J. OPERATOR THEORY
70:2(2013), 375–399
doi: 10.7900/jot.2011sep03.1945

L^1-NORM ESTIMATES OF CHARACTER SUMS DEFINED BY A SIDON SET IN THE DUAL OF A COMPACT KAC ALGEBRA

TOBIAS BLENDEK and JOHANNES MICHALIČEK

Communicated by Kenneth R. Davidson

ABSTRACT. We generalize the following fact to compact Kac algebras: Let G be a compact abelian group, and let f be any trigonometric polynomial on G, whose Fourier transform $\widehat{f}$ vanishes outside of a Sidon set E in the dual, discrete abelian group Γ of G. Then we have $\|f\|_2 \leqslant K_E \|f\|_1$, where K_E is a constant depending only on E. For this generalization, we introduce the notion of Helgason–Sidon sets, which is based on S. Helgason's work on lacunary Fourier series on arbitrary compact groups. We establish the above inequality for all finite linear combinations of characters defined by a Helgason–Sidon set in the set of all minimal central projections.

KEYWORDS: *Compact Kac algebra, Fourier transform, character, Sidon set, strong Sidon set, Helgason–Sidon set.*

MSC (2010): Primary 46L65; Secondary 43A46, 46L89, 81R15, 81R50.

INTRODUCTION

An outstanding problem in the theory of Fourier series was to find estimates for idempotents in $L^1(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle group. In this connection, we mention the so-called *Littlewood conjecture*, whose formulation appeared in [4] the first time. It asserts that there are constants C_1, C_2 such that

$$C_2\sqrt{N} \geqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big|\sum_{k=1}^{N} \mathrm{e}^{\mathrm{i}n_k t}\Big| \,\mathrm{d}t \geqslant C_1 \log(N),$$

where $n_k \in \mathbb{Z}$ for all $k = 1, \ldots, N$ and $N \in \mathbb{N}$. Whereas the upper bound $C_2\sqrt{N}$ is easy to show, the lower bound $C_1 \log(N)$ had remained an open problem for a long time. Eventually, it was solved independently by O.C. McGehee et al. [12] and S.V. Konjagin [8] even for more general coefficients.

In fact, the upper bound $C_2\sqrt{N}$ is not improvable if n_k, $k = 1, \ldots, N$, are elements of a so-called *Sidon set* in $\mathbb{Z}$. In case of an arbitrary compact abelian group G, a Sidon set is a "thin" subset E of the dual, discrete abelian group Γ such that every continuous, complex-valued function on E vanishing at infinity is the restriction to E of a Fourier transform of a function in $L^1(G)$. For locally compact abelian groups, Sidon sets are also known as Helson sets.

The inclusion $L^2(G) \subseteq L^1(G)$ naturally yields $\|f\|_2 \geqslant \|f\|_1$. Now, taking into account Theorem 5.7.7(3) of [13], there is a constant K_E depending only on a given Sidon set E such that we get the inverse inequality

$$\|f\|_2 \leqslant K_E \|f\|_1 \tag{0.1}$$

for every E-polynomial f on G, i.e. for every finite, complex linear combination of continuous characters in Γ such that its Fourier transform $\widehat{f}$ vanishes outside of E. In particular, we obtain for sums of characters $\gamma_k \in E \subseteq \Gamma$, $k = 1, \ldots, N$,

$$\Big\| \sum_{k=1}^{N} \gamma_k \Big\|_2 = \sqrt{N} \leqslant K_E \Big\| \sum_{k=1}^{N} \gamma_k \Big\|_1.$$

In fact, Sidon sets can also be introduced for arbitrary compact groups. According to Theorem 37.7(iv) of [6], corresponding inequalities hold in this case, too.

In this paper, we prove that (0.1) is also valid in a much more general situation:

Let $\mathbb{K} = (M, \Delta, \kappa, \varphi)$ be a compact Kac algebra with the dual, discrete Kac algebra $\widehat{\mathbb{K}} = (\widehat{M}, \widehat{\Delta}, \widehat{\kappa}, \widehat{\varphi})$ of $\mathbb{K}$ and $\{p_i\}_{i \in I}$ the set of all minimal central projections in $\widehat{M}$. First, using the notion of the (generalized) Fourier transform by B.-J. Kahng in [7] for locally compact quantum groups, which were introduced by J. Kustermans and S. Vaes in [9], [10], we regard the inverse Fourier transform of a projection p_i as a character χ_i in M multiplied by the dimension d_i of p_i by showing that it is consistent with the corresponding definition of a character by S.L. Woronowicz in [17], [18] for his compact quantum groups. Next, we generalize the concept of Sidon sets $\mathcal{E}$ in $\{p_i\}_{i \in I}$. Especially, we introduce the notion of Helgason–Sidon sets by adding a "lacunarity" condition, which is due to S. Helgason in [5], to our definition of a Sidon set. So, we get the following main result:

If f is a finite, complex linear combination of characters defined by a Helgason–Sidon set $\mathcal{E}$, then there is a constant $K_{\mathcal{E}}$ depending only on $\mathcal{E}$ such that

$$\|f\|_{2,\varphi} \leqslant K_{\mathcal{E}} \|f\|_{1,\varphi}.$$

In particular, if J is an arbitrary finite subset of I with cardinality $|J|$ such that $\{p_j\}_{j \in J} \subseteq \mathcal{E}$, we obtain

$$\Big\| \sum_{j \in J} \chi_j \Big\|_{2,\varphi} = \sqrt{|J|} \leqslant K_{\mathcal{E}} \Big\| \sum_{j \in J} \chi_j \Big\|_{1,\varphi}.$$

As an outlook, it may be an interesting question, even for groups, if there are any groups without infinite Sidon sets, but which fulfill an inequality similar to (0.1). Probably, one can get some results for so-called Λ_p sets, $p \in (1, \infty)$, see e.g. [6] for further details.

This paper arose from the first author's thesis [1].

1. PRELIMINARIES

Mainly for technical reasons, we formulate all our results for compact Kac algebras. In the following, we briefly summarize its theory. For a detailed exposition of the theory of Kac algebras, we refer to [3]. Furthermore, we give the definition of the (generalized) Fourier transform and its main results from [7], which we use throughout this paper.

Let M be a von Neumann algebra. A *co-product* on M is an injective, unital, normal $*$-homomorphism $\Delta : M \to M\bar{\otimes}M$, where $M\bar{\otimes}M$ denotes the von Neumann algebraic tensor product, which has the following co-associativity property

$$(\Delta \otimes \iota) \circ \Delta = (\iota \otimes \Delta) \circ \Delta,$$

where ι is the identity map of M. The pair (M, Δ) is called a *Hopf–von Neumann algebra*. A *co-involution* κ on (M, Δ) is a normal $*$-anti-automorphism of M such that

$$\kappa^2 = \iota \quad \text{and} \quad \zeta \circ \Delta \circ \kappa = (\kappa \otimes \kappa) \circ \Delta,$$

where $\zeta : M\bar{\otimes}M \to M\bar{\otimes}M$ denotes the flip defined by $\zeta(x_1 \otimes x_2) := x_2 \otimes x_1$ for all $x_1, x_2 \in M$. Then (M, Δ, κ) is called a *co-involutive Hopf–von Neumann algebra*.

Let (M, Δ) be a Hopf–von Neumann algebra and M_* the predual of M, i.e. the Banach space of all σ-weakly continuous linear functionals on M. Then the co-product Δ induces a multiplication $*$ on M_*, which for all $\omega_1, \omega_2 \in M_*$ and $x \in M$ is defined by

$$(x, \omega_1 * \omega_2) := (\Delta(x), \omega_1 \otimes \omega_2). \tag{1.1}$$

In analogy to the ordinary convolution in the predual $L^1(G)$ of $L^\infty(G)$ with a locally compact group G, it is called the *convolution* of ω_1 and ω_2.

Now, let (M, Δ, κ) be a co-involutive Hopf–von Neumann algebra. Then the co-involution κ gives an involution $\circ$ on M_*, which for all $\omega \in M_*$ and $x \in M$ is defined by

$$(x, \omega^\circ) := \overline{(\kappa(x^*), \omega)}.$$

Equipped with the above convolution $*$ and the involution $\circ$, the predual M_* becomes a Banach $*$-algebra.

Let φ be a normal semi-finite faithful (n.s.f.) weight on M. Then one defines

$$\mathfrak{N}_\varphi := \{x \in M : \varphi(x^*x) < \infty\}, \quad \mathfrak{M}_\varphi := \mathfrak{N}_\varphi^*\mathfrak{N}_\varphi \quad \text{and}$$
$$\mathfrak{M}_\varphi^+ := \{x \in M^+ : \varphi(x) < \infty\}.$$

An n.s.f. weight φ on M is called *left invariant* with respect to Δ if

$$(\iota \otimes \varphi)\Delta(x) = \varphi(x)1, \quad x \in M^+.$$

A *Kac algebra* $\mathbb{K}$ is a quadruple $(M, \Delta, \kappa, \varphi)$ consisting of a co-involutive Hopf–von Neumann algebra (M, Δ, κ) and a left invariant weight φ on (M, Δ) such that the following two equations hold:

$$\kappa(\iota \otimes \varphi)(\Delta(x_1^*)(1 \otimes x_2)) = (\iota \otimes \varphi)((1 \otimes x_1^*)\Delta(x_2)) \quad x_1, x_2 \in \mathfrak{N}_\varphi,$$
$$\kappa \circ \sigma_t^\varphi = \sigma_{-t}^\varphi \circ \kappa \quad t \in \mathbb{R},$$

where $(\sigma_t^\varphi)_{t \in \mathbb{R}}$ denotes the modular automorphism group of φ. The weight φ is also called the *left Haar weight*.

If M is commutative, $\mathbb{K} = (M, \Delta, \kappa, \varphi)$ is called *abelian*. If $\zeta \circ \Delta = \Delta$, $\mathbb{K}$ is called *symmetric*. Abelian Kac algebras can be identified with locally compact groups G so that we get $M \cong L^\infty(G)$; if they are additionally symmetric, they can be identified with locally compact abelian groups.

Now, let H_φ be the GNS-space induced by φ with the embedding $\Lambda_\varphi : \mathfrak{N}_\varphi \to H_\varphi$ and the scalar product

$$\langle \Lambda_\varphi(x_1), \Lambda_\varphi(x_2) \rangle := \varphi(x_2^* x_1), \quad x_1, x_2 \in \mathfrak{N}_\varphi.$$

If $B(H_\varphi)$ denotes the set of all bounded linear operators on H_φ, the so-called *fundamental operator* $W \in B(H_\varphi \otimes H_\varphi)$ associated with $\mathbb{K}$ is defined by

$$W(\Lambda_\varphi(x_1) \otimes \Lambda_\varphi(x_2)) := \Lambda_{\varphi \otimes \varphi}(\Delta(x_2)(x_1 \otimes 1)), \quad x_1, x_2 \in \mathfrak{N}_\varphi.$$

Therefore, the *Fourier representation* λ of $\mathbb{K}$ is introduced by

$$\lambda : M_* \to B(H_\varphi), \quad M_* \ni \omega \mapsto (\omega \otimes \iota)(W^*).$$

By means of λ, the von Neumann algebra $\widehat{M}$ is defined by the double commutant of $\lambda(M_*)$, i.e. $\widehat{M} := \lambda(M_*)''$, and the C^*-algebra associated with $\widehat{M}$ by $\widehat{M}_c := \overline{\lambda(M_*)}^{\text{norm}}$. The *dual Fourier representation* $\widehat{\lambda}$ of $\widehat{\mathbb{K}}$ is defined by

$$\widehat{\lambda} : \widehat{M}_* \to B(H_{\widehat{\varphi}}), \quad \widehat{M}_* \ni \theta \mapsto (\iota \otimes \theta)(W) \in M.$$

Overall, in Theorem 4.1.1 of [3], it is obtained a duality theorem, i.e. for every Kac algebra $\mathbb{K} = (M, \Delta, \kappa, \varphi)$, there is a *dual Kac algebra* $\widehat{\mathbb{K}} = (\widehat{M}, \widehat{\Delta}, \widehat{\kappa}, \widehat{\varphi})$ such that the bidual Kac algebra $\widehat{\widehat{\mathbb{K}}}$ is isomorphic to $\mathbb{K}$. This result generalizes the Pontryagin duality theorem for locally compact abelian groups, see e.g. Theorem 1.7.2 of [13].

