TRIDIAGONAL REPRODUCING KERNELS AND SUBNORMALITY

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For John B. Conway, on the occasion of his retirement.

Communicated by Kenneth R. Davidson

ABSTRACT. We consider analytic reproducing kernel Hilbert spaces $\mathcal{H}$ with orthonormal bases of the form $\{(a_n + b_n z)z^n : n \geqslant 0\}$. If $b_n = 0$ for all n, then H is a diagonal space and multiplication by z, M_z, is a weighted shift. Our focus is on providing extensive classes of examples for which M_z is a bounded subnormal operator on a tridiagonal space $\mathcal{H}$ where $b_n \neq 0$. The Aronszajn sum of H and $(1 - z)H$ where H is either the Hardy space or the Bergman space on the disk are two such examples.

KEYWORDS: *Analytic reproducing kernel, subnormal operator, tridiagonal kernel.*

MSC (2010): Primary 47B20; Secondary 47C15.

1. INTRODUCTION AND PRELIMINARIES

The function $K(z, w)$ is *positive semidefinite* (denoted $K \succeq 0$) on the set $E \times E$ if for any finite collection $z_1, z_2, \ldots, z_n$ in E and any complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ the sum

$$\sum_{i,j=1}^{n} \overline{\alpha}_i \alpha_j K(z_i, z_j) \geqslant 0$$

with strict inequality unless all the α_i's are zero. It is well known that if $K \succeq 0$ on E, then the set of functions in z given by

$$\left\{ \sum_{j=1}^{n} \alpha_j K(z, w_j) : \alpha_1, \ldots, \alpha_n \in \mathbb{C}, w_1, \ldots, w_n \in E \right\}$$

has dense span in a Hilbert space $H(K)$ of functions on E with

$$\left\| \sum_{j=1}^{n} \alpha_j K(z, w_j) \right\|^2 = \sum_{i,j=1}^{n} \overline{\alpha}_i \alpha_j K(w_i, w_j).$$

A fundamental property of $H(K)$ is the *Reproducing Property* which states that $f(w) = \langle f(z), K(z, w) \rangle$ for every w in E and f in $H(K)$. Thus evaluation at w is a bounded linear functional for each w in E.

Conversely, it is well known that if F is a Hilbert space of functions defined on E such that evaluation at w is a bounded linear functional for each w in E, then there is a unique K defined on $E \times E$ such that $F = H(K)$. It follows from the reproducing property that $K(z, w) = \overline{K(w, z)}$. Hence if K is analytic in the first variable, then K is coanalytic in the second variable. In this case K is an *analytic kernel*. In later sections of this paper, E will always be the unit disk $\mathbb{D}$ and K will be an analytic kernel.

It is also well known, see N. Aronszajn [3], that if $\{f_n(z)\}$ is an orthonormal basis for a reproducing kernel Hilbert space of functions on E, then

$$K(z, w) = \sum_{n=0}^{\infty} f_n(z)\overline{f_n(w)}$$

for all z, w in E. Moreover if the largest common domain E' of the functions $\{f_n(z)\}$ is larger than E, then the largest natural domain of $H(K)$ is given by

$$\text{Dom}(K) = \Big\{ z \in E' : \sum_{n=0}^{\infty} |f_n(z)|^2 < \infty \Big\}.$$

When K is analytic and E contains the open unit disk, $K(z, w) = \sum_{i,j=0}^{\infty} a_{i,j} z^i \overline{w}^j$ has a Taylor series expansion about $(0, 0)$ with coefficient matrix $A = [a_{i,j}]$. The matrix A is positive and if $A = BB^*$ is any factorization of A, then $H(K)$ is naturally isomorphic to the range space $R(B) = \{B\vec{x} : \vec{x} \in l_+^2\}$ via the map which identifies $B\vec{x}$ with the analytic function f whose Taylor coefficients are the components of $B\vec{x}$. Thus, when B has no kernel, the columns of B correspond to an orthonormal basis for $H(K)$. It should be noted that the matrices are not necessarily bounded, but this is not a problem for the general theory. The interested reader is referred to Adams, McGuire, and Paulsen [2] for more details.

An analytic kernel is tridiagonal if there exists an orthonormal basis of polynomials for $H(K)$ of the form $\{f_n(z) = (a_n + b_n z)z^n : n \geqslant 0\}$ and diagonal if $b_n = 0$ for all n. In this case the coefficient matrix A has bandwidth 3, hence the name tridiagonal, and A can be factored as LL^* where

$$L = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ b_0 & a_1 & 0 & \ddots \\ 0 & b_1 & a_2 & \ddots \\ \vdots & \vdots & b_2 & \ddots \end{pmatrix}.$$

The natural domain $\text{Dom}(K)$ of a tridiagonal kernel is either an open or closed disk about the origin together with at most one point not in the disk. This

was shown in Adams, McGuire [1] in which the properties of M_z were considered. For $a_n > 0$, $H(K)$ contains the polynomials if and only if the sequence

$$\left\{1, \frac{b_n}{a_{n+1}}, \frac{b_n b_{n+1}}{a_{n+1} a_{n+2}}, \frac{b_n b_{n+1} b_{n+2}}{a_{n+1} a_{n+2} a_{n+3}}, \cdots\right\}$$

is square summable for each n. If M_z is bounded, then either $H(K)$ contains the polynomials or $b_n = \lambda a_n$ for all n where λ is a constant. In the latter case $\{(1 + \lambda z) a_n z^n\}$ is a basis for $H(K)$ and M_z is a bounded weighted shift. Also, an example is given in Adams, McGuire [1] in which the polynomials are contained in $H(K)$, yet M_z is not bounded.

Recall that a bounded operator S on a separable complex Hilbert space X is subnormal if there is a normal operator N on a Hilbert space Y containing X such that X is an invariant subspace for N and S is the restriction of N to X. A subnormal operator S is pure if there is no reducing subspace for S on which S is normal. Since our interest is in M_z being a pure subnormal operator on a tridiagonal space, we can assume that the domain is an open disk (see Conway [4], page 398). Additionally, by replacing $K = K(z, w)$ with a dilated kernel $K_r = K(rz, rw)$ we may assume that the domain is the open unit disk $\mathbb{D}$. The details of the isomorphism of $H(K)$ and $H(K_r)$ can be found in [2]. By necessity $H(K)$ contains the polynomials and hence is isomorphic to the closure of the polynomials, denoted $P^2(\mu)$, in $L^2(\mu)$ for some measure μ supported on the closed unit disk.

In Aronszajn [3] it is shown that if $K_1 \succeq 0$ and $K_2 \succeq 0$, then $K_1 + K_2 \succeq 0$ and the space $H(K_1 + K_2)$ consists of the functions $\{f_1 + f_2 : f_1 \in H(K_1), f_2 \in H(K_2)\}$ with norm

$$\|f\|_{K_1 + K_2}^2 = \inf\{\|f_1\|^2 + \|f_2\|^2 : f = f_1 + f_2, f_1 \in H(K_1), f_2 \in H(K_2)\}.$$

We will denote the space $H(K_1 + K_2)$ by $H(K_1)\widehat{\oplus}H(K_2)$. Note that if $H(K_1)$ and $H(K_2)$ are invariant under multiplication by z, then so is $H(K_1)\widehat{\oplus}H(K_2)$.

