, $R \in \mathrm{rad}\mathcal{A}'(A_2)$. Since $\sigma(R) = \{0\}$, thus $\lambda = 0$, so $TS = R \in \mathrm{rad}\mathcal{A}'(A_2)$. ∎

LEMMA 1.3. *Let X be a Banach space and let $A \in B(X)$. Then*

$$\mathcal{A}'(A^{(2)})/\mathrm{rad}\mathcal{A}'(A^{(2)}) \approx M_2(\mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A)).$$

Proof. It is obvious that

$$\mathcal{A}'(A^{(2)}) = \left\{ \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} : T_{ij} \in \mathcal{A}'(A), i,j = 1,2 \right\}.$$

Let

$$J = \left\{ \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} : T_{ij} \in \mathrm{rad}\mathcal{A}'(A), i,j = 1,2 \right\}.$$

Claim 1. $\mathrm{rad}\mathcal{A}'(A^{(2)}) \subseteq J$.

For every $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in \mathrm{rad}\mathcal{A}'(A^{(2)})$, to show $T \in J$, we only need to show $T_{ij} \in \mathrm{rad}\mathcal{A}'(A)$ for $i,j = 1,2$.

Since $\mathrm{rad}\mathcal{A}'(A^{(2)})$ is an ideal of $\mathcal{A}'(A^{(2)})$ and $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{A}'(A^{(2)})$, we have

$$\begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{rad}\mathcal{A}'(A^{(2)}).$$

Therefore $T_{11} \in \mathrm{rad}\mathcal{A}'(A)$. In the similar way, we can conclude that $T_{22}, T_{12}, T_{21} \in \mathrm{rad}\mathcal{A}'(A)$.

Claim 2. $J \subseteq \mathrm{rad}\mathcal{A}'(A^{(2)})$.

Let $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in J$, then $T_{ij} \in \mathrm{rad}\mathcal{A}'(A)$ for $i,j = 1,2$.

For every $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \in \mathcal{A}'(A^{(2)})$, then $S_{11} \in \mathcal{A}'(A)$. Since $\mathrm{rad}\mathcal{A}'(A)$ is an ideal of $\mathcal{A}'(A)$ and $T_{11} \in \mathrm{rad}\mathcal{A}'(A)$, then $T_{11}S_{11} \in \mathrm{rad}\mathcal{A}'(A)$. So $\sigma(T_{11}S_{11}) = \{0\}$. Therefore

$$\sigma\left(\begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \right) = \sigma\left(\begin{pmatrix} T_{11}S_{11} & T_{11}S_{12} \\ 0 & 0 \end{pmatrix} \right) = \{0\},$$

namely $\begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{rad}\mathcal{A}'(A^{(2)})$. Similarly, we obtain

$$\begin{pmatrix} 0 & T_{12} \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ T_{21} & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & T_{22} \end{pmatrix} \in \mathrm{rad}\mathcal{A}'(A^{(2)}).$$

Since $\mathrm{rad}\mathcal{A}'(A^{(2)})$ is a linear space, then $T \in \mathrm{rad}\mathcal{A}'(A^{(2)})$.

By Claim 1 and Claim 2, we conclude that $\mathrm{rad}\mathcal{A}'(A^{(2)}) = J$.

For every $T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \in \mathcal{A}'(A^{(2)})$, then $T_{ij} \in \mathcal{A}'(A)$. Denote by $\overline{\overline{T}}$ the equivalence class of T in $\mathcal{A}'(A^{(2)})/\mathrm{rad}\mathcal{A}'(A^{(2)})$ and denote by $\overline{T}_{ij}$ the equivalence class of T_{ij} in $\mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A)$. Define a map

$$f: \quad \mathcal{A}'(A^{(2)})/\mathrm{rad}\mathcal{A}'(A^{(2)}) \longrightarrow M_2(\mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A)),$$
$$\overline{\overline{T}} \longmapsto \begin{pmatrix} \overline{T}_{11} & \overline{T}_{12} \\ \overline{T}_{21} & \overline{T}_{22} \end{pmatrix}.$$

It is clear that f is well-defined and f is an algebraic isomorphism from $\mathcal{A}'(A^{(2)})$ $/\mathrm{rad}\mathcal{A}'(A^{(2)})$ onto $M_2(\mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A))$. So

$$\mathcal{A}'(A^{(2)})/\mathrm{rad}\mathcal{A}'(A^{(2)}) \approx M_2(\mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A)). \quad \blacksquare$$

LEMMA 1.4. *Let X_1 and X_2 be Banach spaces. Let $A_i \in B(X_i)$ such that*

$$\mathcal{A}'(A_i)/\mathrm{rad}\mathcal{A}'(A_i) \approx \mathbb{C}$$

for $i = 1, 2$ and A_1 is not similar to A_2. Put $A = A_1 \oplus A_2 \oplus A_2$. Then

$$\mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A) \approx \mathbb{C} \oplus M_2(\mathbb{C}).$$

Proof. It is obvious that

$$\mathcal{A}'(A) = \left\{ \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} : \begin{array}{l} T_{11} \in \mathcal{A}'(A_1) \\ T_{ij} \in \mathcal{A}'(A_2), i,j = 2,3 \\ T_{1i}A_2 = A_1T_{1i}, T_{i1}A_1 = A_2T_{i1}, i = 2,3 \end{array} \right\}.$$

Let

$$J = \left\{ \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} : \begin{array}{l} T_{11} \in \mathrm{rad}\mathcal{A}'(A_1) \\ T_{ij} \in \mathrm{rad}\mathcal{A}'(A_2), i,j = 2,3 \\ T_{1i}A_2 = A_1T_{1i}, T_{i1}A_1 = A_2T_{i1}, i = 2,3 \end{array} \right\}.$$

Claim 1. J is an ideal of $\mathcal{A}'(A)$.

It is easy to prove that J is a linear subspace of $\mathcal{A}'(A)$. Let

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \in J \quad \text{and} \quad S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} \in \mathcal{A}'(A).$$

We prove $TS \in J$ in the following. $ST \in J$ can be obtained in the same way. Since

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix}$$
$$= \begin{pmatrix} T_{11}S_{11} + T_{12}S_{21} + T_{13}S_{31} & T_{11}S_{12} + T_{12}S_{22} + T_{13}S_{32} & T_{11}S_{13} + T_{12}S_{23} + T_{13}S_{33} \\ T_{21}S_{11} + T_{22}S_{21} + T_{23}S_{31} & T_{21}S_{12} + T_{22}S_{22} + T_{23}S_{32} & T_{21}S_{13} + T_{22}S_{23} + T_{23}S_{33} \\ T_{31}S_{11} + T_{32}S_{21} + T_{33}S_{31} & T_{31}S_{12} + T_{32}S_{22} + T_{33}S_{32} & T_{31}S_{13} + T_{32}S_{23} + T_{33}S_{33} \end{pmatrix},$$

$$\begin{aligned}(T_{11}S_{1i} + T_{12}S_{2i} + T_{13}S_{3i})A_2 &= T_{11}A_1S_{1i} + T_{12}A_2S_{2i} + T_{13}A_2S_{3i} \\ &= A_1(T_{11}S_{1i} + T_{12}S_{2i} + T_{13}S_{3i})\end{aligned}$$

and

$$\begin{aligned}(T_{i1}S_{11} + T_{i2}S_{21} + T_{i3}S_{31})A_1 &= T_{i1}A_1S_{11} + T_{i2}A_2S_{21} + T_{i3}A_2S_{31} \\ &= A_2(T_{i1}S_{11} + T_{i2}S_{21} + T_{i3}S_{3i})\end{aligned}$$

for $i = 2, 3$, it suffices to prove

$$T_{11}S_{11} + T_{12}S_{21} + T_{13}S_{31} \in \mathrm{rad}\mathcal{A}'(A_1)$$

and

$$T_{i1}S_{1j} + T_{i2}S_{2j} + T_{i3}S_{3j} \in \mathrm{rad}\mathcal{A}'(A_2), \quad i, j = 2, 3.$$

Since $\mathrm{rad}\mathcal{A}'(A_1)$ is an ideal of $\mathcal{A}'(A_1)$, $T_{11} \in \mathrm{rad}\mathcal{A}'(A_1)$ and $S_{11} \in \mathcal{A}'(A_1)$, we have $T_{11}S_{11} \in \mathrm{rad}\mathcal{A}'(A_1)$. Similarly, since $\mathrm{rad}\mathcal{A}'(A_2)$ is an ideal of $\mathcal{A}'(A_2)$, $T_{ik} \in \mathrm{rad}\mathcal{A}'(A_2)$ and $S_{kj} \in \mathcal{A}'(A_2)$, we have $T_{ik}S_{kj} \in \mathrm{rad}\mathcal{A}'(A_2), i, k, j = 2, 3$.

Since $T_{1i}A_2 = A_1T_{1i}$ and $S_{i1}A_1 = A_2S_{i1}$, then $T_{1i}S_{i1} \in \mathrm{rad}\mathcal{A}'(A_1)$ for $i = 2, 3$ by Lemma 1.2. In the same way, we can conclude that $T_{i1}S_{1j} \in \mathrm{rad}\mathcal{A}'(A_2)$ for $i, j = 2, 3$ from $T_{i1}A_1 = A_2T_{i1}$ and $S_{1j}A_2 = A_1S_{1j}$.

From the above, since $\mathrm{rad}\mathcal{A}'(A_1)$ and $\mathrm{rad}\mathcal{A}'(A_2)$ are linear spaces, then $T_{11}S_{11} + T_{12}S_{21} + T_{13}S_{31} \in \mathrm{rad}\mathcal{A}'(A_1)$ and $T_{i1}S_{1j} + T_{i2}S_{2j} + T_{i3}S_{3j} \in \mathrm{rad}\mathcal{A}'(A_2)$, $i, j = 2, 3$.

