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ON COMPACTNESS OF LAPLACE AND STIELTJES TYPE TRANSFORMATIONS IN LEBESGUE SPACES

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ABSTRACT. We obtain criteria for integral transformations of Laplace and Stieltjes type to be compact on Lebesgue spaces of real functions on the semiaxis.

KEYWORDS: *Compactness, boundedness, Lebesgue space, integral operator, Laplace transformation, Stieltjes transformation.*

MSC (2010): 47G10.

INTRODUCTION

Let $0 < r \leqslant \infty$, $I \subseteq [0,\infty) =: \mathbb{R}^+$ and $L^r(I)$ denote the Lebesgue space of all measurable functions f on I such that $\|f\|_{r,I} < \infty$, where

$$\|f\|_{r,I} := \begin{cases} \Big(\int\limits_I |f(t)|^r \mathrm{d}t \Big)^{1/r} & 0 < r < \infty, \\ \operatorname*{ess\,sup}\limits_{t \in I} |f(t)| & r = \infty. \end{cases}$$

If $I = \mathbb{R}^+$ we write $L^r := L^r(\mathbb{R}^+)$, and $\|f\|_r$ means $\|f\|_{r,\mathbb{R}^+}$.

Take $\lambda > 0$, $p \geqslant 1$, $q > 0$ and put $p' := p/(p-1)$. Assume $v \in L^{p'}_{\mathrm{loc}}(0,\infty)$, $w \in L^q_{\mathrm{loc}}(0,\infty)$ are non-negative weight functions. In this article we study compactness properties of an integral transformation $T : L^p \to L^q$

$$(0.1)\qquad Tf(x) := \int\limits_{\mathbb{R}^+} k_T(x,y) f(y) \mathrm{d}y, \quad x \in \mathbb{R}^+,$$

when $k_T(x,y) = \mathrm{e}^{-xy^\lambda} v(y)$ and $k_T(x,y) = w(x)(x^\lambda + y^\lambda)^{-1} v(y)$. In other words, we take as T the Laplace integral operator

$$(0.2)\qquad \mathcal{L}f(x) := \int\limits_{\mathbb{R}^+} \mathrm{e}^{-xy^\lambda} f(y) v(y) \mathrm{d}y, \quad x \in \mathbb{R}^+,$$

with $k_L(x,y) := e^{-xy^\lambda}v(y)$, and the Stieltjes type transformation

$$(0.3) \qquad Sf(x) := w(x)\int_{\mathbb{R}^+}\frac{f(y)v(y)\mathrm{d}y}{x^\lambda+y^\lambda}, \quad x\in\mathbb{R}^+,$$

with $k_S(x,y) := w(x)(x^\lambda+y^\lambda)^{-1}v(y)$. These operators are related to each other by (2.10).

With an appropriate choice of λ, v and w transformations $\mathcal{L}$ and S become special cases of conventional convolution transformation $F(x) = \int_{-\infty}^{\infty} f(t)G(x-t)\mathrm{d}t$, $-\infty < x < \infty$ ([27], Chapter 8, Sections 8.5, 8.6). The Stieltjes type operator (0.3) has also connections with Hilbert's double series theorem (see [21] for details). Some interesting properties and applications of the Laplace type transform (0.2) to differential equations are indicated in Chapter 5 of [12].

In this work we find explicit necessary and sufficient conditions for L^p-L^q-compactness of the operators $\mathcal{L}$ and S expressed in terms of kernels k_L and k_S. The results may be useful for study of characteristic values of the transformations. All cases of summation parameters $p \geqslant 1$ and $q > 0$ are considered. If $0 < p < 1$ and $T : L^p \to L^q$ is compact then $\|T\|_{p\to q} \equiv 0$ (see Theorem 2 of [18]). Note that L^2-L^2 compactness of (0.2) and (0.3) was studied in [25], [26]. We generalize these results for all positive p and q.

Our main method is well-known and consists in splitting an initial operator into a sum of a compact operator and operators with small norms (see e.g. [7], [13], [16]).

The article is organized as follows. Section 2 is devoted to the compactness of the Laplace transformation (0.2). Criteria for the compactness of the Stieltjes operator (0.3) appear in Section 3. Note that the case of negative λ in S ensues from the results for positive λ by simple modification of weight functions v and w.

Throughout the article we use symbols $=:$ and $:=$ for marking new quantities. Products of the form $0\cdot\infty$ are supposed to be equal to 0. An equivalence $A \approx B$ means either $A = c_0 B$ or $c_1 A \leqslant B \leqslant c_2 A$ with c_i, $i = 0,1,2$, depending on λ, p, q only. The symbol $\mathbb{Z}$ denotes integers, χ_E stands for characteristic function of a subset $E \subset \mathbb{R}^+$. We denote $p' := p/(p-1)$ for $p > 1$ and $r := pq/(p-q)$ if $q < p$.

1. COMPACTNESS OF THE LAPLACE TRANSFORM

Denote:

$$V_{c_1}(t) := \int_{c_1}^{t} v^{p'}, \quad A_{\mathcal{L},\langle c_1,c_2\rangle}(t) := (t^{-\lambda} - c_2^{-\lambda})^{1/q}[V_{c_1}(t)]^{1/p'},$$

$$A_{\mathcal{L},\langle c_1,c_2\rangle} := \sup_{c_1<t<c_2} A_{\mathcal{L},\langle c_1,c_2\rangle}(t), \quad A_{\mathcal{L}} := A_{\mathcal{L},\mathbb{R}^+},\ \overline{v}_{c_1}(t) := \operatorname*{ess\,sup}_{c_1<x<t} v(x),$$

$$B_{\mathcal{L},\langle c_1,c_2\rangle} := \Big(\int_{c_1}^{c_2} [t^{-\lambda} - c_2^{-\lambda}]^{r/q}[V_{c_1}(t)]^{r/q'} v^{p'}(t)\mathrm{d}t\Big)^{1/r},$$

$$B_{\mathcal{L}}(t) := t^{-\lambda r/q}[V_0(t)]^{r/q'} v^{p'}(t), \quad B_{\mathcal{L}} := B_{\mathcal{L},\mathbb{R}^+} = \Big(\int_{\mathbb{R}^+} B_{\mathcal{L}}(t)\mathrm{d}t\Big)^{1/r},$$

$$B_q(t) := t^{-\lambda/q} v(t), \quad \overline{B}_q(t) := t^{-\lambda/q}\overline{v}_0(t), \quad B_p := \Big(\int_{\mathbb{R}^+} y^{-\lambda p'} v^{p'}(y)\mathrm{d}y\Big)^{1/p'},$$

