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K_0-MONOID PROPERTIES PRESERVED BY TRACIAL APPROXIMATION

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ABSTRACT. We show that the following K_0-monoid properties of C^*-algebras in the class Ω are inherited by simple unital C^*-algebras in the class $TA\Omega$:
(i) almost unperforated,
(ii) n-comparison,
(iii) cancellation property,
(iv) Riesz decomposition property.

KEYWORDS: *C^*-algebra, tracial approximation, K_0-monoid.*

MSC (2010): 46L05, 46L80, 46L35.

INTRODUCTION

The Elliott conjecture asserts that all nuclear, separable C^*-algebras are classified up to isomorphism by an invariant, called the Elliott invariant. A first version of the Elliott conjecture might be said to have begun with the K-theoretical classification of AF-algebras in [2]. Since then, many classes of C^*-algebras have been found to be classified by the Elliott invariant. Among them, one important class is the class of simple unital AH-algebras. A very important axiomatic version of the classification of AH-algebras without dimension growth was given by H. Lin. Instead of assuming inductive limit structure, he started with a certain abstract approximation property, and showed that C^*-algebras with this abstract approximation property and certain additional properties are AH-algebras without dimension growth. More precisely Lin introduced the class of tracially approximate interval algebras.

Following the notion of Lin on the tracial approximation by interval algebras, G.A. Elliott and Z. Niu in [6] considered tracial approximation by more general C^*-algebras. Let Ω be a class of unital C^*-algebras. Then the class of C^*-algebras which can be tracially approximated by C^*-algebra in Ω, denoted by $TA\Omega$, is defined as follows. A simple unital C^*-algebra A is said to belong to the class $TA\Omega$, if for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any nonzero element $a \geqslant 0$, there exist a nonzero projection $p \in A$ and a C^*-subalgebra B of A with $1_B = p$ and $B \in \Omega$, such that:

(i) $\|xp - px\| < \varepsilon$ for all $x \in F$,

(ii) $pxp \in_\varepsilon B$ for all $x \in F$,

(iii) $1 - p$ is Murray–von Neumann equivalent to a projection in $\overline{aAa}$.

The question of the behavior of C^*-algebra properties under passage from a class Ω to the class $TA\Omega$ is interesting and sometimes important. In fact the property of having tracial states, the property of being of stable rank one, and the property that the strict order on projections is determined by traces were used in the proof of the classification theorem in [6] and [13] by G.A. Elliott and Z. Niu.

In this paper, we show that the following K_0-monoid properties of C^*-algebras in the class Ω are inherited by simple unital C^*-algebras in the class $TA\Omega$:

(i) almost unperforated,

(ii) n-comparison,

(iii) cancellation property,

(iv) Riesz decomposition property.

1. PREMIMINARIES

Let a and b be two positive elements in a C^*-algebra A. We write $[a] \leqslant [b]$ (cf. Definition 3.5.2 in [11]), if there exists a partial isometry $v \in A^{**}$ such that, for every $c \in \mathrm{Her}(a), v^*c, cv \in A, vv^* = P_a$, where P_a is the range projection of a in A^{**}, and $v^*cv \in \mathrm{Her}(b)$. We write $[a] = [b]$ if $v^*\mathrm{Her}(a)v = \mathrm{Her}(b)$. Let n be a positive integer. We write $n[a] \leqslant [b]$, if there are n mutually orthogonal positive elements $b_1, b_2, \dots, b_n \in \mathrm{Her}(b)$ such that $[a] \leqslant [b_i], i = 1, 2, \dots, n$.

Let $0 < \sigma_1 < \sigma_2 \leqslant 1$ be two positive numbers. Define

$$f_{\sigma_1}^{\sigma_2}(t) = \begin{cases} 1 & \text{if } t \geqslant \sigma_2\,, \\ \frac{t-\sigma_1}{\sigma_2-\sigma_1} & \text{if } \sigma_1 \leqslant t \leqslant \sigma_2\,, \\ 0 & \text{if } 0 < t \leqslant \sigma_1\,. \end{cases}$$

Let Ω be a class of unital C^*-algebras. Then the class of C^*-algebras which can be tracially approximated by C^*-algebras in Ω is denoted by $TA\Omega$.

