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ON NORMALIZERS OF C^*-SUBALGEBRAS IN THE CUNTZ ALGEBRA $\mathcal{O}_n$

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ABSTRACT. In this paper we investigate the normalizer $\mathcal{N}_{\mathcal{O}_n}(A)$ of a C^*-subalgebra $A \subset \mathcal{F}_n$ where $\mathcal{F}_n$ is the canonical UHF-subalgebra of type n^∞ in the Cuntz algebra $\mathcal{O}_n$. Under the assumption that the relative commutant $A' \cap \mathcal{F}_n$ is finite-dimensional, we show several facts for normalizers of A. In particular it is shown that the automorphism group $\{\mathrm{Ad}u|_A \ : u \in \mathcal{N}_{\mathcal{F}_n}(A)\}$ has a finite index in $\{\mathrm{Ad}U|_A \ : U \in \mathcal{N}_{\mathcal{O}_n}(A)\}$.

KEYWORDS: *C^*-algebra, Cuntz algebra, normalizers.*

MSC (2010): 46L05.

INTRODUCTION

The purpose of this paper is to investigate the normalizer of C^*-subalgebras in the Cuntz algebra $\mathcal{O}_n$ [1]. Let $\mathcal{F}_n$ be the canonical UHF-subalgebra of $\mathcal{O}_n$. In the paper [4], it is shown that the normalizer group $\mathcal{N}_{\mathcal{O}_n}(\mathcal{F}_n)$ is a subset of $\mathcal{F}_n$. (In [4], more general results are shown.) More generally, if A is a C^*-subalgebra of $\mathcal{F}_n$ with $A' \cap \mathcal{O}_n = \mathbb{C}$, then the normilizer $\mathcal{N}_{\mathcal{O}_n}(A)$ is a subset of $\mathcal{F}_n$. In this paper we investigate the normalizer $\mathcal{N}_{\mathcal{O}_n}(A)$ where A is a C^*-subalgebra of $\mathcal{F}_n$ with a finite-dimensional relative commutant in $\mathcal{F}_n$. In this setting the normalizer $\mathcal{N}_{\mathcal{O}_n}(A)$ is not a subset of $\mathcal{F}_n$ in general. However we can show that the inner automorphism group induced by the elements in $\mathcal{N}_{\mathcal{F}_n}(A)$ has a finite index in the inner automorphism group induced by the elements in $\mathcal{N}_{\mathcal{O}_n}(A)$. In order to show this fact, we show that the relative commutant $A' \cap \mathcal{O}_n$ is also finite-dimensional. As a corollary of our investigation, it is shown that the irreducibility $A' \cap \mathcal{F}_n = \mathbb{C}$ implies that $A' \cap \mathcal{O}_n = \mathbb{C}$. Hence in this case the normalizer group $\mathcal{N}_{\mathcal{O}_n}(A)$ is a subset of $\mathcal{F}_n$.

We would like to explain the motivation of this paper. There is a one-to-one correspondence between all unitaries $\mathcal{U}(\mathcal{O}_n)$ and all endomorphisms $\mathrm{End}(\mathcal{O}_n)$

(all unital $*$-homomorphisms from $\mathcal{O}_n$ to $\mathcal{O}_n$) such that

$$\mathcal{U}(\mathcal{O}_n) \ni u \mapsto \lambda_u \in \mathrm{End}(\mathcal{O}_n)$$

where λ_u is defined by $\lambda_u(S_i) = uS_i$. The endomorphism λ_u is called *localized* if the corresponding unitary is a matrix in the UHF-algebra $\mathcal{F}_n$ [2], [5]. In the paper [9] Szymański showed that the localized endomorphism λ_u is an inner automorphism if and only if u can be written in some special form. As a consequence, if the localized endomorphism λ_u is an inner automorphism, then there exists a unitary $U \in \mathcal{F}_n$ such that $\lambda_u = \mathrm{Ad}U$. Keeping this in mind, we would like to consider the following problem. Let λ_u and λ_v be two localized endomorphisms. If they satisfy $\mathrm{Ad}U \circ \lambda_u = \lambda_v$, what can we say about U? Can we determine such a unitary U? Unfortunately in this paper we cannot say anything about this problem. But we remark that a localized endomorphism has finite index [5], [8]. Therefore the C^*-algebras $\lambda_u(\mathcal{F}_n)' \cap \mathcal{F}_n$ and $\lambda_u(\mathcal{O}_n)' \cap \mathcal{O}_n$ are finite-dimensional. So we expect that our investigation would be helpful on this problem in the future.

