QUASISIMILARITY OF INVARIANT SUBSPACES FOR C_0 OPERATORS WITH MULTIPLICITY TWO

RAPHAËL CLOUÂTRE

Communicated by Kenneth R. Davidson

ABSTRACT. For an operator T of class C_0 with multiplicity two, we show that the quasisimilarity class of an invariant subspace M is determined by the quasisimilarity classes of the restriction $T|M$ and of the compression $T_{M^\perp}$. We also provide a canonical form for the subspace M.

KEYWORDS: *C_0 operators, invariant subspaces, quasisimilarity orbit.*

MSC (2010): 47A45, 47A15.

1. INTRODUCTION

Let $T : \mathcal{H} \to \mathcal{H}$ and $T : \mathcal{H}' \to \mathcal{H}'$ be bounded linear operators on Hilbert spaces. If M and M' are invariant subspaces for T and T' respectively (that is $M \subset \mathcal{H}$ and $M' \subset \mathcal{H}'$ are closed subspaces such that $TM \subset M$ and $T'M' \subset M'$), we say that M is a *quasiaffine transform* of M' if there exists a bounded injective operator with dense range $X : \mathcal{H} \to \mathcal{H}'$ such that $XT = T'X$ and $\overline{XM} = M'$. We write $M \prec M'$ when M is a quasiaffine transform of M'. In that case, we also say that M' lies in the *quasiaffine orbit* of M. When $M \prec M'$ and $M' \prec M$, we say that M and M' are *quasisimilar* and write $M \sim M'$. Quasisimilarity is clearly an equivalence relation on the class of pairs of the form (T, M), where M is an invariant subspace for the bounded linear operator T. In [2], Bercovici raised the basic problem underlying our present investigation: describe the quasisimilarity class of a given invariant subspace for an operator of class C_0 (see definition in Section 2). Theorem 2.12 below (see Section 2) is classical and offers a complete and very simple answer to the problem in the case where the operator has multiplicity one (that is the operator has a cyclic vector). Hence, we are interested in operators of class C_0 with multiplicity higher than one.

In their pioneering work (see [8]), Bercovici and Tannenbaum considered the case where T is a so-called uniform Jordan operator (namely $T = S(\theta) \oplus$

$S(\theta) \oplus \cdots$) with finite multiplicity. In that case, they established that the quasisimilarity class of M is determined by the quasisimilarity class of the restriction $T|M$ (see Section 2). Moreover, the authors observed that for $T = S(z^2) \oplus S(z)$, this classification breaks down, so the corresponding result may fail if T is not uniform. Later on, it was proved in [2] that this classification of invariant subspaces of a uniform Jordan operator holds if and only if $T|M$ has property (P). In general, the quasisimilarity class of an invariant subspace for a uniform Jordan operator is determined by the quasisimilarity classes of the restriction $T|M$ and of the compression $T_{M^\perp}$ (see [3]).

In the context of a non-uniform Jordan operator, much less is known. Related results for general operators of class C_0 can be found in [2], where it is proved that the quasisimilarity class of an invariant subspace is determined by that of $T|M$ if and only if T has property (Q). In case where $T|M$ has multiplicity one, then the weakly quasiaffine orbit of an invariant subspace M is determined by the quasisimilarity classes of $T|M$ and $T_{M^\perp}$ (see [4]).

More recently, nilpotent operators of finite multiplicity have been considered by Li and Müller in [10]. They proved that the quasisimilarity class of M is determined by the quasisimilarity classes of $T|M$ and $T_{M^\perp}$ when either of those operators has multiplicity one. In addition, the authors considered a combinatorial object (a sequence of partitions) known as a *Littlewood–Richardson sequence* which encodes the relationships that must hold between the Jordan models of T, $T|M$ and $T_{M^\perp}$ (see also [9], [6] and [7]). Using these objects, they prove that for multiplicity at least three, the quasisimilarity classes of $T|M$ and $T_{M^\perp}$ are not enough to determine the quasisimilarity class of M (so that our main theorem is sharp as far as multiplicities are concerned). In fact, that information does not even suffice to determine the larger equivalence class of invariant subspaces having a fixed Littlewood–Richardson sequence. However, an easy argument shows that in the case of multiplicity two, the knowledge of the quasisimilarity classes of T, $T|M$ and $T_{M^\perp}$ is enough to determine the Littlewood–Richardson sequence corresponding to M: the so-called Littlewood–Richardson rule can be satisfied in only one way. Hence, this case seems to involve some kind of uniqueness which is not present for higher multiplicities. Our main result confirms and strenghtens this observation, in fact we show that for arbitrary operators of class C_0 with multiplicity two, the quasisimilarity class of an invariant subspace M is determined by that of $T|M$ and $T_{M^\perp}$. We also identify a specific invariant subspace which can serve as a canonical space.

2. BACKGROUND AND PRELIMINARIES

We give here some background concerning operators of class C_0. Let H^∞ be the algebra of bounded holomorphic functions on the open unit disc $\mathbb{D}$. Let $\mathcal{H}$ be a Hilbert space and T a bounded linear operator on $\mathcal{H}$, which we indicate

by $T \in \mathcal{B}(\mathcal{H})$. The operator T is said to be of *class* C_0 if there exists an algebra homomorphism $\Phi : H^\infty \to \mathcal{B}(\mathcal{H})$ with the following properties:

(i) $\|\Phi(u)\| \leqslant u$ for every $u \in H^\infty$;

(ii) $\Phi(p) = p(T)$ for every polynomial p;

(iii) Φ is continuous when H^∞ and $\mathcal{B}(\mathcal{H})$ are given their respective weak* topologies;

(iv) Φ has non-trivial kernel.

We use the notation $\Phi(u) = u(T)$, which is the Sz.-Nagy–Foias H^∞ functional calculus. It is known that $\ker \Phi = m_T H^\infty$ for some inner function m_T called the *minimal function* of T. The minimal function is uniquely determined up to a scalar factor of absolute value one. Given a vector $x \in \mathcal{H}$, we define its minimal function, which we denote by m_x, to be the minimal function of the restriction of the operator T to the invariant subspace $\bigvee\limits_{n=0}^{\infty} T^n x$. A vector $x \in \mathcal{H}$ is said to be *maximal* for T if m_x coincides with m_T up to a scalar factor of absolute value one. A set $E \subset \mathcal{H}$ is said to be *cyclic* for T if $\mathcal{H} = \bigvee\limits_{n=0}^{\infty} T^n E$.

THEOREM 2.1 ([1], Theorem 2.3.6, Theorem 2.3.7). *Let $T \in \mathcal{B}(\mathcal{H})$ be an operator of class C_0. Then the set of maximal vectors for T is a dense G_δ in $\mathcal{H}$. Moreover, given any Banach space $\mathcal{K}$ and any bounded linear operator $A : \mathcal{K} \to \mathcal{H}$ with the property that $A\mathcal{K}$ is a cyclic set for T, the set*

$$\{k \in \mathcal{K} : Ak \text{ is maximal for } T\}$$

is a dense G_δ in $\mathcal{K}$.

We denote by H^2 the Hilbert space of functions

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

holomorphic on the open unit disc, equipped with the norm

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2.$$

Our first lemma is well known.