A Kac algebra $(M, \Delta, \kappa, \varphi)$ is called *compact* if φ is finite, and it is called *discrete* if the predual M_* of M is unital. By Theorem 6.2.2 (Theorem 6.3.2) of [3], compact (discrete) Kac algebras generalize compact (discrete) groups. Moreover, with regard to Theorem 6.3.3 of [3], $\mathbb{K}$ is compact if and only if $\widehat{\mathbb{K}}$ is discrete.

In [17], [18], S.L. Woronowicz introduces the notion of compact quantum groups, which are more general than compact Kac algebras. Nevertheless, the definition of compact quantum groups is so simple that it guarantees the existence of the Haar state φ, whereas its existence has to be assumed in case of compact Kac algebras. Furthermore, in case of compact quantum groups, the overall representation theory remains more or less the same, but the co-involution (antipode map) κ can be unbounded, and we may have $\kappa^2 \neq \iota$. This leads to technical difficulties so that we formulate our results within the framework of compact Kac algebras. For further discussions on compact or discrete quantum groups, respectively, we refer to [11], [15], [16].

For the following, let $\mathbb{K} = (M, \Delta, \kappa, \varphi)$ be always a compact Kac algebra. Then, by Theorem 6.2.1 and Corollary 6.3.4(i) of [3], $\mathbb{K}$ and its dual, discrete Kac algebra $\widehat{\mathbb{K}} = (\widehat{M}, \widehat{\Delta}, \widehat{\kappa}, \widehat{\varphi})$ are *unimodular*, i.e. φ is a κ-invariant trace as well as $\widehat{\varphi}$ is a $\widehat{\kappa}$-invariant trace. Now, see [2], [14], one defines the L^p-spaces $L^p(M, \varphi)$ for all $p \in [1, \infty)$ to be the completion of the set $\{x \in M : \|x\|_{p,\varphi} := \varphi(|x|^p)^{1/p} < \infty\}$ with respect to the L^p-norm $\|\cdot\|_{p,\varphi}$.

Furthermore, according to Theorem 6.2.5(iii),(v) of [3], it is possible to decompose the Fourier representation λ in a direct sum of irreducible, finite-dimensional $*$-representations $(\lambda_i)_{i\in I}$ of M_* with $\lambda_i(\omega) := \lambda(\omega)p_i$ for all $\omega \in M_*$ and $i \in I$, where I denotes an index set and $\{p_i\}_{i\in I}$ denotes the set of all minimal central projections in $\widehat{M}$. Therefore, for every $i \in I$, there exists a Hilbert space H_i with $d_i := \dim H_i < \infty$ such that

$$\widehat{M} \cong \bigoplus_{i\in I} B(H_i) \quad \text{and} \quad \widehat{\varphi} = \sum_{i\in I} d_i \mathrm{Tr}_i, \tag{1.2}$$

where Tr_i denotes the canonical trace on $B(H_i)$ for all $i \in I$. In fact, taking into account Theorem 6.2.6(i),(ii) of [3], every non-degenerate $*$-representation of M_* is decomposable into a direct sum of irreducible, finite-dimensional $*$-representations of M_*, where every irreducible $*$-representation of M_* is automatically finite-dimensional and equivalent to a component of the Fourier representation λ of $\mathbb{K}$. Consequently, λ is the sum of all equivalent classes of irreducible $*$-representations of M_*.

In the situation of a compact Kac algebra $\mathbb{K} = (M, \Delta, \kappa, \varphi)$, we have $\mathfrak{M}_\varphi = \mathfrak{N}_\varphi = M$. For all $x \in M$, we define elements $\omega_x \in M_*$ by $\omega_x := \varphi(\cdot\, x)$. Therefore, ω_x is *square-integrable*, i.e. $\omega_x \in \mathcal{I}$ with

$$\mathcal{I} := \{\omega \in M_* : \exists\, L \geqslant 0 \text{ such that } |\omega(x^*)| \leqslant L\|\Lambda_\varphi(x)\| \;\forall\, x \in \mathfrak{N}_\varphi\},$$

where $\mathcal{I} = L^1(G) \cap L^2(G)$ in case of the abelian Kac algebra $\mathbb{K}_{\mathrm{a}}(G)$ with a locally compact group G. Let $\widehat{\mathbb{K}} = (\widehat{M}, \widehat{\Delta}, \widehat{\kappa}, \widehat{\varphi})$ be the dual, discrete Kac algebra of $\mathbb{K}$. Then one defines

$$\widehat{\mathcal{I}} := \{\theta \in \widehat{M}_* : \exists\, L \geqslant 0 \text{ such that } |\theta(y^*)| \leqslant L\|\Lambda_{\widehat{\varphi}}(y)\| \;\forall\, y \in \mathfrak{N}_{\widehat{\varphi}}\}.$$

Now, we introduce the *Fourier transform* $\mathcal{F}(x) := \lambda(\omega_x) \in \lambda(\mathcal{I}) \subseteq \widehat{M}$ of $x \in M$ and the *inverse Fourier transform* $\mathcal{F}^{-1}(y) := \widehat{\lambda}(\theta_y) \in \widehat{\lambda}(\widehat{\mathcal{I}}) \subseteq M$ of $y \in \mathfrak{M}_{\widehat{\varphi}} \subseteq \widehat{M}$ with $\theta_y := \widehat{\varphi}(\cdot\, y) \in \widehat{\mathcal{I}} \subseteq \widehat{M}_*$, which are formally defined in [7] for all $x \in \widehat{\lambda}(\widehat{\mathcal{I}}) \subseteq M$ and for all $y \in \lambda(\mathcal{I}) \subseteq \widehat{M}$ in case of an arbitrary locally compact quantum group in the von Neumann algebraic setting introduced in [10]. The corresponding C^*-algebraic version of a locally compact quantum group is given in [9], where, in particular, a general duality theorem is shown. This category includes both the Kac algebras in [3] and the compact quantum groups in [17], [18]. But, in contrast to compact quantum groups, the existence of the left Haar weight φ has to be assumed in the definition of locally compact quantum groups in [9], [10].

Moreover, as in case of groups, we have the *Fourier inversion theorem*

$$\mathcal{F}^{-1}(\mathcal{F}(x)) = x \quad \text{and} \quad \mathcal{F}(\mathcal{F}^{-1}(y)) = y \tag{1.3}$$

as well as the *Plancherel formula*

$$\widehat{\varphi}(\mathcal{F}(x_1)^*\mathcal{F}(x_2)) = \varphi(x_1^*x_2) \quad \text{and} \quad \varphi(\mathcal{F}^{-1}(y_1)^*\mathcal{F}^{-1}(y_2)) = \widehat{\varphi}(y_1^*y_2) \tag{1.4}$$

given in [7] when $x = x_1 = x_2 \in \widehat{\lambda}(\widehat{\mathcal{I}}) \subseteq M$ and $y = y_1 = y_2 \in \lambda(\mathcal{I}) \subseteq \widehat{M}$, which can be easily generalized to the above formulas by the standard "polarization identity" technique.

Since φ and $\widehat{\varphi}$ are faithful, we identify $\mathfrak{N}_\varphi$ and $\mathfrak{N}_{\widehat{\varphi}}$ with $\Lambda_\varphi(\mathfrak{N}_\varphi) \subseteq H_\varphi$ and $\Lambda_{\widehat{\varphi}}(\mathfrak{N}_{\widehat{\varphi}}) \subseteq H_{\widehat{\varphi}}$, respectively. Let $\mathcal{F}$ and $\mathcal{F}^{-1}$ also denote the extensions of the Fourier transform and the inverse Fourier transform on H_φ and $H_{\widehat{\varphi}}$, respectively. Then, for all $\xi, \eta \in H_\varphi$, we define a *convolution* $*$ on H_φ by

$$\xi * \eta := \mathcal{F}^{-1}(\mathcal{F}(\xi)\mathcal{F}(\eta)), \tag{1.5}$$

which is based on the convolution $*$ on $\widehat{\lambda}(\widehat{\mathcal{I}}) \subseteq M$ introduced in [7].

Now, for the rest of the paper, we make the following *general assumption*: We assume that $\mathbb{K} = (M, \Delta, \kappa, \varphi)$ is a compact Kac algebra such that $\varphi(1) = 1$. Moreover, λ will always denote the Fourier representation of $\mathbb{K}$ and $\widehat{\mathbb{K}} = (\widehat{M}, \widehat{\Delta}, \widehat{\kappa}, \widehat{\varphi})$ the dual, discrete Kac algebra of $\mathbb{K}$ with the dual Fourier representation $\widehat{\lambda}$ of $\widehat{\mathbb{K}}$. Furthermore, $\mathcal{F}$ and $\mathcal{F}^{-1}$ will always denote the Fourier transform and the inverse Fourier transform, respectively, as well as $\{p_i\}_{i \in I}$ the set of all minimal central projections in $\widehat{M}$ with dimensions d_i.

2. CHARACTERS IN COMPACT KAC ALGEBRAS

Since the p_i, $i \in I$, are projections of the aforementioned type, we have $p_i \in \mathfrak{M}_{\widehat{\varphi}} \subseteq \mathfrak{N}_{\widehat{\varphi}} \subseteq \widehat{M}$, and each p_i is contained in the domain of $\mathcal{F}^{-1}$. Consequently, the expression $\mathcal{F}^{-1}(p_i)$ makes sense and the following definition is valid.

DEFINITION 2.1. For all $i \in I$, we define

$$\chi_i := \frac{\mathcal{F}^{-1}(p_i)}{d_i},$$

and call $\chi_i \in \widehat{\lambda}(\widehat{\mathcal{I}}) \subseteq M$ the *character* of p_i.

Since φ is a state and, taking into account Theorem 6.2.5(ii) of [3], $\lambda(\varphi)$ is a central projection p_0 such that $d_0 = 1$, which projects H_φ onto $\mathbb{C}\Lambda_\varphi(1) = \mathbb{C}1$, we have $\lambda(\varphi) \in \mathfrak{M}_{\widehat{\varphi}} \subseteq \mathfrak{N}_{\widehat{\varphi}} \subseteq \widehat{M}$. Hence, $\lambda(\varphi)$ is contained in the domain of $\mathcal{F}^{-1}$. Consequently, the expression $\mathcal{F}^{-1}(\lambda(\varphi))$ makes sense and the following definition is valid, too.

DEFINITION 2.2. We define a character χ_0 by

$$\chi_0 := \mathcal{F}^{-1}(\lambda(\varphi)),$$

and call $\chi_0 \in \widehat{\lambda}(\widehat{\mathcal{I}}) \subseteq M$ the *one-character* of $\mathbb{K}$.

In order to show that our definition of a character is consistent with the character in [17], [18] for compact quantum groups, we need the following

DEFINITION 2.3. For all $i \in I$, we define an irreducible, finite-dimensional $*$-representation λ_i^c of M_* by

$$\lambda_i^c(\omega_x) := \lambda_i(\omega_{\kappa(x^*)}), \quad x \in M,$$

and call λ_i^c the *conjugate representation* of λ_i.

REMARK 2.4. (i) According to Theorem 6.2.6(ii) of [3], every irreducible $*$-representation of M_* is equivalent to a component λ_i of λ. Hence, it suffices to define the conjugate representation of λ_i for each $i \in I$. Since, by Lemma 6.1.1(i) of [3], the set $\{\omega_x : x \in M\}$ is dense in M_*, Definition 2.3 is valid.

(ii) Regarding λ_i as an element of $B(H_i)\bar{\otimes}M$, we get

$$\lambda_i^c = ({}^*\otimes\kappa)(\lambda_i),$$

where $*$ here denotes the involution on $B(H_i)$. Hence, Definition 2.3 is consistent with the conjugate representation in formula (3.11) of [17] and [18] for compact quantum groups generalizing the corresponding notion for compact groups.

LEMMA 2.5. *For all $i \in I$, we have*

$$\chi_{i^c} = \chi_i^*,$$

where χ_{i^c} denotes the character of the minimal central projection p_{i^c} of the conjugate representation λ_i^c of λ_i.

