

KMS STATES ON THE C^* -ALGEBRAS OF NON-PRINCIPAL GROUPOIDS

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ABSTRACT. We describe KMS-states on the C^* -algebras of étale groupoids in terms of measurable fields of traces on the C^* -algebras of the isotropy groups. We use this description to analyze tracial states on the transformation groupoid C^* -algebras and to give a short proof of recent results of Cuntz, Deninger and Laca on the Toeplitz algebras of the $ax + b$ semigroups of the rings of integers in number fields.

KEYWORDS: *KMS-states, groupoids, Toeplitz algebras, number fields.*

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INTRODUCTION

The problem of classifying KMS-states for various C^* -dynamical systems has been extensively studied since the 1970s. Although it can be approached from different angles, one general result is particularly useful and can be applied to almost all known examples. It is the theorem proved by Renault [12] which states that for the C^* -algebra of an étale principal groupoid G with the dynamics given by an \mathbb{R} -valued 1-cocycle c , there is a one-to-one correspondence between KMS_β -states and quasi-invariant probability measures on $G^{(0)}$ with Radon–Nikodym cocycle $e^{-\beta c}$. Once a C^* -algebra is written as a groupoid C^* -algebra, this theorem allows one, in many cases, to reduce the study of KMS-states to a measure-theoretic problem, to which results and methods of the dynamical systems theory can be applied; see e.g. [3].

Recently, however, several natural examples of C^* -algebras of non-principal groupoids have emerged where the structure of KMS-states is relatively simple, but it cannot be understood in terms of quasi-invariant measures only. One such example is the Toeplitz algebra of the semigroup $\mathbb{N} \rtimes \mathbb{N}^\times$ studied by Laca and Raeburn [8]. In this example one has, for every $\beta > 2$, a unique quasi-invariant measure on $G^{(0)}$ with the required Radon–Nikodym cocycle, but the simplex of

KMS_β -states is isomorphic to the simplex of probability measures on the unit circle. The reason for this structure is that for non-principal groupoids possible extensions of a state on $C_0(G^{(0)})$ to a KMS-state on $C^*(G)$ are determined by a choice of tracial states on the C^* -algebras of the isotropy groups. In the case studied by Laca and Raeburn these isotropy groups turn out to be \mathbb{Z} , and this explains why measures on the circle appear naturally in the classification of KMS-states. This idea was briefly outlined in the preliminary version [11] of this note and in the introduction to [6]. In this extended version we formulate the result more explicitly and in a more general setting than discussed in [11] and [6], and consider more examples.

The proposed strategy for classifying KMS-states on the C^* -algebra $C^*(G)$ of an étale groupoid can therefore be described as follows, see Section 1 for precise statements.

First we have to find all probability measures μ on $G^{(0)}$ with Radon–Nikodym cocycle $e^{-\beta c}$. If the μ -measure of the set of points with non-trivial isotropy is zero, then the only way to extend μ_* to a KMS_β -state is by composing μ_* with the canonical conditional expectation $C^*(G) \rightarrow C_0(G^{(0)})$.

Otherwise all possible extensions of μ_* are obtained by choosing tracial states φ_x on the C^* -algebras $C^*(G_x^x)$ of the isotropy groups. The additional requirements on φ_x are that the field $(\varphi_x)_x$ is essentially G -invariant and μ -measurable.

These requirements imply that it suffices to specify φ_x on a subset intersecting almost every orbit of points with non-trivial isotropy. If the action of G on $G^{(0)}$ has complicated dynamics and a lot of isotropy, this is hardly a simplification and our description of possible extensions of μ_* is probably not very useful. But in many examples the measures μ are concentrated on a set with only countably many points with non-trivial isotropy. In such cases this description allows us to divide classification of KMS-states into two almost disjoint problems: classification of quasi-invariant measures on $G^{(0)}$ with a given Radon–Nikodym cocycle and classification of tracial states on $C^*(G_x^x)$ for countably many points.

This note consists of three sections. Section 1 contains our main general results.

In Section 2 we explain what they mean for transformation groupoids and tracial states. In particular, we will show that classification of tracial states on crossed products by abelian groups reduces completely to a measure-theoretic problem.

In Section 3 we consider the Toeplitz algebras of the $ax + b$ semigroups of the rings of integers in number fields, recently studied by Cuntz, Deninger and Laca [2], and recover the classification of KMS-states obtained in [2]. Our approach clarifies why some computations in [2] resemble those used in the study of the Bost–Connes systems of number fields. It also explains why certain representations play a prominent role in the construction of KMS-states in [2].

1. KMS STATES ON GROUPOID C^* -ALGEBRAS

Let G be a locally compact second countable étale groupoid. We denote the unit space of G by $G^{(0)}$, and the range and source maps $G \rightarrow G^{(0)}$ by r and s , respectively. Recall that being étale means that r and s are local homeomorphisms. For $x \in G^{(0)}$ put

$$G^x = r^{-1}(x), \quad G_x = s^{-1}(x) \quad \text{and} \quad G_x^x = G^x \cap G_x.$$

The C^* -algebra $C^*(G)$ of the groupoid G is the C^* -enveloping algebra of the $*$ -algebra $C_c(G)$ with convolution product

$$(f_1 * f_2)(g) = \sum_{h \in G^r(g)} f_1(h)f_2(h^{-1}g)$$

and involution $f^*(g) = \overline{f(g^{-1})}$.

Let μ be a probability measure on $G^{(0)}$. Assume that for μ -a.e. $x \in G^{(0)}$ we are given a state φ_x on $C^*(G_x^x)$. Denote the generators of $C^*(G_x^x)$ by $u_g, g \in G_x^x$. We say that the field of states $\{\varphi_x\}_{x \in G^{(0)}}$ is μ -measurable if for every $f \in C_c(G)$ the function

$$G^{(0)} \ni x \mapsto \sum_{g \in G_x^x} f(g)\varphi_x(u_g)$$

is μ -measurable; note that this function is always bounded. We do not distinguish between measurable fields which agree for μ -a.e. x .