LEMMA 2.2. *Given $f_1, f_2, \ldots, f_n \in H^2$, there exists an outer function $v \in H^\infty$ such that $f_1 v, f_2 v, \ldots, f_n v$ all belong to H^∞ as well.*

Proof. It suffices to define the absolute value of v on the unit circle, which we take to be

$$\frac{1}{1 + |f_1| + \cdots + |f_n|}. \qquad \blacksquare$$

Recall that given functions $u, v \in H^\infty$, we say that u *divides* v and write $u|v$ if there exists a function $w \in H^\infty$ such that $v = wu$. We write $u \wedge v$ for the

greatest common inner divisor of u and v, and $u \vee v$ for their least common inner multiple. Both of these quantities are determined up to a scalar factor of absolute value one. If u and v are inner functions such that $u|v$ and $v|u$ (or, equivalently, u and v only differ by a scalar factor of absolute value one), we write $u \equiv v$. An inner function $u \in H^\infty$ is said to divide $f \in H^2$ if $f \in uH^2$. We naturally denote by $f \wedge u$ the greatest common inner divisor of f and u. A very useful consequence of Theorem 2.1 is the following.

THEOREM 2.3 ([1], Theorem 3.1.14). *Let $\{f_j\}_{j=0}^\infty \subset H^2$ be a bounded sequence of functions, and let $\theta \in H^\infty$ be an inner function. Then the set of sequences $\{a_j\}_{j=0}^\infty$ in ℓ^1 satisfying the following, is a dense G_δ:*

$$\left(\sum_{j=0}^\infty a_j f_j \right) \wedge \theta \equiv \bigwedge_{j=0}^\infty f_j \wedge \theta.$$

For any inner function $\theta \in H^\infty$, the space $H(\theta) = H^2 \ominus \theta H^2$ is closed and invariant for S^*, the adjoint of the shift operator S on H^2. The operator $S(\theta)$ defined by $S(\theta)^* = S^*|(H^2 \ominus \theta H^2)$ is called a *Jordan block*; it is of class C_0 with minimal function θ. We give some useful properties of these operators; they will be used repeatedly throughout and often without explicit mention.

PROPOSITION 2.4 ([1], Proposition 3.1.10, Corollary 3.1.12). *Let $\theta \in H^\infty$ be an inner function.*

 (i) *The operator $S(\theta)$ has multiplicity one. In fact, $h \in H(\theta)$ is cyclic if and only if $h \wedge \theta \equiv 1$.*

 (ii) *If $\phi \in H^\infty$ is an inner divisor of θ, then $\phi H^2 \ominus \theta H^2$ is an invariant subspace for $S(\theta)$. In fact,*

$$\phi H^2 \ominus \theta H^2 = \operatorname{ran} \phi(S(\theta)) = \ker(\theta/\phi)(S(\theta)).$$

Conversely, any invariant subspace for $S(\theta)$ is of this form.

 (iii) *Let $u \in H^\infty$ be any function and let $X = u(S(\theta))$. Then $\ker X = \ker X^* = \{0\}$ if and only if $u \wedge \theta \equiv 1$.*

The following result follows from the commutant lifting theorem (see [11]).

THEOREM 2.5 ([1], Theorem 3.1.16). *Let θ_0 and θ_1 be two inner functions. Assume that $X : H(\theta_0) \to H(\theta_1)$ satisfies $XS(\theta_0) = S(\theta_1)X$. Then there exists a function $u \in H^\infty$ such that $\theta_1|u\theta_0$, $\|u\| = \|X\|$ and*

$$X = P_{H(\theta_1)}u(S)|H(\theta_0).$$

Conversely, given any function $u \in H^\infty$ satisfying $\theta_1|u\theta_0$, the operator

$$P_{H(\theta_1)}u(S)|H(\theta_0)$$

intertwines $S(\theta_1)$ and $S(\theta_0)$.

A more general family of operators consists of the so-called *Jordan operators*. We will define them here in the case where the Hilbert space on which they act is

separable. These operators are of the form $\bigoplus_{j=0}^{\infty} S(\theta_j)$ where $\{\theta_j\}_{j=0}^{\infty}$ is a sequence of inner functions satisfying $\theta_{j+1} | \theta_j$ for $j \geqslant 0$. The Jordan operators are of fundamental importance in the study of operators of class C_0 as the following theorem illustrates. Recall first that a bounded injective linear operator with dense range is called a *quasiaffinity*. Two operators $T \in \mathcal{B}(\mathcal{H})$ and $T' \in \mathcal{B}(\mathcal{H}')$ are said to be *quasisimilar* if there exist quasiaffinities $X : \mathcal{H} \to \mathcal{H}'$ and $Y : \mathcal{H}' \to \mathcal{H}$ such that $XT = T'X$ and $TY = YT'$. We use the notation $T \sim T'$ to indicate that T and T' are quasisimilar. Recall also that the *multiplicity* of an operator is the smallest cardinality of a cyclic set.

THEOREM 2.6 ([1], Theorem 3.5.1, Corollary 3.5.25). *For any operator T of class C_0 acting on a separable Hilbert space there exists a unique Jordan operator $J = \bigoplus_{j=0}^{\infty} S(\theta_j)$ such that T and J are quasisimilar. Moreover, for each $j \geqslant 0$ we have*

$$\theta_j \equiv \bigwedge \{\phi \in H^{\infty} : T | \overline{\operatorname{ran} \phi(T)} \text{ has multiplicity at most } j\}.$$

The operator J in the previous theorem is called the *Jordan model* of T.

COROLLARY 2.7 ([1], Theorem 3.4.12). *Let T be an operator of class C_0 with Jordan model $\bigoplus_{j=0}^{\infty} S(\theta_j)$. Then T has multiplicity at most n if and only if $\theta_n \equiv 1$.*

THEOREM 2.8 ([1], Corollary 3.1.7, Proposition 3.5.30). *Let T be an operator of class C_0 with Jordan model $\bigoplus_{j=0}^{\infty} S(\theta_j)$. Then T^* is also of class C_0 and its Jordan model is $\bigoplus_{j=0}^{\infty} S(\theta_j^{\sim})$, where for any function $u \in H^{\infty}$ we define $u^{\sim}(z) = \overline{u(\bar{z})}$ for $z \in \mathbb{D}$.*

The next result will be crucial for us; it is usually referred to as the splitting principle.

THEOREM 2.9 ([2], Proposition 1.17). *Let $T \in \mathcal{B}(\mathcal{H})$ be an operator of class C_0 with Jordan model $S(\theta_0) \oplus S(\theta_1)$. Let $K \subset \mathcal{H}$ be an invariant subspace such that $T | K \sim S(\theta_0)$. Let $k \in K$ be a cyclic vector for $(T | K)^*$ and set $K' = \bigvee_{n=0}^{\infty} (T)^{*n} k$, $L = \mathcal{H} \ominus K'$. Then $\mathcal{H} = K \vee L$, $K \cap L = \{0\}$ and $T | L \sim S(\theta_1)$.*

To an operator of class C_0 with finite multiplicity we can associate an inner function called its *determinant*: if $\bigoplus_{j=0}^{\infty} S(\theta_j)$ is the Jordan model of the operator T which has multiplicity n, then $\det T = \theta_0 \theta_1 \cdots \theta_{n-1}$. The following result is helpful when calculating the functions appearing in the Jordan model of an operator. Given an invariant subspace M for an operator T, we denote by $T_{M^{\perp}}$ the compression $P_{M^{\perp}} T | M^{\perp}$.

THEOREM 2.10 ([1], Theorem 7.1.4). *Let T be an operator of class C_0 with finite multiplicity, and let M be an invariant subspace for T. Then $\det T = \det(T|M)$ $\det(T_{M^\perp})$.*

A consequence is the following.

THEOREM 2.11 ([1], Remark 7.1.15, Proposition 7.1.21). *Let $T \in \mathcal{B}(\mathcal{H})$ and $T' \in \mathcal{B}(\mathcal{H}')$ be two operators of class C_0 with finite multiplicities such that $\det T = \det T'$. Let $X : \mathcal{H} \to \mathcal{H}'$ be a bounded linear operator such that $XT = T'X$. Then X is one-to-one if and only if it has dense range.*

We now collect some facts about invariant subspaces for operators of class C_0.