Proof. Let $y \in \mathfrak{M}_{\widehat{\varphi}}$ such that $\theta_y := \widehat{\varphi}(\cdot\, y) \in \widehat{M}_*$. Since $\widehat{\varphi}$ is a $\widehat{\kappa}$-invariant trace, we have $\theta_{\widehat{\kappa}(y^*)} = \theta_y^\circ$ by Lemma 6.1.1(ii) of [3]. Since $p_i \in \mathfrak{M}_{\widehat{\varphi}}$ and since $\widehat{\lambda}$ is a $*$-representation of $\widehat{M}_*$, we get, for all $i \in I$,

$$d_i\chi_{i^c} = \mathcal{F}^{-1}(p_{i^c}) = \widehat{\lambda}(\theta_{p_{i^c}}) = \widehat{\lambda}(\theta_{\widehat{\kappa}(p_i^*)}) = \widehat{\lambda}(\theta_{p_i}^\circ) = \widehat{\lambda}(\theta_{p_i})^* = \mathcal{F}^{-1}(p_i)^* = d_i\chi_i^*.$$

Thus we have $\chi_{i^c} = \chi_i^*$ for all $i \in I$. ∎

LEMMA 2.6. *For all $i \in I$, we have*

$$\chi_i = (\mathrm{Tr}_i \otimes \iota)(\lambda_i^c).$$

Proof. According to [7], we have $\Lambda_\varphi(x) = \Lambda_{\widehat{\varphi}}(\mathcal{F}(x))$ in H_φ for all $x \in M$ and $\Lambda_\varphi(\mathcal{F}^{-1}(y)) = \Lambda_{\widehat{\varphi}}(y)$ in H_φ for all $y \in \mathfrak{M}_{\widehat{\varphi}}$. Hence, on the one hand, we get by Lemma 2.5, for all $x \in M$ and any fixed $i \in I$,

$$\begin{aligned}(d_i\chi_i, \omega_x) &= (\mathcal{F}^{-1}(p_i), \omega_x) = \varphi(\mathcal{F}^{-1}(p_i)x) = \langle \Lambda_\varphi(x), \Lambda_\varphi(\mathcal{F}^{-1}(p_i)^*)\rangle \\ &= \langle \Lambda_\varphi(x), \Lambda_\varphi(\mathcal{F}^{-1}(p_{i^c}))\rangle = \langle \Lambda_{\widehat{\varphi}}(\mathcal{F}(x)), \Lambda_{\widehat{\varphi}}(p_{i^c})\rangle \\ &= \widehat{\varphi}(p_{i^c}^*\mathcal{F}(x)) = \widehat{\varphi}(p_{i^c}\mathcal{F}(x)).\end{aligned}$$

On the other hand, regarding λ_i^c as an element of $B(H_i)\bar{\otimes}M$ and using the decomposition of $\widehat{\varphi}$ into $\sum_{k\in I} d_k \mathrm{Tr}_k$ by (1.2), we obtain, for all $x \in M$ and any fixed $i \in I$,

$$(d_i(\mathrm{Tr}_i \otimes \iota)(\lambda_i^c), \omega_x) = d_i\mathrm{Tr}_i(\lambda_i^c(\omega_x)) = \widehat{\varphi}(\lambda_i^c(\omega_x)) = \widehat{\varphi}(p_{i^c}\lambda(\omega_x)) = \widehat{\varphi}(p_{i^c}\mathcal{F}(x)).$$

Since, by Lemma 6.1.1(i) of [3], the set $\{\omega_x := \varphi(\cdot\, x) : x \in M\}$ is dense in M_*, we finally get, for all $i \in I$,

$$\chi_i = (\mathrm{Tr}_i \otimes \iota)(\lambda_i^c). \quad ∎$$

In accordance with Lemma 2.6, our Definition 2.1 of a character is consistent with the definition on p. 657 of [17] and [18] of a character of a finite-dimensional representation of a compact quantum group generalizing the corresponding notion in the situation of compact groups. Consequently, our characters possess all the classical properties known in case of compact groups, see e.g. [6]. In particular, taking into account Theorem 6.2.6(i),(ii) of [3], for every $i \in I$, there are finite subsets K_i and L_i of I both containing the index 0 such that we have the following decompositions

$$\chi_i^*\chi_i = \sum_{k\in K_i} n_k\chi_k \quad \text{and} \quad \chi_i\chi_i^* = \sum_{l\in L_i} m_l\chi_l, \tag{2.1}$$

where n_k and m_l denote the multiplicities of χ_k and χ_l for all $k \in K_i$ and $l \in L_i$, respectively.

PROPOSITION 2.7. *For all $i, j \in I$, we have*

$$\chi_i * \chi_j = \frac{1}{d_i}\delta_{ij}\chi_i,$$

where δ_{ij} denotes the Kronecker symbol and the convolution $*$ is given by (1.5).

Proof. Applying the Fourier inversion theorem (1.3) and the orthogonality of the set $\{p_i\}_{i\in I}$, we get, for all $i, j \in I$,

$$\begin{aligned}\chi_i * \chi_j &= \mathcal{F}^{-1}(\mathcal{F}(\chi_i)\mathcal{F}(\chi_j)) = \mathcal{F}^{-1}\Big(\mathcal{F}\Big(\frac{\mathcal{F}^{-1}(p_i)}{d_i}\Big)\mathcal{F}\Big(\frac{\mathcal{F}^{-1}(p_j)}{d_j}\Big)\Big) \\ &= \mathcal{F}^{-1}\Big(\frac{1}{d_i}p_i\frac{1}{d_j}p_j\Big) = \frac{1}{d_id_j}\mathcal{F}^{-1}(p_ip_j) = \begin{cases}\frac{1}{d_i}\chi_i & \text{if } i = j, \\ 0 & \text{if } i \neq j.\end{cases} \quad ∎\end{aligned}$$

PROPOSITION 2.8. *For all $i, j \in I$, we have the orthogonality relations*

$$\varphi(\chi_i^* \chi_j) = \delta_{ij} \quad \text{and} \quad \varphi(\chi_i \chi_j^*) = \delta_{ij}.$$

Proof. Using the Plancherel formula (1.4), the orthogonality of the set $\{p_i\}_{i\in I}$ and the decomposition of $\widehat{\varphi}$ into $\sum_{k\in I} d_k \mathrm{Tr}_k$ by (1.2), we obtain, for all $i, j \in I$,

$$\varphi(\chi_i^*\chi_j) = \varphi\Big(\Big(\frac{\mathcal{F}^{-1}(p_i)}{d_i}\Big)^* \frac{\mathcal{F}^{-1}(p_j)}{d_j}\Big) = \frac{1}{d_i d_j}\varphi(\mathcal{F}^{-1}(p_i)^*\mathcal{F}^{-1}(p_j)) = \frac{1}{d_i d_j}\widehat{\varphi}(p_i^* p_j)$$

$$= \frac{1}{d_i d_j}\widehat{\varphi}(p_i\delta_{ij}) = \begin{cases} \frac{1}{d_i^2}\sum_{k\in I} d_k \mathrm{Tr}_k(p_i) = \frac{1}{d_i^2} d_i \mathrm{Tr}_i(p_i) = \frac{1}{d_i^2} d_i^2 = 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Since φ is a trace, we get $\varphi(\chi_i\chi_j^*) = \delta_{ij}$ for all $i, j \in I$, too. ∎

PROPOSITION 2.9. *Let J be a finite subset of I, and let $a_j \in \mathbb{C}$ for all $j \in J$. Then we have, with $f := \sum_{j\in J} a_j \chi_j$,*

$$\|f\|_{2,\varphi} = \sqrt{\sum_{j\in J}|a_j|^2}, \quad \textit{in particular} \quad \Big\|\sum_{j\in J}\mathcal{F}^{-1}(p_j)\Big\|_{2,\varphi} = \sqrt{\sum_{j\in J} d_j^2} \quad \textit{and} \quad \Big\|\sum_{j\in J}\chi_j\Big\|_{2,\varphi} = \sqrt{|J|}.$$

Proof. Applying Proposition 2.8, we get

$$\|f\|_{2,\varphi}^2 = \varphi(|f|^2) = \varphi\Big(\Big(\sum_{j\in J} a_j\chi_j\Big)^* \sum_{k\in J} a_k\chi_k\Big) = \sum_{j\in J}\sum_{k\in J}\overline{a}_j a_k \varphi(\chi_j^*\chi_k) = \sum_{j\in J}|a_j|^2. \quad ∎$$

PROPOSITION 2.10. *Let J be a finite subset of I, and let $a_j \in \mathbb{C}$ for all $j \in J$. Then we have, with $f := \sum_{j\in J} a_j \chi_j$,*

$$\|f\|_{1,\varphi} \leqslant \sqrt{\sum_{j\in J}|a_j|^2}, \quad \textit{in particular} \quad \Big\|\sum_{j\in J}\mathcal{F}^{-1}(p_j)\Big\|_{1,\varphi} \leqslant \sqrt{\sum_{j\in J} d_j^2} \quad \textit{and} \quad \Big\|\sum_{j\in J}\chi_j\Big\|_{1,\varphi} \leqslant \sqrt{|J|}.$$

Proof. Since φ is a state on M, it follows from the Cauchy–Schwarz inequality that, for all $x \in M$,

$$\|x\|_{1,\varphi}^2 = (\varphi(|x|))^2 \leqslant \varphi(1)\varphi(x^*x) = \varphi(|x|^2) = \|x\|_{2,\varphi}^2.$$

Consequently, since $f := \sum_{j\in J} a_j\chi_j \in M$, we get the assertion by Proposition 2.9. ∎

LEMMA 2.11. *For all $i \in I$, we have*

$$\frac{\|\chi_i\|_{2,\varphi}^3}{\|\chi_i\|_{4,\varphi}^2} \leqslant \|\chi_i\|_{1,\varphi}.$$

Proof. Since φ is a trace on $M = \mathfrak{N}_\varphi = \mathfrak{M}_\varphi$, it follows from Hölder's inequality, see Théorème 3.6 of [2], with $p = 3/2$ and $q = 3$ that, for all $i \in I$,

$$\varphi(|\chi_i|^2) \leqslant (\varphi(|\chi_i|))^{2/3}(\varphi(|\chi_i|^4))^{1/3}.$$

Therefore, we get, for all $i \in I$,

$$\|\chi_i\|_{2,\varphi}^3 = (\varphi(|\chi_i|^2))^{3/2} \leqslant \varphi(|\chi_i|)(\varphi(|\chi_i|^4))^{1/2} = \|\chi_i\|_{1,\varphi}\|\chi_i\|_{4,\varphi}^2. \quad \blacksquare$$

PROPOSITION 2.12. *Let $a_i \in \mathbb{C}$ for all $i \in I$. Then we have, for any fixed $i \in I$,*

$$\|a_i\chi_i\|_{4,\varphi} = |a_i|\Big(\sum_{k\in K_i} n_k^2\Big)^{1/4} = |a_i|\Big(\sum_{l\in L_i} m_l^2\Big)^{1/4}, \quad \textit{in particular}$$

$$\|\mathcal{F}^{-1}(p_i)\|_{4,\varphi} = d_i\Big(\sum_{k\in K_i} n_k^2\Big)^{1/4} = d_i\Big(\sum_{l\in L_i} m_l^2\Big)^{1/4} \quad \textit{and}$$

$$\|\chi_i\|_{4,\varphi} = \Big(\sum_{k\in K_i} n_k^2\Big)^{1/4} = \Big(\sum_{l\in L_i} m_l^2\Big)^{1/4}.$$

Proof. Applying Proposition 2.8, we have, for all $i \in I$,

$$\varphi((\chi_i^*\chi_i)^2) = \varphi(\chi_i^*\chi_i\chi_i^*\chi_i) = \varphi((\chi_i^*\chi_i)^*(\chi_i^*\chi_i)) = \varphi\Big(\Big(\sum_{k\in K_i} n_k\chi_k\Big)^*\Big(\sum_{l\in K_i} n_l\chi_l\Big)\Big)$$
$$= \sum_{k\in K_i}\sum_{l\in K_i} n_k n_l \varphi(\chi_k^*\chi_l) = \sum_{k\in K_i} n_k^2.$$

In like manner, we get $\varphi((\chi_i\chi_i^*)^2) = \sum_{l\in L_i} m_l^2$ for all $i \in I$. Since φ is a trace on $M = \mathfrak{N}_\varphi = \mathfrak{M}_\varphi$ and $\chi_i \in M$ for each $i \in I$, we obtain, by Lemma V.2.16 of [14], for all $i \in I$,

$$\varphi((\chi_i^*\chi_i)^2) = \varphi(\chi_i^*\chi_i\chi_i^*\chi_i) = \varphi(\chi_i\chi_i^*\chi_i\chi_i^*) = \varphi((\chi_i\chi_i^*)^2).$$

Hence, it follows that

$$\|a_i\chi_i\|_{4,\varphi}^4 = \varphi((|a_i\chi_i|)^4) = |a_i|^4\varphi((\chi_i^*\chi_i)^2) = |a_i|^4\sum_{k\in K_i} n_k^2 = |a_i|^4\sum_{l\in L_i} m_l^2,$$

and we get the assertion. $\blacksquare$

PROPOSITION 2.13. *For any fixed $i \in I$ and for all $k \in K_i$ and $l \in L_i$, we have*

$$n_k \leqslant d_k \quad \textit{and} \quad m_l \leqslant d_l.$$

In particular, if $d_k = 1$, we have $n_k = 1$. Similarly, if $d_l = 1$, then $m_l = 1$. Thus we get, for all $i \in I$ with $K_i' := K_i\backslash\{0\}$ and $L_i' := L_i\backslash\{0\}$,

$$\chi_i^*\chi_i = \chi_0 + \sum_{k\in K_i'} n_k\chi_k \quad \textit{and} \quad \chi_i\chi_i^* = \chi_0 + \sum_{l\in L_i'} m_l\chi_l.$$

Hence, the one-character χ_0 of $\mathbb{K}$ occurs exactly once in the decompositions of $\chi_i^*\chi_i$ and $\chi_i\chi_i^*$ for all $i \in I$, respectively.