Claim 2. $\mathrm{rad}\mathcal{A}'(A) \subseteq J$.

For every $T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \in \mathrm{rad}\mathcal{A}'(A)$, similarly to the proof of Claim 1 in Lemma 1.3, we can conclude that

$$T_{11} \in \mathrm{rad}\mathcal{A}'(A_1) \quad \text{and} \quad T_{ij} \in \mathrm{rad}\mathcal{A}'(A_2)$$

for $i, j = 2, 3$. So $T \in J$.

Claim 3. $J \subseteq \mathrm{rad}\mathcal{A}'(A)$.

Since J is an ideal of $\mathcal{A}'(A)$, it suffices to prove $\sigma(T) = \{0\}$ for every $T \in J$.

Let $T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \in J$. For every $\lambda \neq 0$, since $T_{11} \in \mathrm{rad}\mathcal{A}'(A_1)$, then $\sigma(T_{11}) = \{0\}$. So $T_{11} - \lambda$ is invertible and

$$(T_{11} - \lambda)^{-1}A_1 = A_1(T_{11} - \lambda)^{-1}.$$

Let $i, j = 2, 3$. Since $T_{1i}A_2 = A_1T_{1i}$, we conclude that

$$(T_{11} - \lambda)^{-1}T_{1i}A_2 = (T_{11} - \lambda)^{-1}A_1T_{1i} = A_1(T_{11} - \lambda)^{-1}T_{1i}.$$

Notice that $T_{j1}A_1 = A_2T_{j1}$, by Lemma 1.2,

$$T_{j1}(T_{11} - \lambda)^{-1}T_{1i} \in \mathrm{rad}\mathcal{A}'(A_2).$$

Since $T_{ji} \in \mathrm{rad}\mathcal{A}'(A_2)$, then

$$T_{ji} - T_{j1}(T_{11} - \lambda)^{-1}T_{1i} \in \mathrm{rad}\mathcal{A}'(A_2).$$

By Claim 2 in Lemma 1.3,

$$\begin{pmatrix} T_{22} - T_{21}(T_{11}-\lambda)^{-1}T_{12} & T_{23} - T_{21}(T_{11}-\lambda)^{-1}T_{13} \\ T_{32} - T_{31}(T_{11}-\lambda)^{-1}T_{12} & T_{33} - T_{31}(T_{11}-\lambda)^{-1}T_{13} \end{pmatrix} \in \mathrm{rad}\mathcal{A}'(A_2^{(2)}).$$

Hence

$$\begin{pmatrix} T_{22} - T_{21}(T_{11}-\lambda)^{-1}T_{12} & T_{23} - T_{21}(T_{11}-\lambda)^{-1}T_{13} \\ T_{32} - T_{31}(T_{11}-\lambda)^{-1}T_{12} & T_{33} - T_{31}(T_{11}-\lambda)^{-1}T_{13} \end{pmatrix} - \lambda$$
$$= \begin{pmatrix} (T_{22}-\lambda) - T_{21}(T_{11}-\lambda)^{-1}T_{12} & T_{23} - T_{21}(T_{11}-\lambda)^{-1}T_{13} \\ T_{32} - T_{31}(T_{11}-\lambda)^{-1}T_{12} & (T_{33}-\lambda) - T_{31}(T_{11}-\lambda)^{-1}T_{13} \end{pmatrix}$$

is invertible.

Now notice that

$$\begin{pmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ 0 & (T_{22}-\lambda) - T_{21}(T_{11}-\lambda)^{-1}T_{12} & T_{23} - T_{21}(T_{11}-\lambda)^{-1}T_{13} \\ 0 & T_{32} - T_{31}(T_{11}-\lambda)^{-1}T_{12} & (T_{33}-\lambda) - T_{31}(T_{11}-\lambda)^{-1}T_{13} \end{pmatrix}$$

and

$$\begin{pmatrix} I & 0 & 0 \\ -T_{21}(T_{11}-\lambda)^{-1} & I & 0 \\ -T_{31}(T_{11}-\lambda)^{-1} & 0 & I \end{pmatrix}$$

are both invertible and

$$\begin{pmatrix} I & 0 & 0 \\ -T_{21}(T_{11}-\lambda)^{-1} & I & 0 \\ -T_{31}(T_{11}-\lambda)^{-1} & 0 & I \end{pmatrix}\begin{pmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{pmatrix}$$
$$= \begin{pmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ 0 & (T_{22}-\lambda) - T_{21}(T_{11}-\lambda)^{-1}T_{12} & T_{23} - T_{21}(T_{11}-\lambda)^{-1}T_{13} \\ 0 & T_{32} - T_{31}(T_{11}-\lambda)^{-1}T_{12} & (T_{33}-\lambda) - T_{31}(T_{11}-\lambda)^{-1}T_{13} \end{pmatrix},$$

then

$$T - \lambda = \begin{pmatrix} T_{11}-\lambda & T_{12} & T_{13} \\ T_{21} & T_{22}-\lambda & T_{23} \\ T_{31} & T_{32} & T_{33}-\lambda \end{pmatrix}$$

is invertible and thus $\sigma(T) = \{0\}$.

By Claim 2 and Claim 3, we conclude that $\mathrm{rad}\mathcal{A}'(A) = J$.

For every $T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} \in \mathcal{A}'(A)$, where $T_{11} \in \mathcal{A}'(A_1)$ and $T_{ij} \in \mathcal{A}'(A_2)$ for $i,j = 2,3$, since $\mathcal{A}'(A_k)/\mathrm{rad}\mathcal{A}'(A_k) \approx \mathbb{C}$ for $k = 1,2$, then $T_{11} = \lambda_{11} + R_{11}$ with $\lambda_{11} \in \mathbb{C}$, $R_{11} \in \mathrm{rad}\mathcal{A}'(A_1)$ and $T_{ij} = \lambda_{ij} + R_{ij}$ with $\lambda_{ij} \in \mathbb{C}$, $R_{ij} \in \mathrm{rad}\mathcal{A}'(A_2)$ for $i,j = 2,3$. So

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix} = \begin{pmatrix} \lambda_{11} & 0 & 0 \\ 0 & \lambda_{22} & \lambda_{23} \\ 0 & \lambda_{32} & \lambda_{33} \end{pmatrix} + \begin{pmatrix} R_{11} & T_{12} & T_{13} \\ T_{21} & R_{22} & R_{23} \\ T_{31} & R_{32} & R_{33} \end{pmatrix},$$

where $\begin{pmatrix} R_{11} & T_{12} & T_{13} \\ T_{21} & R_{22} & R_{23} \\ T_{31} & R_{32} & R_{33} \end{pmatrix} \in J = \mathrm{rad}\mathcal{A}'(A)$. Define a map

$$\begin{array}{rccc} f: & \mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A) & \longrightarrow & \mathbb{C} \oplus M_2(\mathbb{C}), \\ & \overline{T} & \longmapsto & \begin{pmatrix} \lambda_{11} & 0 & 0 \\ 0 & \lambda_{22} & \lambda_{23} \\ 0 & \lambda_{32} & \lambda_{33} \end{pmatrix}, \end{array}$$

where $\overline{T}$ denotes the equivalence class of T in $\mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A)$. It is clear that f is well-defined and f is an algebraic isomorphism from $\mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A)$ onto $\mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A)$. So

$$\mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A) \approx \mathbb{C} \oplus M_2(\mathbb{C}). \quad \blacksquare$$

Similarly to the proof of Lemma 1.4, we can obtain the following

LEMMA 1.5. *Let $\{X_i : 1 \leqslant i \leqslant k\}$ be Banach spaces. Let $A_i \in B(X_i)$ such that $\mathcal{A}'(A_i)/\mathrm{rad}\mathcal{A}'(A_i) \approx \mathbb{C}$ for $i = 1, 2, \ldots, k$ and A_i is not similar to A_j for $i \neq j$. Set $A = \sum\limits_{i=1}^{k} \oplus A_i^{(n_i)}$ on $\sum\limits_{i=1}^{k} \oplus X_i^{(n_i)}$. Then*

$$\mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A) \approx \sum_{i=1}^{k} \bigoplus M_{n_i}(\mathbb{C}).$$

Proof. In fact, we can prove the following results:

$$\mathcal{A}'(A) = \{(T_{ij})_{i,j=1}^{k} : T_{ii} \in \mathcal{A}'(A_i^{(n_i)}), T_{ij}A_j^{(n_j)} = A_i^{(n_i)}T_{ij}, i, j = 1, 2, \ldots, k\}$$

and

$$\mathrm{rad}\mathcal{A}'(A) = \{(T_{ij})_{i,j=1}^{k} : T_{ii} \in \mathrm{rad}\mathcal{A}'(A_i^{(n_i)}), T_{ij}A_j^{(n_j)} = A_i^{(n_i)}T_{ij}, i, j = 1, 2, \ldots, k\},$$

where

$$\mathcal{A}'(A_i^{(n_i)}) = \{(t_{ml})_{m,l=1}^{n_i} : t_{ml} \in \mathcal{A}'(A_i), m, l = 1, 2, \ldots, n_i\}$$

and

$$\mathrm{rad}\mathcal{A}'(A_i^{(n_i)}) = \{(t_{ml})_{m,l=1}^{n_i} : t_{ml} \in \mathrm{rad}\mathcal{A}'(A_i), m, l = 1, 2, \ldots, n_i\}. \quad \blacksquare$$

LEMMA 1.6. *Let $\{X_i : 1 \leqslant i \leqslant k\}$ be Banach spaces. Let $A_i \in B(X_i)$ such that $\mathcal{A}'(A_i)/\mathrm{rad}\mathcal{A}'(A_i) \approx \mathbb{C}$ for $i = 1, 2, \ldots, k$ and A_i is not similar to A_j for $i \neq j$. Set $A = \sum\limits_{i=1}^{k} \oplus A_i^{(n_i)}$ on $\sum\limits_{i=1}^{k} \oplus X_i^{(n_i)}$. Then*

$$\bigvee(\mathcal{A}'(A)) \approx \mathbb{N}_+^{(k)}, \quad K_0(\mathcal{A}'(A)) \approx \mathbb{Z}^{(k)}.$$

Proof. From the proof of Lemma 1.3, we can obtain that

$$\mathcal{A}'(A^{(n)})/\mathrm{rad}\mathcal{A}'(A^{(n)}) \approx M_n(\mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A))$$

for every n.