In Vern Paulsen's [6] notes, the following nice description is given of the space which results by composing a reproducing kernel K on $E \times E$ with a function $\phi : S \to E$.

PROPOSITION 1.1 (Paulsen). *If E and S are sets, $K : E \times E \to \mathbb{C}$ is positive definite and $\phi : S \to E$ is a function, then the function $(K \circ \phi)(z, w) = K(\phi(z), \phi(w))$ is positive definite on $S \times S$,*

$$H(K \circ \phi) = \{f \circ \phi : f \in H(K)\},$$

and $\|g\|_{H(K \circ \phi)} = \inf\{\|f\|_{H(K)} : g = f \circ \phi\}$.

In Section 2 we will produce examples of tridiagonal reproducing kernel Hilbert spaces that arise in part as compositions and for which M_z is a subnormal operator. Interestingly, the spaces are also of the form $H(K_1)\widehat{\oplus}H(K_2)$ where $H(K_1)$ and $H(K_2)$ are unitarily equivalent to diagonal reproducing kernel spaces

for which M_z is subnormal. In particular it is shown that the tridiagonal reproducing kernel Hilbert space $H(K)\widehat{\oplus}(1-z)H(K)$ where $K(z,w) = \frac{1}{1-\overline{w}z}$ is identifiable as a $P^2(\mu)$ space having a close relationship to β, the reciprocal of the square of the Golden Ratio $\frac{1+\sqrt{5}}{2}$. More specifically, if $P_\beta(\theta)$ denotes the Poisson kernel at β and $\mathcal{X}_\beta$ denotes the unit point mass at β, then $d\mu$ is equal to $P_\beta(\theta)\frac{d\theta}{2\pi} + \frac{\beta}{\sqrt{5}}\mathcal{X}_\beta$.

In Section 3, a large class of tridiagonal kernels is introduced for which M_z is subnormal. The approach in this section is to construct measures for which the closure of the polynomials forms spaces for which an orthonormal polynomial basis of the desired tridiagonal form exists. The examples in this section appear to be quite distinct from those in Section 2.

We conclude in Section 4 with some open questions.

Before proceeding to Section 2, we first collect together a few well known examples and facts regarding M_z on diagonal spaces. The reader is referred to Conway [4], Shields [8], and Zhu [10] for details.

Assume $K(z,w) = \sum\limits_{n=0}^{\infty} a_n(\overline{w}z)^n$ has the unit disk $\mathbb{D}$ as its domain and $a_n > 0$ for all n with $a_0 = 1$.

  (i) $\{f_n(z) = \sqrt{a_n}z^n : n \geqslant 0\}$ is an orthonormal basis for $H(K)$.

  (ii) M_z is a unilateral weighted shift with weight sequence $\sqrt{\frac{a_n}{a_{n+1}}}$.

  (iii) M_z is subnormal if and only if there exists a probability measure ν on $[0,1]$ such that for all $n \geqslant 0$,

$$\frac{1}{a_n} = \int\limits_0^1 t^{2n}d\nu(t).$$

In this case $H(K)$ is a measure space with

$$\|f\|_{H(K)}^2 = \frac{1}{2\pi}\int\limits_0^{2\pi}\int\limits_0^1 f(te^{i\theta})d\nu(t)d\theta$$

and the weight sequence $\sqrt{\frac{a_n}{a_{n+1}}}$ increases to 1.

  (iv) If

$$K(z,w) = \sum\limits_{n=0}^{\infty}(\overline{w}z)^n = \frac{1}{1-\overline{w}z},$$

then $\nu(t) = \delta_1(t)$ where δ_1 denotes the unit point mass at 1, $d\mu = \frac{d\nu(t)d\theta}{2\pi}$ is arclength measure on the unit circle, and $H(K)$ is unitarily equivalent to the Hardy space H^2.

  (v) If $K(z,w) = \sum\limits_{n=0}^{\infty}(n+1)(\overline{w}z)^n = \frac{1}{(1-\overline{w}z)^2}$, then $d\nu(t) = 2tdt$, $d\mu(te^{i\theta}) = t\frac{dtd\theta}{\pi}$ is normalized area measure on the unit disk, and $H(K)$ is the Bergman space $L_a^2(\mathbb{D})$.

(vi) For real numbers $s > 1$, if

$$K_s(z, w) = \frac{1}{(1 - \overline{w}z)^s} = \sum_{n=0}^{\infty} \frac{\Gamma(s+n)}{n!\Gamma(s)}(\overline{w}z)^n,$$

then $d\mu(te^{i\theta}) = 2(s-1)t(1-t^2)^{s-2}\frac{dt d\theta}{2\pi}$, and we will denote $H(K_s)$ by $L^2_{a,s}(\mathbb{D})$ and refer to it as the Bergman-s space . Note $L^2_{a,2}(\mathbb{D}) = L^2_a(\mathbb{D})$ is the usual Bergman space.

The reader will note above the connection between the classical moment problem for the reciprocal sequence of the coefficients $\{\frac{1}{a_n}\}$ and the subnormality of M_z. It is well known (see the work of D. Widder and S. Bernstein summarized in D.V. Widder [9]) that the sequence $\frac{1}{a_n}$ is a moment sequence if and only if there exists an h such that $h(n) = \frac{1}{a_n}$ is completely monotonic on $[0, \infty)$ and that in this case the measure can be obtained as $d\nu(t) = g(t)dt$ where $g(t)$ is $\frac{1}{t}$ times the inverse Laplace transform of $h(s/2)$ evaluated at $-\ln(t)$. The presence of a point mass will be signaled by the appearance of the dirac delta function centered at the appropriate point. Note that if a, b, and r are positive numbers, then $\frac{1}{(an+b)^r}$ is an elementary example of a completely monotonic function of n on $[0, \infty)$.

2. SOME COMPOSITION EXAMPLES

The goal of this section is to produce examples of tridiagonal reproducing kernels where M_z on the corresponding space is a bounded subnormal operator and for which there is a fairly close connection to M_z on a diagonal space.

One trivial way to obtain a space on which M_z is subnormal is to take $K(z, w) = L(z)\overline{L(w)}K_0(z, w)$ where L is any analytic function on $\mathbb{D}$, K_0 is a diagonal kernel and M_z on $H(K_0)$ is a subnormal operator. In this case $H(K)$ is simply $L(z)H(K_0) = \{f(z) = L(z)g(z) : g \in H(K_0)\}$ with $\|f\|_{H(K)} = \|g\|_{H(K_0)}$. Clearly M_z on $H(K)$ is unitarily equivalent to M_z on $H(K_0)$. Hence a trivial way to obtain a non-diagonal tridiagonal space on which M_z is subnormal is to simply take $L(z) = az + b$. This connection is too close to the diagonal case and the main focus of this section is to present a nontrivial way to obtain subnormal operators on tridiagonal spaces.