Proof. Using the Fourier inversion theorem (1.3) and the orthogonality of the set $\{p_i\}_{i\in I}$, we get, for any fixed $i \in I$ and for all $k \in K_i$,

$$\mathcal{F}(\chi_i^*\chi_i)p_k = \mathcal{F}\Big(\sum_{j\in K_i} n_j\chi_j\Big)p_k = p_k \sum_{j\in K_i} n_j\mathcal{F}(\chi_j) = p_k \sum_{j\in K_i} n_j\mathcal{F}\Big(\frac{\mathcal{F}^{-1}(p_j)}{d_j}\Big)$$
$$= p_k \sum_{j\in K_i} \frac{n_j}{d_j}p_j = \frac{n_k}{d_k}p_k.$$

Since p_k is the identity operator in $B(H_k)$ for all $k \in K_i$, we obtain

$$\|p_k\|_{\widehat{M}} = \|p_k\|_{B(H_k)} = 1.$$

Regarding Proposition 2.4.6(i) of [3], we have $\|\lambda(\omega)\|_{\widehat{M}} \leqslant \|\omega\|_{M_*}$ for all $\omega \in M_*$. Since φ is a trace on M, the Banach spaces M_* and $L^1(M,\varphi)$ are isometrically isomorphic by Theorem V.2.18 of [14]. Therefore, applying Proposition 2.8, we get, for all $k \in K_i$,

$$\frac{n_k}{d_k} = \frac{n_k}{d_k}\|p_k\|_{\widehat{M}} = \Big\|\frac{n_k}{d_k}p_k\Big\|_{\widehat{M}} = \|\mathcal{F}(\chi_i^*\chi_i)p_k\|_{\widehat{M}} \leqslant \|\mathcal{F}(\chi_i^*\chi_i)\|_{\widehat{M}}\|p_k\|_{\widehat{M}} = \|\mathcal{F}(\chi_i^*\chi_i)\|_{\widehat{M}}$$
$$= \|\lambda(\omega_{\chi_i^*\chi_i})\|_{\widehat{M}} \leqslant \|\omega_{\chi_i^*\chi_i}\|_{M_*} = \|\chi_i^*\chi_i\|_{1,\varphi} = \varphi(\chi_i^*\chi_i) = 1.$$

Hence, we have shown $n_k \leqslant d_k$ for all $k \in K_i$ and any fixed $i \in I$. Similarly, for any fixed $i \in I$ and for all $l \in L_i$, we get the second inequality $m_l \leqslant d_l$. ∎

THEOREM 2.14. *Let $a_i \in \mathbb{C}$ for all $i \in I$. Then we have, for any fixed $i \in I$,*

$$\|a_i\chi_i\|_{1,\varphi} \geqslant \frac{|a_i|}{\sqrt{\sum_{k\in K_i} n_k^2}} = \frac{|a_i|}{\sqrt{\sum_{l\in L_i} m_l^2}}, \quad \textit{in particular}$$
$$\|\mathcal{F}^{-1}(p_i)\|_{1,\varphi} \geqslant \frac{d_i}{\sqrt{\sum_{k\in K_i} n_k^2}} = \frac{d_i}{\sqrt{\sum_{l\in L_i} m_l^2}} \geqslant 1 \quad \textit{and}$$
$$\|\chi_i\|_{1,\varphi} \geqslant \frac{1}{\sqrt{\sum_{k\in K_i} n_k^2}} = \frac{1}{\sqrt{\sum_{l\in L_i} m_l^2}}.$$

Proof. For any fixed $i \in I$, it follows from Lemma 2.11 and Propositions 2.9 and 2.12 that

$$\|a_i\chi_i\|_{1,\varphi} = |a_i|\|\chi_i\|_{1,\varphi} \geqslant |a_i|\frac{\|\chi_i\|_{2,\varphi}^3}{\|\chi_i\|_{4,\varphi}^2} = \frac{|a_i|}{\sqrt{\sum_{k\in K_i} n_k^2}} = \frac{|a_i|}{\sqrt{\sum_{l\in L_i} m_l^2}}.$$

Since the dimension of $\chi_i\chi_i^*$ is equal to d_i^2, we obtain, with regard to the decomposition $\chi_i\chi_i^* = \sum\limits_{l\in L_i} m_l\chi_l$ by (2.1), for all $i \in I$,

$$d_i^2 = \sum_{l\in L_i} m_l d_l.$$

According to Proposition 2.13, for any fixed $i \in I$ and for all $l \in L_i$, we have $m_l \leqslant d_l$. Thus we finally get

$$\frac{d_i}{\sqrt{\sum_{l \in L_i} m_l^2}} = \frac{\sqrt{\sum_{l \in L_i} m_l d_l}}{\sqrt{\sum_{l \in L_i} m_l^2}} \geqslant \frac{\sqrt{\sum_{l \in L_i} m_l^2}}{\sqrt{\sum_{l \in L_i} m_l^2}} = 1. \quad \blacksquare$$

3. CHARACTERS DEFINED BY A SIDON SET

Since best for our purposes, we use the characterization of a Sidon set by Theorem 5.7.3(e) of [13] in case of compact abelian groups or Theorem 37.2(iii) of [6] in case of arbitrary compact groups, respectively, for the following generalization of the concept of a Sidon set to compact Kac algebras.

DEFINITION 3.1. Let E be a subset of I. We call $\mathcal{E} := \{p_i\}_{i \in E}$ a *Sidon set* if for every $\psi \in \Big(\sum\limits_{i \in E} p_i\Big)\widehat{M}_{\mathrm{c}} := \Big\{\Big(\sum\limits_{i \in E} p_i\Big)y : y \in \widehat{M}_{\mathrm{c}}\Big\}$, there is an $\omega \in M_*$ such that, for all $i \in E$,

$$\psi p_i = \lambda(\omega) p_i.$$

We can choose ω in such a way that there is a constant $B_{\mathcal{E}}$ depending only on $\mathcal{E}$ such that

$$\|\omega\|_{M_*} \leqslant B_{\mathcal{E}} \|\psi\|_{\widehat{M}_{\mathrm{c}}}.$$

REMARK 3.2. Since $\widehat{M}_{\mathrm{c}} \cong C_0(\Gamma)$ for locally compact abelian groups with dual group Γ and $\widehat{M}_{\mathrm{c}} \cong \mathfrak{C}_0(\Sigma)$ for arbitrary compact groups with dual object Σ, Definition 3.1 actually includes the above mentioned classical cases.

Now, we modify the Rademacher functions, see e.g. [19], which are used in the proof of (0.1), see Theorem 5.7.7(3) of [13].

DEFINITION 3.3. Let $n \in \mathbb{N}$. For $j = 1, \dots, n$, we call the complex-valued functions $r_j(\cdot)$ on $[0,1]$ defined for all $t \in [(k-1)/4^j, k/4^j]$ and $k = 1, \dots, 4^j$ by

$$r_j(t) := \begin{cases} 1 & \text{if } k \equiv 1 \bmod 4, \\ \mathrm{i} & \text{if } k \equiv 2 \bmod 4, \\ -1 & \text{if } k \equiv 3 \bmod 4, \\ -\mathrm{i} & \text{if } k \equiv 0 \bmod 4, \end{cases}$$

where i denotes the imaginary unit, *modified Rademacher functions*.

LEMMA 3.4. *Let $n \in \mathbb{N}$. For all $i, j, k, l \in \{1, \dots, n\}$, we get*

$$\int_0^1 \overline{r_i(t)} r_j(t) \overline{r_k(t)} r_l(t) \, \mathrm{d}t = \begin{cases} 1 & \text{if } i = j \text{ and } k = l, \\ 1 & \text{if } i = l \text{ and } j = k, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first two cases follow from $\overline{r_j(t)}r_j(t) = 1$ for all $j = 1, \dots, n$ and $t \in [0,1]$. In the third case, the intregrals cancel each other out. ∎

DEFINITION 3.5. Let $x : [0,1] \to M$. For all $p \in [1, \infty)$, we define

$$\|x\|_{p,\varphi,[0,1]} := \Big(\int_0^1 \|x(t)\|_{p,\varphi}^p \,\mathrm{d}t \Big)^{1/p} = \Big(\int_0^1 \varphi(|x(t)|^p) \,\mathrm{d}t \Big)^{1/p}.$$

LEMMA 3.6. *Let $\mathcal{E} = \{p_i\}_{i \in E}$ be a Sidon set with constant $B_{\mathcal{E}}$, and let J be a finite subset of E enumerated by j_m, $m = 1, \dots, n := |J|$. Let $a_{j_m} \in \mathbb{C}$ for all $m = 1, \dots, n$. Let $f := \sum_{m=1}^{n} a_{j_m} \chi_{j_m}$, and let $r_{j_m}(\cdot) := r_m(\cdot)$, $m = 1, \dots, n$, be the modified Rademacher functions on $[0,1]$ as well as $g : [0,1] \to M$ such that $g(t) := \sum_{m=1}^{n} r_{j_m}(t) a_{j_m} \chi_{j_m}$ for all $t \in [0,1]$. Then we get*

$$\|g\|_{1,\varphi,[0,1]} \leqslant B_{\mathcal{E}} \|f\|_{1,\varphi}.$$

Proof. For the following, we write j instead of j_m in order to simplify the notations, i.e. $f = \sum_{j \in J} a_j \chi_j$ and $g(t) = \sum_{j \in J} r_j(t) a_j \chi_j$. For all $t \in [0,1]$, we set

$$\psi(t) := \sum_{k \in J} r_k(t) p_k = \sum_{i \in E} p_i \sum_{k \in J} r_k(t) p_k.$$

Since $p_i \in B(H_i) = \lambda_i(M_*)$ for all $i \in I$, we have $\psi(t) \in \Big(\sum_{i \in E} p_i \Big) \widehat{M}_{\mathrm{c}}$ for all $t \in [0,1]$. Consequently, regarding Definition 3.1, for each such $\psi(t)$ with $t \in [0,1]$, there is an $\omega(t) \in M_*$ such that, for all $i \in E$,

$$\psi(t) p_i = \lambda(\omega(t)) p_i \quad \text{and} \quad \|\omega(t)\|_{M_*} \leqslant B_{\mathcal{E}} \|\psi(t)\|_{\widehat{M}_{\mathrm{c}}}.$$