Recall that the centralizer A_φ of a state φ on a C^* -algebra A is the set of elements $a \in A$ such that $\varphi(ab) = \varphi(ba)$ for all $b \in A$.

THEOREM 1.1. *There is a one-to-one correspondence between states on $C^*(G)$ with the centralizer containing $C_0(G^{(0)})$ and pairs $(\mu, \{\varphi_x\}_x)$ consisting of a probability measure μ on $G^{(0)}$ and a μ -measurable field of states φ_x on $C^*(G_x^x)$. Namely, the state corresponding to $(\mu, \{\varphi_x\}_x)$ is given by*

$$\varphi(f) = \int_{G^{(0)}} \sum_{g \in G_x^x} f(g)\varphi_x(u_g)d\mu(x) \quad \text{for } f \in C_c(G).$$

Proof. Assume φ is a state on $C^*(G)$ with the centralizer containing the algebra $C_0(G^{(0)})$. Let (H, π, ζ) be the corresponding GNS-triple. By Renault’s disintegration theorem [12], [13], the representation π is the integrated form of a representation of G on a measurable, with respect to a measure class $[\nu]$ on $G^{(0)}$, field of Hilbert spaces $H_x, x \in G^{(0)}$. Identifying H with $\int_{G^{(0)}}^\oplus H_x d\nu(x)$, consider

the vector field $(\zeta_x)_x$ that defines ζ . Let μ be the measure on $G^{(0)}$ such that $d\mu(x) = \|\zeta_x\|^2 d\nu(x)$. In other words, μ is the measure defined by the restriction of φ to $C_0(G^{(0)})$. The action of G on $(H_x)_x$ defines, for every x , a representation $\rho_x: C^*(G_x^x) \rightarrow B(H_x)$. For every x with $\zeta_x \neq 0$ denote by φ_x the state $\|\zeta_x\|^{-2}(\rho_x(\cdot)\zeta_x, \zeta_x)$ on $C^*(G_x^x)$.

For every $f \in C_c(G)$ we have

$$\varphi(f) = (\pi(f)\tilde{\zeta}, \tilde{\zeta}) = \int_X \sum_{g \in G^x} f(g)(g\tilde{\zeta}_{s(g)}, \tilde{\zeta}_x) d\nu(x).$$

Therefore to prove the identity in the formulation of the theorem we have to show that for ν -a.e. x and every $g \in G^x \setminus G_x^x$ we have $(g\tilde{\zeta}_{s(g)}, \tilde{\zeta}_x) = 0$. Choose an open subset $U \subset G \setminus G'$, where $G' = \bigcup_x G_x^x$ is the isotropy bundle, such that $r(U) \cap s(U) = \emptyset$. Let $f \in C_c(U)$. For any function $h \in C_c(r(U))$ we have $f * h = 0$. Since h is in the centralizer of φ , we therefore get

$$0 = \varphi(h * f) = \int_{r(U)} h(x) \sum_{g \in G^x} f(g)(g\tilde{\zeta}_{s(g)}, \tilde{\zeta}_x) d\nu(x).$$

Hence $\sum_{g \in G^x} f(g)(g\tilde{\zeta}_{s(g)}, \tilde{\zeta}_x) = 0$ for ν -a.e. $x \in r(U)$. It follows that $(g\tilde{\zeta}_{s(g)}, \tilde{\zeta}_x) = 0$ for ν -a.e. $x \in r(U)$ and all $g \in G^x \cap U$. Since $G \setminus G'$ can be covered by countably many such sets U , we conclude that $(g\tilde{\zeta}_{s(g)}, \tilde{\zeta}_x) = 0$ for ν -a.e. $x \in G^{(0)}$ and all $g \in G^x \setminus G_x^x$.

Conversely, assume we are given a probability measure μ on $G^{(0)}$ and a μ -measurable field of states φ_x on $C^*(G_x^x)$. For every x define a state ψ_x on $C^*(G)$ by

$$\psi_x(f) = \sum_{g \in G_x^x} f(g)\varphi_x(u_g) \quad \text{for } f \in C_c(G).$$

In order to show that ψ_x is indeed a well-defined state, consider the GNS-triple $(K_x, \pi_x, \tilde{\zeta}_x)$ defined by φ_x . Induce π_x to a representation of G and denote by ϑ_x its integrated form. Explicitly, ϑ_x is the representation on the space L_x of functions $\tilde{\zeta}: G_x \rightarrow K_x$ such that

$$\tilde{\zeta}(gh) = \pi_x(u_h^*)\tilde{\zeta}(g) \quad \text{for } g \in G_x \text{ and } h \in G_x^x, \quad \text{and} \quad \sum_{g \in G_x/G_x^x} \|\tilde{\zeta}(g)\|^2 < \infty,$$

given by

$$(\vartheta_x(f)\tilde{\zeta})(g) = \sum_{h \in G^r(g)} f(h)\tilde{\zeta}(h^{-1}g) \quad \text{for } f \in C_c(G).$$

Let $\zeta_x \in L_x$ be the vector defined by $\zeta_x(g) = \pi_x(u_g^*)\tilde{\zeta}_x$ if $g \in G_x^x$ and $\zeta_x(g) = 0$ if $\zeta \in G_x \setminus G_x^x$. Then $\psi_x = (\vartheta_x(\cdot)\zeta_x, \zeta_x)$.

Clearly, $C_0(X)$ is contained in the centralizer of ψ_x ; in fact, for any $f \in C_0(X)$ and $a \in C^*(G)$ we have $\psi_x(fa) = \psi_x(af) = f(x)\psi_x(a)$. By assumption, the map $x \mapsto \psi_x(a)$ is μ -measurable for every $a \in C_c(G)$, hence for every $a \in C^*(G)$. Therefore we can define $\varphi(a) = \int_X \psi_x(a) d\mu(x)$.