$$B_{q',\langle c_1,c_2\rangle} := \Big(\int_{c_1}^{c_2} [t^{-\lambda} - c_2^{-\lambda}]^{q/(1-q)} t^{-\lambda-1}\overline{v}_{c_1}(t)^{q/(1-q)}\mathrm{d}t\Big)^{(1-q)/q},$$

$$B_{q'} = B_{q',\mathbb{R}^+} = \Big(\int_{\mathbb{R}^+} B_{q'}(t)\mathrm{d}t\Big)^{(1-q)/q}, \quad \text{where } B_{q'}(t) := t^{-\lambda/(1-q)-1}\overline{v}_0(t)^{q/(1-q)},$$

$$D_{\langle c_1,c_2\rangle} := c_2^{-\lambda/q}[V_{c_1}(c_2)]^{1/p'}, C_q := \Big(\int_{\mathbb{R}^+} t^{-\lambda}\Big[\int_0^t v(y)\mathrm{d}y\Big]^{q-1} v(t)\mathrm{d}t\Big)^{1/q}.$$

To implement our method for the study of the compactness of the operator $\mathcal{L}$ we start from describing its boundedness properties.

Various conditions were found for the Laplace transformation (0.2) to be bounded in Lebesgue spaces (see e.g. [3], [5]). Convenient for our purposes L^p-L^q criterion for the operator $\mathcal{L}$ was obtained in Theorem 1 of [24] for $1 < p, q < \infty$ and $0 < q \leqslant p < \infty$ (see also Theorem 1 of [17]). Our first statement in the article is its modification for the Laplace operator of the form $f \to \mathcal{L}(f\chi_{\langle c_1,c_2\rangle})$, where $0 \leqslant c_1 < c_2 \leqslant \infty$ and $\langle\cdot,\cdot\rangle$ denotes any of intervals $(\cdot,\cdot)$, $[\cdot,\cdot]$, $[\cdot,\cdot)$ or $(\cdot,\cdot]$. All the rest cases of p, q are also included in the statement.

THEOREM 1.1. (i) *Let $1 < p \leqslant q < \infty$. The operator $\mathcal{L}$ is bounded from $L^p\langle c_1,c_2\rangle$ to L^q if and only if $A_{\mathcal{L},\langle c_1,c_2\rangle} + D_{\langle c_1,c_2\rangle} < \infty$. Moreover, $\|\mathcal{L}\|_{L^p\langle c_1,c_2\rangle\to L^q} \approx A_{\mathcal{L},\langle c_1,c_2\rangle} + D_{\langle c_1,c_2\rangle}$.*

(ii) *If $1 < q < p < \infty$ then $\mathcal{L}$ is $L^p\langle c_1,c_2\rangle$-$L^q$-bounded if and only if $B_{\mathcal{L},\langle c_1,c_2\rangle} + D_{\langle c_1,c_2\rangle} < \infty$, where $\|\mathcal{L}\|_{L^p\langle c_1,c_2\rangle\to L^q} \approx B_{\mathcal{L},\langle c_1,c_2\rangle} + D_{\langle c_1,c_2\rangle}$.*

(iii) *Let $0 < q < 1 < p < \infty$. The Laplace operator $\mathcal{L}$ is bounded from $L^p\langle c_1,c_2\rangle$ to L^q if $B_{\mathcal{L},\langle c_1,c_2\rangle} + D_{\langle c_1,c_2\rangle} < \infty$. If $\mathcal{L} : L^p\langle c_1,c_2\rangle \to L^q$ is bounded then $\|B_q\|_{p',\langle c_1,c_2\rangle} < \infty$. We also have*

$$\|B_q\|_{p',\langle c_1,c_2\rangle} \ll \|\mathcal{L}\|_{L^p\langle c_1,c_2\rangle\to L^q} \ll B_{\mathcal{L},\langle c_1,c_2\rangle} + D_{\langle c_1,c_2\rangle}.$$

(iv) *Let $0 < q < 1 = p$. If $\mathcal{L}$ is L^1-L^q-bounded then* $\operatorname*{ess\,sup}_{t\in\langle c_1,c_2\rangle} B_q(t) < \infty$. *The operator*

$\mathcal{L}$ is bounded from L^1 to L^q if $B_{q',\langle c_1,c_2\rangle} < \infty$ and $D_{\langle c_1,c_2\rangle} < \infty$. Besides, $\operatorname*{ess\,sup}_{t\in\langle c_1,c_2\rangle} B_q(t) \ll \|\mathcal{L}\|_{L^1\langle c_1,c_2\rangle\to L^q} \ll B_{q',\langle c_1,c_2\rangle} + D_{\langle c_1,c_2\rangle}$.

(v) *Let $p = \infty$. If $1 \leqslant q < \infty$ then $\mathcal{L}$ is L^∞-L^q-bounded if and only if $C_q < \infty$ and $\|\mathcal{L}\|_{L^\infty\to L^q} \approx C_q$. For $q < 1$ $\|B_q\|_1 \ll \|\mathcal{L}\|_{L^\infty\to L^q} \ll C_q$, that is $\mathcal{L}$ is bounded if $C_q < \infty$, and $\|B_q\|_1 < \infty$ if $\mathcal{L}$ is bounded.*

(vi) *Let $q = \infty$. If $1 < p \leqslant \infty$ then $\mathcal{L}$ is bounded from L^p to L^∞ if and only if $\|v\|_{p'} < \infty$, where $\|\mathcal{L}\|_{L^p\to L^\infty} = \|v\|_{p'}$. If $p = 1$ then $\|\mathcal{L}\|_{L^1\to L^\infty} = \|v\|_\infty$.*

REMARK 1.2. Notice that $D_{\langle c_1,c_2\rangle} = 0$ if $\langle c_1, c_2\rangle = \mathbb{R}^+$.

Now we are ready to state and prove compactness criteria for $\mathcal{L}$.