Therefore, we get, for all $t \in [0,1]$,

$$\begin{aligned}
\sum_{j \in J} r_j(t) a_j \frac{1}{d_j} p_j &= \sum_{k \in J} r_k(t) p_k \sum_{j \in J} a_j \frac{1}{d_j} p_j = \psi(t) \sum_{j \in J} a_j \frac{1}{d_j} p_j = \sum_{j \in J} a_j \frac{1}{d_j} \psi(t) p_j \\
&= \sum_{j \in J} a_j \frac{1}{d_j} \lambda(\omega(t)) p_j = \lambda(\omega(t)) \sum_{j \in J} a_j \frac{1}{d_j} p_j.
\end{aligned}$$

According to the Fourier inversion theorem (1.3) and the convolution $*$ in M_* by (1.1), we obtain, for all $t \in [0,1]$,

$$\begin{aligned}
\lambda(\omega_{g(t)}) &= \mathcal{F}(g(t)) = \mathcal{F}\Big(\sum_{j \in J} r_j(t) a_j \chi_j \Big) = \mathcal{F}\Big(\sum_{j \in J} r_j(t) a_j \frac{\mathcal{F}^{-1}(p_j)}{d_j} \Big) \\
&= \sum_{j \in J} r_j(t) a_j \frac{1}{d_j} p_j = \lambda(\omega(t)) \sum_{j \in J} a_j \frac{1}{d_j} p_j = \lambda(\omega(t)) \sum_{j \in J} a_j \mathcal{F}\Big(\frac{\mathcal{F}^{-1}(p_j)}{d_j} \Big)
\end{aligned}$$

$$= \lambda(\omega(t))\mathcal{F}\Big(\sum_{j\in J} a_j\chi_j\Big) = \lambda(\omega(t))\lambda(\omega_{\sum_{j\in J} a_j\chi_j}) = \lambda(\omega(t) * \omega_{\sum_{j\in J} a_j\chi_j}).$$

Hence, since λ is faithful, i.e. injective, with regard to Corollary 4.1.3(ii) of [3], we have $\omega_{g(t)} = \omega(t) * \omega_{\sum_{j\in J} a_j\chi_j}$ for all $t \in [0,1]$. Consequently, since the Banach spaces $L^1(M,\varphi)$ and M_* are isometrically isomorphic by Theorem V.2.18 of [14] and since $\|\psi(t)\|_{\widehat{M}_c} = 1$ for each $t \in [0,1]$, it follows that, for all $t \in [0,1]$,

$$\begin{aligned}\|g(t)\|_{1,\varphi} = \|\omega_{g(t)}\|_{M_*} &= \|\omega(t) * \omega_{\sum_{j\in J} a_j\chi_j}\|_{M_*} \leqslant \|\omega(t)\|_{M_*}\|\omega_{\sum_{j\in J} a_j\chi_j}\|_{M_*} \\ &= \|\omega(t)\|_{M_*}\Big\|\sum_{j\in J} a_j\chi_j\Big\|_{1,\varphi} \leqslant B_{\mathcal{E}}\|\psi(t)\|_{\widehat{M}_c}\|f\|_{1,\varphi} = B_{\mathcal{E}}\|f\|_{1,\varphi}.\end{aligned}$$

Since the right side does not depend on t, we finally get

$$\|g\|_{1,\varphi,[0,1]} = \int_0^1 \|g(t)\|_{1,\varphi}\,\mathrm{d}t \leqslant \int_0^1 B_{\mathcal{E}}\|f\|_{1,\varphi}\,\mathrm{d}t = B_{\mathcal{E}}\|f\|_{1,\varphi}. \qquad ∎$$

REMARK 3.7. In fact, in the proof of Lemma 3.6, we even have $\psi(t) \in \Big(\sum_{i\in E} p_i\Big)\mathfrak{Z}(\widehat{M}_c)$ for all $t \in [0,1]$, where $\mathfrak{Z}(\widehat{M}_c)$ denotes the center of $\widehat{M}_c$.

For the following, we write j instead of j_m as in the proof of Lemma 3.6 in order to simplify the notations, i.e. $f = \sum_{j\in J} a_j\chi_j$ and $g(t) = \sum_{j\in J} r_j(t)a_j\chi_j$.

Similar to Lemma 2.11, we need the following

LEMMA 3.8. *Let J be a finite subset of I. Then we have*

$$\frac{\|g\|^3_{2,\varphi,[0,1]}}{\|g\|^2_{4,\varphi,[0,1]}} \leqslant \|g\|_{1,\varphi,[0,1]}.$$

Proof. Since the Rademacher functions $r_j(\cdot)$, $j \in J$, are constant on the intervals $[(k-1)/4^n, k/4^n] \subseteq [0,1]$ for all $k = 1,\dots,4^n$, respectively, the restriction to the interval $[(k-1)/4^n, k/4^n]$ of g for all $k = 1,\dots,4^n$ does not depend on t. Thus, for each $k = 1,\dots,4^n$, we set $g_k := g(t) \in M$ for all $t \in [(k-1)/4^n, k/4^n]$. Now, for all $k = 1,\dots,4^n$, let M_k be the unital, commutative C^*-subalgebra of M generated by $|g_k|$ and the identity $1 \in M$. With regard to Gelfand's theory, M_k can then be identified with the continuous, complex-valued functions $C(\mathrm{spec}(M_k))$ on $\mathrm{spec}(M_k)$, where $\mathrm{spec}(M_k)$ denotes the compact Gelfand space of M_k, which is homeomorphic to the spectrum $\mathrm{spec}(|g_k|)$.

If we denote the restriction to M_k of φ by $\varphi|_{M_k}$ for all $k = 1,\dots,4^n$, we may therefore identify $\varphi|_{M_k}$ with a continuous linear functional on $C(\mathrm{spec}(M_k))$. Hence, Riesz' representation theorem guarantees the existence of a regular Borel measure $\mathrm{d}\mu_k$ on $\mathrm{spec}(M_k)$ such that, for all $x \in M_k \cong C(\mathrm{spec}(M_k))$ and for all

$k = 1, \dots, 4^n$,

$$\varphi|_{M_k}(x) = \int\limits_{\operatorname{spec}(M_k)} x \,\mathrm{d}\mu_k.$$

Now, let $\mathrm{d}t_k$ be the restriction to $[(k-1)/4^n, k/4^n] \subseteq [0,1]$ of Lebesgue measure $\mathrm{d}t$ for all $k = 1, \dots, 4^n$. If we set $A_k := \operatorname{spec}(M_k) \times [(k-1)/4^n, k/4^n]$ for all $k = 1, \dots, 4^n$ with the product measure $\mathrm{d}\mu_k \otimes \mathrm{d}t_k$ and $A := \bigcup\limits_{k=1}^{4^n} A_k$ with the product measure $\mathrm{d}\mu \otimes \mathrm{d}t$, we have

$$\int\limits_0^1 \varphi(|g(t)|)\,\mathrm{d}t = \sum_{k=1}^{4^n} \int\limits_{(k-1)/4^n}^{k/4^n} \int\limits_{\operatorname{spec}(M_k)} |g_k|\,\mathrm{d}\mu_k\,\mathrm{d}t_k = \sum_{k=1}^{4^n} \int\limits_{A_k} |g_k|\,\mathrm{d}\mu_k \otimes \mathrm{d}t_k = \int\limits_A |g(t)|\,\mathrm{d}\mu \otimes \mathrm{d}t.$$

Consequently, according to the modified Rademacher functions, we succeeded in writing $\int\limits_0^1 \varphi(|g(t)|)\,\mathrm{d}t$ as a measure integral. Thus we may apply Hölder's inequality, and we obtain, with $p = 3/2$ and $q = 3$,

$$\int\limits_0^1 \varphi(|g(t)|^2)\,\mathrm{d}t \leqslant \Big(\int\limits_0^1 \varphi(|g(t)|)\,\mathrm{d}t\Big)^{2/3} \Big(\int\limits_0^1 \varphi(|g(t)|^4)\,\mathrm{d}t\Big)^{1/3}.$$

Therefore, we get

$$\begin{aligned} \|g\|_{2,\varphi,[0,1]}^3 &= \Big(\int\limits_0^1 \varphi(|g(t)|^2)\,\mathrm{d}t\Big)^{3/2} \leqslant \Big(\int\limits_0^1 \varphi(|g(t)|)\,\mathrm{d}t\Big)\Big(\int\limits_0^1 \varphi(|g(t)|^4)\,\mathrm{d}t\Big)^{1/2} \\ &= \|g\|_{1,\varphi,[0,1]} \|g\|_{4,\varphi,[0,1]}^2. \quad \blacksquare \end{aligned}$$

LEMMA 3.9. *Let J be a finite subset of I. Then we have*

$$\|g\|_{2,\varphi,[0,1]} = \sqrt{\sum_{j \in J} |a_j|^2}.$$

Proof. Regarding the Plancherel formula (1.4), the orthogonality of the set $\{p_i\}_{i \in I}$ as well as $\overline{r_j(t)} r_j(t) = 1$ for all $j \in J$ and $t \in [0,1]$, we get

$$\begin{aligned} \|g\|_{2,\varphi,[0,1]}^2 &= \int\limits_0^1 \|g(t)\|_{2,\varphi}^2\,\mathrm{d}t = \int\limits_0^1 \varphi(|g(t)|^2)\,\mathrm{d}t = \int\limits_0^1 \varphi(g(t)^* g(t))\,\mathrm{d}t \\ &= \int\limits_0^1 \varphi\Big(\Big(\sum_{j \in J} r_j(t) a_j \chi_j\Big)^* \sum_{k \in J} r_k(t) a_k \chi_k\Big)\,\mathrm{d}t \\ &= \int\limits_0^1 \varphi\Big(\Big(\sum_{j \in J} r_j(t) a_j \frac{\mathcal{F}^{-1}(p_j)}{d_j}\Big)^* \sum_{k \in J} r_k(t) a_k \frac{\mathcal{F}^{-1}(p_k)}{d_k}\Big)\,\mathrm{d}t \end{aligned}$$

$$= \int_0^1 \varphi\Big(\mathcal{F}^{-1}\Big(\sum_{j\in J} r_j(t)a_j \frac{1}{d_j} p_j\Big)^* \mathcal{F}^{-1}\Big(\sum_{k\in J} r_k(t)a_k \frac{1}{d_k} p_k\Big)\Big)\,\mathrm{d}t$$
$$= \int_0^1 \widehat{\varphi}\Big(\Big(\sum_{j\in J} r_j(t)a_j \frac{1}{d_j} p_j\Big)^* \sum_{k\in J} r_k(t)a_k \frac{1}{d_k} p_k\Big)\,\mathrm{d}t$$
$$= \int_0^1 \widehat{\varphi}\Big(\sum_{j\in J} \overline{r_j(t)}\overline{a}_j \frac{1}{d_j} p_j \sum_{k\in J} r_k(t)a_k \frac{1}{d_k} p_k\Big)\,\mathrm{d}t$$
$$= \int_0^1 \widehat{\varphi}\Big(\sum_{j\in J} \overline{r_j(t)}r_j(t)|a_j|^2 \frac{1}{d_j^2} p_j\Big)\,\mathrm{d}t = \int_0^1 \widehat{\varphi}\Big(\sum_{j\in J} |a_j|^2 \frac{1}{d_j^2} p_j\Big)\,\mathrm{d}t.$$

Since $\widehat{\varphi}\Big(\sum\limits_{j\in J} |a_j|^2(1/d_j^2)p_j\Big)$ does not depend on t, it follows from the decomposition of $\widehat{\varphi}$ into $\sum\limits_{i\in I} d_i\mathrm{Tr}_i$ by (1.2) that

$$\int_0^1 \widehat{\varphi}\Big(\sum_{j\in J} |a_j|^2 \frac{1}{d_j^2} p_j\Big)\mathrm{d}t = \widehat{\varphi}\Big(\sum_{j\in J} |a_j|^2 \frac{1}{d_j^2} p_j\Big) = \sum_{j\in J} |a_j|^2 \frac{1}{d_j^2} \widehat{\varphi}(p_j) = \sum_{j\in J} |a_j|^2 \frac{1}{d_j^2} \sum_{i\in I} d_i\mathrm{Tr}_i(p_j)$$
$$= \sum_{j\in J} |a_j|^2 \frac{1}{d_j^2} d_j\mathrm{Tr}_j(p_j) = \sum_{j\in J} |a_j|^2 \frac{d_j^2}{d_j^2} = \sum_{j\in J} |a_j|^2.$$