Define a map $g : \bigvee(\mathcal{A}'(A)) \longrightarrow \bigvee(\mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A))$ by

$$g([P]) = [\overline{P}],$$

where $P \in M_n(\mathcal{A}'(A)) = \mathcal{A}'(A^{(n)})$ is an idempotent and

$$\overline{P} \in \mathcal{A}'(A^{(n)})/\mathrm{rad}\mathcal{A}'(A^{(n)}) \approx M_n(\mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A))$$

denotes the equivalence class of P in $\mathcal{A}'(A^{(n)})/\mathrm{rad}\mathcal{A}'(A^{(n)})$.

In the following, we show that g is well-defined and it is an isomorphism from $\bigvee(\mathcal{A}'(A))$ onto $\bigvee(\mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A))$.

If $[P] = [Q]$, where $P \in \mathcal{A}'(A^{(n)})$ and $Q \in \mathcal{A}'(A^{(m)})$ are idempotents, then there exist $k \geqslant \max\{m, n\}$ and an invertible element $U \in \mathcal{A}'(A^{(k)})$ such that

$$U(P \oplus 0^{(k-n)})U^{-1} = Q \oplus 0^{(k-m)}.$$

So $\overline{U}$ is invertible and

$$\overline{U} \cdot \overline{P \oplus 0^{(k-n)}} \cdot \overline{U}^{-1} = \overline{U(P \oplus 0^{(k-n)})U^{-1}} = \overline{Q \oplus 0^{(k-m)}}.$$

Thus $[\overline{P}] = [\overline{Q}]$, namely $g([P]) = g([Q])$. Hence g is well-defined.

Now we show that g is injective. If $P \in \mathcal{A}'(A^{(n)})$ and $Q \in \mathcal{A}'(A^{(m)})$ are idempotents such that

$$g([P]) = [\overline{P}] = [\overline{Q}] = g([Q]),$$

then there exist $k \geqslant \max\{m, n\}$ and an invertible element

$$\overline{U} \in \mathcal{A}'(A^{(k)})/\mathrm{rad}\mathcal{A}'(A^{(k)})$$

with $U \in \mathcal{A}'(A^{(k)})$ such that

$$\overline{U} \cdot \overline{P \oplus 0^{(k-n)}} \cdot \overline{U}^{-1} = \overline{Q \oplus 0^{(k-m)}}.$$

Since $\overline{U}$ is invertible, there exists $\overline{S} \in \mathcal{A}'(A^{(k)})/\mathrm{rad}\mathcal{A}'(A^{(k)})$ with $S \in \mathcal{A}'(A^{(k)})$ such that

$$\overline{US} = \overline{U} \cdot \overline{S} = \overline{I} = \overline{S} \cdot \overline{U} = \overline{SU}.$$

Then $US = I - R_1$ and $SU = I - R_2$ for some $R_1, R_2 \in \mathrm{rad}\mathcal{A}'(A^{(k)})$. Since $\sigma(R_1) = \sigma(R_2) = \{0\}$, then SU and US are invertible. Therefore U is invertible and thus

$$\overline{U(P \oplus 0^{(k-n)})U^{-1}} = \overline{U} \cdot \overline{P \oplus 0^{(k-n)}} \cdot \overline{U}^{-1} = \overline{Q \oplus 0^{(k-m)}}.$$

So

$$U(P \oplus 0^{(k-n)})U^{-1} = (Q \oplus 0^{(k-m)}) + R$$

for some $R \in \mathrm{rad}\mathcal{A}'(A^{(k)})$. Let $W_1 = 2(Q \oplus 0^{(k-m)}) - I$. Since $\sigma(Q \oplus 0^{(k-m)}) \subseteq \{0, 1\}$, then W_1 is invertible. From the fact that $R \in \mathrm{rad}\mathcal{A}'(A^{(k)})$ and $W_1^{-1} \in \mathcal{A}'(A^{(k)})$, then $RW_1^{-1} \in \mathrm{rad}\mathcal{A}'(A^{(k)})$, so $I + RW_1^{-1}$ is also invertible. Let

$$W = 2(Q \oplus 0^{(k-m)}) - I + R = W_1 + R = (I + RW_1^{-1})W_1.$$

Then W is invertible. Since $P \oplus 0^{(k-n)}$ is an idempotent, then $U(P \oplus 0^{(k-n)})U^{-1}$ is an idempotent, namely $(Q \oplus 0^{(k-m)}) + R$ is an idempotent, so

$$(Q \oplus 0^{(k-m)})^2 + (Q \oplus 0^{(k-m)})R + R(Q \oplus 0^{(k-m)}) + R^2 = (Q \oplus 0^{(k-m)}) + R.$$

Since $Q \oplus 0^{(k-m)}$ is an idempotent, then

$$(Q \oplus 0^{(k-m)})R + R(Q \oplus 0^{(k-m)}) + R^2 = R.$$

So

$$\begin{aligned} W((Q\oplus 0^{(k-m)})+R) &= (Q\oplus 0^{(k-m)})+R(Q\oplus 0^{(k-m)})+2(Q\oplus 0^{(k-m)})R-R+R^2 \\ &= (Q \oplus 0^{(k-m)}) + (Q \oplus 0^{(k-m)})R = (Q \oplus 0^{(k-m)})W. \end{aligned}$$

Thus

$$U(P \oplus 0^{(k-n)})U^{-1} = (Q \oplus 0^{(k-m)}) + R = W^{-1}(Q \oplus 0^{(k-m)})W.$$

Therefore $P \sim_{\mathrm{e}} Q$, namely $[P] = [Q]$. Hence g is injective.

Next we show that g is surjective. For every

$$[\overline{P}] \in \bigvee(\mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A))$$

with $\overline{P} \in \mathcal{A}'(A^{(n)})/\mathrm{rad}\mathcal{A}'(A^{(n)})$, $P \in \mathcal{A}'(A^{(n)})$ and $\overline{P}^2 = \overline{P}$, we have $P^2 - P = R_0$ for some $R_0 \in \mathrm{rad}\mathcal{A}'(A^{(n)})$. Since $A^{(n)} = \sum_{i=1}^{k} \oplus A_i^{(nn_i)}$, by Lemma 1.5, P is of the form $P = B + R$, where $R \in \mathrm{rad}\mathcal{A}'(A^{(n)})$ and $B \in \mathcal{A}'(A^{(n)})$ is a block-diagonal $(\sum_{i=1}^{k} nn_i \times \sum_{i=1}^{k} nn_i)$-matrix over $\mathbb{C}$, namely $B \in \sum_{i=1}^{k} \oplus M_{nn_i}(\mathbb{C})$. Then $\overline{B} = \overline{P}$ and

$$R_0 = P^2 - P = (B+R)^2 - (B+R) = B^2 - B + (BR + RB + R^2 - R).$$

Since $\mathrm{rad}\mathcal{A}'(A^{(n)})$ is an ideal of $\mathcal{A}'(A^{(n)})$, $B \in \mathcal{A}'(A^{(n)})$ and $R, R_0 \in \mathrm{rad}\mathcal{A}'(A^{(n)})$, we can conclude that $B^2 - B \in \mathrm{rad}\mathcal{A}'(A^{(n)})$. By the fact that $B^2 - B$ is a block-diagonal matrix over $\mathbb{C}$, $B^2 = B$. Then we have $g([B]) = [\overline{P}]$. So g is surjective.

Finally, we show that g is a homomorphism. In fact, for $[P], [Q] \in \bigvee(\mathcal{A}'(A))$, where $P \in \mathcal{A}'(A^{(n)})$ and $Q \in \mathcal{A}'(A^{(m)})$ are idempotents, we have

$$\begin{aligned} g([P] + [Q]) &= g([P \oplus Q]) = [\overline{P \oplus Q}] \\ &= [\overline{P} \oplus \overline{Q}] = [\overline{P}] + [\overline{Q}] = g([P]) + g([Q]). \end{aligned}$$

From the above, we have proved that g is an isomorphism, so

$$\bigvee(\mathcal{A}'(A)) \approx \bigvee(\mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A)).$$

By Lemma 1.5,

$$\bigvee(\mathcal{A}'(A)/\mathrm{rad}\mathcal{A}'(A)) \approx \bigvee\Big(\sum_{i=1}^{k} \bigoplus M_{n_i}(\mathbb{C})\Big) \approx \sum_{i=1}^{k} \bigoplus \bigvee(M_{n_i}(\mathbb{C})).$$

Since $\bigvee(M_n(\mathbb{C})) \approx \bigvee(\mathbb{C}) \approx \mathbb{N}_+$ for every n (see [1]), then

$$\bigvee(\mathcal{A}'(A)) \approx \mathbb{N}_+^{(k)}$$

and thus $K_0(\mathcal{A}'(A)) \approx \mathbb{Z}^{(k)}$. ∎

LEMMA 1.7. *Let X be a* H.I. *space and let $T \in B(X)$. Then $\mathcal{A}'(T)/\mathrm{rad}\mathcal{A}'(T) \approx \mathbb{C}$ if and only if $T \in (SI)$.*

Proof. "⇒" If $T \notin (SI)$, there exists a non-trivial idempotent $P \in \mathcal{A}'(T)$, obviously $\sigma(P) = \{0,1\}$. Since $\mathcal{A}'(T)/\mathrm{rad}\mathcal{A}'(T) \approx \mathbb{C}$, there exist $\lambda \in \mathbb{C}$ and $S \in \mathrm{rad}\mathcal{A}'(T)$ such that $P = \lambda + S$. Since $\sigma(S) = \{0\}$, then $\sigma(P) = \{\lambda\}$ is a singleton, a contradiction.