THEOREM 2.12 ([1], Theorem 3.2.13). *Let T be an operator of class C_0 with multiplicity one. Then for every inner divisor θ of m_T, there exists a unique invariant subspace M such that $T|M \sim S(\theta)$, namely $M = \ker \theta(T) = \overline{\mathrm{ran}(m_T/\theta)(T)}$.*

Given an operator $T \in \mathcal{B}(\mathcal{H})$, we denote its commutant by $\{T\}' = \{X \in \mathcal{B}(\mathcal{H}) : XT = TX\}$. Recall that a closed subspace M is said to be *hyperinvariant* for T if it is invariant for every operator $X \in \{T\}'$.

THEOREM 2.13 ([1], Proposition 4.2.1). *Let $\bigoplus_{j=0}^{\infty} S(\theta_j)$ be a Jordan operator. Then a subspace M is hyperinvariant if and only if it is of the form*

$$K = \bigoplus_{j=0}^{\infty} (\psi_j H^2 \ominus \theta_j H^2)$$

where $\psi_j | \theta_j$, $\psi_{j+1} | \psi_j$ and $(\theta_{j+1}/\psi_{j+1}) | (\theta_j/\psi_j)$ for every $j \geqslant 0$.

Let us mention elementary facts about matrices of operators. Here and throughout, we identify a function $u \in H^\infty$ with the multiplication operator $f \mapsto uf$ it defines on H^2.

LEMMA 2.14. *Let $\theta_0, \theta_1 \in H^\infty$ be two inner functions such that $\theta_1 | \theta_0$. Let $\mathcal{A} \subset M_2(H^\infty)$ be the subalgebra of matrices $A = (a_{ij})_{i,j=0,1}$ such that θ_0/θ_1 divides a_{01}. Let $\mathcal{H} = H(\theta_0) \oplus H(\theta_1)$ and $T = S(\theta_0) \oplus S(\theta_1)$. Then the map*

$$\Psi : \mathcal{A} \to \{T\}'$$
$$\Psi(A) = P_\mathcal{H} A | \mathcal{H}$$

is a surjective algebra homomorphism, with

$$\ker \Psi = \left\{ \begin{pmatrix} \theta_0 b_{00} & \theta_0 b_{01} \\ \theta_1 b_{10} & \theta_1 b_{11} \end{pmatrix} : b_{ij} \in H^\infty, i,j = 0,1 \right\}.$$

For every $A \in \mathcal{A}$, there exists $A' \in \mathcal{A}$ such that the operator $\Psi(A')$ satisfies

$$\Psi(A)\Psi(A') = \Psi(A')\Psi(A) = u(T),$$

where $u = \det A$. When $\det A \wedge \theta_0 \equiv 1$ we have that $\Psi(A)$ and $\Psi(A')$ are quasiaffinities.

Proof. Theorem 2.5 ensures that Ψ maps $\mathcal{A}$ onto $\{T\}'$. Moreover, it is clear that Ψ is linear and that it has the announced kernel. It remains to show that it is multiplicative. First note that if $b \in H^\infty$ is divisible by θ_0/θ_1, then

$$P_{H(\theta_0)}bP_{H(\theta_1)}af = P_{H(\theta_0)}b(af - P_{\theta_1 H^2}af) = P_{H(\theta_0)}baf$$

for every $a \in H^\infty, f \in H^2$. Moreover, since $H(\theta_1) \subset H(\theta_0)$, we have

$$P_{H(\theta_1)}bP_{H(\theta_0)}a|H(\theta_0) = P_{H(\theta_1)}P_{H(\theta_0)}bP_{H(\theta_0)}a|H(\theta_0)$$
$$= P_{H(\theta_1)}P_{H(\theta_0)}ba|H(\theta_0) = P_{H(\theta_1)}ba|H(\theta_0)$$

for every $b, a \in H^\infty$. These considerations show that

$$P_{\mathcal{H}}(BA)|\mathcal{H} = (P_{\mathcal{H}}B|\mathcal{H})(P_{\mathcal{H}}A|\mathcal{H})$$

for $A \in M_2(H^\infty)$ and $B \in \mathcal{A}$, so that Ψ is an algebra homomorphism.

Given $A \in \mathcal{A}$, let A' be the (pointwise) algebraic adjoint of A, so that $AA' = A'A = u \operatorname{Id}_{\mathbb{C}^2}$ with $u = \det A$. Note that $a'_{01} = -a_{01}$ so that $A' \in \mathcal{A}$. Set $X = \Psi(A)$ and $X' = \Psi(A')$. We have $X'X = \Psi(A'A) = u(T) = \Psi(AA') = XX'$, that is

$$XX' = XX' = \begin{pmatrix} u(S(\theta_0)) & 0 \\ 0 & u(S(\theta_1)) \end{pmatrix}.$$

When $u \wedge \theta_0 \equiv 1$, the operator $u(T)$ is a quasiaffinity by Proposition 2.4. In particular, this shows that X and X' are quasiaffinities. ∎

Let us recall a relation between operators which is weaker than that of quasisimilarity. Given $T \in \mathcal{B}(\mathcal{H})$ and $T' \in \mathcal{B}(\mathcal{H}')$, we say that T is a *quasiaffine transform* of T' and we write $T \prec T'$, if there exists a quasiaffinity $X : \mathcal{H} \to \mathcal{H}'$ such that $XT = T'X$.

THEOREM 2.15 ([1], Proposition 3.5.32). *Let T and T' be two operators of class C_0. Then $T \prec T'$ is equivalent to $T \sim T'$.*

The next proposition can rephrased as saying that any multiplicity two operator of class C_0 has the so-called property $(*)$ (see [2]).

PROPOSITION 2.16 ([1], Lemma 4.1.11, Proposition 4.1.13). *Let T be an operator of class C_0 with multiplicity two. Let $X \in \{T\}'$ be a quasiaffinity. Then there exists another quasiaffinity $Y \in \{T\}'$ and a function $u \in H^\infty$ such that $XY = YX = u(T)$.*

We obtain a useful consequence.

LEMMA 2.17. *Let T be an operator of class C_0 with multiplicity two. Let M, M' be invariant subspaces for T. Assume that there exist quasiaffinities $X, X' \in \{T\}'$ such that $\overline{XM} = \overline{X'M'}$. Then $M \sim M'$.*

Proof. By Proposition 2.16, there exist quasiaffinities $Y, Y' \in \{T\}'$ and functions $u, u' \in H^\infty$ such that $XY = YX = u(T)$ and $X'Y' = Y'X' = u'(T)$. Hence,

$$\overline{YX'M'} = \overline{YXM} = \overline{u(T)M} \subset M \quad \text{and} \quad \overline{Y'XM} = \overline{Y'X'M'} = \overline{u'(T)M'} \subset M'.$$

On the other hand, by Theorem 2.15 we have that

$$T|M' \sim T|\overline{X'M'} = T|\overline{XM} \sim T|\overline{Y'XM} \quad \text{and} \quad T|M \sim T|\overline{XM} = T|\overline{X'M'} \sim T|\overline{YX'M'}$$

so that Theorem 2.11 implies that $\overline{YX'M'} = M$ and $\overline{Y'XM} = M'$, and we are done. ∎

Let us close this section by proving an elementary fact which motivates our main result.