THEOREM 1.3. (i) *If $1 < p \leqslant q < \infty$ then the operator $\mathcal{L} : L^p \to L^q$ is compact if and only if $A_{\mathcal{L}} < \infty$ and*

$$\text{(i)} \lim_{t\to 0} A_{\mathcal{L}}(t) = 0, \quad \text{(ii)} \lim_{t\to\infty} A_{\mathcal{L}}(t) = 0. \tag{1.1}$$

(ii) *Let $1 \leqslant q < p < \infty$. If $q > 1$ then $\mathcal{L} : L^p \to L^q$ is compact if and only if $B_{\mathcal{L}} < \infty$. When $q = 1$ then $\mathcal{L}$ is compact if and only if $B_p < \infty$.*

(iii) *Let $0 < q < 1 < p < \infty$. The operator $\mathcal{L} : L^p \to L^q$ is compact if $B_{\mathcal{L}} < \infty$. If $\mathcal{L}$ is compact from L^p to L^q then $\|B_q\|_{p'} < \infty$.*

(iv) *Let $0 < q < 1 = p$. $\mathcal{L}$ is compact from L^1 to L^q if $B_{q'} < \infty$. If $\mathcal{L}$ is L^1-L^q-compact then* $\operatorname*{ess\,sup}_{t\in\mathbb{R}^+} B_q(t) < \infty$.

(v) *Let $1 = p \leqslant q < \infty$. The operator $\mathcal{L}$ is L^1-L^q-compact if and only if*

$$\operatorname*{ess\,sup}_{t\in\mathbb{R}^+} \overline{B}_q(t) < \infty \quad \textit{and} \quad \lim_{t\to 0} \overline{B}_q(t) = \lim_{t\to\infty} \overline{B}_q(t) = 0.$$

(vi) *If $1 \leqslant q < p = \infty$ then $\mathcal{L}$ is compact from L^∞ to L^q if and only if $C_q < \infty$.*

(vii) *Let $0 < q < 1$ and $p = \infty$. The operator $\mathcal{L}$ is L^∞-L^q-compact if $C_q < \infty$. If $\mathcal{L}$ is compact from L^∞ to L^q then $\|B_q\|_1 < \infty$.*

(viii) *If $1 < p \leqslant \infty$ then $\mathcal{L}$ is L^p-L^∞-compact if and only if $\|v\|_{p'} < \infty$.*

Proof. (i) *Sufficiency.* Suppose $A_{\mathcal{L}} < \infty$ and $\lim_{t\to 0} A_{\mathcal{L}}(t) = \lim_{t\to\infty} A_{\mathcal{L}}(t) = 0$. Put $0 < a < b < \infty$ and denote $\mathcal{L}_0 f := \mathcal{L}(f\chi_{(a,b)})$, $\mathcal{L}_1 f := \mathcal{L}(f\chi_{[0,a]})$, $\mathcal{L}_2 f := \mathcal{L}(f\chi_{[b,\infty)})$. Obviously,

$$\mathcal{L}f(x) = \sum_{i=0}^{2} \mathcal{L}_i f(x). \tag{1.2}$$

Since $A_{\mathcal{L}} < \infty$ then $\mathcal{L}$ is bounded from L^p to L^q by Theorem 1.1(i). This yields L^p-L^q-boundedness of the operator $\mathcal{L}_0 f$, which is regular.

According to Lemma 4 of [13] the operator $\mathcal{L}_0 : L^p \to L^q$ is compact if

$$M_{\mathcal{L}_0} := \| \| \chi_{(a,b)}(\cdot) k_{\mathcal{L}}(x,\cdot) v(\cdot) \|_{p'} \|_q < \infty.$$

Since $0 < a < b < \infty$ and $v \in L^{p'}_{\mathrm{loc}}(0,\infty)$ we have

$$M^q_{\mathcal{L}_0} \leqslant \frac{1}{a^\lambda q}[V_a(b)]^{q/p'} < \infty. \tag{1.3}$$

Therefore, $\mathcal{L}_0$ is compact from L^p to L^q for any $0 < a < b < \infty$.

Now consider the operators $\mathcal{L}_i$, $i = 1,2$. By Theorem 1.1(i) we have:

$$\|\mathcal{L}_1\|_{L^p\to L^q} \ll \sup_{0\leqslant t\leqslant a} t^{-\lambda/q}[V_0(t)]^{1/p'}, \tag{1.4}$$

$$\|\mathcal{L}_2\|_{L^p\to L^q} \ll \sup_{b\leqslant t<\infty} t^{-\lambda/q}[V_b(t)]^{1/p'}. \tag{1.5}$$

The conditions (1.1) yield

$$\lim_{a\to 0}\sup_{0\leqslant t\leqslant a} t^{-\lambda/q}[V_0(t)]^{1/p'} = 0, \quad \lim_{b\to\infty}\sup_{b\leqslant t<\infty} t^{-\lambda/q}[V_b(t)]^{1/p'} = 0.$$

Together with (1.4) and (1.5) this gives:

$$\lim_{a\to 0}\|\mathcal{L}_1\|_{L^p\to L^q} = 0, \quad \lim_{b\to\infty}\|\mathcal{L}_2\|_{L^p\to L^q} = 0. \tag{1.6}$$

Therefore, (1.2) implies

$$\|\mathcal{L}-\mathcal{L}_0\|_{L^p\to L^q} \leqslant \|\mathcal{L}_1\|_{L^p\to L^q} + \|\mathcal{L}_2\|_{L^p\to L^q}, \tag{1.7}$$

and now the operator $\mathcal{L} : L^p \to L^q$ is compact as a limit of compact operators, when $a \to 0$ and $b \to \infty$.

Necessity. Suppose now $\mathcal{L}$ is compact from L^p to L^q. Then $\mathcal{L}$ is L^p-L^q-bounded and $A_{\mathcal{L}} < \infty$ by Theorem 1.1(i).

To prove (1.1) we assume $\{z_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}^+$ is an arbitrary sequence. To establish the claim (i) in (1.1) suppose $\lim_{k\to\infty} z_k = 0$ and put

$$f_k(t) = \chi_{[0,z_k]}(t)[v(t)]^{p'-1}[V_0(z_k)]^{-1/p}.$$

Since $\|f_k\|_p = 1$ then

$$\Big|\int_{\mathbb{R}^+} f_k(y)g(y)\mathrm{d}y\Big| \leqslant \Big(\int_0^{z_k} |g(y)|^{p'}\mathrm{d}y\Big)^{1/p'} \to 0, \quad k\to\infty,$$

for any $g \in L^{p'}$. Therefore, the sequence $\{f_k\}_{k\in\mathbb{Z}}$ converges weakly to 0 in L^p. Compactness of $\mathcal{L} : L^p \to L^q$ yields strong convergence of $\{\mathcal{L}f_k\}_{k\in\mathbb{Z}}$ in L^q, that is $\lim_{k\to\infty}\|\mathcal{L}f_k\|_q = 0$. Besides, we have

$$\int_0^\infty\Big(\int_0^\infty \mathrm{e}^{-xy^\lambda} f_k(y)v(y)\mathrm{d}y\Big)^q\mathrm{d}x \geqslant \int_0^\infty \mathrm{e}^{-qxz_k^\lambda}\mathrm{d}x[V_0(z_k)]^{q/p'} = \frac{A^q_{\mathcal{L}}(z_k)}{q}.$$

Hence, $\lim_{k\to\infty} A_{\mathcal{L}}(z_k) = 0$, and now (1.1)(i) is proved.