We first consider, for $\beta > 0$, the diagonal kernel

$$K_{\beta,s}(z, w) = \Big(\frac{1 + \beta\overline{w}z}{(1 - \overline{w}z)^s}\Big).$$

resulting from the Aronszajn sum $H(K_s)\widehat{\oplus}\sqrt{\beta}zH(K_s)$. Note that when $\beta = 0$, $K_{\beta,s}(z, w)$ reduces to the previously defined $K_s(z, w)$. Also note that when $s = 1$,

$$K_{\beta,1}(z, w) = 1 + \sum_{n=1}^{\infty}(1 + \beta)(\overline{w}z)^n,$$

when $s = 2$,

$$K_{\beta,2}(z,w) = \sum_{n=0}^{\infty} ((1+\beta)n + 1)(\overline{w}z)^n,$$

and for general $s > 1$,

$$K_{\beta,s}(z,w) = \sum_{n=0}^{\infty} \frac{\Gamma(s+n-1)(n(1+\beta)+s-1)}{n!\Gamma(s)}(\overline{w}z)^n.$$

When $s = 1$ and $s = 2$ it is clear that the reciprocals of the coefficients form moment sequences as $\{\frac{1}{an+b}\}$ is a completely monotonic sequence for positive a and b. The measure in the $s = 1$ case is given by $g_{\beta,1}(t)\frac{dtd\theta}{2\pi}$ where

$$g_{\beta,1}(t) = \left(\frac{\beta}{1+\beta}\delta_0(t) + \frac{1}{1+\beta}\delta_1(t)\right)$$

where δ_x denotes the Dirac delta function positioned at x. The measure in the $s = 2$ case is given by $g_{\beta,2}(t)\frac{dtd\theta}{2\pi}$ where

$$g_{\beta,2}(t) = \frac{2}{1+\beta}t^{(1-\beta)/(1+\beta)}.$$

When $s > 1$, the reciprocals of the coefficients

$$\frac{n!\Gamma(s)}{\Gamma(s+n-1)(n(1+\beta)+s-1)}$$

are recognizable as the product of the completely monotonic sequence $\{\frac{n!\Gamma(s)}{\Gamma(s+n)}\}$ associated with $L_{a,s}^2$ and the sequence $\{\frac{n+s-1}{n(1+\beta)+s-1}\}$. The latter sequence can be expressed as $h(n) = \frac{1}{1+\frac{\beta n}{n+s-1}}$. Since β and $s-1$ are positive, the function $\frac{\beta n}{n+s-1}$ is known to be a complete Bernstein function (see page 218 of Schilling, Song, and Vondraček [7]). By theorem 7.5 on page 63 of [7], h is a Stieltjes function. Since Stieltjes functions are completely monotonic (see Theorem 2.2, page 12, of [7]), h is a completely monotonic function. Thus, the product $\frac{n!\Gamma(s)}{\Gamma(s+n-1)(n(1+\beta)+s-1)}$ is completely monotonic and it follows from Widder [9], page 145, that this is a moment sequence. We will denote the associated measure by $g_{\beta,s}(t)\frac{dtd\theta}{2\pi}$. In the case when s is an integer, by using partial fractions and the inverse Laplace transform, we can explicitly determine

$$g_{\beta,s}(t) = \frac{(-1)^{s-2}2(s-1)!(\beta+1)^{s-2}}{(1+\beta)(s-2-\beta)(s-3-2\beta)\cdots(1-(s-2)\beta)}t^{-(3+\beta-2s)/(1+\beta)}$$

$$+ \sum_{k=0}^{s-3} \frac{(-1)^k 2(s-1)(s-2)}{s-2-k-(k+1)\beta}\binom{s-3}{k}t^{2k+1}.$$

Hence $f \in H(K_{\beta,s})$ has norm given by

$$\|f\|^2_{H(K_{\beta,s})} = \frac{1}{2\pi} \int\limits_0^{2\pi} \int\limits_0^1 |f(te^{i\theta})|^2 g_{\beta,s}(t) dt d\theta$$

and M_z on $H(K_{\beta,s})$ is a subnormal unilateral shift. For convenience we will denote by $\mu_{\beta,s}$ the measure $g_{\beta,s}(t)dtd\theta$.

THEOREM 2.1 (Composition Theorem). *If $\beta \geqslant 0$, $s \geqslant 1$, $a \in \mathbb{C}$, $\alpha \in \mathbb{D}$, $\phi(z) = \frac{z+\alpha}{1+\bar{\alpha}z}$, $\psi(z) = a(1 - \bar{\alpha}z)^{s-1}$, and*

$$K_{\phi,\psi,\beta,s}(z,w) = (\psi \circ \phi)(z)\overline{(\psi \circ \phi)(w)}(K_{\beta,s} \circ \phi)(z,w)$$

$$= \psi(\phi(z))\overline{\psi(\phi(w))} \frac{1 + \beta\phi(z)\overline{\phi(w)}}{(1 - \phi(z)\overline{\phi(w)})^s},$$

then $K_{\phi,\psi,\beta,s}$ is an analytic tridiagonal reproducing kernel on $\mathbb{D}$ for which M_z is a bounded subnormal operator on $H(K_{\phi,\psi,\beta,s})$.

Moreover, there is a constant c and a linear function L such that

$$H(K_{\phi,\psi,\beta,s}) = H(cK_s)\widehat{\oplus}L(z)H(cK_s).$$

Proof. First, note that with $K_{\psi,\beta,s}(z,w) = \psi(z)\overline{\psi(w)}K_{\beta,s}(z,w)$, the Hilbert space $H(K_{\psi,\beta,s})$ consists of the functions

$$\{\psi(z)f(z) : f \in H(K_{\beta,s}) = H(K_s)\widehat{\oplus}\sqrt{\beta}zH(K_s)\}$$

with $\|\psi f\|_{H(K_{\psi,\beta,s})} = \|f\|_{H(K_{\beta,s})} = \|f\|_{H(K_s)\widehat{\oplus}\sqrt{\beta}zH(K_s)}$.

Now consider the reproducing kernel $K_{\phi,\psi,\beta,s} = K_{\psi,\beta,s} \circ \phi$. Since ϕ is injective on $\mathbb{D}$, Proposition 1.1 asserts that $H(K_{\phi,\psi,\beta,s}) = \{f \circ \phi : f \in H(K_{\psi,\beta,s})\}$ and $\|f \circ \phi\|_{H(K_{\phi,\psi,\beta,s})} = \|f\|_{H(K_{\psi,\beta,s})}$. A computation shows that $K_{\phi,\psi,\beta,s}(z,w) = (K_{\psi,\beta,s} \circ \phi)(z,w)$ is equal to

$$\frac{a^2 \left(\frac{1-|\alpha|^2}{1+\bar{\alpha}z}\right)^{s-1} \left(\frac{1-|\alpha|^2}{1+\alpha\overline{w}}\right)^{s-1} \left(\frac{(1+\alpha\overline{w})(1+\bar{\alpha}z)+\beta(\overline{w}+\alpha)(z+\alpha)}{(1+\alpha\overline{w})(1+\bar{\alpha}z)}\right)}{\left(\frac{(1-|\alpha|^2)(1-\overline{w}z)}{(1+\alpha\overline{w}_1)(1+\bar{\alpha}z)}\right)^s}$$

$$= \frac{a^2(1-|\alpha|^2)^{s-2}(1 + \alpha\overline{w} + \bar{\alpha}z + |\alpha|^2\overline{w}z + \beta\overline{w}z + \beta\bar{\alpha}z + \beta\alpha\overline{w} + \beta|\alpha|^2)}{(1 - \overline{w}z)^s}$$

which, if $\beta > 0$, can be expressed as

$$\frac{a^2\beta(1 - |\alpha|^2)^s}{\beta + |\alpha|^2}(1 + L(z)\overline{L(w)})\frac{1}{(1 - \overline{w}z)^s}$$

where

$$L(z) = \frac{\alpha(1+\beta) + (\beta + |\alpha|^2)z}{(1 - |\alpha|^2)\sqrt{\beta}}.$$

Noting that L is a linear function and recalling that $K_s(z,w) = \frac{1}{(1-\overline{w}z)^s}$ is a diagonal kernel, it is now clear that $K_{\phi,\psi,\beta,s}$ is a tridiagonal kernel.