PROPOSITION 2.18. *Let $T \in \mathcal{B}(\mathcal{H})$ and $T' \in \mathcal{B}(\mathcal{H}')$ be operators of class C_0 with finite multiplicities. Assume that $X : \mathcal{H} \to \mathcal{H}'$ is a quasiaffinity such that $XT = T'X$. Let $M \in \mathcal{H}$ be an invariant subspace for T. Then $T'|\overline{XM} \sim T|M$ and $T'_{(XM)^\perp} \sim T_{M^\perp}$.*

Proof. We clearly have $T \prec T'$, so by Theorem 2.15 we find $T \sim T'$. Hence $\det T = \det T'$. If we let $E = \overline{XM}$, it follows from Theorem 2.15 again that $T'|E \sim T|M$, and in particular $\det(T|M) = \det(T'|E)$. Using Theorem 2.10, we find

$$\det(T_{M^\perp}) = \frac{\det T}{\det(T|M)} = \frac{\det T'}{\det(T'|E)} = \det(T'_{E^\perp}).$$

Moreover, we have $X^* E^\perp \subset M^\perp$ and $X^* T'^* = T^* X^*$. Since X^* is injective, we may apply Theorem 2.11 to find $\overline{X^* E^\perp} = M^\perp$. This establishes $T'^*|E^\perp \prec T^*|M^\perp$. By Theorem 2.15, we have $T'^*|E^\perp \sim T^*|M^\perp$ and $T'_{E^\perp} \sim T_{M^\perp}$. ∎

Proposition 2.18 shows in particular that if T is an operator of class C_0 with multiplicity two and M, M' are invariant subspaces for T, then $M \sim M'$ implies $T|M \sim T|M'$ and $T_{M^\perp} \sim T_{M'^\perp}$. Our main theorem says that the converse holds.

3. JORDAN MODEL AND HYPERINVARIANCE

Let us first make a convention. Let $\theta_0, \theta_1 \in H^\infty$ be inner functions such that θ_1 divides θ_0. In the space $H(\theta_0) \oplus H(\theta_1)$, we identify the subspace $H(\theta_0) \oplus \{0\}$ with $H(\theta_0)$ and the subspace $\{0\} \oplus H(\theta_1)$ with $H(\theta_1)$. Given $M \subset H(\theta_0) \oplus H(\theta_1)$ an invariant subspace for the Jordan operator $S(\theta_0) \oplus S(\theta_1)$, it is easy to verify that $\overline{P_{H(\theta_j)}M}$ is invariant for $S(\theta_j)$ for each $j = 0, 1$, so by Proposition 2.4 we can find $\phi_j \in H^\infty$ an inner divisor of θ_j such that $\overline{P_{H(\theta_j)}M} = \phi_j H^2 \ominus \theta_j H^2$. We will use this observation implicitly throughout the remainder of the paper. The following result allows us to focus on a very special kind of subspace M; it is a spiritual cousin of Theorem 3.2 in [4].

THEOREM 3.1. *Let $T = S(\theta_0) \oplus S(\theta_1)$ be a Jordan operator and M be an invariant subspace for T. Then there exists a quasiaffinity $X \in \{T\}'$ such that*

$$\overline{P_{H(\theta_0)}XM} \oplus \overline{P_{H(\theta_1)}XM}$$

is a hyperinvariant subspace for T.

Proof. Let $\xi \in M$ be a maximal vector for $T|M$ and write $\xi = \xi_0 \oplus \xi_1 \in H(\theta_0) \oplus H(\theta_1)$. By Theorem 2.3, we can find non-zero $a_0, a_1 \in \mathbb{C}$ such that

$$\left(a_0 \xi_0 + a_1 \frac{\theta_0}{\theta_1} \xi_1 \right) \wedge \theta_0 \equiv \xi_0 \wedge \frac{\theta_0}{\theta_1} \xi_1 \wedge \theta_0$$

along with non-zero $b_0, b_1 \in \mathbb{C}$ such that

$$(b_0 \xi_0 + b_1 \xi_1) \wedge \theta_1 \equiv \xi_0 \wedge \xi_1 \wedge \theta_1 \quad \text{and}$$

$$\left(a_0 b_1 - b_0 a_1 \frac{\theta_0}{\theta_1} \right) \wedge \theta_0 \equiv a_0 \wedge a_1 \frac{\theta_0}{\theta_1} \wedge \theta_0 \equiv 1 \wedge \frac{\theta_0}{\theta_1} \wedge \theta_0 \equiv 1.$$

Set

$$A = \begin{pmatrix} a_0 & a_1 \theta_0 / \theta_1 \\ b_0 & b_1 \end{pmatrix} \quad \text{and} \quad X = P_{H(\theta_0) \oplus H(\theta_1)} A | (H(\theta_0) \oplus H(\theta_1)).$$

Now, $\det A = a_0 b_1 - b_0 a_1 \theta_0 / \theta_1$, so that X is a quasiaffinity commuting with T by Lemma 2.14. Notice that

$$m_{T|M} \equiv m_\xi \equiv \frac{\theta_0}{\xi_0 \wedge \theta_0} \vee \frac{\theta_1}{\xi_1 \wedge \theta_1} \equiv \frac{\theta_0}{\xi_0 \wedge (\theta_0/\theta_1)\xi_1 \wedge \theta_0}.$$

Therefore, if we let $X\xi = y_0 \oplus y_1 \in H(\theta_0) \oplus H(\theta_1)$ we find

$$y_0 \wedge \theta_0 \equiv \left(a_0 \xi_0 + a_1 \frac{\theta_0}{\theta_1} \xi_1 \right) \wedge \theta_0 \equiv \xi_0 \wedge \frac{\theta_0}{\theta_1} \xi_1 \wedge \theta_0 \equiv \frac{\theta_0}{m_{T|M}} \quad \text{and}$$

$$y_1 \wedge \theta_1 \equiv (b_0 \xi_0 + b_1 \xi_1) \wedge \theta_1 \equiv \xi_0 \wedge \xi_1 \wedge \theta_1$$

so that $y_1 \wedge \theta_1$ divides $\theta_0 / m_{T|M}$.

Consider now the hyperinvariant subspace generated by M,

$$E = \bigvee \{YM : Y \in \{T\}'\}.$$

By Theorem 2.13, E can be written as

$$E = (\psi_0 H^2 \ominus \theta_0 H^2) \oplus (\psi_1 H^2 \ominus \theta_1 H^2)$$

for some inner functions $\psi_0, \psi_1 \in H^\infty$ with the property that $\psi_1 | \psi_0$, $(\theta_1/\psi_1)|(\theta_0/\psi_0)$ and $\psi_j | \theta_j$ for $j = 0, 1$. Note that for $Y \in \{T\}'$, $u \in H^\infty$ and $h \in H(\theta_0) \oplus H(\theta_1)$, we have

$$u(T)Yh = Yu(T)h$$

which shows that $m_{T|M}(T)Yh = 0$ for every $h \in M$ and $Y \in \{T\}'$, whence the minimal functions of $T|E$ and $T|M$ coincide, and

$$\frac{\theta_0}{\psi_0} \equiv m_{T|E} \equiv m_{T|M}.$$

For each $j = 0, 1$, write

$$\overline{P_{H(\theta_j)}XM} = \phi_j H^2 \ominus \theta_j H^2$$

where ϕ_j is an inner divisor of θ_j. Since $XM \subset E$, we have

$$\phi_j H^2 \ominus \theta_j H^2 = \overline{P_{H(\theta_j)}XM} \subset \overline{P_{H(\theta_j)}E} = \psi_j H^2 \ominus \theta_j H^2$$

and thus ψ_j divides ϕ_j for $j = 0, 1$. Notice that $y_0 \in P_{H(\theta_0)}XM$, so ϕ_0 divides $y_0 \wedge \theta_0$. But we established above that $y_0 \wedge \theta_0 \equiv \theta_0/m_{T|M} \equiv \psi_0$, so we find $\phi_0 \equiv \psi_0$. In addition, $y_1 \in P_{H(\theta_1)}XM$ implies that ϕ_1 divides $y_1 \wedge \theta_1$, which in turn divides $\theta_0/m_{T|M}$ as was shown above. Since $\theta_0/m_{T|M} \equiv \psi_0 \equiv \phi_0$, we have that ϕ_1 divides ϕ_0. Finally, using the fact that ψ_1 divides ϕ_1, we find that θ_1/ϕ_1 divides θ_1/ψ_1, which in turn divides $\theta_0/\psi_0 \equiv \theta_0/\phi_0$. Theorem 2.13 completes the proof. ∎

The following is based on Theorem 3.4 of [4]. Properties (iii) and (iv) below are part of the so-called *Weyl identities* (see [5]).