For the proof of (1.1)(ii) we suppose $\lim_{k\to\infty} z_k = \infty$ and put

$$g_k(t) = \chi_{[0,z_k^{-\lambda}]}(t) z_k^{\lambda/q'}.$$

Since $\|g_k\|_{q'} = 1$ we have

$$\Big|\int_0^\infty f(x)g_k(x)\mathrm{d}x\Big| \leqslant \Big(\int_0^{z_k^{-\lambda}} |f(x)|^q \mathrm{d}x\Big)^{1/q} \to 0, \quad k\to\infty,$$

for any $f \in L^q$, which means weak convergence of $\{g_k\}_{k\in\mathbb{Z}}$ in $L^{q'}$. Compactness of $\mathcal{L} : L^p \to L^q$, $1 < p, q < \infty$, implies $L^{q'}$-$L^{p'}$-compactness of the dual operator $\mathcal{L}^* g(y) := v(y)\int_0^\infty \mathrm{e}^{-xy^\lambda} g(x)\mathrm{d}x$. Therefore, $\{\mathcal{L}^* g_k\}_{k\in\mathbb{Z}}$ strongly converges in $L^{p'}$:

$$\lim_{k\to\infty} \|\mathcal{L}^* g_k\|_{p'} = 0. \tag{1.8}$$

We obtain

$$\int_0^\infty v^{p'}(y)\Big(\int_0^\infty \mathrm{e}^{-xy^\lambda} g_k(x)\mathrm{d}x\Big)^{p'}\mathrm{d}y$$

$$\geqslant V_0(z_k)\Big(\int_0^{z_k^{-\lambda}} \mathrm{e}^{-xz_k^\lambda}\mathrm{d}x\Big)^{p'} z_k^{\lambda p'/q'} \geqslant \mathrm{e}^{-p'} V_0(z_k)\Big(\int_0^{z_k^{-\lambda}} \mathrm{d}x\Big)^{p'} z_k^{\lambda p'/q'}$$

$$= \mathrm{e}^{-p'} z_k^{-\lambda p'/q} V_0(z_k) = \mathrm{e}^{-p'} A_{\mathcal{L}}^{p'}(z_k).$$

Together with (1.8) this implies $\lim_{k\to\infty} A_{\mathcal{L}}(z_k) = 0$, and now the condition (1.1)(ii) is fulfilled by the arbitrariness of $\{z_k\}_{k\in\mathbb{Z}}$.

Necessity in (ii), (iii) and (iv) follows by Theorem 1.1 from the hypothesis of compactness and, therefore, boundedness of $\mathcal{L}$.

(ii) *Sufficiency* of the condition $B_{\mathcal{L}} < \infty$ (if $1 < q < p < \infty$) and $B_p < \infty$ (if $q = 1$) for the compactness of $\mathcal{L}$ is provided by Lemma 4 of [13] and Theorem 1.1(ii). Namely, if $1 < q < p < \infty$ then Lemma 4 of [13] yields L^p-L^q-compactness of $\mathcal{L}_0$ (see (1.3)), while norms $\mathcal{L}_1$ and $\mathcal{L}_2$ are estimated by Theorem 1.1(ii) as follows:

$$\begin{aligned} \|\mathcal{L}_1\|_{L^p\to L^q} &\ll \Big(\int_{\mathbb{R}^+} \chi_{[0,a]}(t) B_{\mathcal{L}}(t)\mathrm{d}t\Big)^{1/r}, \\ \|\mathcal{L}_2\|_{L^p\to L^q} &\ll \Big(\int_{\mathbb{R}^+} \chi_{[b,\infty)}(t) B_{\mathcal{L}}(t)\mathrm{d}t\Big)^{1/r}. \end{aligned} \tag{1.9}$$

Thus, $B_{\mathcal{L}} < \infty$ and the estimate (1.7) implies compactness of $\mathcal{L}$ as $a \to 0$, $b \to \infty$. If $q = 1$ then $\mathcal{L}$ is compact by Lemma 4 of [13].

Sufficiency in (iii) and (iv) can be established as follows. Let $B_{\mathcal{L}} < \infty$ if $0 < q < 1 < p < \infty$ and $B_{q'} < \infty$ if $0 < q < 1 = p$. By Theorem 1.1(iii) we obtain the estimate (1.9) for the case $0 < q < 1 < p < \infty$. By the part (iv) of the same theorem we have for $0 < q < 1 = p$:

$$\|\mathcal{L}_2\|_{L^p \to L^q} \ll \Big(\int_{\mathbb{R}^+} \chi_{[b,\infty)}(t) B_{q'}(t) \mathrm{d}t \Big)^{1/r}.$$

Thus, $B_{\mathcal{L}} < \infty$ (or $B_{q'} < \infty$) yields $\|\mathcal{L}_2\|_{L^p \to L^q} \to 0$ as $b \to \infty$.

Now consider the operator $\mathcal{L}_b f := \mathcal{L}_0 f + \mathcal{L}_1 f = L(f\chi_{[0,b]})$. The hypothesis $B_{\mathcal{L}} < \infty$ (or $B_{q'} < \infty$) suffices for the boundedness of $\mathcal{L}$ (see Theorem 1.1). Therefore, $\mathcal{L}_b$ is bounded as well. To prove the compactness of $\mathcal{L}_b$ we shall use an extension of Theorem 5.8 of [11] for the case when an operator K is acting to L^q on the whole $\mathbb{R}^+$. Similar to Theorem 5.8 of [11] we consider first a set $\mathcal{M}_h := \{f \in L^p(a,b) : |f| \leqslant h\}$, where h is an arbitrary positive number. Under this condition and in view of $\mathrm{mes}[0,b] < \infty$ the operator $\mathcal{L}_b$ is bounded from $L^\infty[0,b]$ to L^q. Compactness of $\mathcal{L}_b : \mathcal{M}_h \to L^q$ can be proved similar to Theorem 5.2 of [11]. It remains to note that the rest transformation $\mathcal{L}_b : \{L^p[0,b] \setminus \mathcal{M}_h\} \to L^q$ has a norm tending to 0 as $h \to +\infty$ (see Theorem 5.8 of [11] for details). Thus, $\mathcal{L}_b : L^p[0,b] \to L^q$, $0 < q < 1 \leqslant p < \infty$, is compact as a limit of compact operators, when $h \to +\infty$.