DEFINITION 1.1 ([6]). A simple unital C^*-algebra A is said to belong to the class $TA\Omega$ if for any $\varepsilon > 0$, any finite subset $F \subseteq A$, and any nonzero element $a \geqslant 0$, there exist a nonzero projection $p \in A$ and a C^*-subalgebra B of A with $1_B = p$ and $B \in \Omega$, such that:

(i) $\|xp - px\| < \varepsilon$ for all $x \in F$,

(ii) $pxp \in_\varepsilon B$ for all $x \in F$,

(iii) $[1 - p] \leqslant [a]$.

DEFINITION 1.2 ([8]). Let Ω be a class of unital C^*-algebras. A unital C^*-algebra A is said to have property (III) if for any positive numbers $0 < \sigma_3 < \sigma_4 < \sigma_1 < \sigma_2 < 1$, any $\varepsilon > 0$, any finite subset $F \subseteq A$, any nonzero positive element a, and any integer $n > 0$, there exist a nonzero projection $p \in A$, and a C^*-subalgebra B of A with $B \in \Omega$ and $1_B = p$, such that:

(i) $\|xp - px\| < \varepsilon$ for all $x \in F$,

(ii) $pxp \in_\varepsilon B$ for all $x \in F$, $\|pap\| \geqslant \|a\| - \varepsilon$,

(iii) $n[f_{\sigma_1}^{\sigma_2}((1-p)a(1-p))] \leqslant [f_{\sigma_3}^{\sigma_4}(pap)]$.

LEMMA 1.3 ([6]). *If the class Ω is closed under tensoring with matrix algebras, or closed under taking unital hereditary C^*-subalgebras, then $TA\Omega$ is closed under passing to matrix algebras or unital hereditary C^*-subalgebras.*

THEOREM 1.4 ([8]). *Let Ω be a class of unital C^*-algebras such that Ω is closed under taking unital hereditary C^*-subalgebras and closed taking finite direct sums. Let A be a simple unital C^*-algebra. Then the following are equivalent:*

(i) *$A \in TA\Omega$,*

(ii) *A has property* (III).

We say a C^*-algebra A has the SP-property, if every nonzero hereditary C^*-subalgebra of A contains a nonzero projection.

Call projections $p, q \in M_\infty(A)$ equivalent, denoted $p \sim q$, when there is a partial isometry $v \in M_\infty(A)$ such that $p = v^*v, q = vv^*$. The equivalent classes are denoted by $[\cdot]$ and the set of all these is

$$V(A) := \{[p] : p = p^* = p^2 \in M_\infty(A)\}.$$

Addition in $V(A)$ is defined by

$$[p] + [q] := [\mathrm{diag}(p, q)].$$

$V(A)$ becomes an abelian monoid, we call $V(A)$ the K_0-monoid of A.

All abelian monoids have a natural pre-order, the algebraic ordering, defined as follows: if $x, y \in M$, we write $x \leqslant y$ if there is a z in M such that $x + z = y$. In the case of $V(A)$, the algebraic ordering is given by Murray–von Neumann subequivalence, that is, $[p] \leqslant [q]$ if and only if there is a projection $p' \leqslant q$ such that $p \sim p'$. We also write, as is customary, $p \preceq q$ to mean that p is subequivalent to q.

If $x, y \in M$, we will write $x \leqslant^* y$ if there is a nonzero element z in M, such that $x + z = y$.

Let us recall that an element u in a monoid M is an order unit provided $u \neq 0$ and, for any x in M, there is $n \in \mathbb{N}$ such that $x \leqslant nu$.

We say that a monoid M is conical if $x + y = 0$ only when $x = y = 0$. Note that, for any C^*-algebra A, the monoid $V(A)$ is conical.

We say that a monoid M has the cancellation property when it satisfies the statement that for any $a, b, c \in M$, $a + c = b + c$ implies that $a = b$.

We remind the reader that a monoid M is almost unperforated if whenever $(k+1)x \leqslant ky$ for $k \in \mathbb{N}$, it follows that $x \leqslant y$. More generally, M has n-comparison if where $x, y_0, y_1, y_2, \ldots, y_n$ are elements in M such that $x <_s y_j$ for all $j = 0, 1, \ldots, n$, then $x \leqslant y_0 + y_1 + \cdots + y_n$. Here $x <_s y$ means that $(k+1)x \leqslant ky$ for some nature number k. It follows immediately from the definitions that M is almost unperforated if and only if M has 0-comparison.

We say that a monoid M satisfies the Riesz decomposition property if whenever $a \leqslant b_1 + b_2$ in M, there exist $a_1, a_2 \in M$ such that $a = b_1 + b_2$ and $a_i \leqslant b_i$ for $i = 1, 2$.