1. MAIN RESULTS

The Cuntz algebra $\mathcal{O}_n$ is the C^*-algebra generated by isometries $S_1, \ldots, S_n$ satisfying $\sum\limits_{i=1}^{n} S_i {S_i}^* = 1$. The gauge action γ_z ($z \in \mathbb{T}$) on $\mathcal{O}_n$ is defined by $\gamma_z(S_i) = zS_i$. Let $\mathcal{F}_n$ be the fixed point algebra of the gauge action. This algebra is isomorphic to the UHF-algebra of type n^∞. So $\mathcal{F}_n$ has the unique tracial state τ. We have a conditional expectation $E : \mathcal{O}_n \to \mathcal{F}_n$ defined by

$$E(x) = \int_{\mathbb{T}} \gamma_z(x)\mathrm{d}z.$$

The canonical shift φ is defined by $\varphi(x) = \sum\limits_{i=1}^{n} S_i x S_i^*$. It is easy to see that $S_i x = \varphi(x) S_i$ and $x {S_i}^* = {S_i}^* \varphi(x)$. For each $x \in \mathcal{O}_n$, we have the Fourier expansion

$$x = \sum_{k=1}^{\infty} S_1^{*k} x_{-k} + x_0 + \sum_{k=1}^{\infty} x_k S_1^k$$

where $x_k = E(x S_1^{*k})$, $x_{-k} = E(S_1^k x)$ and $x_0 = E(x)$. (The right-hand side converges in the Hilbert space generated by the GNS-representation with respect to $\tau \circ E$.) For example, if $x = S_1 S_2 S_5^* S_8^{*2} S_3^*$, then $S_1^2 x \in \mathcal{F}_n$ and $x = S_1^{*2}(S_1^2 x) = S_1^{*2} E(S_1^2 x)$.

For the inclusion of C^*-algebras $A \subset B$ with a common unit, the normalizer group is defined by

$$\mathcal{N}_B(A) = \{u \in B : uAu^* = A,\ uu^* = u^*u = 1\}.$$

For a unitary operator u, we define the inner automorphism by $\mathrm{Ad}u(x) = uxu^*$. We denote by $\mathrm{Ad}u|_A$ the restriction of $\mathrm{Ad}u$ to A.

The following two theorems are the main results of this paper.

THEOREM 1.1. *Let A be a C^*-subalgebra of $\mathcal{F}_n$. If the relative commutant $A' \cap \mathcal{F}_n$ is finite-dimensional, then the algebra $A' \cap \mathcal{O}_n$ is also finite-dimensional.*

THEOREM 1.2. *Let A be as above. We consider two subgroups of the automorphism group $\mathrm{Aut}(A)$ as follows.*

$$G = \{\mathrm{Ad}U|_A \ : U \in \mathcal{N}_{\mathcal{O}_n}(A)\}, \quad H = \{\mathrm{Ad}u|_A \ : u \in \mathcal{N}_{\mathcal{F}_n}(A)\}.$$

Then H is a subgroup of G with finite index.

We need some preparations to show these theorems.