PROPOSITION 3.2. *Let $T = S(\theta_0) \oplus S(\theta_1)$ be a Jordan operator. Let M be an invariant subspace for T. For each $j = 0, 1$, write*

$$\overline{P_{H(\theta_j)}M} = \phi_j H^2 \ominus \theta_j H^2$$

where ϕ_j is an inner divisor of θ_j. Assume that $\phi_1|\phi_0$ and $(\theta_1/\phi_1)|(\theta_0/\phi_0)$. Let $S(\alpha_0) \oplus S(\alpha_1)$ be the Jordan model of $T|M$ and $S(\beta_0) \oplus S(\beta_1)$ be the Jordan model of $T_{M^\perp}$. Then:
  (i) $\theta_0\theta_1 = \alpha_0\alpha_1\beta_0\beta_1$;
  (ii) $\phi_0 \equiv \theta_0/\alpha_0$ and $\phi_1 \equiv \beta_1$;
  (iii) $(\theta_1/\beta_1)|\alpha_0$;
  (iv) $\beta_1|(\theta_0/\alpha_0)$ and $\beta_1|(\theta_1/\alpha_1)$.

Proof. Using the decomposition $\mathcal{H} = M \oplus M^\perp$ to compute the determinant, we find by Theorem 2.10 that

$$\theta_0\theta_1 = \det T = \det(T|M)\det(T_{M^\perp}) = \alpha_0\alpha_1\beta_0\beta_1$$

which is (i). Assume for the moment that (ii) holds. Then, (iii) is equivalent to our assumption $(\theta_1/\phi_1)|(\theta_0/\phi_0)$. Moreover, $\phi_1|\phi_0$ is equivalent to the first part of (iv). Note now that if $h = h_0 \oplus h_1 \in M$, then by choice of ϕ_1 we have $(\theta_1/\phi_1)(T)h \in H(\theta_0) \oplus \{0\}$, so in fact the operator

$$(T|M)|\overline{\mathrm{ran}(\theta_1/\phi_1)(T|M)}$$

has multiplicity at most 1. By Theorem 2.6, we conclude that $\alpha_1|(\theta_1/\phi_1)$ so the second half of (iv) also follows from (ii). Hence, it only remains to identify ϕ_0 and ϕ_1, that is to show (ii).

Since $T|M$ has multiplicity two, we can find vectors $\xi = \xi_0 \oplus \xi_1 \in M$ and $\eta = \eta_0 \oplus \eta_1 \in M$ with $\xi_j, \eta_j \in H(\theta_j)$ for $j = 0, 1$, such that

$$M = \bigvee_{n=0}^{\infty} \{T^n\xi, T^n\eta\}.$$

It is easy to see that $\phi_j \equiv \xi_j \wedge \eta_j \wedge \theta_j$ and

$$\overline{P_{H(\theta_j)}M} = \bigvee_{n=0}^{\infty} \{S(\theta_j)^n \xi_j, S(\theta_j)^n \eta_j\}$$

for $j = 0, 1$. Now, we have

$$S(\theta_j)|\overline{P_{H(\theta_j)}M} \sim S(\theta_j/\phi_j)$$

so that θ_j/ϕ_j is the minimal function of $S(\theta_j)|\overline{P_{H(\theta_j)}M}$. Since for every $u \in H^\infty$ we have

$$u(T) = u(S(\theta_0)) \oplus u(S(\theta_1)),$$

it is clear that the minimal function of $T|M$, namely α_0, is equal to the least common inner multiple of the minimal functions of $S(\theta_j)|\overline{P_{H(\theta_j)}M}$ for $j = 0, 1$, which is θ_0/ϕ_0. Hence, $\alpha_0 \equiv \theta_0/\phi_0$.

Consider the operators $Y = P_{H(\theta_0)}|M$ and $Z = P_{H(\theta_1)}|M$. It is straightforward to verify that

$$YT_{M\ominus\ker Y} = (S(\theta_0)|\overline{YM})(Y|(M \ominus \ker Y)) \quad \text{and}$$

$$ZT_{M\ominus\ker Z} = (S(\theta_1)|\overline{ZM})(Z|(M \ominus \ker Z)).$$

Thus, by virtue of Theorem 2.15, we get

$$T_{M\ominus\ker Y} \sim S(\theta_0)|\overline{P_{H(\theta_0)}M} \sim S(\theta_0/\phi_0) \quad \text{and}$$

$$T_{M\ominus\ker Z} \sim S(\theta_1)|\overline{P_{H(\theta_1)}M} \sim S(\theta_1/\phi_1).$$

Notice now that

$$\ker Y = M \cap (0 \oplus H(\theta_1)) = 0 \oplus (\rho H^2 \ominus \theta_1 H^2) \quad \text{and}$$

$$\ker Z = M \cap (H(\theta_0) \oplus 0) = (\sigma H^2 \ominus \theta_0 H^2) \oplus 0$$

for some inner functions $\rho, \sigma \in H^\infty$ such that ρ divides θ_1 and σ divides θ_0. We now calculate the determinant using the decomposition $M = \ker Y \oplus (M \ominus \ker Y)$, namely

$$\alpha_0 \alpha_1 = \det(T|M) = \det(T|\ker Y)\det(T_{M\ominus\ker Y}) = \frac{\theta_0 \theta_1}{\phi_0 \rho}.$$

Using the fact that $\alpha_0 \equiv \theta_0/\phi_0$, we find $\alpha_1 \equiv \theta_1/\rho$. On the other hand, writing $M = \ker Z \oplus (M \ominus \ker Z)$ and computing determinants, we find

$$\alpha_0 \alpha_1 = \det(T|M) = \det(T|\ker Z)\det(T_{M\ominus\ker Z}) = \frac{\theta_0 \theta_1}{\sigma \phi_1}.$$

whence

$$\sigma = \frac{\theta_0 \theta_1}{\alpha_0 \alpha_1 \phi_1}.$$

Using once again that $\alpha_0 \equiv \theta_0/\phi_0$, we get

$$\sigma \equiv \rho \frac{\phi_0}{\phi_1}.$$

Note that by assumption ϕ_1 divides ϕ_0, so that ρ divides σ. Now, β_0 is the minimal function of $T_{M^\perp}$, and thus is the greatest common inner divisor of the functions $u \in H^\infty$ such that $u(T)(H(\theta_0) \oplus H(\theta_1)) \subset M$. Given such a function u, we have

$$u(T)(P_{H(\theta_0)}1 \oplus 0) = P_{H(\theta_0)}u \oplus 0 \in M \cap (H(\theta_0) \oplus 0) = \ker Z$$

and thus σ divides u. Conversely, using that ρ divides σ, we have

$$\sigma(T)(H(\theta_0) \oplus H(\theta_1)) = \operatorname{ran} \sigma(S(\theta_0)) \oplus \operatorname{ran} \sigma(S(\theta_1)) \subset \operatorname{ran} \sigma(S(\theta_0)) \oplus \operatorname{ran} \rho(S(\theta_1))$$

$$= (\sigma H^2 \ominus \theta_0 H^2) \oplus (\rho H^2 \ominus \theta_1 H^2) = \ker Z \vee \ker Y \subset M.$$

Hence, $\beta_0 \equiv \sigma$. Finally, using the decomposition $\mathcal{H} = M \oplus M^\perp$ to compute the determinant, we find

$$\theta_0 \theta_1 = \det T = \det(T|M) \det(T_{M^\perp}) = \alpha_0 \alpha_1 \beta_0 \beta_1$$

and thus

$$\beta_1 \equiv \frac{\theta_0 \theta_1}{\alpha_0 \alpha_1 \beta_0} \equiv \frac{\phi_0 \rho}{\sigma} \equiv \phi_1. \quad \blacksquare$$

We close this section by providing a type of converse to Proposition 3.2. Let us first establish an elementary lemma.