Finally, it is easy to see that if $(\mu, \{\varphi_x\}_x)$ and $(\tilde{\mu}, \{\tilde{\varphi}_x\}_x)$ define the same state then $\mu = \tilde{\mu}$ and $\varphi_x = \tilde{\varphi}_x$ for μ -a.e. x . ■

Let $E: C^*(G) \rightarrow C_0(G^{(0)})$ be the canonical conditional expectation. Given a probability measure μ on $G^{(0)}$, consider the μ -measurable field of states consisting of canonical traces on $C^*(G_x^x)$. The corresponding state on $C^*(G)$ is $\mu_* \circ E$.

COROLLARY 1.2. *Let μ be a probability measure on $G^{(0)}$. Assume that the points of $G^{(0)}$ with non-trivial isotropy form a set of μ -measure zero. Then $\varphi = \mu_* \circ E$ is the unique state on $C^*(G)$ such that the centralizer of φ contains $C_0(G^{(0)})$ and $\varphi|_{C_0(G^{(0)})} = \mu_*$.*

A short proof of this corollary, not relying on the disintegration theorem, can be obtained along the same lines as the proof of Proposition 1.1 in [4].

Let c be a continuous \mathbb{R} -valued 1-cocycle on G , that is, a continuous homomorphism $c: G \rightarrow \mathbb{R}$. It defines a one-parameter group of automorphisms of $C^*(G)$ by $\sigma_t^c(f)(g) = e^{itc(g)}f(g)$.

Recall that a measure μ on $G^{(0)}$ is called quasi-invariant with Radon–Nikodym cocycle e^c if $d\mu_r/d\mu_s = e^c$, where the measures μ_r and μ_s on G are defined by

$$\int_G f d\mu_r = \int_{G^{(0)}} \sum_{g \in G^x} f(g) d\mu(x), \quad \int_G f d\mu_s = \int_{G^{(0)}} \sum_{g \in G_x} f(g) d\mu(x) \quad \text{for } f \in C_c(G).$$

Equivalently, for every open set $U \subset G$ such that r and s are injective on U we have

$$\frac{dT_*\mu}{d\mu}(x) = e^{c(g_x)} \quad \text{for } x \in s(U),$$

where $g_x \in U$ is the unique element such that $s(g_x) = x$ and where $T: r(U) \rightarrow s(U)$ is the homeomorphism defined by $T(r(g_x)) = x$.

Now recall that if σ is a strongly continuous one-parameter group of automorphisms of a C^* -algebra A and $\beta \in \mathbb{R}$, then a state φ on A is called a σ -KMS $_\beta$ -state if φ is σ -invariant and $\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$ for a dense set of σ -analytic elements $a, b \in A$.

THEOREM 1.3. *Let c be a continuous \mathbb{R} -valued 1-cocycle on G , σ^c the dynamics on $C^*(G)$ defined by c , and $\beta \in \mathbb{R}$. Then there exists a one-to-one correspondence between σ^c -KMS $_\beta$ -states on $C^*(G)$ and pairs $(\mu, \{\varphi_x\}_x)$ consisting of a probability measure μ on $G^{(0)}$ and a μ -measurable field of states φ_x on $C^*(G_x^x)$ such that:*

- (i) μ is quasi-invariant with Radon–Nikodym cocycle $e^{-\beta c}$;
- (ii) $\varphi_x(u_g) = \varphi_{r(h)}(u_{hg^{h^{-1}}})$ for μ -a.e. x and all $g \in G_x^x$ and $h \in G_x$; in particular, φ_x is tracial for μ -a.e. x ;
- (iii) $\varphi_x(u_g) = 0$ for μ -a.e. x and all $g \in G_x^x \setminus c^{-1}(0)$.

Note that if $\beta \neq 0$ and μ is a quasi-invariant measure with Radon–Nikodym cocycle $e^{-\beta c}$ then $G_x^x \subset c^{-1}(0)$ for μ -a.e. x . Therefore condition (iii) is redundant in this case, but it is still useful to keep it in mind. On the level of KMS-states this corresponds to the following fact: if φ is a state such that $\varphi(ab) = \varphi(b\sigma_{i\beta}(a))$ for $\beta \neq 0$, then φ is automatically σ -invariant.

Proof of Theorem 1.3. Since the centralizer of any σ^c -KMS-state φ contains the algebra $C_0(G^{(0)})$, by Theorem 1.1 any such state is defined by a pair $(\mu, \{\varphi_x\}_x)$ consisting of a probability measure μ on $G^{(0)}$ and a μ -measurable field of states φ_x on $C^*(G_x^x)$. Therefore we just have to check that given such a pair $(\mu, \{\varphi_x\}_x)$ the corresponding state is a KMS_β -state if and only if conditions (i)–(iii) are satisfied.

It is easy to see that condition (iii) is equivalent to σ^c -invariance. We will show that (i) and (ii) together are equivalent to $\varphi(f_1 * f_2) = \varphi(f_2 * \sigma_{i\beta}^c(f_1))$ for all $f_1, f_2 \in C_c(G)$.

Assume the equality $\varphi(f_1 * f_2) = \varphi(f_2 * \sigma_{i\beta}^c(f_1))$ holds for all $f_1, f_2 \in C_c(G)$. Fix an open set $U \subset G$ such that r and s are injective on U . Let $x \mapsto h^x$ be the inverse of $r: U \rightarrow r(U)$, and $x \mapsto h_x$ the inverse of $s: U \rightarrow s(U)$. Denote by $T: r(U) \rightarrow s(U)$ the homeomorphism defined by $Tx = s(h^x)$, so that $h_{Tx} = h^x$. For $f_1 \in C_c(U)$ and $f_2 \in C_c(G)$ we have

$$(f_1 * f_2)(g) = \begin{cases} f_1(h^x)f_2((h^x)^{-1}g) & \text{if } x = r(g) \in r(U), \\ 0 & \text{if } r(g) \notin r(U), \end{cases}$$

$$(f_2 * \sigma_{i\beta}^c(f_1))(g) = \begin{cases} e^{-\beta c(h_x)} f_1(h_x)f_2(gh_x^{-1}) & \text{if } x = s(g) \in s(U), \\ 0 & \text{if } s(g) \notin s(U). \end{cases}$$