Summing up we can claim that (0.2) is compact from L^p to L^q, $0 < q < 1 \leqslant p < \infty$, on the strength of $\|\mathcal{L}_2\|_{L^p \to L^q} \to 0$, when $b \to \infty$, compactness of $\mathcal{L}_b$ and in view of $\|\mathcal{L} - \mathcal{L}_b\|_{L^p \to L^q} = \|\mathcal{L}_2\|_{L^p \to L^q}$.

(v) *Sufficiency.* Suppose $\operatorname*{ess\,sup}_{t \in \mathbb{R}^+} \overline{B}_q(t) < \infty$ and

$$\text{(i) } \lim_{t \to 0} \overline{B}_q(t) = 0, \quad \text{(ii) } \lim_{t \to \infty} \overline{B}_q(t) = 0. \tag{1.10}$$

For $0 < a < b < \infty$ we put $\mathcal{L}_a f(x) := \mathrm{e}^{-xa^\lambda} \int_a^b f(y) v(y) \mathrm{d}y$, $x \in \mathbb{R}^+$, and note that $\mathcal{L}_a$ is the operator of rank 1 with $\|\mathcal{L}_a\|_{L^1 \to L^q} = q^{-1/q} a^{-\lambda/q} \overline{v}_a(b) < \infty$. Besides, $\mathcal{L}_a$ is a majorant for the operator $\mathcal{L}_0$, which is L^1-L^q-bounded with the norm estimated as follows:

$$\|\mathcal{L}_0\|_{L^1 \to L^q} = \|\mathcal{L}\|_{L^1(a,b) \to L^q} = q^{-1/q} \operatorname*{ess\,sup}_{a<t<b} \overline{B}_q(t) =: q^{-1/q} M < \infty.$$

Suppose $\{f_n\}_{n \in \mathbb{Z}}$ is an arbitrary bounded sequence in $L^1(a,b)$ and assume $\{f_{n_k}\}$ is its Cauchy subsequence, that is for any $\varepsilon_0 > 0$ there exists $N(\varepsilon_0)$ such that $\|f_{n_k} - f_{m_k}\|_{1,(a,b)} < \varepsilon_0$, when $n_k, m_k > N(\varepsilon_0)$. Put

$$E_{n_k,m_k}(\varepsilon) := \{x \in \mathbb{R}^+ : |\mathcal{L}_0 f_{n_k}(x) - \mathcal{L}_0 f_{m_k}(x)| > \varepsilon\}.$$

We have for any $\varepsilon > 0$:

$$\int_{E_{n_k,m_k}(\varepsilon)} \mathrm{d}x \leqslant \varepsilon^{-1} \int_{\mathbb{R}^+} \Big| \int_a^b \mathrm{e}^{-xy^\lambda} [f_{n_k}(y) - f_{m_k}(y)] v(y) \mathrm{d}y \Big| \mathrm{d}x$$

$$\leqslant \varepsilon^{-1}\int_a^b y^{-\lambda}|f_{n_k}(y)-f_{m_k}(y)|v(y)\mathrm{d}y \leqslant \varepsilon^{-1}Ma^{-\lambda/q'}\|f_{n_k}-f_{m_k}\|_{1,(a,b)}.$$

If $\varepsilon_0=\varepsilon\delta a^{\lambda/q'}M^{-1}$ then $\mu_{L^q}(E_{n_k,m_k}(\varepsilon))<\delta$ as $n_k,m_k>N(\varepsilon,\delta)$ for any $\varepsilon>0$, $\delta>0$. Therefore, $\mathcal{L}_0$ is compact in measure. Thus, $\mathcal{L}_0$ is L^1-L^q-compact as a transformation majorated by the compact operator $\mathcal{L}_a$ (see Chapter 2, Section 5.6 of [11] for details). Further, by Chapter XI, Section 1.5, Theorem 4 of [10]

$$\|\mathcal{L}_1\|_{L^1\to L^q}=q^{-1/q}\operatorname*{ess\,sup}_{t\in[0,a]}B_q(t)\leqslant q^{-1/q}\sup_{t\in[0,a]}\overline{B}_q(t),$$
$$\|\mathcal{L}_2\|_{L^1\to L^q}\leqslant q^{-1/q}\operatorname*{ess\,sup}_{t\in[b,\infty)}t^{-\lambda/q}\overline{v}_b(t)\leqslant q^{-1/q}\sup_{t\in[b,\infty)}\overline{B}_q(t).$$

Since (1.10) are fulfilled we can state that $\mathcal{L}_1$ and $\mathcal{L}_2$ are operators with small norms, when $a\to 0$, $b\to\infty$. Together with compactness of $\mathcal{L}_0$ this implies the compactness of $\mathcal{L}$ from L^1 to L^q for all $1\leqslant q<\infty$.

Necessity. Suppose $\mathcal{L}$ is L^1-L^q-compact. Then the claim $\operatorname*{ess\,sup}_{t\in\mathbb{R}^+}\overline{B}_q(t)<\infty$ follows from Theorem 3.2 of [23] and Chapter XI, Section 1.5, Theorem 4 of [10] (see also Remark 1.2). As for necessity of (1.10)(i), note that

$$\mathcal{L}f=\mathcal{L}(f\chi_{[0,x^{-1/\lambda}]})+\mathcal{L}(f\chi_{[x^{-1/\lambda},\infty)}):=\mathcal{L}_xf+\mathcal{L}^xf,$$

where $\mathcal{L}_x$ and $\mathcal{L}^x$ are compact. Besides, (1.10)(i) is equivalent to

$$\lim_{k\to-\infty}2^{-\lambda k/q}\overline{v}_0(2^k)=0. \tag{1.11}$$

Now suppose the contrary. Then, similar to p. 84 of [7], given $\gamma\in(0,1)$ there is a sequence $k_j\to-\infty$, some $\varepsilon>0$ and functions $f_{k_j}\geqslant 0$, $\|f_{k_j}\|_{L^1}\leqslant 1$, such that $\int_0^{2^{k_j}} f_m(y)v(y)\mathrm{d}y\geqslant\gamma\,\overline{v}_0(2^{k_j})$ and $2^{-k_j\lambda/q}\overline{v}_0(2^{k_j})\geqslant\varepsilon$. By continuity of the integral, there are $\beta_{k_j}\in(0,2^{k_j})$ such that $\int_{\beta_{k_j}}^{2^{k_j}} f_{k_j}(y)v(y)\mathrm{d}y\geqslant\gamma^2\,\overline{v}_0(2^{k_j})$. Set $F_{k_j}=f_{k_j}\chi_{(\beta_{k_j},2^{k_j})}$. Then we have for k_i and k_j such that $2^{k_i+1}<\beta_{k_j}$:

$$\begin{aligned}\|\mathcal{L}_xF_{k_i}-\mathcal{L}_xF_{k_j}\|_q^q&=\int_{\mathbb{R}^+}\Big|\int_0^{x^{-1/\lambda}}\mathrm{e}^{-xy^\lambda}[F_{k_i}(y)-F_{k_j}(y)]v(y)\mathrm{d}y\Big|^q\mathrm{d}x\\&=\lambda\int_{\mathbb{R}^+}s^{-\lambda-1}\Big|\int_0^s\mathrm{e}^{-(y/s)^\lambda}[F_{k_i}(y)-F_{k_j}(y)]v(y)\mathrm{d}y\Big|^q\mathrm{d}s\\&=:\|\widetilde{\mathcal{L}}_xF_{k_i}-\widetilde{\mathcal{L}}_xF_{k_j}\|_q^q\geqslant\|\chi_{(2^{k_i},2^{k_i+1})}(\widetilde{\mathcal{L}}_xF_{k_i}-\widetilde{\mathcal{L}}_xF_{k_j})\|_q^q\end{aligned}$$

$$= \|\chi_{(2^{k_i},2^{k_i+1})}\widetilde{\mathcal{L}}_x F_{k_i}\|_q^q = \lambda \int\limits_{2^{k_i}}^{2^{k_i+1}} s^{-\lambda-1}\Big(\int\limits_{\beta_{k_i}}^{2^{k_i}} \mathrm{e}^{-(y/s)^\lambda} f_{k_i}(y)v(y)\mathrm{d}y\Big)^q \mathrm{d}s$$

$$\geqslant \mathrm{e}^{-1}\lambda \int\limits_{2^{k_i}}^{2^{k_i+1}} s^{-\lambda-1}\mathrm{d}s\Big(\int\limits_{\beta_{k_i}}^{2^{k_i}} f_{k_i}(y)v(y)\mathrm{d}y\Big)^q$$

$$\geqslant \gamma^{2q}\frac{2^\lambda-1}{2^\lambda \mathrm{e}}2^{-\lambda k_i}[\overline{v}_0(2^{k_i})]^q \geqslant \gamma^{2q}\frac{2^\lambda-1}{2^\lambda \mathrm{e}}\varepsilon^q > 0,$$

and, therefore, $\mathcal{L}_x$ is not compact.

Necessity of (1.10)(ii) can be established by the similar way obtaining a contradiction with the compactness of $\mathcal{L}^x$. Another way to prove (1.10)(ii) for $q > 1$ is analogous to the proof of necessity (1.1)(ii) in the part (i) of this theorem.

Proof of (vi)–(viii) rests on Theorem 5.2 of [11] and Theorem 2.21 of [1]. ∎

REMARK 1.4. $\mathcal{L}$ cannot be compact from L^1 to L^∞ for any v.

2. COMPACTNESS OF THE STIELTJES TRANSFORM

Criteria for S to be bounded in Lebesgue spaces were found in [2], [9], [21]. But their particular form is not suitable for our purposes. We will use other boundedness conditions, which directly follow from properties of Hardy operator

$$Hf(x) := x^{-\lambda}w(x)\int\limits_0^x f(y)v(y)\mathrm{d}y$$

and the relation

$$\frac{1}{2}[Hf(x) + H^*f(x)] \leqslant Sf(x) \leqslant Hf(x) + H^*f(x), \quad f \geqslant 0, \tag{2.1}$$

with dual to H transformation $H^*f(x) := w(x)\int_x^\infty f(y)y^{-\lambda}v(y)\mathrm{d}y$ (see Theorem 2.1 and [6], [14], [15], [22], [23] for details). Add some notations:

$$\mathcal{V}_t(\infty) := \int\limits_t^\infty \frac{v^{p'}(y)\mathrm{d}y}{y^{\lambda p'}}, \quad W_{c_1}(t) := \int\limits_{c_1}^t w^q, \quad \mathcal{W}_t(c_2) := \int\limits_t^{c_2}\frac{w^q(x)\mathrm{d}x}{x^{\lambda q}}.$$

THEOREM 2.1. (i) *If $1 < p \leqslant q < \infty$ then the operator S is bounded from L^p to L^q if and only if $A_H + A_{H^*} < \infty$, where $\|S\|_{L^p\to L^q} \approx A_H + A_{H^*}$ and*

$$A_H := \sup_{t\in\mathbb{R}^+} A_H(t) := \sup_{t\in\mathbb{R}^+}[V_0(t)]^{1/p'}[\mathcal{W}_t(\infty)]^{1/q}, \tag{2.2}$$

$$A_{H^*} := \sup_{t\in\mathbb{R}^+} A_{H^*}(t) := \sup_{t\in\mathbb{R}^+}[\mathcal{V}_t(\infty)]^{1/p'}[W_0(t)]^{1/q}. \tag{2.3}$$

(ii) *If* $0 < q < 1 < p < \infty$ *or* $1 < q < p < \infty$ *then* $S : L^p \to L^q$ *is bounded if and only if* $B_H + B_{H^*} < \infty$, *where* $\|S\|_{L^p \to L^q} \approx B_H + B_{H^*}$ *and*

$$B_H := \Big(\int_{\mathbb{R}^+} [V_0(t)]^{r/p'} [\mathcal{W}_t(\infty)]^{r/p} t^{-\lambda q} w^q(t) \mathrm{d}t \Big)^{1/r},$$

$$B_{H^*} := \Big(\int_{\mathbb{R}^+} [\mathcal{V}_t(\infty)]^{r/p'} [W_0(t)]^{r/p} w^q(t) \mathrm{d}t \Big)^{1/r}.$$

(iii) *Let* $0 < q \leqslant 1 = p$. *If* $0 < q < 1$ *then* S *is* L^1-L^q*-bounded if and only if* $B_{1,H} + B_{1,H^*} < \infty$, *where* $\|S\|_{L^1 \to L^q} \approx B_{1,H} + B_{1,H^*}$ *and*

$$B_{1,H} := \Big(\int_{\mathbb{R}^+} \overline{v}_0(t)^{q/(1-q)} [\mathcal{W}_t(\infty)]^{q/(1-q)} t^{-\lambda q} w^q(t) \mathrm{d}t \Big)^{(1-q)/q},$$

$$B_{1,H^*} := \Big(\int_{\mathbb{R}^+} [t^{-\lambda} \overline{v}_t(\infty)]^{q/(1-q)} [W_0(t)]^{q/(1-q)} w^q(t) \mathrm{d}t \Big)^{(1-q)/q}.$$