LEMMA 3.3. *Let $T = S(\theta_0) \oplus S(\theta_1)$ be a Jordan operator. Assume that $\phi_0, \phi_1 \in H^\infty$ are inner divisors of θ_0 and θ_1 respectively, with the additional property that either $(\theta_1/\phi_1)|(\theta_0/\phi_0)$ or $(\theta_0/\phi_0)|(\theta_1/\phi_1)$. Set*

$$\xi = (P_{H(\theta_0)}\phi_0) \oplus (P_{H(\theta_1)}\phi_1) \in H(\theta_0) \oplus H(\theta_1).$$

Then

$$\bigvee_{n=0}^{\infty} T^n \xi \subset \{(P_{H(\theta_0)}\phi_0 g) \oplus (P_{H(\theta_1)}\phi_1 g) : g \in H^2\}.$$

Proof. Let $y = y_0 \oplus y_1 \in \bigvee_{n=0}^{\infty} T^n \xi$. We can find a sequence of polynomials $\{r_n\}_n$ with the property that for each $j = 0, 1$, we have

$$y_j = \lim_{n \to \infty} r_n(S(\theta_j))(P_{H(\theta_j)}\phi_j).$$

Notice now that for each $n \geqslant 0$, we have

$$r_n(S(\theta_j))(P_{H(\theta_j)}\phi_j) = P_{H(\theta_j)}r_n \phi_j = \phi_j P_{H(\theta_j/\phi_j)}r_n$$

$$= \phi_j(S(\theta_j))(P_{H(\theta_j)}r_n) \in \operatorname{ran} \phi_j(S(\theta_j)).$$

Proposition 2.4 implies that the range of $\phi_j(S(\theta_j))$ is closed, so there exist $g_j \in H(\theta_j)$ with the property that $y_j = \phi_j(S(\theta_j))g_j = \phi_j P_{H(\theta_j/\phi_j)}g_j$. We have for $j = 0, 1$ that

$$\phi_j P_{H(\theta_j/\phi_j)}g_j = \lim_{n \to \infty} \phi_j P_{H(\theta_j/\phi_j)}r_n$$

and since $1/\phi_j = \overline{\phi}_j$ we find that

(3.1) $$P_{H(\theta_j/\phi_j)}g_j = \lim_{n \to \infty} P_{H(\theta_j/\phi_j)}r_n.$$

If $(\theta_1/\phi_1)|(\theta_0/\phi_0)$, we have $H(\theta_1/\phi_1) \subset H(\theta_0/\phi_0)$ and (3.1) implies that

$$P_{H(\theta_1/\phi_1)}g_0 = P_{H(\theta_1/\phi_1)}g_1.$$

Hence

$$y_1 = \phi_1 P_{H(\theta_1/\phi_1)}g_1 = \phi_1 P_{H(\theta_1/\phi_1)}g_0 = P_{H(\theta_1)}\phi_1 g_0$$

so that

$$y_0 \oplus y_1 \in \{(P_{H(\theta_0)}\phi_0 g) \oplus (P_{H(\theta_1)}\phi_1 g) : g \in H^2\}.$$

If, on the other hand, $(\theta_0/\phi_0)|(\theta_1/\phi_1)$, we have $H(\theta_0/\phi_0) \subset H(\theta_1/\phi_1)$ and (3.1) implies that $P_{H(\theta_0/\phi_0)}g_1 = P_{H(\theta_0/\phi_0)}g_0$. Hence

$$y_0 = \phi_0 P_{H(\theta_0/\phi_0)}g_0 = \phi_0 P_{H(\theta_0/\phi_0)}g_1 = P_{H(\theta_0)}\phi_0 g_1$$

so that

$$y_0 \oplus y_1 \in \{(P_{H(\theta_0)}\phi_0 g) \oplus (P_{H(\theta_1)}\phi_1 g) : g \in H^2\}. \quad \blacksquare$$

PROPOSITION 3.4. *Let* $\theta_j, \alpha_j, \beta_j \in H^\infty$ *be inner functions such that* $\alpha_j|\theta_j$ *and* $\beta_j|\theta_j$ *for* $j = 0, 1$. *Assume that:*
   (i) $\theta_1|\theta_0$, $\beta_1|\beta_0$ *and* $\alpha_1|\alpha_0$;
   (ii) $(\theta_1/\beta_1)|\alpha_0$;
   (iii) $\beta_1|((\theta_0/\alpha_0) \wedge (\theta_1/\alpha_1))$;
   (iv) $\theta_0\theta_1 = \alpha_0\alpha_1\beta_0\beta_1$.
   Let $T = S(\theta_0) \oplus S(\theta_1)$. *If we set*

$$\xi = (P_{H(\theta_0)}(\theta_0/\alpha_0)) \oplus (P_{H(\theta_1)}\beta_1) \quad and \quad \eta = 0 \oplus (P_{H(\theta_1)}(\theta_1/\alpha_1)),$$

then

$$N = \bigvee_{n=0}^{\infty} \{T^n\xi, T^n\eta\}$$

is an invariant subspace for T *with the property that*

$$\overline{P_{H(\theta_0)}N} = (\theta_0/\alpha_0)H^2 \ominus \theta_0 H^2, \quad \overline{P_{H(\theta_1)}N} = \beta_1 H^2 \ominus \theta_1 H^2, \quad T|N \sim S(\alpha_0) \oplus S(\alpha_1)$$

and $T_{N^\perp} \sim S(\beta_0) \oplus S(\beta_1)$.

   Proof. Let $\phi_0 = \theta_0/\alpha_0, \phi_1 = \beta_1, \psi_0 = \theta_0$ and $\psi_1 = \theta_1/\alpha_1$. We have $\eta = (P_{H(\theta_0)}\psi_0) \oplus (P_{H(\theta_1)}\psi_1) = 0 \oplus (P_{H(\theta_1)}\psi_1)$ and $\xi = (P_{H(\theta_0)}\phi_0) \oplus (P_{H(\theta_1)}\phi_1)$. It is manifest in view of (ii) that $(\theta_1/\phi_1)|(\theta_0/\phi_0)$ and we have trivially that $(\theta_0/\psi_0)|(\theta_1/\psi_1)$. By Lemma 3.3, we have that

$$\bigvee_{n=0}^{\infty} T^n\xi \subset \{(P_{H(\theta_0)}\phi_0 g) \oplus (P_{H(\theta_1)}\phi_1 g) : g \in H^2\} \quad and$$

$$\bigvee_{n=0}^{\infty} T^n\eta \subset \{(P_{H(\theta_0)}\psi_0 g) \oplus (P_{H(\theta_1)}\psi_1 g) : g \in H^2\} = \{0 \oplus (P_{H(\theta_1)}\psi_1 g) : g \in H^2\}.$$

We want to show that the sets appearing on the right-hand sides intersect trivially. Suppose then that $P_{H(\theta_0)}\phi_0 g = 0$ and $P_{H(\theta_1)}\phi_1 g = P_{H(\theta_1)}\psi_1 h$ for some $g, h \in H^2$.