Therefore the equality $\varphi(f_1 * f_2) = \varphi(f_2 * \sigma_{i\beta}^c(f_1))$ reads as

$$(1.1) \quad \int_{r(U)} f_1(h^x) \sum_{g \in G_x^x} f_2((h^x)^{-1}g) \varphi_x(u_g) d\mu(x) \\ = \int_{s(U)} e^{-\beta c(h_x)} f_1(h_x) \sum_{g \in G_x^x} f_2(gh_x^{-1}) \varphi_x(u_g) d\mu(x).$$

Let $f \in C_c(s(U))$. Apply the above identity to the functions f_1 and f_2 defined by $f_1(h_x) = f(x)$ for $x \in s(U)$ and $f_2 = f_1^*$. Since $f_1(h^x) = f_1(h_{Tx}) = f(Tx)$ for $x \in r(U)$, we get

$$\int_{r(U)} |f(Tx)|^2 d\mu(x) = \int_{s(U)} e^{-\beta c(h_x)} |f(x)|^2 d\mu(x).$$

Since this is true for all U on which r and s are injective and all $f \in C_c(s(U))$, we see that μ is quasi-invariant with Radon–Nikodym cocycle $e^{-\beta c}$, so condition (i) is satisfied. But then (1.1), for arbitrary $f_1 \in C_c(U)$ and $f_2 \in C_c(G)$, can be written as

$$\int_{r(U)} f_1(h^x) \sum_{g \in G_x^x} f_2((h^x)^{-1}g) \varphi_x(u_g) d\mu(x) \\ = \int_{r(U)} f_1(h_{Tx}) \sum_{g \in G_{Tx}^{Tx}} f_2(gh_{Tx}^{-1}) \varphi_{Tx}(u_g) d\mu(x).$$

Using that $h_{Tx} = h^x$ and $G_x^x h^x = h^x G_{Tx}^{Tx}$, this, in turn, can be written as

$$\begin{aligned} \int_{r(U)} f_1(h^x) \sum_{g \in G_x^x} f_2((h^x)^{-1}g) \varphi_x(u_g) d\mu(x) \\ = \int_{r(U)} f_1(h^x) \sum_{g \in G_x^x} f_2((h^x)^{-1}g) \varphi_{Tx}(u_{(h^x)^{-1}gh^x}) d\mu(x). \end{aligned}$$

Since this equality holds for all $f_1 \in C_c(U)$ and $f_2 \in C_c(G)$, we conclude that for μ -a.e. $x \in r(U)$ we have $\varphi_x(u_g) = \varphi_{Tx}(u_{(h^x)^{-1}gh^x})$ for all $g \in G_x^x$. As G can be covered by countably many open sets U such that r and s are injective on U , it follows that $\varphi_x(u_g) = \varphi_{r(h)}(u_{hg h^{-1}})$ for μ -a.e. x and all $g \in G_x^x$ and $h \in G_x$, so condition (ii) is also satisfied.

Conversely, if conditions (i) and (ii) are satisfied, then we see from the above computations that $\varphi(f_1 * f_2) = \varphi(f_2 * \sigma_{i\beta}^c(f_1))$ for all $f_1 \in C_c(U)$ and $f_2 \in C_c(G)$, where $U \subset G$ is any open set such that r and s are injective on U . Hence

$$\varphi(f_1 * f_2) = \varphi(f_2 * \sigma_{i\beta}^c(f_1))$$

for all $f_1, f_2 \in C_c(G)$. ■

Given a probability measure μ on $G^{(0)}$ with Radon–Nikodym cocycle $e^{-\beta c}$, the simplest way to extend the state μ_* to a σ^c -KMS $_\beta$ -state on $C^*(G)$ is by composing μ_* with the conditional expectation $E: C^*(G) \rightarrow C_0(G^{(0)})$. As we already noted, this corresponds to taking the canonical traces on $C^*(G_x^x)$ for φ_x . If the points with non-trivial isotropy have measure zero, then $\mu_* \circ E$ is the unique σ^c -KMS $_\beta$ -state extending μ_* . In general, to extend μ_* we have to find measurable fields of tracial states satisfying properties (ii) and (iii) above. If there are many points with non-trivial isotropy, this is a difficult problem and it is not clear how useful the description in terms of fields of traces is. A simple example of such a rich isotropy bundle structure is the transformation groupoid of the action of the infinite symmetric group S_∞ on $\{0, 1\}^\infty$. In many cases, however, the measure μ is concentrated on a set with only countably many points with non-trivial isotropy, and then the above result gives a complete description of possible extensions of μ_* .

COROLLARY 1.4. *Suppose μ is a quasi-invariant probability measure on $G^{(0)}$ with Radon–Nikodym cocycle $e^{-\beta c}$. Assume there exists a sequence $\{O_n\}_{n=1}^N$, $N \in \mathbb{N} \cup \{+\infty\}$, of different orbits of the action of G on $G^{(0)}$ such that $\mu(O_n) > 0$ for every n , and almost all points in $G^{(0)} \setminus \bigcup_n O_n$ have trivial isotropy. Choose a point $x_n \in O_n$ for every n . Then there is a one-to-one correspondence between σ^c -KMS $_\beta$ -states on $C^*(G)$ extending the state μ_* on $C_0(G^{(0)})$ and sequences of tracial states τ_n on $C^*(G_{x_n}^{x_n})$ such that $\tau_n(u_g) = 0$ for every $g \in G_{x_n}^{x_n} \setminus c^{-1}(0)$.*

Proof. Every orbit O_n is a countable set, so the measurability assumption is satisfied for any choice of states φ_x on $C^*(G_x^x)$ for $x \in O_n$. Since the action of G on O_n is transitive, the map $\{\varphi_x\}_{x \in O_n} \mapsto \varphi_{x_n}$ is a bijection between sequences of states such that $\varphi_x(u_g) = \varphi_{r(h)}(u_{hg}h^{-1})$ for every $x \in O_n$ and all $g \in G_x^x$ and $h \in G_x$ and the set of tracial states on $C^*(G_{x_n}^{x_n})$. ■

2. TRACES ON CROSSED PRODUCTS

Let X be a locally compact second countable space. Assume a countable group Γ acts on X by homeomorphisms. Then $C_0(X) \rtimes \Gamma$ is the C^* -algebra of the transformation groupoid $X \times \Gamma$. We will apply the results of the previous section to describe tracial states on $C_0(X) \rtimes \Gamma$.