Altogether, we have shown the assertion. ∎

LEMMA 3.10. *Let J be a finite subset of I. Then we have*

$$\|g\|_{4,\varphi,[0,1]}^4 \leqslant \varphi\Big(\Big(\sum_{j\in J} |a_j|^2 \chi_j^*\chi_j\Big)^2\Big) + \varphi\Big(\Big(\sum_{j\in J} |a_j|^2 \chi_j\chi_j^*\Big)^2\Big).$$

Proof. Since $\mathbb{K} = (M, \Delta, \kappa, \varphi)$ is compact, we have $\varphi \in M_*$. Consequently, the integral $\int_0^1 \mathrm{d}t$ and φ can be exchanged. Since φ is a trace on $M = \mathfrak{N}_\varphi = \mathfrak{M}_\varphi$, we therefore infer from Lemma V.2.16 of [14], Lemma 3.4 and Proposition 2.12 that

$$\|g\|_{4,\varphi,[0,1]}^4$$
$$= \int_0^1 \|g(t)\|_4^4\,\mathrm{d}t = \int_0^1 \varphi(|g(t)|^4)\,\mathrm{d}t = \int_0^1 \varphi((g(t)^*g(t))^2)\,\mathrm{d}t$$
$$= \int_0^1 \varphi\Big(\Big(\sum_{i\in J} r_i(t)a_i\chi_i\Big)^* \sum_{j\in J} r_j(t)a_j\chi_j \Big(\sum_{k\in J} r_k(t)a_k\chi_k\Big)^* \sum_{l\in J} r_l(t)a_l\chi_l\Big)\,\mathrm{d}t$$

$$= \varphi\Big(\sum_{i\in J}\overline{a}_i\chi_i^*\sum_{j\in J}a_j\chi_j\sum_{k\in J}\overline{a}_k\chi_k^*\sum_{l\in J}a_l\chi_l\int_0^1\overline{r_i(t)}r_j(t)\overline{r_k(t)}r_l(t)\,\mathrm{d}t\Big)$$

$$= \varphi\Big(\sum_{j\in J}|a_j|^2\chi_j^*\chi_j\sum_{k\in J}|a_k|^2\chi_k^*\chi_k\Big) + \varphi\Big(\sum_{i\in J}\overline{a}_i\chi_i^*\Big(\sum_{j\in J}|a_j|^2\chi_j\chi_j^*\Big)a_i\chi_i\Big) - \varphi\Big(\sum_{i\in J}|a_i|^4\chi_i^*\chi_i\chi_i^*\chi_i\Big)$$

$$= \varphi\Big(\sum_{j\in J}|a_j|^2\chi_j^*\chi_j\sum_{k\in J}|a_k|^2\chi_k^*\chi_k\Big) + \varphi\Big(\sum_{i\in J}\overline{a}_i\chi_i^*\Big(\sum_{j\in J}|a_j|^2\chi_j\chi_j^*\Big)a_i\chi_i\Big) - \sum_{i\in J}\|a_i\chi_i\|_{4,\varphi}^4$$

$$= \varphi\Big(\sum_{j\in J}|a_j|^2\chi_j^*\chi_j\sum_{k\in J}|a_k|^2\chi_k^*\chi_k\Big) + \varphi\Big(\sum_{i\in J}\overline{a}_i\chi_i^*\Big(\sum_{j\in J}|a_j|^2\chi_j\chi_j^*\Big)a_i\chi_i\Big) - \sum_{i\in J}|a_i|^4\sum_{k\in K_i}n_k^2.$$

$$\leqslant \varphi\Big(\sum_{j\in J}|a_j|^2\chi_j^*\chi_j\sum_{k\in J}|a_k|^2\chi_k^*\chi_k\Big) + \varphi\Big(\sum_{i\in J}\overline{a}_i\chi_i^*\Big(\sum_{j\in J}|a_j|^2\chi_j\chi_j^*\Big)a_i\chi_i\Big)$$

$$= \varphi\Big(\Big(\sum_{j\in J}|a_j|^2\chi_j^*\chi_j\Big)^2\Big) + \sum_{i\in J}|a_i|^2\sum_{j\in J}|a_j|^2\varphi(\chi_i^*\chi_j\chi_j^*\chi_i)$$

$$= \varphi\Big(\Big(\sum_{j\in J}|a_j|^2\chi_j^*\chi_j\Big)^2\Big) + \sum_{i\in J}|a_i|^2\sum_{j\in J}|a_j|^2\varphi(\chi_j\chi_j^*\chi_i\chi_i^*)$$

$$= \varphi\Big(\Big(\sum_{j\in J}|a_j|^2\chi_j^*\chi_j\Big)^2\Big) + \varphi\Big(\Big(\sum_{j\in J}|a_j|^2\chi_j\chi_j^*\Big)^2\Big). \qquad \blacksquare$$

PROPOSITION 3.11. *Let J be a finite subset of I. Then we get*

$$\varphi\Big(\Big(\sum_{j\in J}|a_j|^2\chi_j^*\chi_j\Big)^2\Big) = \sum_{i\in I}\Big(\sum_{j\in J}|a_j|^2 n_{i_j}\Big)^2,$$

where n_{i_j} denotes the multiplicity of χ_i in the decomposition of $\chi_j^*\chi_j$ for all $j\in J$. Similarly, we get

$$\varphi\Big(\Big(\sum_{j\in J}|a_j|^2\chi_j\chi_j^*\Big)^2\Big) = \sum_{i\in I}\Big(\sum_{j\in J}|a_j|^2 m_{i_j}\Big)^2,$$

where m_{i_j} denotes the multiplicity of χ_i in the decomposition of $\chi_j\chi_j^*$ for all $j\in J$.

Proof. Using the Plancherel formula (1.4) and the Fourier inversion theorem (1.3), we get

$$\varphi\Big(\Big(\sum_{j\in J}|a_j|^2\chi_j^*\chi_j\Big)^2\Big)$$

$$= \varphi\Big(\Big(\sum_{j\in J}|a_j|^2\chi_j^*\chi_j\Big)^*\Big(\sum_{j\in J}|a_j|^2\chi_j^*\chi_j\Big)\Big) = \widehat{\varphi}\Big(\mathcal{F}\Big(\sum_{j\in J}|a_j|^2\chi_j^*\chi_j\Big)^*\mathcal{F}\Big(\sum_{j\in J}|a_j|^2\chi_j^*\chi_j\Big)\Big)$$

$$= \widehat{\varphi}\Big(\mathcal{F}\Big(\sum_{j\in J}|a_j|^2\sum_{k\in K_j}n_k\chi_k\Big)^*\mathcal{F}\Big(\sum_{j\in J}|a_j|^2\sum_{k\in K_j}n_k\chi_k\Big)\Big)$$

$$= \widehat{\varphi}\Big(\mathcal{F}\Big(\sum_{j\in J}|a_j|^2\sum_{k\in K_j}\frac{n_k}{d_k}d_k\chi_k\Big)^*\mathcal{F}\Big(\sum_{j\in J}|a_j|^2\sum_{k\in K_j}\frac{n_k}{d_k}d_k\chi_k\Big)\Big)$$
$$= \widehat{\varphi}\Big(\Big(\sum_{j\in J}|a_j|^2\sum_{k\in K_j}\frac{n_k}{d_k}\mathcal{F}(\mathcal{F}^{-1}(p_k))^*\Big)\Big(\sum_{j\in J}|a_j|^2\sum_{k\in K_j}\frac{n_k}{d_k}\mathcal{F}(\mathcal{F}^{-1}(p_k))\Big)\Big)$$
$$= \widehat{\varphi}\Big(\Big(\sum_{j\in J}|a_j|^2\sum_{k\in K_j}\frac{n_k}{d_k}p_k^*\Big)\Big(\sum_{j\in J}|a_j|^2\sum_{k\in K_j}\frac{n_k}{d_k}p_k\Big)\Big) = \widehat{\varphi}\Big(\Big(\sum_{j\in J}|a_j|^2\sum_{k\in K_j}\frac{n_k}{d_k}p_k\Big)^2\Big).$$

Arranging the characters according to their occurrence in the decompositions of the products $\chi_j^*\chi_j$ for all $j \in J$ by (2.1), it therefore follows from the orthogonality of the set $\{p_i\}_{i\in I}$ and the decomposition of $\widehat{\varphi}$ into $\sum_{l\in I} d_l \mathrm{Tr}_l$ by (1.2) that

$$\widehat{\varphi}\Big(\Big(\sum_{j\in J}|a_j|^2\sum_{k\in K_j}\frac{n_k}{d_k}p_k\Big)^2\Big)$$
$$= \widehat{\varphi}\Big(\sum_{i\in I}\frac{1}{d_i^2}p_i\Big(\sum_{j\in J}|a_j|^2 n_{i_j}\Big)^2\Big) = \sum_{l\in I} d_l \mathrm{Tr}_l\Big(\sum_{i\in I}\frac{1}{d_i^2}p_i\Big(\sum_{j\in J}|a_j|^2 n_{i_j}\Big)^2\Big)$$
$$= \sum_{i\in I} d_i\frac{1}{d_i^2}\Big(\sum_{j\in J}|a_j|^2 n_{i_j}\Big)^2\mathrm{Tr}_i(p_i) = \sum_{i\in I}\frac{d_i^2}{d_i^2}\Big(\sum_{j\in J}|a_j|^2 n_{i_j}\Big)^2 = \sum_{i\in I}\Big(\sum_{j\in J}|a_j|^2 n_{i_j}\Big)^2.$$

Altogether, we have shown the first assertion. Similarly, we get the second one. ∎

THEOREM 3.12. *Let $\mathcal{E} = \{p_i\}_{i\in E}$ be a Sidon set with constant $B_{\mathcal{E}}$, and let J be a finite subset of E. Let $a_j \in \mathbb{C}$ for all $j \in J$, and let $f := \sum_{j\in J} a_j\chi_j$.*

(i) *If, for all $j \in J$,*

$$\chi_j^*\chi_j = \chi_0 \quad \textit{and} \quad \chi_j\chi_j^* = \chi_0,$$

where χ_0 denotes the one-character of $\mathbb{K}$, we get

$$\|f\|_{2,\varphi} = \sqrt{\sum_{j\in J}|a_j|^2} \leqslant \sqrt{2}B_{\mathcal{E}}\|f\|_{1,\varphi}, \quad \textit{in particular}$$
$$\Big\|\sum_{j\in J}\mathcal{F}^{-1}(p_j)\Big\|_{2,\varphi} = \sqrt{\sum_{j\in J}d_j^2} \leqslant \sqrt{2}B_{\mathcal{E}}\Big\|\sum_{j\in J}\mathcal{F}^{-1}(p_j)\Big\|_{1,\varphi} \quad \textit{and}$$
$$\Big\|\sum_{j\in J}\chi_j\Big\|_{2,\varphi} = \sqrt{|J|} \leqslant \sqrt{2}B_{\mathcal{E}}\Big\|\sum_{j\in J}\chi_j\Big\|_{1,\varphi}.$$

(ii) *If, for all $j \in J$,*

$$\chi_j^*\chi_j = \chi_0 + \chi_j^{(1)} \quad \textit{and} \quad \chi_j\chi_j^* = \chi_0 + \chi_j^{(2)},$$

where χ_0 denotes the one-character of $\mathbb{K}$ as well as $\chi_j^{(1)}$ and $\chi_j^{(2)}$ are characters such that, for all $j, j' \in J$ with $j \neq j'$,

$$\chi_j^{(1)} \neq \chi_{j'}^{(1)} \quad \textit{and} \quad \chi_j^{(2)} \neq \chi_{j'}^{(2)},$$

we get

$$\sqrt{\frac{\left(\sum_{j\in J}|a_j|^2\right)^3}{\left(\sum_{j\in J}|a_j|^2\right)^2+\sum_{j\in J}|a_j|^4}}\leqslant\sqrt{2}B_{\mathcal{E}}\|f\|_{1,\varphi},\quad \textit{in particular}$$

$$\sqrt{\frac{\left(\sum_{j\in J}d_j^2\right)^3}{\left(\sum_{j\in J}d_j^2\right)^2+\sum_{j\in J}d_j^4}}\leqslant\sqrt{2}B_{\mathcal{E}}\Big\|\sum_{j\in J}\mathcal{F}^{-1}(p_j)\Big\|_{1,\varphi}\quad \textit{and}$$

$$\sqrt{\frac{|J|^2}{|J|+1}}\leqslant\sqrt{2}B_{\mathcal{E}}\Big\|\sum_{j\in J}\chi_j\Big\|_{1,\varphi}.$$