2. MAIN SECTION

THEOREM 2.1. *Let Ω be a class of unital stably finite C^*-algebras such that for any $B \in \Omega$ the K_0-monoid $V(B)$ is almost unperforated. Then the K_0-monoid $V(A)$ is almost unperforated for any simple unital C^*-algebra $A \in TA\Omega$.*

Proof. We need to show that $x \leqslant y$ in $V(A)$ whenever $(k+1)x \leqslant ky$ for some integer $k > 0$. By Lemma 1.3 we may assume that $x = [p], y = [q]$ for some projections $p, q \in \mathrm{proj}(A)$. For $F = \{p, q\}$, any $\varepsilon > 0$, since $A \in TA\Omega$, there exist a projection $r \in A$ and a C^*-subalgebra $B \subseteq A$ with $B \in \Omega, 1_B = r$ such that:

(i) $\|xr - rx\| < \varepsilon$ for all $x \in F$,

(ii) $rxr \in_\varepsilon B$ for all $x \in F$.

By (i) and (ii) there exist projections $p_1, q_1 \in B$ and $p_2, q_2 \in (1-r)A(1-r)$ such that

$$\|p - p_1 - p_2\| < \varepsilon, \quad \|q - q_1 - q_2\| < \varepsilon.$$

Therefore we have

$$[p] = [p_1] + [p_2], \quad [q] = [q_1] + [q_2].$$

We also have

$$(k+1)[p_1] \leqslant k[q_1], \quad (k+1)[p_2] \leqslant k[q_2].$$

Since $B \in \Omega$ and $V(B)$ is almost unperforated, we have $[p_1] \leqslant [q_1]$.

If $[p_1] = [q_1]$, then we have $(k+1)[p_1] \leqslant k[q_1] = k[p_1] \leqslant^* (k+1)[p_1]$. So $\bigoplus_{n=1}^{k+1} p_1$ is equivalent to a proper subprojection of itself, and this contradicts to the stable finiteness of A (A has stably finite, because C^*-algebras in Ω are stably finite).

Therefore we have $[p_1] \leqslant^* [q_1]$. Since $[p_1] \leqslant^* [q_1]$, there exists a nonzero projection s such that $[p_1] + [s] = [q_1]$. For $G = \{p_2, q_2, s\}$, any $\varepsilon > 0$, since $A \in TA\Omega$, there exist a projection $m \in A$ and a C^*-subalgebra $C \subseteq A$ with $C \in \Omega, 1_C = m$ such that:

(i′) $\|xm - mx\| < \varepsilon$ for all $x \in G$,

(ii′) $mxm \in_\varepsilon C$ for all $x \in G$, (iii′) $[1-m] \leqslant [s]$.

By (i′) and (ii′) there exist projections $p_3, q_3 \in C$ and $p_4, q_4 \in (1-m)A(1-m)$ such that

$$\|p_2 - p_3 - p_4\| < \varepsilon, \quad \|q_2 - q_3 - q_4\| < \varepsilon.$$

Therefore we have

$$[p_2] = [p_3] + [p_4], \quad [q_2] = [q_3] + [q_4].$$

We also have

$$(k+1)[p_3] \leqslant k[q_3], \quad (k+1)[p_4] \leqslant k[q_4].$$

Since $C \in \Omega$ and $V(C)$ is almost unperforated, we have $[p_3] \leqslant [q_3]$. By (iii′) $[p_4] \leqslant [1-m] \leqslant [s]$, therefore

$$\begin{aligned}[p] &= [p_1] + [p_2] = [p_1] + [p_3] + [p_4] \leqslant [p_1] + [q_3] + [p_4] \\ &\leqslant [p_1] + [q_3] + [s] \leqslant [q_1] + [q_3] \leqslant [q]. \qquad \blacksquare\end{aligned} \tag{2.1}$$

THEOREM 2.2. *Let Ω be a class of unital stably finite C^*-algebras such that for any $B \in \Omega$ the K_0-monoid $V(B)$ has n-comparison. Then the K_0-monoid $V(A)$ has n-comparison for any simple unital C^*-algebra $A \in TA\Omega$.*