"⇐" Firstly we show that $\sigma(S)$ is a singleton for every $S \in \mathcal{A}'(T)$. Since X is a H.I. space, $\sigma(S)$ is an at most countable set. If $\sigma(S)$ is not a singleton, then there exists an open-and-closed proper subset τ of $\sigma(S)$. Let $E(\tau)$ be the spectral projection of S corresponding to τ. Then $E(\tau)$ is a non-trivial idempotent. Since $S \in \mathcal{A}'(T)$, $E(\tau) \in \mathcal{A}'(T)$, which contradicts to $T \in (SI)$. So $\sigma(S)$ is a singleton.

In the following, we prove that $\mathcal{A}'(T)/\mathrm{rad}\mathcal{A}'(T) \approx \mathbb{C}$. We only need to show that there exists $\lambda \in \mathbb{C}$ such that $S - \lambda \in \mathrm{rad}\mathcal{A}'(T)$ for every $S \in \mathcal{A}'(T)$. From the above, we can assume $\sigma(S) = \{\lambda\}$. For every $S' \in \mathcal{A}'(T)$, we have $(S-\lambda)S', S'(S-\lambda) \in \mathcal{A}'(T)$. So $\sigma((S-\lambda)S')$ and $\sigma(S'(S-\lambda))$ are all singletons from the above. Since $\sigma((S-\lambda)S')\backslash\{0\} = \sigma(S'(S-\lambda))\backslash\{0\}$, we can conclude that $\sigma((S-\lambda)S') = \sigma(S'(S-\lambda)) = \{t\}$ for some $t \in \mathbb{C}$. If $t \neq 0$, then both $(S-\lambda)S'$ and $S'(S-\lambda)$ are invertible. Hence $S-\lambda$ is invertible, which contradicts to $\sigma(S) = \{\lambda\}$. It follows that $\sigma((S-\lambda)S') = \{0\}$. Since $S' \in \mathcal{A}'(T)$ is arbitrary, $S - \lambda \in \mathrm{rad}\mathcal{A}'(T)$. Then $\mathcal{A}'(T)/\mathrm{rad}\mathcal{A}'(T) \approx \mathbb{C}$. ∎

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. "⇒" Since X is a H.I. space and $A_i \in (SI)$, then

$$\mathcal{A}'(A_i)/\mathrm{rad}\mathcal{A}'(A_i) \approx \mathbb{C},$$

for $i = 1,2$ by Lemma 1.7. Since $A_1 \sim A_2$, then $A \sim A_1^{(2)}$. So

$$K_0(\mathcal{A}'(A)) \approx K_0(\mathcal{A}'(A_1^{(2)})) \approx \mathbb{Z}$$

by Lemma 1.6.

"⇐" If A_1 is not similar to A_2, then $K_0(\mathcal{A}'(A)) \approx \mathbb{Z}^{(2)}$ by Lemma 1.6, a contradiction. ∎

2. SMALL COMPACT PERTURBATION OF STRONGLY IRREDUCIBLE OPERATORS

In this section we discuss the small compact perturbation of strongly irreducible operators on separable H.I. spaces.

THEOREM 2.1. *Let X be a separable* H.I. *space and let $T \in B(X)$ with connected spectrum $\sigma(T)$. Then, for given $\varepsilon > 0$, there exists a compact operator $K \in B(X)$ with $\|K\| < \varepsilon$ such that $T + K \in (SI)$.*

Before we begin the proof of Theorem 2.3, we need some lemmas. The following lemma is useful, but its proof is a routing work. Thus, we are not going to prove it.

LEMMA 2.2. *Let X be a* H.I. *space and let $T \in B(X)$ with $\sigma(T) = \{0\}$. If T satisfies one of the following conditions*

(i) $\dim \ker T < \infty$,

(ii) $\dim(X/\overline{\mathrm{ran} T}) < \infty$,

then for given $\varepsilon > 0$, there exists a compact operator $K \in B(X)$ with $\|K\| < \varepsilon$ such that $T + K \in (SI)$.

In order to overcome the essential difficulty (i.e. $\dim \ker T = \infty$ and $\dim(X/\overline{\mathrm{ran} T}) = \infty$), we need a series of lemmas. Firstly, we need the concept of quasi-complementary subspaces. Two closed subspaces X_1, X_2 of a Banach space X are said to be quasi-complementary if $X_1 \cap X_2 = \{0\}$ and $X = \overline{X_1 + X_2}$. In this case, each of X_1 and X_2 is called a quasi-complement of the other of them. Corollary 8.2 of [17] shows that every closed subspace of separable Banach spaces admits a quasi-complement.

LEMMA 2.3. *Let X be an infinite dimensional separable Banach space, let X_1 be a closed subspace of X with $\dim(X/X_1) = \infty$ and let X_2 be a quasi-complement of X_1. Then there exist $\{f_i\} \subseteq X_1^{\perp} \subseteq X^*$ and $\{x_i\} \subseteq X_2$ such that $f_i(x_j) = \delta_{ij}$ and $\overline{\mathrm{span}\{f_i\}}^{w^*} = X_1^{\perp}$, where $\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases}$ and $X_1^{\perp} = \{f \in X^* : f(x) = 0$ for all $x \in X_1\}$.*

Proof. Since $\dim(X/X_1) = \infty$, then $\dim X_2 = \infty$. Since X is separable, X/X_1 is separable. Then $(X/X_1)^*$ is w*-separable, so $X_1^{\perp}$ is also w*-separable. Hence there exists $\{g_i\} \subseteq X_1^{\perp}$, $g_i \neq 0$ such that $\overline{\mathrm{span}\{g_i\}}^{w^*} = X_1^{\perp}$.

Let $f_1 = g_1$. Then $f_1 \in X_1^{\perp}$. Therefore there exists $y_1 \in X_2$ such that $f_1(y_1) = 1$. Otherwise $f_1|_{X_2} = 0$. By the fact that $f_1|_{X_1} = 0$ and $X = \overline{X_1 + X_2}$, $f_1 = g_1 = 0$, which is a contradiction.

Set $x_1 = y_1$. Then $x_1 \in X_2$ and $f_1(x_1) = 1$.

Let $h_2 = \min\{n : g_n \notin \mathrm{span}\{f_1\}\}$.

Let $f_2 = g_{h_2} - g_{h_2}(x_1) f_1$. Then $f_2(x_1) = 0$. Since $f_2 \in X_1^{\perp}$, there exists $y_2 \in X_2$ such that $f_2(y_2) = 1$. Otherwise $f_2|_{X_2} = 0$. By the fact that $f_2 \in X_1^{\perp}$ and $X = \overline{X_1 + X_2}$, $f_2 = 0$, which contradicts to the choice of h_2.

Set $x_2 = y_2 - f_1(y_2) x_1$. Then $x_2 \in X_2$, $f_1(x_2) = 0$ and $f_2(x_2) = 1$.

Let $h_3 = \min\{n : g_n \notin \mathrm{span}\{f_1, f_2\}\}$. It is obvious that $h_3 > h_2$.

Let $f_3 = g_{h_3} - g_{h_3}(x_1)f_1 - g_{h_3}(x_2)f_2$. Then $f_3(x_1) = f_3(x_2) = 0$. Since $f_3 \in X_1^{\perp}$, there exists $y_3 \in X_2$ such that $f_3(y_3) = 1$.

Set $x_3 = y_3 - f_1(y_3)x_1 - f_2(y_3)x_2$. Then $x_3 \in X_2$, $f_1(x_3) = f_2(x_3) = 0$ and $f_3(x_3) = 1$.

Continuing in the same way, we can obtain $\{f_i\} \subseteq X_1^{\perp} \subseteq X^*$ and $\{x_i\} \subseteq X_2$ such that $f_i(x_j) = \delta_{ij}$. From the choice of h_i, we can see that $g_i \in \operatorname{span}\{f_j : 1 \leqslant j \leqslant k\}$ for $h_k \leqslant i < h_{k+1}$, where $k \geqslant 1$ and $h_1 = 1$. Then $\overline{\operatorname{span}\{f_i\}}^{w^*} = X_1^{\perp}$ by the fact that $\overline{\operatorname{span}\{g_i\}}^{w^*} = X_1^{\perp}$. ∎

LEMMA 2.4. *Let X be an infinite dimensional separable Banach space, let X_1 be a closed subspace of X with $\dim(X/X_1) = \infty$ and let X_2 be a quasi-complement of X_1. Then there exists a compact operator $K \in B(X)$ such that $\sigma(K) = \{0\}$, $\ker K = X_1$ and $\overline{\operatorname{ran} K} \subseteq X_2$.*

Proof. By Lemma 2.5, there exist $\{f_i\} \subseteq X_1^{\perp} \subseteq X^*$ and $\{x_i\} \subseteq X_2$ such that $f_i(x_j) = \delta_{ij}$ and $\overline{\operatorname{span}\{f_i\}}^{w^*} = X_1^{\perp}$. Let

$$K = \sum_{i=1}^{\infty} a_i f_i \otimes x_{i+1},$$

where $a_i = 1/2^i \|f_i\| \|x_{i+1}\|$. Then $K \in B(X)$ is a compact operator and $\overline{\operatorname{ran} K} \subseteq X_2$.