For $x \in X$, denote by Γ_x the stabilizer of x in Γ . Let μ be a probability measure on X . According to our definition, a field of states φ_x on $C^*(\Gamma_x)$ is μ -measurable if, for every $g \in \Gamma$, the function $x \mapsto \varphi_x(u_g)$ is μ -measurable on the closed set of points fixed by g . This can also be formulated as follows.

Every state φ_x extends to a state ψ_x on $C^*(\Gamma)$ such that $\psi_x(u_g) = 0$ for every $g \notin \Gamma_x$. This can be proved using induction, exactly as in the proof of Theorem 1.1, so we will write $\text{Ind}_{\Gamma_x}^{\Gamma} \varphi_x$ for ψ_x . Then $\{\varphi_x\}_x$ is μ -measurable if and only if the map $x \mapsto \text{Ind}_{\Gamma_x}^{\Gamma} \varphi_x$ from X into the state space $S(C^*(\Gamma))$ of $C^*(\Gamma)$ is μ -measurable, where $S(C^*(\Gamma))$ is considered with the Borel structure defined by the weak* topology.

Applying Theorem 1.3 to the transformation groupoid $X \times \Gamma$ and the zero cocycle, we get the following result.

THEOREM 2.1. *There is a one-to-one correspondence between tracial states on $C_0(X) \rtimes \Gamma$ and pairs $(\mu, \{\varphi_x\}_x)$ consisting of a probability measure μ on X and a μ -measurable field of states φ_x on $C^*(\Gamma_x)$ such that:*

- (i) μ is Γ -invariant;
- (ii) $\varphi_x(u_g) = \varphi_{hx}(u_{hg}h^{-1})$ for μ -a.e. x and all $g \in \Gamma_x$ and $h \in \Gamma$.

Equivalently, we can say that to define a tracial state we need a Γ -invariant probability measure μ on X and a field of tracial states φ_x on $C^*(\Gamma_x)$ such that the map $x \mapsto \text{Ind}_{\Gamma_x}^{\Gamma} \varphi_x \in S(C^*(\Gamma))$ is μ -measurable and Γ -equivariant, where $g \in \Gamma$ acts on $S(C^*(\Gamma))$ by mapping a state ψ into $\psi \circ (\text{Ad } u_g)^{-1}$.

The above result was obtained using Renault’s disintegration theorem. In the case of transformation groupoids this theorem is a simple consequence of standard results on disintegration of representations of C^* -algebras, see for example Chapter IV in [14]. But, in fact, in this case the groupoid picture can be bypassed altogether. In order to show this we will need the following observation, which in a way goes back to Appendix in [9].

LEMMA 2.2. *For any state φ on a unital C^* -algebra A , there exists a unique state Φ on $A_\varphi^{op} \otimes_{\max} A$ such that $\Phi(a \otimes b) = \varphi(ab)$ for all $a \in A_\varphi$ and $b \in A$.*

Proof. We may assume that $A \subset B(H)$ and φ is defined by a cyclic vector $\xi \in H$. Assume first that ξ is separating for A'' . Let J be the corresponding modular conjugation. Define a representation π of $A_\varphi^{op} \otimes_{\max} A$ on H by

$$\pi(a \otimes b) = Ja^*Jb = bJa^*J.$$

If $a \in A_\varphi$, then a commutes with the modular operator, hence $Ja\xi = a^*\xi$. Therefore $\pi(a \otimes b)\xi = ba\xi$, so that $\Phi := (\pi(\cdot), \xi, \xi)$ is the required state.

In the general case we will show that a representation π of $A_\varphi^{op} \otimes_{\max} A$ such that $\pi(a \otimes b)\xi = ba\xi$ always exists. For $a \in A_\varphi$ and $b, c \in A$ we have

$$(ba\xi, c\xi) = (b\xi, ca^*\xi).$$

It follows that for every $a \in A_\varphi$ there exists a well-defined operator $\rho(a)$ on $A\xi$ such that $\rho(a)b\xi = ba\xi$, and then ρ is a representation of A_φ^{op} on $A\xi$. Since $\rho(u)$ is unitary for unitary u , this is a representation by bounded operators, so it extends to a representation of A_φ^{op} on H . Its image commutes with A , so we can define a representation π of $A_\varphi^{op} \otimes_{\max} A$ on H by $\pi(a \otimes b) = \rho(a)b$. Then $\pi(a \otimes b)\xi = ba\xi$. ■

Assume now that φ is a state on $A = C_0(X) \rtimes \Gamma$ with the centralizer containing $C_0(X)$. Using the above lemma define a state Φ on $C_0(X) \otimes A$. Denote by j the canonical homomorphism $C^*(\Gamma) \rightarrow M(A)$. Extending Φ to the multiplier algebra and composing this extension with

$$\text{id} \otimes j: C_0(X) \otimes C^*(\Gamma) \rightarrow C_0(X) \otimes M(A),$$

we get a state Ψ on $C_0(X) \otimes C^*(\Gamma)$ such that

$$\Psi(f \otimes a) = \varphi(fj(a)) \quad \text{for } f \in C_0(X) \text{ and } a \in C^*(\Gamma).$$

Disintegrating Ψ with respect to $C_0(X)$ we get a probability measure μ on X and states ψ_x on $C^*(\Gamma)$ such that $\Psi = \int_X^\oplus \psi_x d\mu(x)$, that is,

$$\Psi(f \otimes u_g) = \int_X f(x)\psi_x(u_g)dv(x) \quad \text{for } f \in C_0(X) \text{ and } g \in \Gamma.$$

Therefore $\varphi_x = \psi_x|_{C^*(\Gamma_x)}$ are exactly the states given by Theorem 1.1.