LEMMA 1.3. *For $X \in A' \cap \mathcal{O}_n$, we set $x_k = E(XS_1^{*k})$ and $x_{-k} = E(S_1^k X)$. Then for any $a \in A$ we have $ax_k = x_k \varphi^k(a)$, $x_{-k}a = \varphi^k(a)x_{-k}$ and $x_k x_k^*$, $x_{-k}^* x_{-k} \in A' \cap \mathcal{F}_n$*

Proof. For any $a \in A$, we see that

$$\begin{aligned} ax_k &= aE(XS_1^{*k}) = E(aXS_1^{*k}) = E(XaS_1^{*k}) \\ &= E(XS_1^{*k}\varphi^k(a)) = E(XS_1^{*k})\varphi^k(a) = x_k\varphi^k(a) \end{aligned}$$

and therefore

$$x_k x_k^* a = x_k(a^* x_k)^* = x_k(x_k \varphi^k(a)^*)^* = x_k \varphi^k(a) x_k^* = a x_k x_k^*.$$

In the same way we also have $x_{-k}a = \varphi^k(a)x_{-k}$ and $x_{-k}^* x_{-k} a = a x_{-k}^* x_{-k}$. ∎

LEMMA 1.4. *There is a positive integer N satisfying the following properties. For any integer $k \geqslant N$ and any element $X \in A' \cap \mathcal{O}_n$, we have $x_k = E(XS_1^{*k}) = 0$ and $x_{-k} = E(S_1^k X) = 0$.*

Proof. We compute

$$x_k^* x_k = E(XS_1^{*k})^* E(XS_1^{*k}) \leqslant E(S_1^k X^* X S_1^{*k}) \leqslant \|X\|^2 E(S_1^k S_1^{*k}) = \|X\|^2 S_1^k S_1^{*k}.$$

Let $R = \mathcal{F}_n''$ be the hyperfinite II_1-factor. We take the polar decomposition $x_k = v_k|x_k|$ in R. Then the above computation shows that $v_k^* v_k \leqslant S_1^k S_1^{*k}$. On the other hand, since $x_k x_k^*$ is an element of the finite-dimensional C^*-algebra $A' \cap \mathcal{F}_n$, we have $v_k v_k^* \in A' \cap \mathcal{F}_n$. Since the C^*-algebra $A' \cap \mathcal{F}_n$ is finite-dimensional, there is a positive number c satisfying $\tau(p) \geqslant c$ for any non-zero projection $p \in A' \cap \mathcal{F}_n$. We can take a positive integer N satisfying $\tau(S_1^k S_1^{*k}) = 1/n^k < c$ for any $k \geqslant N$. Then we see that $\tau(v_k v_k^*) = \tau(v_k^* v_k) \leqslant \tau(S_1^k S_1^{*k}) < c$ and hence $v_k v_k^* = 0$. So we conclude that $x_k = 0$ for $k \geqslant N$. In the same way we also have $x_{-k} = 0$ for $k \geqslant N$. ∎

LEMMA 1.5. *Let N be the positive integer in the previous lemma. For any $X \in A' \cap \mathcal{O}_n$, we have*

$$X = \sum_{k=1}^{N} S_1^{*k} x_{-k} + x_0 + \sum_{k=1}^{N} x_k S_1^k$$

where $x_k = E(XS_1^{*k})$, $x_{-k} = E(S_1^k X)$ and $x_0 = E(X)$.

Proof. We have the Fourier expansion

$$X = \sum_{k=1}^{\infty} S_1^{*k} x_{-k} + x_0 + \sum_{k=1}^{\infty} x_k S_1^k.$$

Thus by the previous lemma, we are done. ∎

LEMMA 1.6. *We define the isomorphism π_k on A by*

$$\pi_k(x) = \begin{pmatrix} x & 0 \\ 0 & \varphi^k(x) \end{pmatrix}, \quad x \in A.$$

Then we have

$$\begin{pmatrix} 0 & x_k \\ x_k^* & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_{-k}^* \\ x_{-k} & 0 \end{pmatrix} \in \pi_k(A)' \cap M_2(\mathcal{F}_n)$$

where $x_k = E(XS_1^{*k})$, $x_{-k} = E(S_1^k X)$ for $X \in A' \cap \mathcal{O}_n$.