The first relation implies that $g \in (\theta_0/\phi_0)H^2$. Since $(\theta_1/\phi_1)|(\theta_0/\phi_0)$, we have that $\theta_1|(\phi_1\theta_0/\phi_0)$ and thus

$$P_{H(\theta_1)}\psi_1 h = P_{H(\theta_1)}\phi_1 g = 0.$$

We have therefore established that

$$\{(P_{H(\theta_0)}\phi_0 g) \oplus (P_{H(\theta_1)}\phi_1 g) : g \in H^2\} \cap \{(P_{H(\theta_0)}\psi_0 g) \oplus (P_{H(\theta_1)}\psi_1 g) : g \in H^2\} = \{0\}$$

whence $K_\xi \cap K_\eta = \{0\}$, where $K_\xi = \bigvee_{n=0}^{\infty} T^n \xi$ and $K_\eta = \bigvee_{n=0}^{\infty} T^n \eta$.

It is straightforward to verify that the minimal function of ξ is α_0 (by (ii)), and that the minimal function of η is α_1. Define $N = K_\xi \vee K_\eta$. Using the fact that $K_\xi \cap K_\eta = \{0\}$, it is easy to see that

$$T|N \sim T|K_\xi \oplus T|K_\eta \sim S(\alpha_0) \oplus S(\alpha_1).$$

It follows from (iii) that $\phi_0|\psi_0$ and $\phi_1|\psi_1$, thus $\overline{P_{H(\theta_j)}N} = \phi_j H^2 \ominus \theta_j H^2$ for $j = 0, 1$. Finally, suppose the Jordan model of $T_{N^\perp}$ is equal to $S(\gamma_0) \oplus S(\gamma_1)$. By Theorem 3.2 (along with (ii) and (iii)), we find that $\gamma_1 = \phi_1 = \beta_1$. By Theorem 2.10 we have

$$\theta_0\theta_1 = \det T = \det(T|N)\det(T_{N^\perp}) = \alpha_0\alpha_1\gamma_0\beta_1$$

so that property (iv) implies $\gamma_0 = \beta_0$. This completes the proof. ∎

4. CLASSIFICATION THEOREM

We take the final step towards our classification result.

PROPOSITION 4.1. *Let $T = S(\theta_0) \oplus S(\theta_1)$ be a Jordan operator. Let M and M' be invariant subspaces for T such that $T|M \sim T|M'$ and $T_{M^\perp} \sim T_{M'^\perp}$. Let $S(\alpha_0) \oplus S(\alpha_1)$ be the Jordan model of $T|M$ and $T|M'$, and $S(\beta_0) \oplus S(\beta_1)$ be the Jordan model of $T_{M^\perp}$ and $T_{M'^\perp}$. For $j = 0, 1$, let*

$$\overline{P_{H(\theta_j)}M} = \phi_j H^2 \ominus \theta_j H^2 \quad and \quad \overline{P_{H(\theta_j)}M'} = \phi_j' H^2 \ominus \theta_j H^2,$$

where ϕ_j and ϕ_j' are inner divisors of θ_j. Assume that $\phi_1|\phi_0$ and $(\theta_1/\phi_1)|(\theta_0/\phi_0)$, along with $\phi_1'|\phi_0'$ and $(\theta_1/\phi_1')|(\theta_0/\phi_0')$. Then there exists a quasiaffinity $X \in \{T\}'$ such that $M = \overline{XM'}$.

Proof. By Proposition 3.2, we have that $\phi_0 \equiv \theta_0/\alpha_0 \equiv \phi_0'$ and $\phi_1 \equiv \beta_1 \equiv \phi_1'$, so that

$$\overline{P_{H(\theta_j)}M} = \overline{P_{H(\theta_j)}M'}$$

for $j = 0, 1$. Define

$$E = \overline{P_{H(\theta_0)}M} \oplus \overline{P_{H(\theta_1)}M} = \overline{P_{H(\theta_0)}M'} \oplus \overline{P_{H(\theta_1)}M'},$$

which is hyperinvariant for T by Theorem 2.13. Hence, E contains M and M' along with any image of those subspaces under an operator lying in the commutant of T.

By Theorem 2.1, we may choose $\xi = \xi_0 \oplus \xi_1 \in M$ a maximal vector for $T|M$ with the additional property that ξ_j is maximal for $S(\theta_j)|\overline{P_{H(\theta_j)}M}$ for each $j = 0, 1$. Similarly, we may choose $\xi' = \xi_0' \oplus \xi_1' \in M'$ a maximal vector for $T|M'$ with the additional property that ξ_j' is maximal for $S(\theta_j)|\overline{P_{H(\theta_j)}M'}$ for each $j = 0, 1$. By Lemma 2.2, there exists an outer function $v \in H^\infty$ such that $v\xi_0, v\xi_1, v\xi_0'$ and $v\xi_1'$ all belong to H^∞. Since v is outer and θ_0 is inner, we have $v \wedge \theta_0 \equiv 1$, so that $v(S(\theta_0)), v(S(\theta_1))$ and $v(T)$ are quasiaffinities by Proposition 2.4, and thus $v(T)\xi$ and $v(T)\xi'$ have the same maximality properties as ξ and ξ' respectively. Consequently, upon replacing $\xi \in M$ and $\xi' \in M'$ by $v(T)\xi \in M$ and $v(T)\xi' \in M'$ respectively, we may further assume that

$$\xi = (P_{H(\theta_0)}\phi_0 f_0) \oplus (P_{H(\theta_1)}\phi_1 f_1) \quad \text{and} \quad \xi' = (P_{H(\theta_0)}\phi_0 f_0') \oplus (P_{H(\theta_1)}\phi_1 f_1')$$

where $f_0, f_1, f_0', f_1' \in H^\infty$ satisfy $f_j \wedge \theta_0 \equiv f_j' \wedge \theta_0 \equiv 1$ for $j = 0, 1$. Define

$$Y = \begin{pmatrix} f_0(S(\theta_0)) & 0 \\ 0 & f_1(S(\theta_1)) \end{pmatrix}, \quad Y' = \begin{pmatrix} f_0'(S(\theta_0)) & 0 \\ 0 & f_1'(S(\theta_1)) \end{pmatrix}.$$

It is clear $Y'\xi = Y\xi'$. Lemma 2.14 implies that $Y, Y' \in \{T\}'$ are quasiaffinities and that there exist quasiaffinities $Z, Z' \in \{T\}'$ such that $ZY = YZ = (f_0 f_1)(T)$ and $Z'Y' = Y'Z' = (f_0' f_1')(T)$. If we set $K = \bigvee_{n=0}^{\infty} T^n\xi \subset M$ and $K' = \bigvee_{n=0}^{\infty} T^n\xi' \subset M'$, then $Z'Y\xi' = Z'Y'\xi = (f_0' f_1')(T)\xi \in K$, so that $\overline{Z'YK'} \subset K$. By virtue of Theorem 2.15, we see that $T|\overline{Z'YK'} \sim T|K' \sim S(\alpha_0)$, so Theorem 2.12 implies that $\overline{Z'YK'} = K$. In particular, $K \subset \overline{Z'YM'}$. In addition, we have $\overline{Z'YM'} \subset E$ because of the hyperinvariance of E.