The above argument suggests yet another way of looking at pairs $(\mu, \{\varphi_x\}_x)$ consisting of a probability measure μ and a μ -measurable field of states on $C^*(\Gamma_x)$: any such pair defines a state $\Psi = \int_X^\oplus \text{Ind}_{\Gamma_x}^\Gamma \varphi_x d\mu(x)$ on $C_0(X) \otimes C^*(\Gamma)$, and this way we get all states $\Psi = \int_X^\oplus \psi_x d\mu(x)$ such that $\psi_x(u_g) = 0$ for μ -a.e. $x \in X$ and all $g \notin \Gamma_x$.

For abelian groups this point of view combined with Theorem 2.1 allow us to completely reduce the classification of tracial states to a measure-theoretic problem.

COROLLARY 2.3. *If Γ is abelian, there is a bijection between tracial states τ on $C_0(X) \rtimes \Gamma$ and probability measures $\nu = \int_X^\oplus \nu_x d\mu(x)$ on $X \times \widehat{\Gamma}$ such that*

(i) ν is invariant with respect to the action of Γ on the first factor of $X \times \widehat{\Gamma}$; equivalently, μ is Γ -invariant and $\nu_x = \nu_{gx}$ for μ -a.e. $x \in X$ and every $g \in \Gamma$;

(ii) ν_x is Γ_x^\perp -invariant for μ -a.e. $x \in X$.

Namely, the trace τ corresponding to such a measure ν is given by

$$\tau(fu_g) = \int_{X \times \widehat{\Gamma}} f(x)\chi(g)d\nu(x, \chi) \quad \text{for } f \in C_0(X) \text{ and } g \in \Gamma.$$

Proof. We only need to note that if φ_x is the state on $C^*(\Gamma) = C(\widehat{\Gamma})$ defined by ν_x then $\varphi_x(u_g) = 0$ for all $g \notin \Gamma_x$ if and only if ν_x is Γ_x^\perp -invariant. ■

Note that in this case it is particularly easy to check that the state τ corresponding to ν exists. Indeed, define a representation ρ of $C_0(X) \rtimes \Gamma$ on the space $L^2(X \times \widehat{\Gamma}, d\nu)$ by

$$(\rho(f)\zeta)(x, \chi) = f(x)\zeta(x, \chi), \quad (\rho(u_g)\zeta)(x, \chi) = \chi(g)\zeta(g^{-1}x, \chi)$$

and consider the function $\xi \equiv 1$ in $L^2(X \times \widehat{\Gamma}, d\nu)$. Then $\tau = (\rho(\cdot)\xi, \xi)$.

COROLLARY 2.4. *If Γ is abelian, there exists a one-to-one correspondence between extremal tracial states on $C_0(X) \rtimes \Gamma$ and triples (H, χ, μ) , where H is a subgroup of Γ , χ is a character of H and μ is an ergodic Γ -invariant probability measure μ on X such that $\Gamma_x = H$ for μ -a.e. $x \in X$. Namely, the trace corresponding to (H, χ, μ) is given by*

$$\tau(fu_g) = \begin{cases} \chi(g) \int_X f(x)d\mu(x) & \text{if } g \in H, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Extremal tracial states correspond to extremal probability measures ν on $X \times \widehat{\Gamma}$ with properties (i) and (ii) in Corollary 2.3. If ν is extremal, then its projection μ onto X is an extremal Γ -invariant measure, that is, μ is ergodic. But then ν must be of the form $\mu \times \lambda$ for a probability measure λ on $\widehat{\Gamma}$. Furthermore, for every $g \in \Gamma$ the set $X_g = \{x : gx = x\}$ is Γ -invariant, so either X_g or its complement has measure zero. In other words, on a subset of full measure we have $\Gamma_x = H$ for a subgroup $H \subset \Gamma$. Hence λ is H^\perp -invariant, so it can be considered as a measure on $\widehat{\Gamma}/H^\perp = \widehat{H}$. As ν is extremal, λ is an extremal probability measure on \widehat{H} , that is, $\lambda = \delta_\chi$ for some $\chi \in \widehat{H}$.

Conversely, any measure ν of the form $\mu \times \delta_\chi$, where μ is ergodic and $\chi \in \widehat{H}$, where H is the common stabilizer of the points in a subset of X of full measure, is an extremal measure with properties (i) and (ii). ■

REMARK 2.5. If τ is an extremal tracial state and (H, χ, μ) is the corresponding triple, then there is a non-canonical isomorphism

$$\pi_\tau(C_0(X) \rtimes \Gamma)'' \cong L^\infty(X, \mu) \rtimes \Gamma/H.$$

Namely, extend χ to a character $\tilde{\chi}$ of Γ . Then the required isomorphism ρ is given by $\rho(f) = f \in L^\infty(X, \mu)$, $\rho(u_g) = \tilde{\chi}(g)u_{\bar{g}}$, where \bar{g} is the image of g in Γ/H . Note that since $L^\infty(X, \mu) \rtimes \Gamma/H$ is a factor, this provides a direct proof of the extremality of τ .

Finally, consider the case $\Gamma = \mathbb{Z}$ explicitly used in [6]. Denote by $T: X \rightarrow X$ the homeomorphism corresponding to $1 \in \mathbb{Z}$. For $n \geq 0$ denote by $X_n \subset X$ the subset of points of period n (so X_0 is the set of aperiodic points). Then any measure ν on $X \times \mathbb{T}$ with properties (i) and (ii) from Corollary 2.3 decomposes into a sum of measures satisfying the same properties and concentrated on $X_n \times \mathbb{T}$ for some n .

If ν is concentrated on X_0 then $\nu = \mu \times \lambda$, where λ is the Lebesgue measure, and the corresponding trace is $\mu_* \circ E$, where $E: C_0(X) \rtimes \mathbb{Z} \rightarrow C_0(X)$ is the canonical conditional expectation.