Now we show that $X_1 = \ker K$. On the one hand, $X_1 \subseteq \ker K$ by the fact that $\{f_i\} \subseteq X_1^{\perp}$. On the other hand, if $Kx = 0$, namely $\sum_{j=1}^{\infty} a_j f_j(x) x_{j+1} = 0$, then

$$f_i(x) = \frac{1}{a_i} f_{i+1}\Big(\sum_{j=1}^{\infty} a_j f_j(x) x_{j+1}\Big) = 0$$

for each i. Hence $x \in (\overline{\operatorname{span}\{f_i\}}^{w^*})_{\perp} = (X_1^{\perp})_{\perp} = X_1$, where $A_{\perp} = \{x \in X : f(x) = 0 \text{ for all } f \in A\}$ for a subset A of X^*. So $\ker K \subseteq X_1$. Therefore $\ker K = X_1$.

Finally, we show that $\sigma(K) = \{0\}$. Since K is compact, it is enough to prove that $K - \lambda$ is injective for every $\lambda \neq 0$.

If $Kx = \lambda x$ for some $x \in X$, then $K^n x = \lambda^n x$. Therefore

$$x = \frac{1}{\lambda^n} K^n x = \frac{1}{\lambda^n} \sum_{i=1}^{\infty} (a_i \cdots a_{i+n-1}) f_i(x) x_{i+n} \in \overline{\operatorname{span}\{x_i : i \geqslant n+1\}}.$$

Hence $f_n(x) = 0$ for every n. Then $x \in (\overline{\operatorname{span}\{f_i\}}^{w^*})_{\perp} = (X_1^{\perp})_{\perp} = X_1$. By the fact that $x \in \overline{\operatorname{ran} K}$ and $\overline{\operatorname{ran} K} \cap X_1 \subseteq X_2 \cap X_1 = \{0\}$, we have $x = 0$. Therefore $K - \lambda$ is injective. It follows that $\sigma(K) = \{0\}$. ∎

LEMMA 2.5. *Let X be a separable* H.I. *space and let $T \in B(X)$ with $\sigma(T) = \{0\}$. If $\dim(X/\overline{\operatorname{ran} T}) = \infty$, then for given $\varepsilon > 0$, there exists a compact operator $K \in B(X)$*

with $\|K\| < \varepsilon$ *such that* $\sigma(T+K) = \{0\}$ *and*

$$\bigcup_{i=1}^{\infty} \ker(T+K)^i + \operatorname{ran}(T+K) \neq X.$$

Proof. Since X is separable, $\overline{\operatorname{ran} T}$ is a quasi-complementary subspace in X. Then there exists a closed subspace X_0 of X such that $\overline{\operatorname{ran} T} \cap X_0 = \{0\}$ and $X = \overline{\overline{\operatorname{ran} T} + X_0}$. Since $\dim(X/\overline{\operatorname{ran} T}) = \infty$, then $\dim X_0 = \infty$. By Lemma 2.4, there exists a compact operator $K \in B(X)$ with $\|K\| < \varepsilon$ such that $\sigma(K) = \{0\}$, $\ker K = \overline{\operatorname{ran} T}$ and $\overline{\operatorname{ran} K} \subseteq X_0$.

In the following, we show that the operator $T+K$ satisfies this lemma.

Firstly we show that $\sigma(T+K) = \{0\}$. Since X is a H.I. space and $\sigma(T) = \{0\}$, then T is a strictly singular operator. And since K is a compact operator, then $T+K$ is also a strictly singular operator. By Remark 0.3, for every $\lambda \neq 0$, $T+K-\lambda$ is a Fredholm operator with index 0. Therefore in order to prove $\sigma(T+K) = \{0\}$, it suffices to show that $T+K-\lambda$ is injective for every $\lambda \neq 0$.

If $(T+K)x = \lambda x$ for some $x \in X$, by the fact that $Tx \in \overline{\operatorname{ran} T} = \ker K$,

$$K(Kx) = K(T+K)x = \lambda(Kx).$$

Therefore $Kx = 0$ by $\sigma(K) = \{0\}$, thus $Tx = \lambda x$. Since $\sigma(T) = \{0\}$, then $x = 0$. Thus $T+K-\lambda$ is injective.

Next we show that

$$\bigcup_{i=1}^{\infty} \ker(T+K)^i \subseteq \overline{\operatorname{ran} T}.$$

It only needs to show that $\ker(T+K)^i \subseteq \overline{\operatorname{ran} T}$ for every i.

We proceed by induction.

For every $x \in \ker(T+K)$, since $Tx + Kx = 0$, then

$$Tx = -Kx \in \overline{\operatorname{ran} T} \cap \overline{\operatorname{ran} K} \subseteq \overline{\operatorname{ran} T} \cap X_0 = \{0\}.$$

Therefore $Kx = 0$, namely $x \in \ker K = \overline{\operatorname{ran} T}$. Thus $\ker(T+K) \subseteq \overline{\operatorname{ran} T}$.

Suppose that $\ker(T+K)^j \subseteq \overline{\operatorname{ran} T}$. We show that $\ker(T+K)^{j+1} \subseteq \overline{\operatorname{ran} T}$. For every $x \in \ker(T+K)^{j+1}$,

$$Tx + Kx \in \ker(T+K)^j \subseteq \overline{\operatorname{ran} T},$$

so $Kx \in \overline{\operatorname{ran} T} \cap \overline{\operatorname{ran} K} = \{0\}$, thus $x \in \ker K = \overline{\operatorname{ran} T}$. This shows that $\ker(T+K)^{j+1} \subseteq \overline{\operatorname{ran} T}$.

Now we have proved $\ker(T+K)^i \subseteq \overline{\operatorname{ran} T}$ for every i, so $\bigcup_{i=1}^{\infty} \ker(T+K)^i \subseteq \overline{\operatorname{ran} T}$.

Since $\operatorname{ran}(T+K) \subseteq \operatorname{ran} T + \operatorname{ran} K$, then

$$\bigcup_{i=1}^{\infty} \ker(T+K)^i + \operatorname{ran}(T+K) \subseteq \overline{\operatorname{ran} T} + \operatorname{ran} K.$$

Therefore in order to show that $\bigcup_{i=1}^{\infty} \ker(T+K)^i + \operatorname{ran}(T+K) \neq X$, it is enough to show that $\overline{\operatorname{ran}T} + \operatorname{ran}K \neq X$.

Since K is a compact operator, it follows that $X_0 \nsubseteq \operatorname{ran}K$. We choose $y \in X_0 \backslash \operatorname{ran}K$, then $y \notin \overline{\operatorname{ran}T} + \operatorname{ran}K$. In fact, if $y = y_1 + y_2 \in \overline{\operatorname{ran}T} + \operatorname{ran}K$ for $y_1 \in \overline{\operatorname{ran}T}$ and $y_2 \in \operatorname{ran}K$, then $y - y_2 = y_1 \in \overline{\operatorname{ran}T} \cap X_0 = \{0\}$. Hence $y = y_2 \in \operatorname{ran}K$, which is a contradiction. ∎

LEMMA 2.6. *Let X be a Banach space, let $T \in B(X)$ with $\sigma(T) = \{0\}$ and let $(M, N) \in \operatorname{Red}(T)$ with $\dim M < \infty$. If $\bigcup_{i=1}^{\infty} \ker T^i + \operatorname{ran}T \neq X$, then $\bigcup_{i=1}^{\infty} \ker(T|_N)^i + \operatorname{ran}T|_N \neq N$.*

Proof. Since $\sigma(T) = \{0\}$, then $\sigma(T|_M) = \{0\}$. Therefore, by $\dim M < \infty$, $T|_M$ is a nilpotent operator, thus $\bigcup_{i=1}^{\infty} \ker(T|_M)^i = M$. If $\bigcup_{i=1}^{\infty} \ker(T|_N)^i + \operatorname{ran}T|_N = N$, then

$$\bigcup_{i=1}^{\infty} \ker(T|_M)^i + \operatorname{ran}T|_M + \bigcup_{i=1}^{\infty} \ker(T|_N)^i + \operatorname{ran}T|_N = M + N = X.$$

It is obvious that

$$\bigcup_{i=1}^{\infty} \ker T^i + \operatorname{ran}T = \bigcup_{i=1}^{\infty} \ker(T|_M)^i + \bigcup_{i=1}^{\infty} \ker(T|_N)^i + \operatorname{ran}T|_M + \operatorname{ran}T|_N = X,$$

which is a contradiction, so

$$\bigcup_{i=1}^{\infty} \ker(T|_N)^i + \operatorname{ran}T|_N \neq N. \quad ∎$$

LEMMA 2.7. *Let X be a* H.I. *space and let $T \in B(X)$ with $\sigma(T) = \{0\}$. If there exists a $(M, N) \in \operatorname{Red}(T)$ such that $\dim M = n < \infty$, $T|_N \in (SI)$ and $\bigcup_{i=1}^{\infty} \ker(T|_N)^i + \operatorname{ran}T|_N \neq N$, then for given $\varepsilon > 0$, there exists a compact operator $K \in B(X)$ with $\|K\| < \varepsilon$ such that $T + K \in (SI)$.*

Proof. Step 1. If $n = 0$, then $T \in (SI)$. We can take $K = 0$.