Proof. This is an immediate consequence of the relations $ax_k = x_k\varphi^k(a)$ and $x_{-k}a = \varphi^k(a)x_{-k}$ for $a \in \mathcal{O}_n$. ∎

LEMMA 1.7. *The C^*-algebra $\pi_k(A)' \cap M_2(\mathcal{F}_n)$ is finite-dimensional.*

Proof. We set

$$B = \pi_k(A)' \cap M_2(\mathcal{F}_n), \quad e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in B, \quad f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1 - e.$$

Then we see that $eBe \simeq A' \cap \mathcal{F}_n$ and $fBf \simeq \varphi^k(A)' \cap \mathcal{F}_n \simeq M_{n^k}(\mathbb{C}) \otimes (A' \cap \mathcal{F}_n)$. So both algebras eBe and fBf are finite-dimensional and hence B is finite-dimensional. Indeed the von Neumann algebras $eB''e$ and $fB''f$ are finite-dimensional. So the center of B'' is finite-dimensional. Therefore we may assume that B'' is a factor. Then the finite-dimensionality of $eB''e$ and $fB''f$ ensures that B'' is finite-dimensional. Therefore B is finite-dimensional. ∎

Proof of Theorem 1.1. Consider the vector space

$$V_k = \{x_k = E(XS_1^{*k}) : X \in A' \cap \mathcal{O}_n\}.$$

Since the map

$$V_k \ni x \mapsto \begin{pmatrix} 0 & x \\ x^* & 0 \end{pmatrix} \in \pi_k(A)' \cap M_2(\mathcal{F}_n)$$

is injective and $\mathbb{R}$-linear, the vector space V_k is finite-dimensional. In the same way the vector space

$$V_{-k} = \{x_k = E(S_1^k X) : X \in A' \cap \mathcal{O}_n\}$$

is also finite-dimensional. On the other hand, the element $x_0 = E(X)$ belongs to the finite-dimensional C^*-algebra $A' \cap \mathcal{F}_n$. Combining these with Lemma 1.5, we see that $A' \cap \mathcal{O}_n$ is finite-dimensional. ∎

PROPOSITION 1.8. *There exists an orthogonal family of minimal projections $e_1, \dots, e_l \in A' \cap \mathcal{O}_n$ satisfying the following:*

(i) $\sum_{i=1}^{l} e_i = 1$ *and* $e_1, \dots, e_l \in A' \cap \mathcal{F}_n$.

(ii) *There are integers* $k_1, \dots, k_l$ *such that* $\mathrm{Ad}u_z(x) = \gamma_z(x)$ *for* $x \in A' \cap \mathcal{O}_n$ *where* $u_z = z^{k_1}e_1 + \cdots + z^{k_l}e_l$.

Proof. Since $A' \cap \mathcal{O}_n$ is finite-dimensional and γ-invariant, there exists an orthogonal family of minimal projections $e_1, \dots, e_l \in A' \cap \mathcal{O}_n$ and integers $k_1, \dots, k_l$ such that $\mathrm{Ad}u_z(x) = \gamma_z(x)$ where $u_z = z^{k_1}e_1 + \cdots + z^{k_l}e_l$ and $x \in A' \cap \mathcal{O}_n$. Then $e_i \in (A' \cap \mathcal{O}_n)^\gamma = A' \cap \mathcal{F}_n$. ∎

COROLLARY 1.9. *If A is an irreducible C^*-subalgebras of $\mathcal{F}_n$, then we have $A' \cap \mathcal{O}_n = \mathbb{C}$.*

Proof. By the previous proposition, we know that there are minimal projections in $A' \cap \mathcal{O}_n$ such that they belong to $A' \cap \mathcal{F}_n$. Thus we are done. ∎

In the rest of this paper we frequently use the projections $e_1, \dots, e_l \in A' \cap \mathcal{F}_n$ and the unitary $u_z = z^{k_1}e_1 + \cdots + z^{k_l}e_l$ in the above proposition.

REMARK 1.10. The Bratteli diagram of the inclusion $A' \cap \mathcal{F}_n \subset A' \cap \mathcal{O}_n$ has a special form. These two algebras have a common family of minimal projections. So for each vertex corresponding to a direct summand of $A' \cap \mathcal{F}_n$, there is only one edge which starts at this vertex. For example, if $A' \cap \mathcal{F}_n = \mathbb{C}$, then $A' \cap \mathcal{O}_n = \mathbb{C}$. If $A' \cap \mathcal{F}_n = \mathbb{C} \oplus \mathbb{C}$, then $A' \cap \mathcal{O}_n$ is isomorphic to either $\mathbb{C} \oplus \mathbb{C}$ or $M_2(\mathbb{C})$.