If $\beta = 0$, then $K_{\phi,\psi,\beta,s}$ reduces to

$$K_{\phi,\psi,\beta,s}(z,w) = \frac{a^2(1-|\alpha|^2)^{s-2}|\alpha|^2}{(1-\overline{w}z)^s}\left(\frac{1}{\overline{\alpha}}+z\right)\left(\frac{1}{\alpha}+\overline{w}\right)$$

and $H(K_{\phi,\psi,\beta,s})$ is the trivial case of a linear function times the diagonal space $H(K_s)$.

Assuming $\beta > 0$ and letting $c = \frac{a^2\beta(1-|\alpha|^2)^s}{\beta+|\alpha|^2}$ we note that

$$H(K_{\phi,\psi,\beta,s}) = H(cK_s)\widehat{\oplus}L(z)H(cK_s)$$

is invariant under multiplication by z since both of the summands are. Since evaluation is a continuous linear functional, the closed graph theorem implies M_z is bounded on $H(K_{\phi,\psi,\beta,s})$.

In the preliminary remarks to this theorem we observed that M_z on $H(K_{\beta,s})$ is subnormal with measure $\mu_{\beta,s}$. Since $H(K_{\psi,\beta,s}) = \psi(z)H(K_{\beta,s})$ with the norm of $h = \psi f$ given by

$$\|h\|^2_{H(K_{\psi,\beta,s})} = \|f\|^2_{H(K_{\beta,s})} = \int_{\mathbb{D}}|f|^2\mathrm{d}\mu_{\beta,s} = \int_{\mathbb{D}}|h(z)|^2\frac{\mathrm{d}\mu_{\beta,s}(z)}{|\psi(z)|^2},$$

M_z is bounded on $H(K_{\psi,\beta,s})$ and unitarily equivalent to M_z on $H(K_{\beta,s})$. Recall $h \in H(K_{\phi,\psi,\beta,s})$ is uniquely represented by $h = f \circ \phi$ for some $f \in H(K_{\psi,\beta,s})$. With the change of variables $z = \phi(\zeta)$ at the appropriate step, we see that

$$\|h\|^2_{H(K_{\phi,\psi,\beta,s})} = \|f\circ\phi\|^2_{H(K_{\phi,\psi,\beta,s})} = \|f\|^2_{H(K_{\psi,\beta,s})} = \int_{\mathbb{D}}|f(z)|^2\mathrm{d}\mu_{\psi,\beta,s}(z)$$

$$= \int_{\mathbb{D}}|f(\phi(\zeta))|^2|\phi'(\zeta)|\mathrm{d}\mu_{\psi,\beta,s}(\zeta) = \int_{\mathbb{D}}|h(\zeta)|^2|\phi'(\zeta)|\mathrm{d}\mu_{\psi,\beta,s}(\zeta)$$

where $\mathrm{d}\mu_{\psi,\beta,s}$ is the measure associated with the space $H(K_{\psi,\beta,s})$. Thus $H(K_{\phi,\psi,\beta,s})$ is a measure space and M_z is a subnormal operator. ∎

REMARK 2.2. In the composition theorem just completed, the choice of ψ is almost, but not quite, unique for a given α. If $\psi(z) = a(1-\frac{1}{\alpha}z)^{s-1}$, then

$$K_{\phi,\psi,\beta,s}(z,w) = \frac{a^2\beta(1-|\alpha|^2)^s|\alpha|^{2(1-s)}}{\beta+|\alpha|^2}(1+L(z)\overline{L(w)})\frac{(\overline{w}z)^{s-1}}{(1-\overline{w}z)^s}$$

where L is the same as in the composition theorem. The essential effect of this is to replace the diagonal kernel $K_s(z,w) = \frac{1}{(1-\overline{w}z)^s}$ with the diagonal kernel $\frac{(\overline{w}z)^{s-1}}{(1-\overline{w}z)^s}$. This simply replaces the role of the space $H(K_s)$ with the space $z^{s-1}H(K_s)$ and results in a very similar tridiagonal kernel for which M_z is subnormal. This is the only other choice of ψ which will lead via the above composition method to a tridiagonal kernel for a given $\alpha \in \mathbb{D}$.

REMARK 2.3. If $A_1 = r_1 e^{i\theta_1}$ and $A_2 = r_2 e^{i\theta_2}$ are any two complex numbers and $L(z) = A_1 + A_2 z = r_1 e^{i\theta_1} + r_2 e^{i\theta_2} z$, then

$$L(z)\overline{L(w)} = (r_1 + r_2 e^{i(\theta_2 - \theta_1)} z)(r_1 + r_2 e^{-i(\theta_2 - \theta_1)} \overline{w})$$

implies A_1 can be taken to be real. By a rotation of z and w by $(\theta_1 - \theta_2)$, we can assume A_2 is also real in our representation of $K_{\phi,\psi,\beta,s}$. It is noteworthy that if A_1 and A_2 are any given real numbers and s is a fixed positive integer, then there exists $\alpha \in \mathbb{D}$ and $\beta > 0$ such that

$$K_{\phi,\psi,\beta,s}(z,w) = (1 + (A_1 + A_2 z)(A_1 + A_2 \overline{w}))K_s(z,w).$$

In fact

$$\alpha = \frac{1 + A_1^2 + A_2^2 - \sqrt{A_1^4 - 2A_1^2(A_2^2 - 1) + (A_2^2 + 1)^2}}{2A_1 A_2} \quad \text{and}$$

$$\beta = A_2^2 - A_1^2 + \alpha(1 + A_1^2 - A_2^2)\frac{A_1}{A_2}.$$

Thus a correspondence exists between the reproducing kernel sum spaces $H(K_s) \widehat{\oplus} L(z)H(K_s)$ and the composition spaces $H(K_{\phi,\psi,\beta,s})$.

Next we are going to look at two examples from a dual perspective. The first is through the theorem just proved. The second is a different approach that illuminates the relationship between the composition and the reproducing kernel sum from the point of view of the matricial range space of the tridiagonal kernels.