Proof. We need to show that $x \leqslant y_0 + y_1 + \cdots + y_n$ in $V(A)$ whenever $(k_i + 1)x \leqslant k_i y_i$ for some integer $k_i > 0$ and for all $0 \leqslant i \leqslant n$. Note that k_i can be chosen to be the same for all y_i, that is, set $k = (k_0+1)(k_1+1)\cdots(k_n+1) - 1$, then $(k+1)x \leqslant k y_i$ for all $0 \leqslant i \leqslant n$. By Lemma 1.3, we may assume that $x = [p], y_i = [q_i]$ for some projections $p, q_i \in \mathrm{proj}(A)$ and for all $0 \leqslant i \leqslant n$. For $F = \{p, q_0, q_1, \ldots, q_n\}$, any $\varepsilon > 0$, since $A \in TA\Omega$, there exist a projection $r \in A$ and a C^*-subalgebra $B \subseteq A$ with $B \in \Omega, 1_B = r$ such that:

(i) $\|xr - rx\| < \varepsilon$ for all $x \in F$,

(ii) $rxr \in_{\varepsilon} B$ for all $x \in F$.

By (i) and (ii) there exist projections $p_1, q_{01}, q_{11}, \ldots, q_{n1} \in B$ and $p_2, q_{02}, q_{12}, \ldots, q_{n2} \in (1-r)A(1-r)$ such that:

$$\|p - p_1 - p_2\| < \varepsilon, \quad \|q_i - q_{i1} - q_{i2}\| < \varepsilon,$$

for all $0 \leqslant i \leqslant n$. Therefore we have

$$[p] = [p_1] + [p_2], \quad [q_i] = [q_{i1}] + [q_{i2}].$$

We also have

$$(k+1)[p_1] \leqslant k[q_{i1}], \quad (k+1)[p_2] \leqslant k[q_{i2}].$$

Since $B \in \Omega$ and $V(B)$ has n-comparison, we have $[p_1] \leqslant [q_{01}] + [q_{11}] + \cdots + [q_{n1}]$.

If $[p_1] = [q_{01}] + [q_{11}] + \cdots + [q_{n1}]$, then we have $(k+1)[p_1] \leqslant^* k([q_{01}] + [q_{11}] + \cdots + [q_{n1}]) \leqslant^* (k+1)([q_{01}] + [q_{11}] + \cdots + [q_{n1}]) = (k+1)[p_1]$. So $\bigoplus_{n=1}^{k+1} p_1$ is equivalent to a proper subprojection of itself, and this contradicts to the stable finiteness of A (A has stably finite, because C^*-algebras in Ω are stably finite).

Therefore we have $[p_1] \leqslant^* [q_{01}] + [q_{11}] + \cdots + [q_{n1}]$. Since $[p_1] \leqslant^* [q_{01}] + [q_{11}] + \cdots + [q_{n1}]$, there exists a nonzero projection s such that $[p_1] + [s] = [q_{01}] + [q_{11}] + \cdots + [q_{n1}]$. For $G = \{p_2, q_{02}, q_{12}, \cdots q_{n2}, s\}$, any $\varepsilon > 0$, since $A \in TA\Omega$, there exist a projection $m \in A$ and a C^*-subalgebra $C \subseteq A$ with $C \in \Omega, 1_C = m$ such that:

(i′) $\|xm - mx\| < \varepsilon$ for all $x \in G$,
(ii′) $mxm \in {}_{\varepsilon}C$ for all $x \in G$,
(iii′) $[1 - m] \leqslant [s]$.

By (i′) and (ii′) there exist projections $p_3, q_{03}, q_{13}, \ldots, q_{n3} \in C$ and $p_4, q_{04}, q_{14}, \ldots, q_{n4} \in (1-m)A(1-m)$ such that

$$\|p_2 - p_3 - p_4\| < \varepsilon, \quad \|q_{i2} - q_{i3} - q_{i4}\| < \varepsilon.$$

Therefore we have

$$[p_2] = [p_3] + [p_4], \quad [q_{i2}] = [q_{i3}] + [q_{i4}].$$

We also have

$$(k+1)[p_3] \leqslant k[q_{i3}], \quad (k+1)[p_4] \leqslant k[q_{i4}].$$

Since $C \in \Omega$ and $V(C)$ has n-comparison, we have $[p_3] \leqslant [q_{03}] + [q_{13}] + \cdots + [q_{n3}]$.