LEMMA 1.11. *Let $U \in \mathcal{O}_n$ be a unitary satisfying $UAU^* \subset \mathcal{F}_n$. Then we have:*

(i) $U^*\gamma_z(U) \in A' \cap \mathcal{O}_n$.

(ii) *There exists a unitary $w \in A' \cap \mathcal{O}_n$ and integers $m_1, \dots, m_l$ such that $\gamma_z(Uwe_i) = z^{m_i}Uwe_i$*

Proof. For any $a \in A$, since $UaU^* \in \mathcal{F}_n$, we see that

$$\gamma_z(U)a\gamma_z(U)^* = \gamma_z(UaU^*) = UaU^*.$$

Thus we have $U^*\gamma_z(U)\in A'\cap\mathcal{O}_n$. It is easy to see that the family $\{U^*\gamma_z(U)u_z\}_{z\in\mathbb{T}}$ is a unitary group. Indeed since $U^*\gamma_z(U)\in A'\cap\mathcal{O}_n$, we see that

$$\begin{aligned}U^*\gamma_{z_1}(U)u_{z_1}U^*\gamma_{z_2}(U)u_{z_2} &= U^*\gamma_{z_1}(U)\mathrm{Ad}u_{z_1}(U^*\gamma_{z_2}(U))u_{z_1z_2}\\ &= U^*\gamma_{z_1}(U)\gamma_{z_1}(U^*)\gamma_{z_1z_2}(U)u_{z_1z_2} = U^*\gamma_{z_1z_2}(U)u_{z_1z_2}.\end{aligned}$$

Since $\{U^*\gamma_z(U)u_z\}_{z\in\mathbb{T}}$ is a unitary group in the finite-dimensional C^*-algebra $A'\cap\mathcal{O}_n$, we can take a unitary $w\in A'\cap\mathcal{O}_n$ and integers $n_1,\dots,n_l$ such that $w^*U^*\gamma_z(U)u_zw = z^{n_1}e_1+\cdots+z^{n_l}e_l$. Then we see that

$$\begin{aligned}\gamma_z(Uwe_i) &= \gamma_z(U)u_zwu_z^*e_i = \{Uw(z^{n_1}e_1+\cdots+z^{n_l}e_l)w^*u_z^*\}u_zwu_z^*e_i\\ &= Uw(z^{n_1}e_1+\cdots+z^{n_l}e_l)u_z^*e_i = z^{n_i-k_i}Uwe_i. \quad ∎\end{aligned}$$

REMARK 1.12. By the previous lemma, we know that the Fourier expansion of U can be written down as a finite sum. Indeed if $m_i>0$, then $Uwe_i = (Uwe_iS_1^{*m_i})S_1^{m_i}$ and $Uwe_iS_1^{*m_i}\in\mathcal{F}_n$. On the other hand if $m_i<0$, then $Uwe_j = S_1^{*-m_i}(S_1^{-m_i}Uwe_j)$ and $S_1^{-m_i}Uwe_j\in\mathcal{F}_n$. Therefore the Fourier expansion of Uw is a finite sum. Combining this with Lemma 1.5, we can show that the Fourier expansion of U is a finite sum.

PROPOSITION 1.13. *For any normalizer $U\in\mathcal{N}_{\mathcal{O}_n}(A)$, there exist unitaries $v\in A'\cap\mathcal{F}_n$ and $w\in A'\cap\mathcal{O}_n$ satisfying*

$$vUw\in\mathcal{N}_{\mathcal{O}_n}(e_1\mathcal{F}_ne_1\oplus\cdots\oplus e_l\mathcal{F}_ne_l).$$