(iii) *If, for all $j\in J$,*

$$\chi_j^*\chi_j=\chi_0+\sum_{k\in K_j'}\chi_k\quad\textit{and}\quad\chi_j\chi_j^*=\chi_0+\sum_{l\in L_j'}\chi_l,$$

where χ_0 denotes the one-character of $\mathbb{K}$, such that, for any fixed $j,j'\in J$ with $j\neq j'$ as well as for all $k\in K_j'$, $k'\in K_{j'}'$ and for all $l\in L_j'$, $l'\in L_{j'}'$,

$$\chi_k\neq\chi_{k'}\quad\textit{and}\quad\chi_l\neq\chi_{l'},$$

we get

$$\sqrt{\frac{\left(\sum_{j\in J}|a_j|^2\right)^3}{\left(\sum_{j\in J}|a_j|^2\right)^2+\sum_{j\in J}(d_j^2-1)|a_j|^4}}\leqslant\sqrt{2}B_{\mathcal{E}}\|f\|_{1,\varphi},\quad \textit{in particular}$$

$$\sqrt{\frac{\left(\sum_{j\in J}d_j^2\right)^3}{\left(\sum_{j\in J}d_j^2\right)^2+\sum_{j\in J}(d_j^2-1)d_j^4}}\leqslant\sqrt{2}B_{\mathcal{E}}\Big\|\sum_{j\in J}\mathcal{F}^{-1}(p_j)\Big\|_{1,\varphi}\quad \textit{and}$$

$$\sqrt{\frac{|J|^3}{|J|^2+\sum_{j\in J}(d_j^2-1)}}\leqslant\sqrt{2}B_{\mathcal{E}}\Big\|\sum_{j\in J}\chi_j\Big\|_{1,\varphi}.$$

Proof. (i) Since $\chi_j^*\chi_j=\chi_0$ and $\chi_j\chi_j^*=\chi_0$ for all $j\in J$, we have $n_{0_j}=m_{0_j}=1$ for all $j\in J$, while $n_{i_j}=m_{i_j}=0$ for all $i\neq 0$ and $j\in J$. Hence, regarding Lemma 3.10 and Proposition 3.11, we get

$$\begin{aligned}\|g\|_{4,\varphi,[0,1]}^4&\leqslant\varphi\Big(\Big(\sum_{j\in J}|a_j|^2\chi_j^*\chi_j\Big)^2\Big)+\varphi\Big(\Big(\sum_{j\in J}|a_j|^2\chi_j\chi_j^*\Big)^2\Big)\\&=\sum_{i\in I}\Big(\sum_{j\in J}|a_j|^2n_{i_j}\Big)^2+\sum_{i\in I}\Big(\sum_{j\in J}|a_j|^2m_{i_j}\Big)^2\end{aligned}$$

$$= \Big(\sum_{j\in J}|a_j|^2\Big)^2 + \Big(\sum_{j\in J}|a_j|^2\Big)^2 = 2\Big(\sum_{j\in J}|a_j|^2\Big)^2.$$

Together with Lemmas 3.9, 3.8 and 3.6, it follows that

$$\frac{\Big(\sum_{j\in J}|a_j|^2\Big)^{3/2}}{\Big(2\Big(\sum_{j\in J}|a_j|^2\Big)^2\Big)^{1/2}} \leqslant \frac{\|g\|^3_{2,\varphi,[0,1]}}{\|g\|^2_{4,\varphi,[0,1]}} \leqslant \|g\|_{1,\varphi,[0,1]} \leqslant B_{\mathcal{E}}\|f\|_{1,\varphi}.$$

Consequently, by Proposition 2.9, we have shown the assertion.

(ii) By Proposition 2.13, χ_0 occurs exactly once in each decomposition of $\chi_j^*\chi_j$ and $\chi_j\chi_j^*$ for all $j \in J$. Thus, as in (i), we have $n_{0_j} = m_{0_j} = 1$ for all $j \in J$. Hence, $\chi_j^{(1)} \neq \chi_0$ and $\chi_j^{(2)} \neq \chi_0$ for all $j \in J$ such that $n_j^{(1)} = m_j^{(2)} = 1$ for all $j \in J$, while all other multiplicities are zero. Since $\chi_j^{(1)} \neq \chi_{j'}^{(1)}$ and $\chi_j^{(2)} \neq \chi_{j'}^{(2)}$ for all $j, j' \in J$ such that $j \neq j'$, we infer from Lemma 3.10 and Proposition 3.11 that

$$\begin{aligned}
&\|g\|^4_{4,\varphi,[0,1]}\\
&\leqslant \varphi\Big(\Big(\sum_{j\in J}|a_j|^2\chi_j^*\chi_j\Big)^2\Big) + \varphi\Big(\Big(\sum_{j\in J}|a_j|^2\chi_j\chi_j^*\Big)^2\Big) = \sum_{i\in I}\Big(\sum_{j\in J}|a_j|^2 n_{i_j}\Big)^2 + \sum_{i\in I}\Big(\sum_{j\in J}|a_j|^2 m_{i_j}\Big)^2\\
&= \Big(\sum_{j\in J}|a_j|^2\Big)^2 + \sum_{j\in J}(|a_j|^2)^2 + \Big(\sum_{j\in J}|a_j|^2\Big)^2 + \sum_{j\in J}(|a_j|^2)^2 = 2\Big(\Big(\sum_{j\in J}|a_j|^2\Big)^2 + \sum_{j\in J}|a_j|^4\Big).
\end{aligned}$$

Together with Lemmas 3.9, 3.8 and 3.6, it follows that

$$\frac{\Big(\sum_{j\in J}|a_j|^2\Big)^{3/2}}{\Big(2\Big(\Big(\sum_{j\in J}|a_j|^2\Big)^2 + \sum_{j\in J}|a_j|^4\Big)\Big)^{1/2}} \leqslant \frac{\|g\|^3_{2,\varphi,[0,1]}}{\|g\|^2_{4,\varphi,[0,1]}} \leqslant \|g\|_{1,\varphi,[0,1]} \leqslant B_{\mathcal{E}}\|f\|_{1,\varphi}.$$

(iii) Similar to (ii), we infer from Lemma 3.10 and Proposition 3.11 that

$$\begin{aligned}
\|g\|^4_{4,\varphi,[0,1]} &\leqslant \varphi\Big(\Big(\sum_{j\in J}|a_j|^2\chi_j^*\chi_j\Big)^2\Big) + \varphi\Big(\Big(\sum_{j\in J}|a_j|^2\chi_j\chi_j^*\Big)^2\Big)\\
&= \sum_{i\in I}\Big(\sum_{j\in J}|a_j|^2 n_{i_j}\Big)^2 + \sum_{i\in I}\Big(\sum_{j\in J}|a_j|^2 m_{i_j}\Big)^2\\
&= \Big(\sum_{j\in J}|a_j|^2\Big)^2 + \sum_{j\in J}\sum_{k\in K'_j}(|a_j|^2)^2 + \Big(\sum_{j\in J}|a_j|^2\Big)^2 + \sum_{j\in J}\sum_{l\in L'_j}(|a_j|^2)^2\\
&= \Big(\sum_{j\in J}|a_j|^2\Big)^2 + \sum_{j\in J}|K'_j||a_j|^4 + \Big(\sum_{j\in J}|a_j|^2\Big)^2 + \sum_{j\in J}|L'_j||a_j|^4.
\end{aligned}$$

Since the dimension of $\chi_j^*\chi_j$ and $\chi_j\chi_j^*$ is equal to d_j^2, respectively, the index sets K'_j and L'_j are greatest if χ_k and χ_l are characters of one-dimensional projections for all $k \in K'_j$ and $l \in L'_j$. Therefore, for all $j \in J$, we can estimate the cardinalities $|K'_j|$ and $|L'_j|$ from above by

$$|K'_j| \leqslant d_j^2 - 1 \quad \text{and} \quad |L'_j| \leqslant d_j^2 - 1.$$

Thus we have

$$\|g\|_{4,\varphi,[0,1]}^4 \leqslant 2\Big(\Big(\sum_{j\in J}|a_j|^2\Big)^2 + \sum_{j\in J}(d_j^2-1)|a_j|^4\Big).$$

Together with Lemmas 3.9, 3.8 and 3.6, it follows that

$$\frac{\Big(\sum_{j\in J}|a_j|^2\Big)^{3/2}}{\Big(2\Big(\Big(\sum_{j\in J}|a_j|^2\Big)^2 + \sum_{j\in J}(d_j^2-1)|a_j|^4\Big)\Big)^{1/2}} \leqslant \frac{\|g\|_{2,\varphi,[0,1]}^3}{\|g\|_{4,\varphi,[0,1]}^2} \leqslant \|g\|_{1,\varphi,[0,1]} \leqslant B_{\mathcal{E}}\|f\|_{1,\varphi}. \qquad \blacksquare$$

REMARK 3.13. Let G be a compact group with the dual object Σ of G. For each $\sigma \in \Sigma$, let $U^{(\sigma)} \in \sigma$ be a continuous, irreducible, unitary representation of G with the representation space H_σ. According to Definition 4.1 of [5], a subset $E \subseteq \Sigma$ is called *lacunary* if the two following conditions are satisfied:

(i) Whenever (α, β) and (γ, δ) are different pairs from E, i.e. the corresponding characters $\chi_{\alpha\oplus\beta}$ and $\chi_{\gamma\oplus\delta}$ are different from each other, then the representations $U^{(\alpha)} \otimes U^{(\beta)}$ and $U^{(\gamma)} \otimes U^{(\delta)}$ are disjoint, i.e. no irreducible component of $U^{(\alpha)} \otimes U^{(\beta)}$ is equivalent to an irreducible component of $U^{(\gamma)} \otimes U^{(\delta)}$.

(ii) There is a constant K such that, for all $\alpha, \beta \in E$,

$$n_{\alpha,\beta} < K,$$

where $n_{\alpha,\beta}$ denotes the number of all irreducible components in $U^{(\alpha)} \otimes U^{(\beta)}$ counted with multiplicity.

In general, regarding p. 447 and Theorem 37.10 of [6], a lacunary subset of Σ is not a Sidon set.

Now, for all $j \in J$, the special decompositions of the products $\chi_j^*\chi_j$ and $\chi_j\chi_j^*$ in Theorem 3.12 are motivated by the above property (i) of a lacunary subset.

REMARK 3.14. (i) Let G be a compact abelian group with the dual group Γ of G. Then we have, for all $\gamma \in \Gamma$,

$$\overline{\gamma}\gamma = \gamma\overline{\gamma} = |\gamma|^2 = 1_G,$$

where 1_G denotes the one-character of G. Consequently, all continuous characters $\gamma \in \Gamma$ fulfill the special decompositions in Theorem 3.12(i).

(ii) Let $\mathfrak{I}$ be a non-empty index set. For each $\iota \in \mathfrak{I}$, let H_ι be a finite-dimensional Hilbert space, and let $\mathfrak{U}(H_\iota)$ denote the unitary group consisting of all unitary operators on H_ι. Then $\mathcal{G} := \prod_{\iota\in\mathfrak{I}} \mathfrak{U}(H_\iota)$ is a compact group under the product topology. For each $\iota \in \mathfrak{I}$, the projection π_ι of $\mathcal{G}$ onto $\mathfrak{U}(H_\iota)$ is a continuous, irreducible, unitary representation of $\mathcal{G}$. By Remark 37.5 of [6], the set $\{\pi_\iota : \iota \in \mathfrak{I}\}$ is a Sidon set in the dual object $\Sigma_{\mathcal{G}}$ of $\mathcal{G}$. For each $\iota \in \mathfrak{I}$, let $\overline{\pi}_\iota$ denote the conjugate representation of π_ι. Then, by 29.46(b) of [6], we have, for all $\iota \in \mathfrak{I}$,

$$\overline{\pi}_\iota \otimes \pi_\iota = \pi_{\mathcal{G}} \oplus \pi_\iota^{(1)}$$

with the trivial representation $\pi_{\mathcal{G}}$ of $\mathcal{G}$ and a continuous, irreducible, $(d_\iota^2 - 1)$-dimensional, unitary representation $\pi_\iota^{(1)}$ of $\mathcal{G}$. Consequently, it follows for the corresponding character χ_ι of π_ι that, for all $\iota \in \mathfrak{I}$,

$$\overline{\chi}_\iota \chi_\iota = 1_{\mathcal{G}} + \chi_\iota^{(1)},$$

where $1_{\mathcal{G}}$ denotes the one-character of $\mathcal{G}$. If $\iota, \iota' \in \mathfrak{I}$ such that $\iota \neq \iota'$, we have $\chi_\iota \neq \chi_{\iota'}$, in particular $\chi_\iota^{(1)} \neq \chi_{\iota'}^{(1)}$. Thus, the characters χ_ι, $\iota \in \mathfrak{I}$, fulfill the special decompositions in Theorem 3.12(ii). In fact, by p. 788 of [5], the Sidon set $\{\pi_\iota : \iota \in J\}$ is also lacunary.