By (iii′) we have $[p_4] \leqslant [1-m] \leqslant [s]$, therefore:

$$\begin{aligned} [p] &= [p_1] + [p_2] = [p_1] + [p_3] + [p_4] \leqslant [p_1] + [q_{03}] + [q_{13}] + \cdots + [q_{n3}] + [p_4] \\ &\leqslant [p_1] + [q_{03}] + [q_{13}] + \cdots + [q_{n3}] + [s] \\ &\leqslant [q_{01}] + [q_{11}] + \cdots + [q_{n1}] + [q_{03}] + [q_{13}] + \cdots + [q_{n3}] \qquad (2.2) \\ &\leqslant [q_0] + [q_1] + \cdots + [q_n]. \quad \blacksquare \end{aligned}$$

THEOREM 2.3. *Let Ω be a class of unital C^*-algebras such that for any $B \in \Omega$ the K_0-monoid $V(B)$ has the Riesz decomposition property. Then the K_0-monoid $V(A)$ has the Riesz decomposition property for any simple unital C^*-algebra $A \in TA\Omega$.*

Proof. We need to show that if $a \leqslant b_1 + b_2$ in $V(A)$, then there exist $a_1, a_2 \in V(A)$ such that $a = a_1 + a_2$ and $a_1 \leqslant b_1$, $a_2 \leqslant b_2$ in $V(A)$. One may assume that $a \leqslant^* b_1 + b_2$. Otherwise, one has the decomposition right away and there is nothing need to proof. By Lemma 1.3, without loss of generality we may assume that $a = [p], b_1 = [q_1], b_2 = [q_2]$, where $p, q_1, q_2 \in \mathrm{proj}(A)$. For any $\varepsilon > 0$, any $F = \{p, q_1, q_2\}$, since $A \in TA\Omega$, there are projection $r \in A$ and C^*-subalgebra B of A with $1_B = r$, $B \in \Omega$, such that:

(i) $\|xr - rx\| < \varepsilon$ for all $x \in F$,
(ii) $rxr \in {}_{\varepsilon}B$ for all $x \in F$.

By functional calculus, there are projections $p', q_1', q_2' \in B$ and projections $p'', q_1'', q_2'' \in (1-r)A(1-r)$ such that:

$$[p] = [p'] + [p''], \quad [q_1] = [q_1'] + [q_1''], \quad [q_2] = [q_2'] + [q_2''],$$
$$[p'] \leqslant^* [q_1'] + [q_2'], \quad [p''] \leqslant [q_1''] + [q_2''].$$

Since $B \in \Omega$ and $V(B)$ the Riesz decomposition property, there exist projection $p_1', p_2' \in B$ such that $[p'] = [p_1'] + [p_2']$ and $[p_1'] \leqslant [q_1'], [p_2'] \leqslant [q_2']$.

Since $[p] = [p_1'] + [p_2'] \leqslant^* [q_1'] + [q_2']$, and $[p_1'] \leqslant [q_1'], [p_2'] \leqslant [q_2']$, therefore we have $[p_1'] \leqslant^* [q_1']$ or $[p_2'] \leqslant^* [q_2']$. Without loss of generality we may assume that $[p_1'] \leqslant^* [q_1']$, there exists a nonzero projection $m \in A$ such that $[p_1'] + [m] = [q_1']$. For $G = \{p'', q_1'', q_2''\}$, and any $\varepsilon > 0$, there exist a C^*-subalgebra C of A and projection $t \in A$ with $1_C = t$ and $C \in \Omega$ such that:

(i′) $\|xt - tx\| < \varepsilon$ for all $x \in G$,

(ii′) $txt \in {}_{\varepsilon}C$ for all $x \in G$,

(iii′) $[1-t] \leqslant [m]$.

By functional calculus, there are projections $p''', q_1''', q_2''' \in C$ and $p'''', q_1'''', q_2'''' \in (1-t)A(1-t)$ such that:

$$[p''] = [p'''] + [p''''], \quad [q_1''] = [q_1'''] + [q_1''''], \quad [q_2] = [q_2'''] + [q_2''''],$$
$$[p'''] \leqslant [q_1'''] + [q_2'''], \quad [p''''] \leqslant [q_1''''] + [q_2''''].$$

Since $C \in \Omega$ and $V(C)$ has the Riesz decomposition property, there exist projections $p_1''', p_2''' \in C$ such that

$$[p'''] = [p_1'''] + [p_2'''], \quad [p_1'''] \leqslant [q_1'''], [p_2'''] \leqslant [q_2'''].$$

Therefore we have

$$\begin{aligned}[p] &= [p'] + [p''] = [p'] + [p'''] + [p''''] = [p_1'] + [p_2'] + [p_1'''] + [p_2'''] + [p''''] \\ &= ([p_1'] + [p_1'''] + [p'''']) + ([p_2'] + [p_2''']).\end{aligned}$$