Step 2. If $n > 0$, we have $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$, where $T_1 = T|_M$ and $T_2 = T|_N$. Without loss of generality, we can assume that T_1 admits a (lower triangular) Jordan block representation with respect to some basis $\{e_i\}_{i=1}^n$ of M, namely

$$\begin{cases} T_1 e_i = e_{i+1} & i = 1, \ldots, n-1, \\ T_1 e_n = 0. \end{cases}$$

It is obvious that there exists $f \in X^*$ satisfying

$$\begin{cases} f(e_i) = 0 \quad i = 1, \dots, n-1, \\ f(e_n) = 1, \\ f|_N = 0. \end{cases}$$

Put

$$X_0 = \bigcup_{i=1}^{\infty} \ker T_2^i + \operatorname{ran} T_2.$$

Then $X_0 \neq N$. We choose $y \in N \setminus X_0$ with $\|y\| = \varepsilon/2\|f\|$. Let $K = f \otimes y$. Then $K \in B(X)$ is a compact operator with $\|K\| < \varepsilon$. Denote $f_1 = f|_M$. Then

$$K = \begin{pmatrix} 0 & 0 \\ f_1 \otimes y & 0 \end{pmatrix} \quad \text{and} \quad T + K = \begin{pmatrix} T_1 & 0 \\ f_1 \otimes y & T_2 \end{pmatrix}.$$

In the following, we show that $T + K \in (SI)$.

Otherwise there exists a non-trivial $(M'', N'') \in \operatorname{Red}(T+K)$ with $0 < \dim M'' < \infty$. Then $(T+K)|_{M''}$ admits a Jordan standard form representation. We can select a subspace M' of M'' such that $M'' = M' \oplus M'''$, M' and M''' are invariant subspaces of $T+K$ and $(T+K)|_{M'}$ admits a (lower triangular) Jordan block representation with respect to some basis $\{e_i'\}_{i=1}^m$ of M', namely

$$\begin{cases} (T+K)|_{M'} e_i' = e_{i+1}' \quad i = 1, \dots, m-1, \\ (T+K)|_{M'} e_m' = 0. \end{cases}$$

Let $N' = M''' \oplus N''$. Then $(M', N') \in \operatorname{Red}(T+K)$.

Put

$$e_1' = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n + x,$$

where $x \in N, a_i \in \mathbb{C}, i = 1, 2, \dots, n$.

We proceed the proof by considering the following three cases.

Case 1. $m > n$. Then

$$\begin{aligned} e_2' &= (T+K)e_1' = a_1 e_2 + a_2 e_3 + \cdots + a_{n-1} e_n + a_n y + T_2 x, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ e_n' &= (T+K)e_{n-1}' = a_1 e_n + a_2 y + \cdots + a_n T_2^{n-2} y + T_2^{n-1} x, \\ e_{n+1}' &= (T+K)e_n' = a_1 y + a_2 T_2 y + \cdots + a_n T_2^{n-1} y + T_2^n x, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ e_m' &= (T+K)e_{m-1}' = a_1 T_2^{m-n-1} y + a_2 T_2^{m-n} y + \cdots + a_n T_2^{m-2} y + T_2^{m-1} x, \\ 0 &= (T+K)e_m' = a_1 T_2^{m-n} y + a_2 T_2^{m-n+1} y + \cdots + a_n T_2^{m-1} y + T_2^m x. \end{aligned}$$

Hence

$$a_1 y + a_2 T_2 y + \cdots + a_n T_2^{n-1} y + T_2^n x \in \ker T_2^{m-n},$$

so $a_1 y \in \ker T_2^{m-n} + \operatorname{ran} T_2 \subseteq X_0$. By the fact that $y \notin X_0$, we have $a_1 = 0$. Therefore

$$a_2 T_2^{m-n+1} y + \cdots + a_n T_2^{m-1} y + T_2^m x = 0.$$

In the similar way, we can obtain $a_2 = a_3 = \cdots = a_n = 0$, so $e'_i = T_2^{i-1} x \in N, 1 \leqslant i \leqslant m$ and $T_2^m x = 0$.

Case 2. $m = n$. Then

$$\begin{aligned}
e'_2 &= (T+K)e'_1 = a_1 e_2 + \cdots + a_{n-1} e_n + a_n y + T_2 x,\\
&\cdots\cdots\cdots\cdots\cdots\cdots,\\
e'_n &= (T+K)e'_{n-1} = a_1 e_n + a_2 y + \cdots + a_n T_2^{n-2} y + T_2^{n-1} x,\\
0 &= (T+K)e'_n = a_1 y + a_2 T_2 y + \cdots + a_n T_2^{n-1} y + T_2^n x.
\end{aligned}$$

Since $y \notin \operatorname{ran} T_2$, then $a_1 = 0$. Therefore

$$a_2 y + a_3 T_2 y + \cdots + a_n T_2^{n-2} y + T_2^{n-1} x \in \ker T_2,$$

so $a_2 y \in \ker T_2 + \operatorname{ran} T_2 \subseteq X_0$. By the fact that $y \notin X_0$, we have $a_2 = 0$. In the similar way, we can conclude $a_3 = \cdots = a_n = 0$, so $e'_i = T_2^{i-1} x \in N, 1 \leqslant i \leqslant n$ and $T_2^n x = 0$.

Case 3. $m < n$. Then

$$\begin{aligned}
e'_2 &= (T+K)e'_1 = a_1 e_2 + \cdots + a_{n-1} e_n + a_n y + T_2 x,\\
&\cdots\cdots\cdots\cdots\cdots\cdots,\\
e'_m &= (T+K)e'_{m-1} = a_1 e_m + \cdots + a_{n-m+1} e_n + a_{n-m+2} y + \cdots + a_n T_2^{m-2} y + T_2^{m-1} x,\\
0 &= (T+K)e'_m a_1 e_{m+1} + \cdots + a_{n-m} e_n + a_{n-m+1} y + a_{n-m+2} T_2 y + \cdots + a_n T_2^{m-1} y + T_2^m x.
\end{aligned}$$

Hence

$$\begin{aligned}
-(a_1 e_{m+1} + \cdots + a_{n-m} e_n) &= a_{n-m+1} y + a_{n-m+2} T_2 y + \cdots + a_n T_2^{m-1} y + T_2^m x\\
&\in M \cap N = \{0\}.
\end{aligned}$$

Then $a_1 e_{m+1} + \cdots + a_{n-m} e_n = 0$ and

$$a_{n-m+1} y + a_{n-m+2} T_2 y + \cdots + a_n T_2^{m-1} y + T_2^m x = 0.$$

Since $\{e_i\}_{i=1}^n$ is a basis of M, then $a_1 = a_2 = \cdots = a_{n-m} = 0$. And similar to the proof of Case 2, we can obtain $a_{n-m+1} = \cdots = a_n = 0$, so $e'_i = T_2^{i-1} x \in N, 1 \leqslant i \leqslant m$ and $T_2^m x = 0$.

From the above proof, we have $e'_i = T_2^{i-1} x \in N, 1 \leqslant i \leqslant m$ and $T_2^m x = 0$. Then

$$\begin{cases} T_2 e'_i = e'_{i+1} & i = 1, \ldots, m-1,\\ T_2 e'_m = 0. \end{cases}$$

So $M' \subseteq N$ and $T_2 M' \subseteq M'$.

Let $Y = N \cap N'$. Then $N = M' \oplus Y$. In fact, for every $x \in N$, since $X = M' \oplus N'$, assume that $x = m + n$ with $m \in M', n \in N'$. Since $M' \subseteq N$, then

$n = x - m \in N \cap N' = Y$. Thus $N = M' + Y$. From $M' \cap Y \subseteq M' \cap N' = \{0\}$, we can conclude that $N = M' \oplus Y$.

Since $(T+K)|_N = T_2$, then

$$T_2 Y = (T+K)Y \subseteq (T+K)N' \subseteq N'.$$

It is obvious that

$$T_2 Y \subseteq T_2 N \subseteq N.$$

Thus

$$T_2 Y \subseteq N \cap N' = Y.$$

Therefore $(M', Y) \in \mathrm{Red}(T_2)$. It is clear that (M', Y) is non-trivial, which contradicts to $T_2 \in (SI)$, so $T + K \in (SI)$. ∎

LEMMA 2.8. *Let X be an infinite dimensional separable Banach space and let X_1 be an infinite dimensional closed subspace of X. Then there exist $\{f_i\} \subseteq X^*$ and $\{x_i\} \subseteq X_1$ such that $\overline{\mathrm{span}\{f_i\}}^{w^*} = X^*$, $\overline{\mathrm{span}\{x_i\}} = X_1$ and $f_i(x_j) = \delta_{ij}$.*

Proof. Since X is separable, then X_1 is also separable. By Proposition 1.f.3 of [16], there exist $\{h'_i\} \subseteq X_1^*$ with $\|h'_i\| = 1$ and $\{x_i\} \subseteq X_1$ such that $\overline{\mathrm{span}\{h'_i\}}^{w^*} = X_1^*$, $\overline{\mathrm{span}\{x_i\}} = X_1$ and $h'_i(x_j) = \delta_{ij}$. Let $h_i \in X^*$ with $h_i|_{X_1} = h'_i$ and $\|h_i\| = 1$.

Since X/X_1 is separable, then $(X/X_1)^*$ is w*-separable. So $X_1^\perp$ is also w*-separable. Thus there exist $\{g_i\} \subseteq X_1^\perp$, $g_i \neq 0$, such that $\overline{\mathrm{span}\{g_i\}}^{w^*} = X_1^\perp$.

We divide the set of natural number into countable mutually disjoint infinite subsets, that is, $\mathbb{N} = \bigcup_{i=1}^{\infty} N_i$, where N_i is an infinite subset of $\mathbb{N}$ and $N_i \cap N_j = \varnothing$ for $i \neq j$. Write $N_i = \{n_{i1}, n_{i2}, \ldots\}$ and let $f_{n_{ij}} = j g_i + h_{n_{ij}}$.

If there exists x such that $f_{n_{ij}}(x) = 0$ for every i, j, namely $j g_i(x) + h_{n_{ij}}(x) = 0$, then $g_i(x) = 0$ for every i. Otherwise $g_{i_0}(x) \neq 0$ for some i_0. It follows that $\|x\| \geqslant |h_{n_{i_0 j}}(x)| = j|g_{i_0}(x)|$ for every j, which is a contradiction. So $x \in (\overline{\mathrm{span}\{g_i\}}^{w^*})_\perp = (X_1^\perp)_\perp = X_1$. Since $\overline{\mathrm{span}\{h'_i\}}^{w^*} = X_1^*$ and $h'_{n_{ij}}(x) = h_{n_{ij}}(x) = -j g_i(x) = 0$ for every i, j, then $x = 0$. Therefore $\overline{\mathrm{span}\{f_i\}}^{w^*} = X^*$.