If $q = 1$ *then*

$$\|S\|_{L^1 \to L^1} \approx \sup_{t \in \mathbb{R}^+} \overline{v}_0(t) \int_t^\infty x^{-\lambda} w(x) \mathrm{d}x + \sup_{t \in \mathbb{R}^+} \overline{v}_t(\infty) t^{-\lambda} \int_0^t w(x) \mathrm{d}x.$$

Now we put

$$\Lambda := \Big(\int_{\mathbb{R}^+} \Big(\int_{\mathbb{R}^+} \frac{w(x)\mathrm{d}x}{x^\lambda + y^\lambda} \Big)^{p'} v^{p'}(y) \mathrm{d}y \Big)^{1/p'},$$

$$S_H := \sup_{t \in \mathbb{R}^+} \overline{v}_0(t) [\mathcal{W}_t(\infty)]^{1/q}, \quad S_{H^*} := \sup_{t \in \mathbb{R}^+} \overline{v}_t(\infty) t^{-\lambda} [W_0(t)]^{1/q},$$

$$S_{H,a}(t) := \overline{v}_0(t) [\mathcal{W}_t(a)]^{1/q}, \quad S_{H^*,a}(t) := \overline{v}_t(a) t^{-\lambda} [W_0(t)]^{1/q},$$

$$S_{H,b}(t) := \overline{v}_b(t) [\mathcal{W}_t(\infty)]^{1/q}, \quad S_{H^*,b}(t) := \overline{v}_t(\infty) t^{-\lambda} [W_b(t)]^{1/q}.$$

The following theorem is devoted to the compactness criteria for the Stieltjes transformation $S : L^p \to L^q$.

THEOREM 2.2. (i) *If* $1 < p \leqslant q < \infty$ *then* $S : L^p \to L^q$ *is compact if and only if* $A_H + A_{H^*} < \infty$ *and*

$$\text{(2.4)} \qquad \text{(i)} \lim_{t \to 0} [A_H(t) + A_{H^*}(t)] = 0, \quad \text{(ii)} \lim_{t \to \infty} [A_H(t) + A_{H^*}(t)] = 0.$$

(ii) *Let* $0 < q < p < \infty$ *and* $p > 1$. *If* $q \neq 1$ *then* S *is compact if and only if* $B_H + B_{H^*} < \infty$. *If* $q = 1$ *then* S *is* L^p-L^1*-compact if and only if* $\Lambda < \infty$.

(iii) *If* $0 < q < 1 = p$ *then* S *is* L^p-L^q*-compact if and only if* $B_{1,H} + B_{1,H^*} < \infty$.

(iv) *If* $p = 1 \leqslant q < \infty$ *then the operator* $S : L^p \to L^q$ *is compact if and only if* $S_H + S_{H^*} < \infty$ *and* $\lim_{a \to 0} \sup_{0 < t < a} [S_{H,a}(t) + S_{H^*,a}(t)] = \lim_{b \to \infty} \sup_{b < t < \infty} [S_{H,b}(t) + S_{H^*,b}(t)] = 0.$

Proof. (i) Let $1 < p \leqslant q < \infty$ and suppose $A_H + A_{H^*} < \infty$ (see (2.2) and (2.3)). Besides, assume that the conditions (2.4) are fulfilled. It is known (see [7], [20]) that these conditions guaranty L^p-L^q-compactness, $1 < p \leqslant q < \infty$, of the operator $H + H^*$, which is majorating the transformation S (see the relation (2.1)). From here the compactness of $S : L^p \to L^q$ ensues by Theorem 5.10 of [11].

The condition $A_H + A_{H^*} < \infty$ and the equalities (2.4) are also necessary for L^p-L^q-compactness of S, when $1 < p \leqslant q < \infty$, by standard arguments for the Hardy operators H and H^*.

(ii), (iii) Let $0 < q < p < \infty$ and $p \geqslant 1$. If $q \neq 1$ we suppose $B_H + B_{H^*} < \infty$ for $p > 1$ and $B_{1,H} + B_{1,H^*} < \infty$ for the case $p = 1$. Compactness of S in the case $p > 1$ is guaranteed by $B_H + B_{H^*} < \infty$ (see [4]). If $p = 1$ and $B_{1,H} + B_{1,H^*} < \infty$ the compactness of S can be stated similarly to sufficiency of the conditions (iv) in Theorem 1.3.

If $q = 1$ then S is compact by Lemma 4 of [13] provided $\Lambda < \infty$.

Necessity of $B_H + B_{H^*} < \infty$ in (ii) and $B_{1,H} + B_{1,H^*} < \infty$ in (iii) ensues from the compactness and, therefore, boundedness of S.

(iv) It remains to consider $p = 1 \leqslant q < \infty$. Suppose $S_H + S_{H^*} < \infty$,

$$\text{(2.5)}\quad \text{(i)} \lim_{a\to 0} \sup_{0<t<a} [S_{H,a}(t) + S_{H^*,a}(t)] = 0, \quad \text{(ii)} \lim_{b\to\infty} \sup_{b<t<\infty} [S_{H,b}(t) + S_{H^*,b}(t)] = 0,$$

and prove sufficiency of these assumptions for the L^1-L^q-compactness of S. In view of (2.5) given $\varepsilon > 0$ there exist $0 < r < R < \infty$ such that

$$\sup_{0<t<r} S_{H,r} < \frac{\varepsilon}{7}, \quad \sup_{0<t<r} S_{H^*,r} < \frac{\varepsilon}{7}, \tag{2.6}$$

$$\sup_{R<t<\infty} S_{H,R} < \frac{\varepsilon}{7}, \quad \sup_{R<t<\infty} S_{H^*,R} < \frac{\varepsilon}{7}. \tag{2.7}$$