EXAMPLE 2.4. The Aronszajn sum $H^2 \widehat{\oplus} (1-z)H^2$ can be realized from the composition theorem by taking $\alpha = \frac{1}{2}(3 - \sqrt{5})$, $\beta = -\alpha$, $s = 2$, $a = \frac{1}{\sqrt{(1-|\alpha|^2)(\alpha+1)}}$, and $\psi(z) = a(1 - \overline{\alpha}z)$. The resulting measure is $P_\beta(\theta)\frac{d\theta}{2\pi} + \frac{\beta}{\sqrt{5}}\mathcal{X}_\beta$ where $P_\beta(\theta)$ denotes the Poisson kernel $\frac{1-|\beta|^2}{|1+\overline{\beta}e^{i\theta}|^2}$ and $\mathcal{X}_\beta$ denotes the unit point mass at β. Notice that $\beta = -\alpha$ is the reciprocal of the square of the Golden Ratio.

Now, from the matricial perspective, consider

$$K(z,w) = \frac{1 + (1-z)(1-\overline{w})}{1 - \overline{w}z} = v(z)Av(w)^*$$

$$= (1, z, z^2, \ldots) \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots \\ -1 & 3 & -1 & 0 & \ddots \\ 0 & -1 & 3 & -1 & \ddots \\ 0 & 0 & -1 & 3 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ \overline{w} \\ \overline{w}^2 \\ \vdots \end{pmatrix}.$$

Factoring $A = LL^*$ where

$$L = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ b_0 & a_1 & 0 & \cdots \\ 0 & b_1 & a_2 & \ddots \\ 0 & 0 & b_2 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

we arrive at the recursion $a_0 = \sqrt{2}$, $b_n = \frac{-1}{a_n}$, and $a_n^2 + \frac{1}{a_{n-1}^2} = 3$. This is easily solved to produce $a_n = \frac{\alpha_{2(n+1)}}{\alpha_{2n}}$ where α_n is the n-th Fibonacci number. Note the zeros of $f_n(z)$ are 0 and $\frac{-a_n}{b_n} = a_n^2$. The latter converges to ϕ^2 where $\phi = \frac{1+\sqrt{5}}{2}$ is the Golden Ratio. The negative reciprocal of this is the α above.

EXAMPLE 2.5. Similarly, the Aronszajn sum $L_a^2(\mathbb{D}) \widehat{\oplus} (1-z) L_a^2(\mathbb{D})$ can be realized from the composition theorem by taking $\alpha = \frac{1}{2}(3 - \sqrt{5})$, $\beta = -\alpha$, $s = 2$, $a = \frac{1}{\sqrt{(1-|\alpha|^2)(\alpha+1)}}$, and $\psi(z) = a(1 - \bar{\alpha}z)$. Note that

$$K(z,w) = \frac{1 + (1-z)(1-\bar{w})}{(1 - \bar{w}z)^2} = v(z) A v(w)^*$$

$$= (1, z, z^2, \ldots) \begin{pmatrix} 2 & -1 & 0 & 0 & \cdots \\ -1 & 5 & -2 & 0 & \ddots \\ 0 & -2 & 8 & -3 & \ddots \\ 0 & 0 & -3 & 11 & \ddots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 \\ \bar{w} \\ \bar{w}^2 \\ \vdots \end{pmatrix}.$$

Factoring $A = LL^*$ as in the previous example

$$L = \begin{pmatrix} a_0 & 0 & 0 & \cdots \\ b_0 & a_1 & 0 & \cdots \\ 0 & b_1 & a_2 & \ddots \\ 0 & 0 & b_2 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix},$$

we arrive at the recursion $a_0 = \sqrt{2}$, $b_n = \frac{-(n+1)}{a_n}$, and

$$a_n^2 + \frac{n^2}{a_{n-1}^2} = 2 + 3n.$$

Alternatively, setting $a_n = \sqrt{\frac{t_{n+1}}{t_n}}$,

$$t_{n+1} = (2 + 3n)t_n - n^2 t_{n-1}.$$

This type of non-constant coefficient linear recursion is difficult to solve directly. However, working in reverse the transformation in the composition theorem with $\zeta = S(z) = \frac{z-\alpha}{1-\alpha z}$, we see that $K(z, \overline{w})$

$$= K(S^{-1}(\zeta), S^{-1}(\overline{w})) = Q(\zeta)\overline{Q(\omega)}\left[\frac{1 + \frac{\alpha^2 + (1-\alpha)^2}{1+(1-\alpha)^2}\overline{\omega}\zeta}{1 - \overline{\omega}\zeta}\right],$$

where

$$Q(\zeta) = \frac{\sqrt{1+(1-\alpha)^2}}{1-\alpha^2}(1+\alpha\zeta).$$

So S^{-1} transforms K to a new kernel which is a linear function times a diagonal space which has associated measure $g(t)\mathrm{d}t$ with

$$g(t) = \left(1 + \frac{1}{\sqrt{5}}\right)t^{1/\sqrt{5}}.$$

The resulting transformed measure on $H(K)$ is given by

$$\|h\|^2_{H(K)} = \frac{\alpha}{\pi}\int\limits_0^{2\pi}\int\limits_0^1 |h(re^{i\theta})|^2\left|\frac{re^{i\theta}-\alpha}{1-\alpha re^{i\theta}}\right|^{1/\sqrt{5}}\frac{r\mathrm{d}r\mathrm{d}\theta}{|re^{i\theta}-\alpha||1-\alpha re^{i\theta}|}.$$

Of particular interest is the recursion $a_n^2 + \frac{n^2}{a_{n-1}^2} = 2 + 3n$ with $a_0 = \sqrt{2}$ which is now seen to have solution expressed as

$$a_n = \sqrt{\frac{(n+1)\int\limits_0^{2\pi}\int\limits_0^1 r^{n+1}e^{i(n+1)\theta}\left(\sqrt{2}-\frac{1}{\sqrt{2}}re^{-i\theta}\right)\left|\frac{re^{i\theta}-\frac{1}{\phi^2}}{1-\frac{1}{\phi^2}re^{i\theta}}\right|^{1/\sqrt{5}}\frac{r\mathrm{d}r\mathrm{d}\theta}{\left|re^{i\theta}-\frac{1}{\phi^2}\right|\left|1-\frac{1}{\phi^2}re^{i\theta}\right|}}{\int\limits_0^{2\pi}\int\limits_0^1 r^n e^{in\theta}\left(\sqrt{2}-\frac{1}{\sqrt{2}}re^{-i\theta}\right)\left|\frac{re^{i\theta}-\frac{1}{\phi^2}}{1-\frac{1}{\phi^2}re^{i\theta}}\right|^{1/\sqrt{5}}\frac{r\mathrm{d}r\mathrm{d}\theta}{\left|re^{i\theta}-\frac{1}{\phi^2}\right|\left|1-\frac{1}{\phi^2}re^{i\theta}\right|}}}.$$

While this is neither an elegant nor a direct solution, the authors know of no other method of solving this type of recursion.

3. A CLASS OF SUBNORMAL OPERATORS

We begin this section with a result which gives a characterization of K being a tridiagonal kernel.