Proof. By the previous lemma we have $\gamma_z(Uwe_i) = z^{m_i}Uwe_i$. Then we get $\gamma_z(Uwe_iw^*U^*) = Uwe_iw^*U^*$ and hence $Uwe_iw^*U^*\in\mathcal{F}_n$. Since $UA'\cap\mathcal{O}_nU^* = A'\cap\mathcal{O}_n$, we have $Uwe_iw^*U^*\in A'\cap\mathcal{F}_n$. Thus $\{Uwe_iw^*U^*\}_i$ is a family of minimal projections in the finite-dimensional C^*-algebra $A'\cap\mathcal{F}_n$. So we can find a unitary $v\in A'\cap\mathcal{F}_n$ satisfying $vUwe_iw^*U^*v^*=e_j$. Since $\gamma_z(vUwe_i)=v\gamma_z(Uwe_i)= z^{m_i}vUwe_i$ for any $x\in\mathcal{F}_n$, we see that $\gamma_z\circ\mathrm{Ad}vUw(e_ixe_i)=\mathrm{Ad}vUw(e_ixe_i)$. Therefore $(vUw)e_i\mathcal{F}_ne_i(vUw)^*\subset e_j\mathcal{F}_ne_j$. On the other hand, since $\gamma_z(w^*U^*v^*e_j) = \gamma_z(vUwe_i)^* = (z^{m_i}Uwe_i)^* = z^{-m_i}w^*U^*v^*e_j$, we also have $(vUw)^*e_j\mathcal{F}_ne_j(vUw)\subset e_i\mathcal{F}_ne_i$. Therefore we have

$$\begin{aligned}&\mathrm{Ad}vUw(e_1\mathcal{F}_ne_1\oplus\cdots\oplus e_l\mathcal{F}_ne_l)\subset e_1\mathcal{F}_ne_1\oplus\cdots\oplus e_l\mathcal{F}_ne_l\,,\\ &\mathrm{Ad}w^*U^*v^*(e_1\mathcal{F}_ne_1\oplus\cdots\oplus e_l\mathcal{F}_ne_l)\subset e_1\mathcal{F}_ne_1\oplus\cdots\oplus e_l\mathcal{F}_ne_l\,,\end{aligned}$$

and hence

$$vUw\in\mathcal{N}_{\mathcal{O}_n}(e_1\mathcal{F}_ne_1\oplus\cdots\oplus e_k\mathcal{F}_ne_k). \quad ∎$$

REMARK 1.14. The normalizer $\mathcal{N}_{\mathcal{O}_n}(\mathcal{F}_n)$ is a subset of $\mathcal{F}_n$. However the structure of $\mathcal{N}_{\mathcal{O}_n}(e_1\mathcal{F}_ne_1\oplus\cdots\oplus e_l\mathcal{F}_ne_l)$ is not simple in general. See Examples 1.17 and 1.18.

LEMMA 1.15. *Let $e\in\mathcal{F}_n$ be a projection. If a partial isometry $u\in\mathcal{O}_n$ satisfies $u^*u=uu^*=e$ and $ue\mathcal{F}_neu^*=e\mathcal{F}_ne$, then we have $u\in\mathcal{F}_n$.*

Proof. Since $u^*\gamma_z(u) \in (e\mathcal{F}_n e)' \cap e\mathcal{O}_n e = e(\mathcal{F}_n' \cap \mathcal{O}_n)e = \mathbb{C}e$, we have $\gamma_z(u) = z^m u$ for some integer m. We will show $m = 0$. Suppose that $m > 0$. Set $v = uS_1^{*m}$. Then we have $\gamma_z(v) = v$ and hence $v \in \mathcal{F}_n$. Then we compute $v^*v = S_1^m e S_1^{*m} = \varphi^m(e)S_1^m S_1^{*m}$ and $vv^* = uu^* = e$. So we see that $\tau(e) = \tau(vv^*) = \tau(v^*v) = \tau(\varphi^m(e)S_1^m S_1^{*m}) = \tau(e) \times \tau(S_1^m S_1^{*m}) = (1/n^m)\tau(e) < \tau(e)$. This is a contradiction. On the other hand, if $m < 0$, we have $\gamma_z(u^*) = z^{-m}u^*$. So we get a contradiction in the same way. ∎