Since $g_i \in X_1^\perp$ and $h_i(x_j) = \delta_{ij}$, it is obvious that $f_i(x_j) = \delta_{ij}$. ∎

LEMMA 2.9. *Let X be an infinite dimensional separable Banach space and let X_1 be an infinite dimensional closed subspace of X. Then there exists a compact operator $K \in B(X)$ satisfying $\sigma_{\mathrm{p}}(K) = \varnothing$ and $\mathrm{ran} K \subseteq X_1$.*

Proof. By Lemma 2.8, there exist $\{f_i\} \subseteq X^*$ and $\{x_i\} \subseteq X_1$ such that $f_i(x_j) = \delta_{ij}$ and $\overline{\mathrm{span}\{f_i\}}^{w^*} = X^*$. Let

$$K = \sum_{i=1}^{\infty} a_i f_i \otimes x_{i+1},$$

where $a_i = 1/2^i \|f_i\| \|x_{i+1}\|$. Then $K \in B(X)$ is a compact operator and $\operatorname{ran} K \subseteq X_1$.

In the following we show that $\sigma_{\mathrm{p}}(K) = \emptyset$. For every $\lambda \in \mathbb{C}$, if there exists $x \in X$ such that $\sum_{i=1}^{\infty} a_i f_i(x) x_{i+1} = Kx = \lambda x$, then

$$0 = f_1\Big(\sum_{i=1}^{\infty} a_i f_i(x) x_{i+1}\Big) = \lambda f_1(x) \quad \text{and} \quad a_{j-1} f_{j-1}(x) = f_j\Big(\sum_{i=1}^{\infty} a_i f_i(x) x_{i+1}\Big) = \lambda f_j(x)$$

for $j \geqslant 2$.

Step 1. If $\lambda = 0$, then $f_{j-1}(x) = (\lambda / a_{j-1}) f_j(x) = 0$ for $j \geqslant 2$. Since $\overline{\operatorname{span}\{f_i\}}^{\mathrm{w}^*} = X^*$, then $x = 0$.

Step 2. If $\lambda \neq 0$, then $f_1(x) = 0$ and $f_2(x) = (1/\lambda) a_1 f_1(x) = 0$, inductively, $f_i(x) = 0$ for every i. Since $\overline{\operatorname{span}\{f_i\}}^{\mathrm{w}^*} = X^*$, then $x = 0$.

From the above, $\sigma_{\mathrm{p}}(K) = \emptyset$. ∎

LEMMA 2.10. *Let X be a* H.I. *space and let $T \in B(X)$ with $\sigma(T) = \{0\}$. If there exists no $(M, N) \in \operatorname{Red}(T)$ such that $\dim M = n < \infty$ and $T|_N \in (SI)$, then for given $\varepsilon > 0$, there exists a compact operator $K \in B(X)$ with $\|K\| < \varepsilon$ such that $\sigma(T+K) = \{0\}$ and*

$$\dim(\ker(T+K)/(\ker(T+K) \cap \overline{\operatorname{ran}(T+K)})) = \infty.$$

Proof. By hypothesis, $T \notin (SI)$. By the definition of strongly irreducible operators and the definition of H.I. spaces, there exists a non-trivial $(X'_{11}, X'_{12}) \in \operatorname{Red}(T)$ with $0 < \dim X'_{11} < \infty$. Then $T|_{X'_{11}}$ admits a Jordan standard form representation. We can select a subspace X_{11} of X'_{11} such that $X'_{11} = X_{11} \oplus X''_{11}$, X_{11} and X''_{11} are invariant subspaces of T and $T|_{X_{11}}$ admits a (lower triangular) Jordan block representation. Let $X_{12} = X''_{11} \oplus X'_{12}$. Then $(X_{11}, X_{12}) \in \operatorname{Red}(T)$. Let

$$T = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{12} \end{pmatrix} \begin{matrix} X_{11} \\ X_{12} \end{matrix}.$$

By hypothesis, $T_{12} \notin (SI)$. In the similar way, we can assume that

$$T_{12} = \begin{pmatrix} T_{21} & 0 \\ 0 & T_{22} \end{pmatrix} \begin{matrix} X_{21} \\ X_{22} \end{matrix}$$

with $0 < \dim X_{21} < \infty$ satisfies that $T_{12}|_{X_{21}}$ admits a (lower triangular) Jordan block representation. By hypothesis again, $T_{22} \notin (SI)$. Otherwise $(X_{11} \oplus X_{21}, X_{22}) \in \operatorname{Red}(T)$ satisfies $\dim(X_{11} \oplus X_{21}) < \infty$ and $T|_{X_{22}} = T_{22} \in (SI)$, which is a contradiction. Continuing in the same way, we can assume that

$$T = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{12} \end{pmatrix} \begin{matrix} X_{11} \\ X_{12} \end{matrix}, \quad T_{12} = \begin{pmatrix} T_{21} & 0 \\ 0 & T_{22} \end{pmatrix} \begin{matrix} X_{21} \\ X_{22} \end{matrix},$$

$$T_{22} = \begin{pmatrix} T_{31} & 0 \\ 0 & T_{32} \end{pmatrix} \begin{matrix} X_{31} \\ X_{32} \end{matrix}, \quad \cdots\cdots\cdots$$

where $X_{i1} = [e_{ij}]_{j=1}^{n_i}$ and

$$\begin{cases} T(e_{ij}) = e_{i,j+1} & j = 1,2,\dots,n_i - 1, \\ T(e_{in_i}) = 0, & \end{cases} \quad i = 1,2,\dots.$$

For every i,

$$\begin{cases} \text{if } n_i = 1, \text{ let } K_i' = 0, \text{ the operator from } X_{i1} \text{ into } X_{i+1,1}, \\ \text{if } n_i > 1, \text{ let } K_i' \text{ be the operator from } X_{i1} \text{ into } X_{i+1,1}, \end{cases} \quad \text{satisfies}$$

$$\begin{cases} K_i'(e_{i,n_i-1}) = e_{i+1,1}, \\ K_i'(e_{ij}) = 0, & j \neq n_i - 1, 1 \leqslant j \leqslant n_i. \end{cases}$$

Let $K_i'' \in B(X)$ be the natural extension of K_i' to X, that is,

$$\begin{cases} K_i''|_{X_{j1}} = 0 & j = 1,2,\dots,i-1, \\ K_i''|_{X_{i1}} = K_i', \\ K_i''|_{X_{i2}} = 0. \end{cases}$$

Let $a_i = \begin{cases} 0 & K_i' = 0, \\ \varepsilon/2^i \|K_i''\| & K_i' \neq 0. \end{cases}$ Denote $K_i = a_i K_i'$. Put

$$K = \sum_{i=1}^{\infty} a_i K_i''.$$

Then $K \in B(X)$ is a compact operator with $\|K\| < \varepsilon$.

For each $m \geqslant 1$,

$$(2.1) \quad T+K = \begin{pmatrix} T_{11} & & & & & & & \\ K_1 & \ddots & & & & & & \\ & \ddots & T_{2k-2,1} & & & & & \\ & & K_{2k-2} & T_{2k-1,1} & & & & \\ & & & K_{2k-1} & T_{2k,1} & & & \\ & & & & K_{2k} & T_{2k+1,1} & & \\ & & & & & K_{2k+1} & \ddots & \\ & & & & & & \ddots & T_{m1} & \\ & & & & & & & K_m & T_{m2} + Q_m \end{pmatrix},$$

where $Q_m = \Big(\sum_{i=m+1}^{\infty} a_i K_i''\Big)|_{X_{m2}}$.

In the following, we show that $T + K$ satisfies this lemma.

Since

$$(2.2)\quad T+\sum_{i=1}^{m} a_i K_i'' = \begin{pmatrix} T_{11} & & & & & & & & \\ K_1 & \ddots & & & & & & & \\ & \ddots & T_{2k-2,1} & & & & & & \\ & & K_{2k-2} & T_{2k-1,1} & & & & & \\ & & & K_{2k-1} & T_{2k,1} & & & & \\ & & & & K_{2k} & T_{2k+1,1} & & & \\ & & & & & K_{2k+1} & \ddots & & \\ & & & & & & \ddots & T_{m1} & \\ & & & & & & & K_m & T_{m2} \end{pmatrix},$$

then

$$\sigma\Big(T+\sum_{i=1}^{m}a_iK_i''\Big) \subseteq \Big(\bigcup_{i=1}^{m}\sigma(T_{i1})\Big) \cup \sigma(T_{m2}) = \sigma(T) = \{0\}.$$

Hence $\sigma\Big(T+\sum_{i=1}^{m}a_iK_i''\Big) = \{0\}$ is connected.

Since

$$T+\sum_{i=1}^{m}a_iK_i'' \to T+K$$

as $m \to \infty$ and the set $\{T \in B(X) : \sigma(T) \text{ is connected}\}$ is a closed subset of $B(X)$, then $\sigma(T+K)$ is connected. Since X is a H.I. space and $\sigma(T) = \{0\}$, then T is a strictly singular operator. Since K is a compact operator, then $T+K$ is also a strictly singular operator. By Remark 0.3, $\sigma(T+K)$ is an at most countable set and $0 \in \sigma(T+K)$. Since $\sigma(T+K)$ is not only connected but also at most countable, then $\sigma(T+K)$ is a singleton. So $\sigma(T+K) = \{0\}$.