PROPOSITION 3.1. *An analytic reproducing kernel K on the unit disk is a tridiagonal kernel such that $H(K)$ contains the polynomials if and only if for $m \geqslant n+1$,*

$$\kappa_n = \frac{\langle z^n, z^m \rangle}{\langle z^{n+1}, z^m \rangle}$$

is independent of m and the sequence

$$\left\{\frac{1}{a_0}, \frac{\kappa_0}{a_1}, \frac{\kappa_0\kappa_1}{a_2}, \frac{\kappa_0\kappa_1\kappa_2}{a_3}, \ldots\right\}$$

is an l^2 sequence where $\frac{1}{a_n} = \|z^n - \kappa_n z^{n+1}\|$.

Proof. Suppose that $K(z, w) = \sum\limits_{n=0}^{\infty} f_n(z)\overline{f_n(w)}$ is an analytic reproducing kernel on the unit disk where $f_n(z) = a_n z^n + b_n z^{n+1}$ form an orthonormal basis and $H(K)$ contains the polynomials. Recall that $z^m \in H(K)$ for all $m \geqslant 0$ implies that the sequence

$$\left\{1, \frac{b_m}{a_{m+1}}, \frac{b_m b_{m+1}}{a_{m+1}a_{m+2}}, \frac{b_m b_{m+1} b_{m+2}}{a_{m+1}a_{m+2}a_{m+3}}, \ldots\right\}$$

is square summable for each m (see [1]). Writing $z^m = \sum\limits_{n=0}^{\infty} c_n f_n(z)$ and observing that the derivatives of z^m at 0 must match the derivatives of the series at 0, we see that $c_n = 0$ for $0 \leqslant n < m$. Thus $z^m = \sum\limits_{n=m}^{\infty} c_n f_n(z)$. Hence $\langle f_n, z^m \rangle = \langle a_n z^n + b_n z^{n+1}, z^m \rangle = 0$ for $m > n$ which implies that $\kappa_n = \frac{\langle z^n, z^m \rangle}{\langle z^{n+1}, z^n \rangle} = -\frac{b_n}{a_n}$ for all $m > n$. Noting that $\|z^n + \kappa_n z^{n+1}\|^2$

$$= \left\langle z^n + \frac{b_n}{a_n}z^{n+1}, z^n + \frac{b_n}{a_n}z^{n+1}\right\rangle = \left\langle \frac{1}{a_n}f_n(z), \frac{1}{a_n}f_n(z)\right\rangle = \frac{1}{|a_n|^2},$$

we conclude that

$$\left\{\frac{1}{a_0}, \frac{\kappa_0}{a_1}, \frac{\kappa_0\kappa_1}{a_2}, \frac{\kappa_0\kappa_1\kappa_2}{a_3}, \ldots\right\}$$

is an l^2 sequence.

Conversely, assume $\kappa_n = \frac{\langle z^n, z^m \rangle}{\langle z^{n+1}, z^n \rangle}$ is independent of $m > n$ and

$$\left\{\frac{1}{a_0}, \frac{\kappa_0}{a_1}, \frac{\kappa_0\kappa_1}{a_2}, \frac{\kappa_0\kappa_1\kappa_2}{a_3}, \ldots\right\}$$

is an l^2 sequence. Let $g_n(z) = z^n - \kappa_n z^{n+1}$ and note that for $m > n$,

$$\langle g_n(z), z^m \rangle = \langle z^n, z^m \rangle - \kappa_n \langle z^{n+1}, z^m \rangle = 0.$$

It follows that $\{g_n\}$ is an orthogonal set and $\{f_n\} = \left\{\frac{g_n}{\|g_n\|}\right\}$ is an orthonormal basis for the tridiagonal kernel $K(z, w) = \sum\limits_{n=0}^{\infty} f_n(z)\overline{f_n(w)}$. Since

$$\left\{\frac{1}{a_0}, \frac{\kappa_0}{a_1}, \frac{\kappa_0\kappa_1}{a_2}, \frac{\kappa_0\kappa_1\kappa_2}{a_3}, \ldots\right\}$$

is an l^2 sequence, the polynomials are in $H(K)$. ∎

In Examples 2.2 and 2.3, κ_n converges to $\lambda = \frac{3-\sqrt{5}}{2}$ which is the reciprocal of the square of the Golden Ratio. In the next family of examples the measure is constructed to satisfy Proposition 3.1.

LEMMA 3.2. *Let a and b be real numbers, $p > -1$, and $|\lambda| > 1$. If*

$$u(r,\theta) = (a + (r-1)b)r^p + 2r^p \Re\Big\{ \sum_{j=1}^{\infty} \Big[\frac{a - (1-r)(b+2|j|a)}{\lambda^{|j|}} \Big] r^{|j|} e^{ij\theta} \Big\}$$

is a non-negative function on $\mathbb{D}$, then $d\mu(re^{i\theta}) = u(r,\theta)\frac{drd\theta}{2\pi}$ is a bounded Borel measure on $\mathbb{D}$ for which

$$\frac{\langle z^n, z^m \rangle_{L^2(\mu)}}{\langle z^{n+1}, z^m \rangle_{L^2(\mu)}} = \frac{1}{\lambda} \frac{(2n+p+2)a - b}{(2n+p+4)a - b}.$$

Proof. First note that $u(r,\theta)$ is real-valued, bounded, and absolutely convergent for $re^{i\theta} \in \overline{\mathbb{D}}$ since $|\lambda| > 1$. Given $d u(r,\theta) = u(r,\theta)\frac{d\theta dr}{2\pi}$ defined as above and assuming $m \geqslant n$, we see that formally

$$\langle z^n, z^m \rangle_{L^2(\mu)} = \int_0^1 \int_0^{2\pi} r^{m+n} e^{-i(m-n)\theta} u(r,\theta) \frac{d\theta dr}{2\pi}$$

$$= \int_0^1 r^{m+n} \Big(\frac{a - (1-r)(b + 2(m-n)a)}{\lambda^{m-n}} \Big) r^{m-n+p} dr$$

$$= \frac{1}{\lambda^{m-n}} \int_0^1 r^{2m+p}[a(1 - 2(m-n)) - b] + r^{2m+p+1}[b + 2(m-n)a] dr$$

$$= \frac{1}{\lambda^{m-n}} \frac{(2n+p+2)a - b}{(2m+p+1)(2m+p+2)}.$$

Consequently, if $m > n$, then

$$\kappa_n = \frac{\langle z^n, z^m \rangle}{\langle z^{n+1}, z^m \rangle} = \frac{1}{\lambda} \frac{(2n+p+2)a - b}{(2n+p+4)a - b}$$

is independent of m. Moreover, note that as long as $a \neq 0$, κ_n is not a constant. ∎

REMARK 3.3. Next, note that

$$u(r,\theta) = (a + (r-1)b)r^p + 2\Re\Big\{ \sum_{j=1}^{\infty} r^p(a + (r-1)b + 2a(r-1)j) \Big(\frac{re^{i\theta}}{\lambda} \Big)^j \Big\}$$

$$= (a + (r-1)b)r^p + 2r^p \Re\Big\{ \frac{re^{i\theta}}{\lambda} \Big(\frac{a + (r-1)b}{1 - \frac{re^{i\theta}}{\lambda}} + \frac{2(r-1)a}{(1 - \frac{re^{i\theta}}{\lambda})^2} \Big) \Big\}.$$

If a and b are chosen so that $a + (r-1)b$ is positive for $0 \leqslant r \leqslant 1$ and $|\lambda|$ is taken to be sufficiently large, then $\frac{u(r,\theta)}{r^p}$ is easily seen to be a positive function on

$\mathbb{D}$. Hence, there is a wide range for the values a, b, and λ such that $\mathrm{d}u(r,\theta) = u(r,\theta)\frac{\mathrm{d}\theta\mathrm{d}r}{2\pi}$ is a positive measure on the disk.