Now we divide S into a sum $Sf = S_{r,R}f + \sum_{i=1}^{2} [S_{r,i}f + S_{R,i}f]$ of compact operators $S_{r,R}f := \chi_{(r,R)}S(f\chi_{(r,R)})$, $S_{r,1}f := \chi_{[0,R)}S(f\chi_{([0,r]})$, $S_{R,1}f := \chi_{[R,\infty)}S(f\chi_{[0,R)})$, $S_{r,2}f := \chi_{[0,r]}S(f\chi_{(r,\infty)})$ and $S_{R,2}f := \chi_{(r,\infty)}S(f\chi_{[R,\infty)})$. To confirm the compactness of these operators we shall use a combination of Theorem 2.21 of [1] and Corollary 5.1 of [8]. That is we need to show that for a given $\varepsilon > 0$ there exist $\delta > 0$ and points $0 < s < t < \infty$ such that for almost all $y \in \mathbb{R}^+$ and for every $h > 0$ with $h < \delta$

$$\text{(2.8)}\quad \text{(i) } J_s^q(y) := \int_0^s |\mathbf{k}_S(x,y)|^q \mathrm{d}x < \varepsilon^q, \quad \text{(ii) } J_t^q(y) := \int_t^\infty |\mathbf{k}_S(x,y)|^q \mathrm{d}x < \varepsilon^q,$$

where $\mathbf{k}_S(x,y) := w(x)k_S(x,y)v(y)$, and

$$J_h^q(y) := \int_{\mathbb{R}^+} |\mathbf{k}_S(x+h,y) - \mathbf{k}_S(x,y)|^q \mathrm{d}x < \varepsilon^q. \tag{2.9}$$

We start from $\mathcal{S} := S_{r,1} + S_{r,R} + S_{R,2}$. Suppose $h < \delta(\varepsilon)$ and write

$$J_{h,\mathcal{S}}(y) = v(y)\Big(\int\limits_{\mathbb{R}^+} w^q(x)\Big[\frac{1}{x^\lambda + y^\lambda} - \frac{1}{(x+h)^\lambda + y^\lambda}\Big]^q \mathrm{d}x\Big)^{1/q}.$$

For simplicity consider the case $\lambda = 1$ and denote

$$I^q_{(c_1,c_2)}(y,h) := \int\limits_0^r \frac{w^q(x)\mathrm{d}x}{(x+y)^q(x+y+h)^q}.$$

We have

$$\begin{aligned} J_{h,\mathcal{S}}(y) = {} & h\chi_{[0,r)}(y)v(y)I_{(0,r)}(y,h) + h\chi_{[0,r)}(y)v(y)I_{(r,R)}(y,h) \\ & + h\chi_{(r,R)}(y)v(y)I_{(r,R)}(y,h) + h\chi_{[R,\infty)}(y)v(y)I_{(r,R)}(y,h) \\ & + h\chi_{[R,\infty)}(y)v(y)I_{(R,\infty)}(y,h) =: \sum_{i=1}^5 J_{h,i}(y). \end{aligned}$$

The conditions (2.6) and (2.7) imply $J_{h,1}(y) \leqslant 2\varepsilon/7$, and $J_{h,5}(y) \leqslant 2\varepsilon/7$. To estimate $J_{h,2}(y)$ note that

$$J_{h,2}(y) \leqslant hr^{-1}\overline{v}_0(r)[\mathcal{W}_r(\infty)]^{1/q} \leqslant hr^{-1}S_H.$$

From here, with $\delta = \varepsilon r/7S_H$ we obtain $J_{h,2} \leqslant \varepsilon/7$. Analogously, $J_{h,4} \leqslant \varepsilon/7$ if $\delta = \varepsilon r/7S_{H^*}$. For $J_{h,3}$ note that $\overline{v}_r(R)[W_r(R)]^{1/q} < M < \infty$ provided $w \in L^q_{\text{loc}}(0,\infty)$ and $v \in L^\infty_{\text{loc}}(0,\infty)$. Therefore, $J_{h,3}(y) \leqslant hMr^{-2}$ and $J_{h,3}(y) \leqslant \varepsilon/7$ if $\delta = \varepsilon r^2/7M$.

Summing up, we obtain $J_{h,\mathcal{S}}(y) < \varepsilon$ for almost all $y \in \mathbb{R}^+$, that is the condition (2.9) is satisfied. Fulfillment of the claims (2.8) ensues from (2.6) and (2.7) with $s = r$ and $t = R$. Thus, the sum $S_{r,1} + S_{r,R} + S_{R,2}$ is compact.

Compactness of the operator $S_{r,2}$ can be demonstrated as follows. The condition (2.8)(ii) is automatically fulfilled with $t = r$. To demonstrate (2.8)(i) note that $\|S_{r,2}\|_{L^1\to L^q} \approx \overline{v}_r(\infty)r^{-\lambda}[W_0(r)]^{1/q} \leqslant S_{H^*} < \infty$. Hence, given $\varepsilon > 0$ there exists $0 < s \leqslant r$ such that $J_{s,S_{r,2}}(y) < \varepsilon$. The condition (2.9) may be shown analogously with $\delta = \varepsilon r^\lambda/S_{H^*}$. Similar arguments work for the operator $S_{R,1}$.

Necessity of the conditions $[S_H + S_{H^*}] < \infty$ and (2.5) follow from Lemma 1, Theorem 1 of [7] and the relation (2.1). ∎

REMARK 2.3. In some cases the compactness of S can be established through the Laplace operator (0.2). Indeed, by the factorization

$$S = \mathcal{L}^*_w\mathcal{L}_v \tag{2.10}$$

with $\mathcal{L}_v \equiv \mathcal{L}$ and $\mathcal{L}^*_w f(x) := w(y)\int\limits_{\mathbb{R}^+} \mathrm{e}^{-xy^\lambda} f(x)\mathrm{d}x$ we are able to state compactness of $S : L^p \to L^q$ if the conditions of Theorem 1.3 are fulfilled for either $\mathcal{L}_v : L^p \to L^\theta$ or $\mathcal{L}_w : L^{q'} \to L^{\theta'}$ of the form $\mathcal{L}_w f(x) := \int\limits_{\mathbb{R}^+} \mathrm{e}^{-xy^\lambda} f(y)w(y)\mathrm{d}y$. Here the parameter θ' is such that $\theta' = \theta/(\theta-1)$ for any $\theta > 1$. In particular, if $w \equiv v$ and $p = q' \leqslant q = p'$ then $S : L^p \to L^{p'}$ is compact if and only if $\sup_{t\in\mathbb{R}^+} A(t) := \sup_{t\in\mathbb{R}^+} t^{-\lambda/2}[V_0(t)]^{1/p'} < \infty$ and $\lim_{t\to 0} A(t) = \lim_{t\to\infty} A(t) = 0$.

REMARK 2.4. Since S is two-weighted then its compactness criteria for $p = \infty$ and/or $q = \infty$ can be derived from Theorem 2.2, excluding the case $p = 1, q = \infty$, when $S : L^1 \to L^\infty$ is never compact.

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