If μ is concentrated on X_n , $n \geq 1$, then ν is a $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ -invariant measure on $X_n \times \mathbb{T}$, where the second factor $\mathbb{Z}/n\mathbb{Z}$ acts on \mathbb{T} by rotations. Consider the simplest case where μ is concentrated on the orbit of a point x of period n . Then $\mu = n^{-1} \sum_{k=0}^{n-1} \delta_{T^k x}$ and $\nu = \mu \times \lambda$, where λ is a measure that is invariant under the rotation by $2\pi/n$ degrees (we will say that λ is n -rotation invariant). The corresponding trace can be written as follows.

The $*$ -homomorphism $\rho: C_0(X) \rightarrow C(\mathbb{Z}/n\mathbb{Z})$, $\rho(f)(k) = f(T^k x)$, extends to a $*$ -homomorphism

$$\rho: C_0(X) \rtimes \mathbb{Z} \rightarrow C(\mathbb{Z}/n\mathbb{Z}) \rtimes \mathbb{Z}.$$

Passing to the dual groups we can identify $C(\mathbb{Z}/n\mathbb{Z}) \rtimes \mathbb{Z}$ with $C(\mathbb{T}) \rtimes \mathbb{Z}/n\mathbb{Z}$. By composing the canonical conditional expectation $C(\mathbb{T}) \rtimes \mathbb{Z}/n\mathbb{Z} \rightarrow C(\mathbb{T}) = C^*(\mathbb{Z})$ with ρ we then get a completely positive map

$$E_x: C_0(X) \rtimes \mathbb{Z} \rightarrow C^*(\mathbb{Z}), \quad E_x(fu^m) = \frac{1}{n} \left(\sum_{k=0}^{n-1} f(T^k x) \right) u^m,$$

where $u = u_1 \in M(C_0(X) \rtimes \mathbb{Z})$ is the canonical unitary implementing T , so $ufu^* = f(T^{-1}\cdot)$. The measure λ defines a state λ_* on $C^*(\mathbb{Z})$ by $\lambda_*(u^m) = \int_{\mathbb{T}} z^m d\lambda(z)$. Then $\lambda_* \circ E_x$ is the required tracial state on $C_0(X) \rtimes \mathbb{Z}$.

It follows that in the case when there are only countably many periodic orbits Corollary 2.3 for $\Gamma = \mathbb{Z}$ can be formulated as follows.

COROLLARY 2.6. Assume a homeomorphism T of X has at most countably many periodic orbits O_n . For every n choose $x_n \in O_n$. Then any tracial state τ on $C_0(X) \rtimes \mathbb{Z}$

has a unique decomposition

$$\tau = \mu_* \circ E + \sum_n \lambda_{n*} \circ E_{x_n},$$

where μ is a T -invariant measure on X such that $\mu(O_n) = 0$ for every n , λ_n is an $|O_n|$ -rotation invariant measure on \mathbb{T} , and $\mu(X) + \sum_n \lambda_n(\mathbb{T}) = 1$. Conversely, any such collection of measures μ, λ_n defines a tracial state.

3. KMS STATES ON THE TOEPLITZ ALGEBRAS OF $ax + b$ SEMIGROUPS

In a recent paper Cuntz, Deninger and Laca [2] have studied the Toeplitz algebra of the $ax + b$ semigroup of the ring of integers in a number field. In this section we will show how to recover their classification of KMS-states using our general framework. In addition to illustrating the general theory, our goal is to clarify a connection between the rather involved analysis in [2] and that of the Bost–Connes systems of number fields.

We will follow the notation in [7] rather than the one in [2]. Let K be a number field with the ring of integers \mathcal{O} . Denote by V_K the set of places of K , and by $V_{K,f} \subset V_K$ the subset of finite places. For every place v denote by K_v the completion of K at v . For $v \in V_{K,f}$, let \mathcal{O}_v be the closure of \mathcal{O} in K_v . We write $v|\infty$ when v is infinite and denote by $K_\infty = \prod_{v|\infty} K_v$ the completion of K at

all infinite places. The adèle ring \mathbb{A}_K is the restricted product, as v ranges over all places, of the rings K_v , with respect to $\mathcal{O}_v \subset K_v$ for $v \in V_{K,f}$. When the product is taken only over finite places v , we get the ring $\mathbb{A}_{K,f}$ of finite adèles; we then have $\mathbb{A}_K = K_\infty \times \mathbb{A}_{K,f}$. The ring of integral adèles is $\widehat{\mathcal{O}} = \prod_{v \in V_{K,f}} \mathcal{O}_v \subset \mathbb{A}_{K,f}$. Denote by

$N_K: \mathbb{A}_{K,f}^* \rightarrow (0, +\infty)$ the absolute norm.

Let $\mathfrak{T}[\mathcal{O}]$ be the Toeplitz algebra of the semigroup $\mathcal{O} \rtimes \mathcal{O}^\times$ [2], where \mathcal{O}^\times is the semigroup of nonzero elements in \mathcal{O} . Although $\mathfrak{T}[\mathcal{O}]$ can be defined in terms of generators and relations, we will use a presentation of $\mathfrak{T}[\mathcal{O}]$ as a groupoid C^* -algebra. For this consider the space Ω_K defined as the quotient of $\mathbb{A}_{K,f} \times \mathbb{A}_{K,f} / \widehat{\mathcal{O}}^*$ by the equivalence relation

$$(r, a) \sim (s, b) \iff a = b \text{ and } r - s \in a\widehat{\mathcal{O}}.$$

In other words, Ω_K consists of pairs (r, a) with $a \in \mathbb{A}_{K,f} / \widehat{\mathcal{O}}^*$ and $r \in \mathbb{A}_{K,f} / a\widehat{\mathcal{O}}$. It is a locally compact space with the quotient topology. Denote by $\Omega_{\mathcal{O}}$ the compact open subset of Ω_K consisting of pairs (r, a) with $a \in \widehat{\mathcal{O}} / \widehat{\mathcal{O}}^*$ and $r \in \widehat{\mathcal{O}} / a\widehat{\mathcal{O}}$. The group $K \rtimes K^*$ acts on $\mathbb{A}_{K,f} \times \mathbb{A}_{K,f}$ by $(n, k)(r, a) = (n + kr, ka)$. This action defines an action of $K \rtimes K^*$ on Ω_K . By Propositions 5.1 and 5.2 in [2] there is a canonical isomorphism

$$\mathfrak{T}[\mathcal{O}] \cong \mathbf{1}_{\Omega_{\mathcal{O}}} (C_0(\Omega_K) \rtimes K \rtimes K^*) \mathbf{1}_{\Omega_{\mathcal{O}}}.$$

Therefore $\mathfrak{T}[\mathcal{O}]$ is the C*-algebra of the reduction of the transformation groupoid $\Omega_K \times (K \rtimes K^*)$ by $\Omega_{\mathcal{O}}$. The homomorphism $K \rtimes K^* \rightarrow \mathbb{R}, (n, k) \mapsto -\log N_K(k)$, defines a 1-cocycle on the groupoid, which in turn defines a dynamics σ on $\mathfrak{T}[\mathcal{O}]$.