(iii) For any fixed $i \in \{0, 1/2, 1, 3/2, \ldots\}$, let χ_i denote the character corresponding to the representation $T^{(i)}$ of $\mathfrak{SU}(2)$ in accordance with 29.13 of [6]. Then it follows from Theorem 29.26 of [6] that

$$\chi_i^* \chi_i = \chi_i \chi_i = \sum_{k=0}^{2i} \chi_k = \chi_0 + \sum_{k=1}^{2i} \chi_k,$$

where χ_0 is the one-character of $\mathfrak{SU}(2)$. Hence, for all $k = 1, \ldots, 2i$, the characters χ_k from the decompositions of $\chi_i^* \chi_i$ and $\chi_j^* \chi_j$, such that $i < j$, are equal. Consequently, the character χ_i does not fulfill the special decomposition of Theorem 3.12(iii). In fact, by p. 789 of [5], the dual object $\Sigma_{\mathfrak{SU}(2)}$ of $\mathfrak{SU}(2)$ has no infinite lacunary subset.

Furthermore, for the compact groups $\mathfrak{SO}(3)$, $\mathfrak{U}(2)$ and $\mathfrak{O}(3)$, we get similar results such that the special decompositions in Theorem 3.12(iii) are not valid, respectively.

Combining a special case of property (ii) of a lacunary subset in the dual object of a compact group in Remark 3.13 with Definition 3.1 of a Sidon set, we make the following

DEFINITION 3.15. Let $\mathcal{E} = \{p_i\}_{i \in E}$ be a Sidon set. We call $\mathcal{E}$ a *Helgason–Sidon set* if there is a constant $C \geqslant 1$ such that, for all $i \in E$,

$$\sum_{k \in K_i} n_k \leqslant C. \tag{3.1}$$

REMARK 3.16. Obviously, the characters from Remark 3.14(i) and (ii) fulfill inequality (3.1), whereas in Remark 3.14(iii) we get a constant C_i depending on the character χ_i.

PROPOSITION 3.17. *Let $\mathcal{E} = \{p_i\}_{i \in E}$ be a Sidon set. Then the following assertions are equivalent:*

(i) *$\mathcal{E}$ is a Helgason–Sidon set.*

(ii) *There is a constant* $C \geqslant 1$ *such that, for all* $i \in E$,

$$\sum_{l \in L_i} m_l \leqslant C.$$

(iii) *There is a constant* $C \geqslant 1$ *such that, for all* $i \in E$,

$$\|\chi_i\|_{4,\varphi}^4 = \sum_{k \in K_i} n_k^2 = \sum_{l \in L_i} m_l^2 \leqslant C.$$

Proof. (i) $\Rightarrow$ (iii) There is a constant $C \geqslant 1$ such that, for all $i \in E$,

$$\sum_{k \in K_i} n_k^2 \leqslant \Big(\sum_{k \in K_i} n_k\Big)^2 \leqslant C^2.$$

The rest of assertion (iii) follows from Proposition 2.12.

(iii) $\Rightarrow$ (i) There is a constant $C \geqslant 1$ such that, for all $i \in E$,

$$\sum_{k \in K_i} n_k \leqslant \sum_{k \in K_i} n_k^2 \leqslant C.$$

(ii) $\Leftrightarrow$ (iii) In like manner, we obtain this equivalence. ∎

DEFINITION 3.18. Let $\mathcal{E} = \{p_i\}_{i \in E}$ be a Sidon set. We call $\mathcal{E}$ a *strong Sidon set* if there is a constant $C \geqslant 1$ such that, for all $i \in E$,

$$d_i \leqslant C.$$

PROPOSITION 3.19. *A strong Sidon set is also a Helgason–Sidon set.*

Proof. Let $\mathcal{E} = \{p_i\}_{i \in E}$ be a strong Sidon set. Then there is a constant $C \geqslant 1$ such that $d_i \leqslant C$ for all $i \in E$. Using the fact that the dimension of $\chi_i^* \chi_i$ is equal to d_i^2 and regarding the decomposition $\chi_i^* \chi_i = \sum_{k \in K_i} n_k \chi_k$ by (2.1), we get, for all $i \in E$,

$$\sum_{k \in K_i} n_k \leqslant \sum_{k \in K_i} n_k d_k = d_i^2 \leqslant C^2. \quad ∎$$

THEOREM 3.20. *Let* $\mathcal{E} = \{p_i\}_{i \in E}$ *be a Helgason–Sidon set (strong Sidon set) with constant* $B_{\mathcal{E}}$, *and let* J *be a finite subset of* E. *Let* $a_j \in \mathbb{C}$ *for all* $j \in J$, *and let* $f := \sum_{j \in J} a_j \chi_j$. *Then there is a constant* $C \geqslant 1$ *such that*

$$\|f\|_{2,\varphi} = \sqrt{\sum_{j \in J} |a_j|^2} \leqslant \sqrt{2} C B_{\mathcal{E}} \|f\|_{1,\varphi}, \quad \textit{in particular}$$

$$\Big\| \sum_{j \in J} \mathcal{F}^{-1}(p_j) \Big\|_{2,\varphi} = \sqrt{\sum_{j \in J} d_j^2} \leqslant \sqrt{2} C B_{\mathcal{E}} \Big\| \sum_{j \in J} \mathcal{F}^{-1}(p_j) \Big\|_{1,\varphi} \quad \textit{and}$$

$$\Big\| \sum_{j \in J} \chi_j \Big\|_{2,\varphi} = \sqrt{|J|} \leqslant \sqrt{2} C B_{\mathcal{E}} \Big\| \sum_{j \in J} \chi_j \Big\|_{1,\varphi}.$$

Proof. By Proposition 3.19, we need only to consider the case that $\mathcal{E} = \{p_i\}_{i\in E}$ is a Helgason–Sidon set. Then, according to Proposition 3.17, there are constants $C_1, C_2 \geqslant 1$ such that, for all $j \in J \subseteq E$,

$$\sum_{k\in K_j} n_k \leqslant C_1 \quad \text{and} \quad \sum_{l\in L_j} m_l \leqslant C_2.$$

Let $C := \max(C_1, C_2)$. By rearranging the occurring sums, we thus infer from Lemma 3.10 and Proposition 3.11 that

$$\begin{aligned}\|g\|_{4,\varphi,[0,1]}^4 &\leqslant \varphi\Big(\Big(\sum_{j\in J}|a_j|^2\chi_j^*\chi_j\Big)^2\Big) + \varphi\Big(\Big(\sum_{j\in J}|a_j|^2\chi_j\chi_j^*\Big)^2\Big)\\ &= \sum_{i\in I}\Big(\sum_{j\in J}|a_j|^2 n_{i_j}\Big)^2 + \sum_{i\in I}\Big(\sum_{j\in J}|a_j|^2 m_{i_j}\Big)^2\\ &\leqslant \Big(\sum_{i\in I}\sum_{j\in J}|a_j|^2 n_{i_j}\Big)^2 + \Big(\sum_{i\in I}\sum_{j\in J}|a_j|^2 m_{i_j}\Big)^2\\ &= \Big(\sum_{j\in J}|a_j|^2\sum_{k\in K_j} n_k\Big)^2 + \Big(\sum_{j\in J}|a_j|^2\sum_{l\in L_j} m_l\Big)^2 \leqslant 2C^2\Big(\sum_{j\in J}|a_j|^2\Big)^2.\end{aligned}$$

Together with Lemmas 3.9, 3.8 and 3.6, it follows that

$$\frac{\Big(\sum_{j\in J}|a_j|^2\Big)^{3/2}}{\sqrt{2}C\sum_{j\in J}|a_j|^2} \leqslant \frac{\|g\|_{2,\varphi,[0,1]}^3}{\|g\|_{4,\varphi,[0,1]}^2} \leqslant \|g\|_{1,\varphi,[0,1]} \leqslant B_{\mathcal{E}}\|f\|_{1,\varphi}.$$

Consequently, by Proposition 2.9, we have shown the assertion. ∎

REFERENCES

[1] T. BLENDEK, *Normen von Projektionen im Dual einer kompakten Kac-Algebra*, Ph.D. Dissertation, University of Hamburg, Hamburg 2010.

[2] J. DIXMIER, Formes linéaires sur un anneau d'opérateurs, *Bull. Soc. Math. France* **81**(1953), 9–39.

[3] M. ENOCK, J.-M. SCHWARTZ, *Kac Algebras and Duality of Locally Compact Groups*, Springer-Verlag, Berlin-Heidelberg-New York 1992.

[4] G.H. HARDY, J.E. LITTLEWOOD, A new proof of a theorem on rearrangements, *J. London Math. Soc.* **23**(1948), 163–168.

[5] S. HELGASON, Lacunary Fourier series on noncommutative groups, *Proc. Amer. Math. Soc.* **9**(1958), 782–790.

[6] E. HEWITT, K.A. ROSS, *Abstract Harmonic Analysis*. II, Springer-Verlag, Berlin-Heidelberg-New York 1970.

[7] B.-J. KAHNG, Fourier transform on locally compact quantum groups, *J. Operator Theory* **64**(2010), 69–87.

[8] S.V. KONJAGIN, On the Littlewood problem [Russian], *Izv. Akad. Nauk. U.S.S.R. Ser. Mat.* **45**(1981), 243–265.

[9] J. KUSTERMANS, S. VAES, Locally compact quantum groups, *Ann. Sci. École Norm. Sup. (4)* **33**(2000), 837–934.

[10] J. KUSTERMANS, S. VAES, Locally compact quantum groups in the von Neumann algebraic setting, *Math. Scand.* **92**(2003), 68–92.

[11] A. MAES, A. VAN DAELE, Notes on compact quantum groups, *Nieuw Arch. Wisk. (4)* **16**(1998), 73–112.

[12] O.C. MCGEHEE, L. PIGNO, B. SMITH, Hardy's inequality and the L^1 norm of exponential sums, *Ann. of Math.* **113**(1981), 613–618.

[13] W. RUDIN, *Fourier Analysis on Groups*, Interscience Publ., a division of John Wiley and Sons, New York-London 1962.

[14] M. TAKESAKI, *Theory of Operator Algebras*. I, Springer-Verlag, Berlin-Heidelberg-New York 2003.

[15] A. VAN DAELE, The Haar measure on a compact quantum group, *Proc. Amer. Math. Soc.* **123**(1995), 3125–3128.

[16] A. VAN DAELE, Discrete quantum groups, *J. Algebra* **180**(1996), 431–444.

[17] S.L. WORONOWICZ, Compact matrix pseudogroups, *Comm. Math. Phys.* **111**(1987), 613–665.

[18] S.L. WORONOWICZ, Compact quantum groups, in *Symétries quantiques (Les Houches, 1995)*, North-Holland, Amsterdam 1998, pp. 845–884.

[19] A. ZYGMUND, *Trigonometric Series*. I, Cambridge Univ. Press, Cambridge 1959.

TOBIAS BLENDEK, DEPARTMENT OF MATHEMATICS AND STATISTICS, HELMUT SCHMIDT UNIVERSITY HAMBURG, HOLSTENHOFWEG 85, 22043 HAMBURG, GERMANY
E-mail address: tobias.blendek@gmx.de

JOHANNES MICHALIČEK, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAMBURG, BUNDESSTRASSE 55, 20146 HAMBURG, GERMANY
E-mail address: michalicek@math.uni-hamburg.de

Received September 3, 2011; revised August 13, 2012.