It is obvious that $e_{kn_k} \in \ker(T+K)$. In order to show that

$$\dim(\ker(T+K)/(\ker(T+K)\cap\overline{\operatorname{ran}(T+K)})) = \infty,$$

it suffices to show that $\{e_{2k,n_{2k}} + (\ker(T+K)\cap\overline{\operatorname{ran}(T+K)})\}_{k=1}^{\infty}$ is linearly independent in $\ker(T+K)/(\ker(T+K)\cap\overline{\operatorname{ran}(T+K)})$.

If

$$\sum_{k=1}^{l} b_k(e_{2k,n_{2k}} + (\ker(T+K)\cap\overline{\operatorname{ran}(T+K)})) = 0$$

for some $\{b_k\}_{k=1}^{l} \subseteq \mathbb{C}$, then

$$\sum_{k=1}^{l} b_k e_{2k,n_{2k}} \in \overline{\operatorname{ran}(T+K)},$$

so there exist

$$x^{(n)} = x_1^{(n)} + \cdots + x_{2k-2}^{(n)} + x_{2k-1}^{(n)} + x_{2k}^{(n)} + x_{2k+1}^{(n)} + \cdots + x_{2l+1}^{(n)} + y^{(n)}, n \in \mathbb{N}$$

such that

$$(T+K)x^{(n)} \to \sum_{k=1}^{l} b_k e_{2k,n_{2k}}$$

as $n \to \infty$, where $\begin{cases} x_i^{(n)} \in X_{i1} & i = 1,2,\ldots,2l+1, \\ y^{(n)} \in X_{2l+1,2}. \end{cases}$

For each $1 \leqslant k \leqslant l$, according to the matrix (2.2) and the construction of K_i, we can obtain the following assertions:

$$K_{2k-2}x_{2k-2}^{(n)} + T_{2k-1,1}x_{2k-1}^{(n)} \to 0 \quad (n \to \infty), \tag{2.3}$$

$$K_{2k-1}x_{2k-1}^{(n)} + T_{2k,1}x_{2k}^{(n)} \to b_k e_{2k,n_{2k}} \quad (n \to \infty), \tag{2.4}$$

$$K_{2k}x_{2k}^{(n)} + T_{2k+1,1}x_{2k+1}^{(n)} \to 0 \quad (n \to \infty), \tag{2.5}$$

where $K_0 = 0$ and $x_0^{(n)} = 0$.

We prove $b_k = 0$ by two cases.

Case 1. $K_{2k} \neq 0$. By the construction of K_{2k}, $n_{2k} > 1$. We may assume that

$$x_{2k}^{(n)} \in \alpha_{2k}^{(n)} e_{2k,n_{2k}-1} + \operatorname{span}\{e_{2k,j} : j \neq n_{2k}-1, 1 \leqslant j \leqslant n_{2k}\},$$

where $\alpha_{2k}^{(n)} \in \mathbb{C}$. By (2.4), $\alpha_{2k}^{(n)} \to b_k$ as $n \to \infty$. And by (2.5), $\alpha_{2k}^{(n)} \to 0$ as $n \to \infty$, so $b_k = 0$.

Case 2. $K_{2k} = 0$. By the construction of K_{2k}, $T_{2k,1} = 0$.

If $K_{2k-1} = 0$, by (2.4), $b_k = 0$.

If $K_{2k-1} \neq 0$, by the construction of K_{2k-1}, $n_{2k-1} > 1$. We may assume that

$$x_{2k-1}^{(n)} \in \alpha_{2k-1}^{(n)} e_{2k-1,n_{2k-1}-1} + \operatorname{span}\{e_{2k-1,j} : j \neq n_{2k-1}-1, 1 \leqslant j \leqslant n_{2k-1}\},$$

where $\alpha_{2k-1}^{(n)} \in \mathbb{C}$. By (2.3), $\alpha_{2k-1}^{(n)} \to 0$ as $n \to \infty$. And by (2.4), $\alpha_{2k-1}^{(n)} \to b_k$ as $n \to \infty$, so $b_k = 0$.

In a word, $b_k = 0$ for each $1 \leqslant k \leqslant l$, so

$$\{e_{2k,n_{2k}} + (\ker(T+K) \cap \overline{\operatorname{ran}(T+K)})\}_{k=1}^{\infty}$$

is linearly independent in $\ker(T+K)/(\ker(T+K) \cap \overline{\operatorname{ran}(T+K)})$. ∎

LEMMA 2.11. *Let X be a separable* H.I. *space and let $T \in B(X)$ with $\sigma(T) = \{0\}$. If* $\dim(\ker T/(\ker T \cap \overline{\operatorname{ran} T})) = \infty$, *then for given $\varepsilon > 0$, there exists a compact operator $K \in B(X)$ with $\|K\| < \varepsilon$ such that $T + K \in (SI)$.*

Proof. Since X is separable, then $\ker T$ is separable. Thus $\ker T \cap \overline{\operatorname{ran} T}$ is a quasi-complementary subspace in $\ker T$, so there exists a closed subspace X_1 of $\ker T$ such that

$$X_1 \cap (\ker T \cap \overline{\operatorname{ran} T}) = \{0\} \quad \text{and} \quad \ker T = \overline{X_1 + (\ker T \cap \overline{\operatorname{ran} T})}.$$

Since $\dim(\ker T/(\ker T \cap \overline{\operatorname{ran} T})) = \infty$, then $\dim X_1 = \infty$. By Lemma 2.9, there exists a compact operator $K \in B(X)$ with $\|K\| < \varepsilon$ such that $\sigma_p(K) = \emptyset$ and $\operatorname{ran} K \subseteq X_1$.

In the following, we show that $\sigma_p(T+K) = \emptyset$.

For every $\lambda \neq 0$, if $(T+K)x = \lambda x$ for some $x \in X$, by the fact that $Kx \in \operatorname{ran} K \subseteq X_1 \subseteq \ker T$,

$$T(Tx) = T(T+K)x = \lambda Tx.$$

Since $\sigma(T) = \{0\}$ and thus $Tx = 0$, then $Kx = \lambda x$. And since $\sigma_p(K) = \emptyset$, then $x = 0$. We conclude that $\lambda \notin \sigma_p(T+K)$.

If $(T+K)x = 0$, then

$$Tx = -Kx \in X_1 \cap (\ker T \cap \overline{\operatorname{ran} T}) = \{0\}.$$

Thus $Kx = 0$. Since $\sigma_p(K) = \emptyset$, then $x = 0$. Hence $0 \notin \sigma_p(T+K)$.

From the above, $\sigma_p(T+K) = \emptyset$. By the definition of strongly irreducible operators and the definition of H.I. spaces, it is easy to see that $T + K \in (SI)$. ∎

We are now in a position to prove Theorem 2.3.

Proof of Theorem 2.3. Since X is a H.I. space, the operator T can be written as a sum of a scalar and a strictly singular operator. By Remark 0.3, $\sigma(T)$ is an at most countable set. Since $\sigma(T)$ is connected, we conclude that $\sigma(T)$ is a singleton. Suppose that $\sigma(T) = \{\lambda\}$. Then $\sigma(T - \lambda) = \{0\}$. We consider the following two cases:

Case 1. $\dim(X/\overline{\operatorname{ran}(T-\lambda)}) < \infty$. By Lemma 2.4, there exists a compact operator $K \in B(X)$ with $\|K\| < \varepsilon$ such that $T - \lambda + K \in (SI)$. Then $T + K \in (SI)$.

Case 2. $\dim(X/\overline{\operatorname{ran}(T-\lambda)}) = \infty$. By Lemma 2.5, there exists a compact operator $K_1 \in B(X)$ with $\|K_1\| < \varepsilon/3$ such that $\sigma(T - \lambda + K_1) = \{0\}$ and

$$\bigcup_{i=1}^{\infty} \ker(T-\lambda+K_1)^i + \operatorname{ran}(T-\lambda+K_1) \neq X.$$

To finish the proof, we consider the following two cases again:

Case 2.1. There exists a $(M, N) \in \operatorname{Red}(T - \lambda + K_1)$ with $\dim M < \infty$ and $(T-\lambda+K_1)|_N \in (SI)$. By Lemma 2.6 and by the fact that $\bigcup_{i=1}^{\infty} \ker(T-\lambda+K_1)^i + \operatorname{ran}(T-\lambda+K_1) \neq X$,

$$\bigcup_{i=1}^{\infty} \ker((T-\lambda+K_1)|_N)^i + \operatorname{ran}(T-\lambda+K_1)|_N \neq N.$$

Then $T - \lambda + K_1$ satisfies the condition of Lemma 2.7. Therefore there exists a compact operator $K_2 \in B(X)$ with $\|K_2\| < \varepsilon/3$ such that $T - \lambda + K_1 + K_2 \in (SI)$, so $T + K_1 + K_2 \in (SI)$. We take $K = K_1 + K_2$.

Case 2.2. There exists no $(M, N) \in \operatorname{Red}(T - \lambda + K_1)$ with $\dim M < \infty$ and $(T-\lambda+K_1)|_N \in (SI)$. From Lemma 2.10, there exists a compact operator $K_3 \in B(X)$ with $\|K_3\| < \varepsilon/3$ such that $\sigma(T - \lambda + K_1 + K_3) = \{0\}$ and $\dim(\ker(T -$

$\lambda + K_1 + K_3)/(\ker(T - \lambda + K_1 + K_3) \cap \overline{\operatorname{ran}(T - \lambda + K_1 + K_3)})) = \infty$. And from Lemma 2.11, there exists a compact operator $K_4 \in B(X)$ with $\|K_4\| < \varepsilon/3$ such that $T - \lambda + K_1 + K_3 + K_4 \in (SI)$, so $T + K_1 + K_3 + K_4 \in (SI)$. We take $K = K_1 + K_3 + K_4$. ∎

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