LEMMA 1.16. *Let B be the abelian C^*-algebra generated by $e_1, \ldots, e_l$. Then we have*

$$\mathcal{N}_{\mathcal{O}_n}(A) \cap \mathcal{N}_{\mathcal{O}_n}(B) \subset \mathcal{N}_{\mathcal{O}_n}(e_1\mathcal{F}_n e_1 \oplus \cdots \oplus e_k\mathcal{F}_n e_k).$$

Proof. The proof is essentially the same as that of Lemma 1.11 and Proposition 1.13. For any $U \in \mathcal{N}_{\mathcal{O}_n}(A) \cap \mathcal{N}_{\mathcal{O}_n}(B)$, since $U \in \mathcal{N}_{\mathcal{O}_n}(B)$, we have $U^*\gamma_z(U) \in B'$ and hence $U^*\gamma_z(U)u_z \in B'$. Therefore we can take $w = 1$ in the proof of Lemma 1.11. Then since $U \in \mathcal{N}_{\mathcal{O}_n}(B)$, we have $Uwe_iw^*U^* = Ue_iU^* = e_j$ and hence we can take $v = 1$ in the proof of Proposition 1.13. Thus by Proposition 1.13, we have $U \in \mathcal{N}_{\mathcal{O}_n}(e_1\mathcal{F}_n e_1 \oplus \cdots \oplus e_k\mathcal{F}_n e_k)$. ∎

Proof of Theorem 1.2. We can choose a finite family of unitaries $U_1, \ldots, U_N \in \mathcal{N}_{\mathcal{O}_n}(A) \cap \mathcal{N}_{\mathcal{O}_n}(B) \subset \mathcal{N}_{\mathcal{O}_n}(e_1\mathcal{F}_n e_1 \oplus \cdots \oplus e_k\mathcal{F}_n e_k)$ satisfying the following. For any $V \in \mathcal{N}_{\mathcal{O}_n}(A) \cap \mathcal{N}_{\mathcal{O}_n}(B)$, there exists U_i such that $\mathrm{Ad}V = \mathrm{Ad}U_i$ on B.

For any $U \in \mathcal{N}_{\mathcal{O}_n}(A)$, by Proposition 1.13 there exist unitaries $v \in A' \cap \mathcal{F}_n$ and $w \in A' \cap \mathcal{O}_n$ satisfying $vUw \in \mathcal{N}_{\mathcal{O}_n}(e_1\mathcal{F}_n e_1 \oplus \cdots \oplus e_k\mathcal{F}_n e_k)$. Then since $vUw \in \mathcal{N}_{\mathcal{O}_n}(A) \cap \mathcal{N}_{\mathcal{O}_n}(B)$, we can take U_i satisfying $\mathrm{Ad}U_i^* vUw = \mathrm{id}$ on B. Combining this with the fact that $U_i \in \mathcal{N}_{\mathcal{O}_n}(e_1\mathcal{F}_n e_1 \oplus \cdots \oplus e_k\mathcal{F}_n e_k)$ we see that $U_i^* vUwe_j \in \mathcal{N}_{e_j\mathcal{O}_n e_j}(e_j\mathcal{F}_n e_j) \subset \mathcal{F}_n$ and hence $U_i^* vUw \in \mathcal{N}_{\mathcal{F}_n}(A)$. Here we used Lemma 1.15. Therefore we see that $\mathrm{Ad}U|_A = \mathrm{Ad}vUw|_A \in (\mathrm{Ad}U_i|_A)H$. This implies that the index $[G : H]$ is finite. ∎

EXAMPLE 1.17. Let e be a projection in $\mathcal{F}_n$. Consider the C^*-algebra $A = e\mathcal{F}_n e \oplus (1-e)\mathcal{F}_n(1-e)$. Here we remark that $A' \cap \mathcal{F}_n = \mathbb{C}e \oplus \mathbb{C}(1-e)$. We will show that $\mathcal{N}_{\mathcal{O}_n}(A) \subset \mathcal{F}_n$ and hence $G = H$. This can be shown by K-theoretic argument as follows.