Since $[p'''] \leqslant [1-t] \leqslant [m]$, we have

$$[p_1'] + [p_1'''] + [p''''] \leqslant [p_1'] + [q_1'''] + [m] \leqslant [q_1'] + [q_1'''] \leqslant [q_1]$$

and $[p_2'] + [p_2'''] \leqslant [q_2]$. Set $[p_1] = [p_1'] + [p_1'''] + [p'''']$, $[p_2] = [p_2'] + [p_2''']$, we have $[p] = [p_1] + [p_2]$ and $[p_1] \leqslant [q_1]$, $[p_2] \leqslant [q_2]$. ∎

THEOREM 2.4. *Let Ω be a class of unital C^*-algebras such that for any $B \in \Omega$ the K_0-monoid $V(B)$ has the cancellation property. Then the K_0-monoid $V(A)$ has the cancellation property for any simple unital C^*-algebra $A \in TA\Omega$.*

Proof. We need to show that $a + c = b + c$ implies that $a = b$ where $a, b, c \in V(A)$.

We may assume that $a = [p], b = [q], c = [e]$ where $p, q, e \in M_j(A)$ for sufficiently large integer j and p, q, e are projections. By Lemma 1.3, we may assume that $p, q, e \in A$.

For any $1/2 > \varepsilon > 0$, $F = \{p, q, e\}$, since $A \in TA\Omega$, by Theorem 1.4, there exist a projection $r \in A$ and a C^*-subalgebra $B \subseteq A$ with $B \in \Omega, 1_B = r$ such that:

(i) $\|xr - rx\| < \varepsilon$ for all $x \in F$,

(ii) $rxr \in_\varepsilon B$ for all $x \in F$.

By functional calculus, there exist projections $p_1, q_1, e_1 \in B$ and projections $p_2, q_2, e_2 \in (1-r)A(1-r)$ such that:

$$\|p - p_1 - p_2\| < \varepsilon, \quad \|q - q_1 - q_2\| < \varepsilon, \quad \|e - e_1 - e_2\| < \varepsilon.$$

Therefore we have

$$[p] = [p_1] + [p_2], \quad [e] = [e_1] + [e_2], \quad [q] = [q_1] + [q_2],$$
$$[p_1] + [e_1] = [q_1] + [e_1], \quad [p_2] + [e_2] = [q_2] + [e_2].$$

Since $V(B)$ has the cancellation property, therefore $[p_1] = [q_1]$ in $V(B)$. For any $1/2 > \varepsilon > 0$, $F_1 = \{p_2, q_2, e_2\}$, there exist a projection $s \in C$ and a C^*-subalgebra $C \subseteq A$ with $1_C = s$ and $C \in \Omega$ such that:

(i′) $\|xr - rx\| < \varepsilon$ for all $x \in F$,

(ii′) $rxr \in_\varepsilon C$ for all $x \in F$,

(iii′) $2[1 - s] \leqslant [p_1]$.

By functional calculus, there exist projections $p_{21}, q_{21}, e_{21} \in C$ and projections $p_{22}, q_{22}, e_{22} \in (1-s)A(1-s)$ such that

$$\|p_2 - p_{21} - p_{22}\| < \varepsilon, \quad \|q_2 - q_{21} - q_{22}\| < \varepsilon, \quad \|e_2 - e_{21} - e_{22}\| < \varepsilon,$$

then

$$[p_2] = [p_{21}] + [p_{22}], \quad [q_2] = [q_{21}] + [q_{22}], \quad [e_2] = [e_{21}] + [e_{22}], \quad \text{and}$$
$$[p_{21}] + [e_{21}] = [q_{21}] + [e_{21}], \quad [p_{22}] + [e_{22}] = [q_{22}] + [e_{22}].$$

Since $V(C)$ has the cancellation property, therefore $[p_{21}] = [q_{21}]$ in $V(C)$. Since $2[e_{22}] \leqslant 2[1-s] \leqslant [p_1]$, there is a nonzero partial isometry $v \in A$ such that $v^*v = e_{22}, vv^* \leqslant p_1$, therefore:

$$\begin{aligned}[p] &= [p_1 - vv^*] + [p_{21}] + [p_{22}] + [vv^*] = [p_1 - vv^*] + [p_{21}] + [p_{22}] + [e_{22}] \\ &= [p_1 - vv^*] + [p_{21}] + [q_{22}] + [e_{22}] = [p_1 - vv^*] + [p_{21}] + [q_{22}] + vv^* \\ &= [p_1] + [p_{21}] + [q_{22}] = [q]. \quad \blacksquare\end{aligned} \tag{2.3}$$

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