THEOREM 3.4. *If a and b are real numbers such that $a + (r-1)b$ is positive for $0 \leqslant r \leqslant 1$, $p > -1$, and*

$$u(r,\theta) = (a + (r-1)b)r^p + 2r^p\Re\left\{ \sum_{j=1}^{\infty} \left[\frac{a - (1-r)(b+2|j|a)}{\lambda^{|j|}} \right] r^{|j|}\mathrm{e}^{\mathrm{i}j\theta} \right\},$$

where λ is a complex number sufficiently large in modulus so that $\mathrm{d}\mu(r\mathrm{e}^{\mathrm{i}\theta}) = u(r,\theta)\frac{\mathrm{d}r\mathrm{d}\theta}{2\pi}$ is a positive Borel measure on $\mathbb{D}$, then $P^2(\mu)$ is an analytic tridiagonal reproducing kernel Hilbert space of functions on the unit disk $\mathbb{D}$ for which M_z is a subnormal operator.

Proof. By Lemma 3.2 and Remark 3.3 above, $\frac{u(r,\theta)}{r^p}$ is a positive function on $\mathbb{D}$ and $\mathrm{d}\mu(r\mathrm{e}^{\mathrm{i}\theta}) = u(r,\theta)\frac{\mathrm{d}r\mathrm{d}\theta}{2\pi}$ is a positive bounded Borel measure on $\mathbb{D}$ for which

$$\kappa_n = \frac{\langle z^n, z^m \rangle_{L^2(\mu)}}{\langle z^{n+1}, z^m \rangle_{L^2(\mu)}} = \frac{1}{\lambda}\frac{(2n+p+2)a - b}{(2n+p+4)a - b}.$$

If $w \in \mathbb{D} \setminus \{0\}$, $B(w,r_1)$ denotes the disk of radius r_1 about w, and p is a polynomial, then the mean-value theorem implies

$$|p(w)| = \left| \frac{1}{r_1} \int_{B(w,r_1)} p(r\mathrm{e}^{\mathrm{i}\theta})\frac{\mathrm{d}\theta\mathrm{d}r}{2\pi} \right| = \left| \frac{1}{r_1} \int_0^{r_1}\int_0^{2\pi} p(w+r\mathrm{e}^{\mathrm{i}\theta})\frac{\mathrm{d}\theta\mathrm{d}r}{2\pi} \right|$$

$$= \left| \frac{1}{r_1} \int_0^{r_1}\int_0^{2\pi} p(w+r\mathrm{e}^{\mathrm{i}\theta})\sqrt{u(w+r\mathrm{e}^{\mathrm{i}\theta})}\frac{1}{\sqrt{u(w+r\mathrm{e}^{\mathrm{i}\theta})}}\frac{\mathrm{d}\theta\mathrm{d}r}{2\pi} \right|.$$

Since $p\sqrt{u}$ and $\frac{1}{\sqrt{u}}$ are square-integrable functions on $B(w,r_1)$ with respect to $\frac{\mathrm{d}\theta\mathrm{d}r}{2\pi}$, Holder's inequality gives

$$|p(w)| \leqslant \frac{1}{r_1}\left(\int_{B(w,r_1)} |p\sqrt{u}|^2\frac{\mathrm{d}\theta\mathrm{d}r}{2\pi} \right)^{1/2}\left(\int_{B(w,r_1)} |\frac{1}{\sqrt{u}}|^2\frac{\mathrm{d}\theta\mathrm{d}r}{2\pi} \right)^{1/2} \leqslant C_w\|p\|_\mu$$

where C_w is a constant dependent on w. When $w = 0$, a similar argument can be applied to an annulus $A(r_0,r_1)$ about 0 of inner radius r_0 and outer radius r_1 to obtain

$$|p(0)| \leqslant \frac{1}{r_1 - r_0}\left(\int_{A(r_0,r_1)} |p\sqrt{u}|^2\frac{\mathrm{d}\theta\mathrm{d}r}{2\pi} \right)^{1/2}\left(\int_{A(r_0,r_1)} |\frac{1}{\sqrt{u}}|^2\frac{\mathrm{d}\theta\mathrm{d}r}{2\pi} \right)^{1/2} \leqslant C_w\|p\|_\mu.$$

Thus for each $w \in \mathbb{D}$, evaluation at w extends to a unique bounded linear functional for which there is a unique function $K_w \in P^2(\mu)$ such that $\langle p, K_w \rangle = p(w)$ for all polynomials. Note that $\|K_w\| \leqslant C_w$ is uniformly bounded for w in any

small disk B compactly contained in $\mathbb{D}$. By Proposition 7.6, page 63, of Conway [4], every point in the unit disk is an analytic bounded point evaluation and hence $P^2(\mu)$ is an analytic reproducing kernel Hilbert space of functions on $\mathbb{D}$.

From the proof of Lemma 3.2,

$$\langle z^n, z^n \rangle = \frac{(2n+p+2)a - b}{(2n+p+1)(2n+p+2)} \approx \frac{a}{2n+p+4} \quad \text{and}$$

$$\langle z^n, z^{n+1} \rangle = \frac{(2n+p+2)a - b}{\lambda(2n+p+3)(2n+p+2)} \approx \frac{a}{2n+p+3}.$$

Since $\kappa_n \approx \frac{1}{\lambda}$,

$$\|z^n - \kappa_n z^{n+1}\|^2 = \|z^n\|^2 + \kappa_n^2 \|z^{n+1}\|^2 - 2\Re\{\kappa_n \langle z^n, z^{n+1} \rangle\}$$

$$\approx \frac{a}{2n+p+1} + \frac{1}{\lambda^2}\frac{a}{2n+p+3} - \frac{2}{\lambda^2}\frac{a}{2n+p+3} \approx \frac{a}{2n+p+1}\left(1 - \frac{1}{\lambda^2}\right).$$

Thus $a_n = \frac{1}{\|z^n - \kappa_n z^{n+1}\|} \approx c\sqrt{n}$ and

$$\left\{ \frac{1}{a_0}, \frac{\kappa_0}{a_1}, \frac{\kappa_0 \kappa_1}{a_2}, \frac{\kappa_0 \kappa_1 \kappa_2}{a_3}, \dots \right\}$$

is seen to be an l^2 sequence as its terms are approximately a constant times $\frac{1}{\lambda^{n-1}\sqrt{n}}$ where $|\lambda| > 1$. By Proposition 3.1, $P^2(\mu)$ is a tridiagonal reproducing kernel Hilbert space for which the functions $f_n(z) = \frac{z^n - \kappa_n z^{n+1}}{\|z^n - \kappa_n z^{n+1}\|}$ form an orthonormal basis for $P^2(\mu)$. Since μ is a bounded measure on a compact set, M_z is a bounded subnormal operator on $P^2(\mu)$ and the proof is complete. ∎

While the class of examples in the above theorem appear to be distinctly different than the examples in Section 2, it is possible that the examples are some type of composition with a diagonal kernel for which multiplication by z is a subnormal operator. The authors leave this as an open question.