For any $U \in \mathcal{N}_{\mathcal{O}_n}(A)$, if $UeU^* = e$, it follows from Lemma 1.15 that $U \in \mathcal{F}_n$. So we consider the case $UeU^* = 1 - e$. Since $U^*\gamma_z(U)e \in (e\mathcal{F}_n e)' \cap eO_n e = e(\mathcal{F}_n' \cap O_n)e = \mathbb{C}e$, we have $\gamma_z(U)e = z^m Ue$ for some integer m. We will show $m = 0$. Suppose that $m > 0$. Set $v = UeS_1^{*m}$. Then we have $\gamma_z(v) = v$ and hence $v \in \mathcal{F}_n$. Then we compute $v^*v = S_1^m e S_1^{*m} = \varphi^m(e)S_1^m S_1^{*m}$ and $vv^* = UeU^* = 1 - e$. So we see that $1 - \tau(e) = \tau(vv^*) = \tau(v^*v) = \tau(\varphi^m(e)S_1^m S_1^{*m}) = \tau(e) \times \tau(S_1^m S_1^{*m}) = (1/n^m)\tau(e)$. Since $\mathcal{F}_n$ is the UHF-algebra of type n^∞, we can write $\tau(e) = q/n^p$. So we get $1 - q/n^p = (1/n^m)(q/n^p)$ and hence

$$n^{m+p} = q(1 + n^m).$$

This is impossible. Indeed, consider the prime factorization $n = p_1^{k_1} \times \cdots \times p_n^{k_n}$. Then we have

$$(p_1^{k_1} \times \cdots \times p_n^{k_n})^{m+p} = q(1 + (p_1^{k_1} \times \cdots \times p_n^{k_n})^m).$$

Therefore we must have

$$1 + (p_1^{k_1} \times \cdots \times p_n^{k_n})^m = p_1^{l_1} \times \cdots \times p_n^{l_n}.$$

However this cannot occur because the left hand side has the remainder 1 when dividing by p_1.

EXAMPLE 1.18. We can write

$$\mathcal{O}_2 \supset \mathcal{F}_2 = M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \otimes \cdots .$$

Consider two projections

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes 1 \otimes 1 \otimes \cdots \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes 1 \otimes \cdots .$$

Since $\varphi(e) S_1 S_1{}^* \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ and $\tau(\varphi(e) S_1 S_1{}^*) = 1/4$, there exists a partial isometry $v \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ such that $v^* v = \varphi(e) S_1 S_1{}^*$ and $vv^* = f$. We set

$$U = vS_1 + (vS_1)^* + (1 - e - f).$$

Then it is easy to see that

$$U \in \mathcal{N}_{\mathcal{O}_n}(e\mathcal{F}_n e \oplus f\mathcal{F}_n f \oplus (1 - e - f)\mathcal{F}_n(1 - e - f)).$$

We let $A = e\mathcal{F}_n e \oplus f\mathcal{F}_n f \oplus (1 - e - f)\mathcal{F}_n(1 - e - f)$. Since $\tau(UeU^*) = \tau(f) = 1/4 \neq 1/2 = \tau(e)$, we have

$$\mathrm{Ad}U|_A \notin \{\mathrm{Ad}u|_A \ : u \in \mathcal{N}_{\mathcal{F}_n}(A)\}.$$

Therefore we see that $G \neq H$.

REMARK 1.19. If A is of the form $A = e_1 \mathcal{F}_n e_1 \oplus \cdots \oplus e_l \mathcal{F}_n e_l$, then we have $A' \cap \mathcal{F}_n = A' \cap \mathcal{O}_n = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_l$. On the other hand, in Remark 1.10 we see that the Bratteli diagram of the inclusion $A' \cap \mathcal{F}_n \subset A' \cap \mathcal{O}_n$ has a special form. So we might expect that $A' \cap \mathcal{F}_n = A' \cap \mathcal{O}_n$. However this is wrong in general. Indeed there exists a C^*-subalgebra $A \subset \mathcal{F}_n$ with finite index such that $A' \cap \mathcal{F}_n \neq A' \cap \mathcal{O}_n$. We can take $A = \lambda_u(\mathcal{F}_n)$ where λ_u is a localized endomorphism. See [4], [5].

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