Choose now a cyclic vector $k \in K$ for $(T|K)^*$; the fact that $(T|K)^*$ has multiplicity one follows from Theorem 2.8. Define the following subspaces:

$$K_M = \bigvee_{n=0}^{\infty} (T|M)^{*n}k, \quad K_{M'} = \bigvee_{n=0}^{\infty} (T|\overline{Z'YM'})^{*n}k, \quad K_E = \bigvee_{n=0}^{\infty} (T|E)^{*n}k,$$

along with $L_M = M \ominus K_M$, $L_{M'} = \overline{Z'YM'} \ominus K_{M'}$ and $F = E \ominus K_E$. We claim now that $L_M \subset F$ and $L_{M'} \subset F$. Indeed, let $h \in L_M \subset M \subset E$. Then, for any $n \geqslant 0$, we have

$$\langle h, (T|E)^{*n}k \rangle = \langle h, P_E T^{*n}k \rangle = \langle h, P_M T^{*n}k \rangle = \langle h, (T|M)^{*n}k \rangle = 0$$

since $L_M \perp K_M$. A similar computation shows that $L_{M'} \subset F$.

Note that the Jordan model of $T|E$ is easy to identify given the form of E and the fact that θ_1/ϕ_1 divides θ_0/ϕ_0:

$$T|E \sim S(\theta_0/\phi_0) \oplus S(\theta_1/\phi_1) = S(\alpha_0) \oplus S(\theta_1/\beta_1).$$

Also, by Theorem 2.15 we have $T|\overline{Z'YM'} \sim T|M' \sim S(\alpha_0) \oplus S(\alpha_1)$. We now apply Theorem 2.9 to $T|M, T|\overline{Z'YM'}$ and $T|E$ for the subspace $K \subset M \cap \overline{Z'YM'} \cap E$. We have that

$$L_M \cap K = L_{M'} \cap K = F \cap K = \{0\},$$

$$M = K \vee L_M, \quad \overline{Z'YM'} = K \vee L_{M'}, \quad E = K \vee F \quad \text{and}$$

$$T|L_M \sim T|L_{M'} \sim S(\alpha_1), \quad T|F \sim S(\theta_1/\beta_1).$$

Since $L_M, L_{M'} \subset F$ and $T|F$ has multiplicity one, Theorem 2.12 implies that $L_M = L_{M'}$, whence $M = K \vee L_M = K \vee L_{M'} = \overline{Z'YM'}$. Setting $X = Z'Y$ completes the proof. ∎

We can now establish the existence of a canonical space.

THEOREM 4.2. *Let $T \in \mathcal{B}(\mathcal{H})$ be an operator of class C_0 with Jordan model $S(\theta_0) \oplus S(\theta_1)$. Let $M \subset \mathcal{H}$ be an invariant subspace for T such that $S(\alpha_0) \oplus S(\alpha_1)$ is the Jordan model of $T|M$ and $S(\beta_0) \oplus S(\beta_1)$ is the Jordan model of $T_{M^\perp}$. Let $N \subset H(\theta_0) \oplus H(\theta_1)$ be the smallest invariant subspace for $S(\theta_0) \oplus S(\theta_1)$ containing*

$$\xi = (P_{H(\theta_0)}(\theta_0/\alpha_0)) \oplus (P_{H(\theta_1)}\beta_1) \quad \text{and} \quad \eta = 0 \oplus (P_{H(\theta_1)}(\theta_1/\alpha_1)).$$

Then M is quasisimilar to N.

Proof. Let $J = S(\theta_0) \oplus S(\theta_1)$. By Theorem 2.6, we can find quasiaffinities $A : \mathcal{H} \to H(\theta_0) \oplus H(\theta_1)$ and $B : H(\theta_0) \oplus H(\theta_1) \to \mathcal{H}$ such that $AT = JA, BJ = TB$. Applying Theorem 3.1, we obtain a quasiaffinity $X \in \{J\}'$ such that

$$\overline{P_{H(\theta_0)}XAM} \oplus \overline{P_{H(\theta_1)}XAM}$$

is a hyperinvariant subspace for J. By virtue of Proposition 2.18, we have $J|\overline{XAM} \sim T|M$ and $J_{(XAM)^\perp} \sim T_{M^\perp}$. By Theorem 2.13, Proposition 3.2, Proposition 3.4 and Proposition 4.1, we can find a quasiaffinity $Y \in \{J\}'$ such that $\overline{YXAM} = N$, and thus $M \prec N$. Now, it is obvious that $N \prec \overline{BN}$, whence $M \prec \overline{BN}$. But then Lemma 2.17 implies that $\overline{BN} \sim M$, and thus $N \prec M$. This completes the proof. ∎

The main result of the paper is now easily proved.

THEOREM 4.3. *Let T be an operator of class C_0 with multiplicity two. Let M, M' be two invariant subspaces for T. Then $M \sim M'$ if and only if $T|M \sim T|M'$ and $T_{M^\perp} \sim T_{M'^\perp}$.*

Proof. One direction follows from Proposition 2.18 and the sentence immediately following its proof. Assume now that $T|M \sim T|M'$ and $T_{M^\perp} \sim T_{M'^\perp}$. Then Theorem 4.2 implies the existence of an invariant subspace N for the Jordan model of T such that $M \sim N$ and $M' \sim N$, so we are done. ∎

REFERENCES

[1] H. BERCOVICI, *Operator Theory and Arithmetic in H^∞*, Math. Surveys Monogr., vol. 26, Amer. Math. Soc., Providence, RI 1988.

[2] H. BERCOVICI, The quasisimilarity orbits of invariant subspaces, *J. Funct. Anal.* **95**(1991), 344–363.

[3] H. BERCOVICI, T. SMOTZER, Quasisimilarity of invariant subspaces for uniform Jordan operators of infinite multiplicity, *J. Funct. Anal.* **140**(1996), 87–99.

[4] H. BERCOVICI, T. SMOTZER, Classification of cyclic invariant subspaces of Jordan operators, in *Recent Advances in Operator Theory*, Oper. Theory Adv. Appl., vol. 124, Birkhäuser, Basel 2001, pp. 131–143.

[5] H. BERCOVICI, W.S. LI, Invariant subspaces with extremal structure for operators of class C_0, in *Operator Theory, Structured Matrices, and Dilations*, Theta Ser. Adv. Math., vol. 7, Theta, Bucharest 2007, pp. 115–123,

[6] H. BERCOVICI, W.S. LI, T. SMOTZER, A continuous version of the Littlewood–Richardson rule and its application to invariant subspaces, *Adv. Math.* **134**(1998), 278–293.

[7] H. BERCOVICI, W.S. LI, T. SMOTZER, Continuous versions of the Littlewood–Richardson rule, selfadjoint operators, and invariant subspaces, *J. Operator Theory* **54**(2005), 69–92.

[8] H. BERCOVICI, A. TANNENBAUM, The invariant subspaces of a uniform Jordan operator, *J. Math. Anal. Appl.* **156**(1991), 220–230.

[9] W.S. LI, V. MÜLLER, Littlewood–Richardson sequences associated with C_0-operators, *Acta Sci. Math. (Szeged)* **64**(1998), 609–625.

[10] W.S. LI, V. MÜLLER, Invariant subspaces of nilpotent operators and LR-sequences, *Integral Equations Operator Theory* **34**(1999), 197–226.

[11] D. SARASON, Generalized interpolation in H^∞, *Trans. Amer. Math. Soc.* **127**(1967), 179–203.

RAPHAËL CLOUÂTRE, DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, 831 EAST 3RD STREET, BLOOMINGTON, IN 47405, U.S.A.
    E-mail address: rclouatr@indiana.edu