4. OPEN QUESTIONS

(i) In the composition examples, can we determine the limit of

$$\kappa_n = \frac{\langle z^n, z^m \rangle}{\langle z^{n+1}, z^m \rangle}?$$

(ii) Is M_z subnormal if the nontrivial zero of $f_n(z) = (a_n + b_n z)z^n$ diverges to infinity? In particular suppose $f_n(z) = (n + 1 + z)z^n$? In this case $K(z, w) = \frac{1 + \overline{w}z}{(1-\overline{w}z)^3} + \frac{\overline{w}+z}{(1-\overline{w}z)^2} + \frac{\overline{w}z}{(1-\overline{w}z)}$, while for $m \geqslant n$, $\langle z^n, z^m \rangle = \sum_{j=m+1}^{\infty} \frac{n!m!(-1)^{(m-n)}}{(j!)^2}$, and $\kappa_n = \frac{\langle z^n, z^m \rangle}{\langle z^{n+1}, z^m \rangle} = \frac{-1}{n+1}$.

The following table consists of examples of such kernels for which M_z passes several numerical tests for subnormality (polynomial hyponormality). The tests consisted of checking positivity of commutators of the operator as well as some polynomials of the operator.

More precisely, the positivity of the self commutator of M_z was checked by a computation of the first 25 sub-determinants of size $k \times k$. The precision was set to 80 and the size of the original matrix for M_z was 80×80. Additionally, the positivity of several sub-determinants of the self commutator of M_ϕ were checked with $\phi(z) = -z + 3z^2 + z^3 + 12z^4$. A few other polynomials were randomly chosen. The symbolic algebra package Mathematica was used for the computations.

The computations suggest that large classes of tridiagonal kernels exist for which M_z is a subnormal operator and which are not of the types described in Sections 2 and 3.

a_n	b_n	range of k	conjecture
$(n+1)^2$	$n+k$	$-3 \leqslant k \leqslant 3$	subnormal?
$(n+2)^2$	$n+k$	$-7 \leqslant k \leqslant 8$	subnormal?
$(n+3)^2$	$n+k$	$-12 \leqslant k \leqslant 15$	subnormal?
$(n+4)^2$	$n+k$	$-18 \leqslant k \leqslant 22$	subnormal?
$(n+5)^2$	$n+k$	$-26 \leqslant k \leqslant 31$	subnormal?
$(n+6)^2$	$n+k$	$-35 \leqslant k \leqslant 41$	subnormal?
$(n+7)^2$	$n+k$	$-45 \leqslant k \leqslant 52$	subnormal?
$(n+8)^2$	$n+k$	$-57 \leqslant k \leqslant 64$	subnormal?
$(n+9)^2$	$n+k$	$-69 \leqslant k \leqslant 78$	subnormal?
$(n+10)^2$	$n+k$	$-83 \leqslant k \leqslant 93$	subnormal?
$(n+11)^2$	$n+k$	$-99 \leqslant k \leqslant 109$	subnormal?
$(n+1)^4$	$(n+k)^2$	$-3 \leqslant k \leqslant 3$	subnormal?
$(n+2)^4$	$(n+k)^2$	$-7 \leqslant k \leqslant 8$	subnormal?
$(n+1)^5$	$(n+k)^2$	$-6 \leqslant k \leqslant 5$	subnormal?
$(n+1)^6$	$(n+k)^2$	$-9 \leqslant k \leqslant 8$	subnormal?
$(n+1)^6$	$n+k$	$-84 \leqslant k \leqslant 83$	subnormal?
$\sqrt{n+1}$	$\sqrt[4]{n+1}$	none	subnormal?
$1+\sqrt{n+1}$	$\sqrt[4]{n+1}$	none	subnormal?

(iii) Assume $K_1(z,w) = \sum a_n(\overline{w}z)^n$ and $K_2(z,w) = \sum b_n(\overline{w}z)^n$ satisfy:

(a) $\lim \frac{a_n}{a_{n+1}} = \lim \frac{b_n}{b_{n+1}} = 1$;

(b) $\lim a_n = \lim b_n = +\infty$;

(c) $\frac{1}{a_n} = \int\limits_0^1 t^{2n} d\nu_1(t)$ and $\frac{1}{b_n} = \int\limits_0^1 t^{2n} d\nu_2(t)$ for all n.

Note $\nu_1\{1\} = \nu_2\{1\} = 0$, $H(K_i) = P^2(\mu_i)$ where $d\mu_i(re^{i\theta}) = \nu_i(r)\frac{dr d\theta}{2\pi}$, and M_z is a bounded subnormal operator. So condition (a) ensures $\mathbb{D}$ is a common domain and condition (b) ensures the measures live only on $\mathbb{D}$ and not on the boundary.

Conjecture. Under the above conditions, multiplication by z on $H(K_1 + K_2) = H(K_1)\widehat{\oplus}H(K_2)$ is a subnormal operator.

This is equivalent to any of the below:

(a) The parallel sum of $\frac{1}{a_n}$ and $\frac{1}{b_n}$ defined as

$$\left(\left(\frac{1}{a_n}\right)^{-1} + \left(\frac{1}{b_n}\right)^{-1}\right)^{-1} = \frac{1}{a_n + b_n}$$

is a moment sequence. (The parallel sum has an extensive history in electric engineering applications and is simply half the harmonic mean.)

(b) The harmonic mean of $\frac{1}{a_n}$ and $\frac{1}{b_n}$ defined as

$$2\left(\left(\frac{1}{a_n}\right)^{-1} + \left(\frac{1}{b_n}\right)^{-1}\right)^{-1} = \frac{2}{a_n + b_n}$$

is a moment sequence.

(c) The Hankel matrix below must be positive for all n

$$H_{\{\frac{1}{a_n+b_n}\}} = \begin{pmatrix} \frac{1}{a_0+b_0} & \frac{1}{a_1+b_1} & \cdots & \frac{1}{a_n+b_n} \\ \frac{1}{a_1+b_1} & \frac{1}{a_2+b_2} & \cdots & \frac{1}{a_{n+1}+b_{n+1}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{1}{a_n+b_n} & \frac{1}{a_{n+1}+b_{n+1}} & \cdots & \frac{1}{a_{2n}+b_{2n}} \end{pmatrix}.$$

(d) For each $k \geqslant 0$,

$$(-1)^k \Delta^k \left(\frac{1}{a_n + b_n}\right) \geqslant 0$$

where Δ^k denotes the k-th finite difference.

(iv) *General question.* If K_1 and K_2 are measure spaces on a common open domain Ω with the measures supported only on Ω, under what conditions is the space $H(K_1)\widehat{\oplus}H(K_2)$ with kernel $K_1 + K_2$ a measure space?

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