According to our general scheme, in order to classify σ -KMS $_{\beta}$ -states on $\mathfrak{T}[\mathcal{O}]$ we first have to find probability measures μ on $\Omega_{\mathcal{O}}$ with Radon–Nikodym cocycle N_K^{β} . In other words, we are looking for measures μ such that

$$\mu((n, k)Y) = N_K(k)^{\beta} \mu(Y)$$

for Borel $Y \subset \Omega_{\mathcal{O}}$ and $(n, k) \in K \rtimes K^*$ such that $(n, k)Y \subset \Omega_{\mathcal{O}}$. Since any translationally \mathcal{O} -invariant measure on $\widehat{\mathcal{O}}$ is a Haar measure, it is not difficult to show, see the proof of Proposition 2.1 in [6], that any measure μ as above must be the image under the projection $\widehat{\mathcal{O}} \times \widehat{\mathcal{O}}/\widehat{\mathcal{O}}^* \rightarrow \Omega_{\mathcal{O}}$ of a measure $m \times \nu$, where m is the normalized Haar measure on $\widehat{\mathcal{O}}$ and ν is a probability measure on $\widehat{\mathcal{O}}/\widehat{\mathcal{O}}^*$. Furthermore, as $m(k \cdot) = N_K(k)m$, the above condition on μ is satisfied if and only if

$$(3.1) \quad \nu(kY) = N_K(k)^{\beta-1} \nu(Y)$$

for Borel $Y \subset \widehat{\mathcal{O}}/\widehat{\mathcal{O}}^*$ and $k \in K^*$ such that $kY \subset \widehat{\mathcal{O}}/\widehat{\mathcal{O}}^*$. Clearly, there are no such measures for $\beta < 1$, so there are no σ -KMS $_{\beta}$ -states on $\mathfrak{T}[\mathcal{O}]$ for $\beta < 1$. The case $\beta = 1$ is also easy: the only such measure ν is concentrated at 0. We will denote this measure by ν_1 . Assume now that $\beta > 1$. A more general problem is studied in [7]. Namely, we have the following equivalent form of Theorem 2.5 in [7].

THEOREM 3.1. *Let $K_+^* \subset K^*$ be the subgroup of totally positive elements. Consider probability measures ν on $\widehat{\mathcal{O}}$ such that*

$$(3.2) \quad \nu(kY) = N_K(k)^{\beta-1} \nu(Y) \text{ for Borel } Y \subset \widehat{\mathcal{O}} \text{ and } k \in K_+^* \text{ such that } kY \subset \widehat{\mathcal{O}}.$$

Then

(i) *for every $1 < \beta \leq 2$ there exists a unique probability measure on $\widehat{\mathcal{O}}$ satisfying (3.2);*

(ii) *for every $\beta > 2$ and $a \in \mathbb{A}_{K,f}^*$ there exists a unique probability measure on $\widehat{\mathcal{O}}$ satisfying (3.2) that is concentrated on $\overline{K_+^*}a \cap \widehat{\mathcal{O}}$; this way we get, for every $\beta > 2$, a one-to-one correspondence between extremal probability measures satisfying (3.2) and points of the compact group $\mathbb{A}_{K,f}^*/\overline{K_+^*}$; in particular, the simplex of measures satisfying (3.2) is canonically isomorphic to the simplex of probability measures on $\mathbb{A}_{K,f}^*/\overline{K_+^*}$, and any such measure is concentrated on $\mathbb{A}_{K,f}^* \cap \widehat{\mathcal{O}}$.*

Here $\overline{K_+^*}$ denotes the closure of K_+^* in $\mathbb{A}_{K,f}^*$.

We now apply this theorem to classify measures satisfying (3.1). Clearly, there is a one-to-one correspondence between such measures and $\widehat{\mathcal{O}}^*$ -invariant measures satisfying (3.2).

Let us first consider the case $\beta > 2$. We will need only one conclusion from Theorem 3.1(ii): any measure ν satisfying (3.1) is concentrated on $(\mathbb{A}_{K,f}^* \cap \widehat{\mathcal{O}}) / \widehat{\mathcal{O}}^*$. Recall that $\mathbb{A}_{K,f}^* / \widehat{\mathcal{O}}^*$ can be identified with the group J_K of fractional ideals, while K^* / \mathcal{O}^* is the group of principal fractional ideals. Therefore if h_K is the class number of K , then the action of K^* by multiplication on $\mathbb{A}_{K,f}^* / \widehat{\mathcal{O}}^*$ has exactly h_K orbits. If $\mathfrak{a} \in \mathbb{A}_{K,f}^* / \widehat{\mathcal{O}}^* = J_K$, then obviously there exists at most one probability measure satisfying (3.1) that is concentrated on $K^* \mathfrak{a} \cap \widehat{\mathcal{O}} / \widehat{\mathcal{O}}^*$. Such a measure indeed exists:

$$\nu_{\mathfrak{a},\beta} = \frac{1}{\zeta(\beta - 1, [\mathfrak{a}])} \sum_{\mathfrak{b} \in J_K^+ \cap [\mathfrak{a}]} N_K(\mathfrak{b})^{-(\beta-1)} \delta_{\mathfrak{b}},$$

where $[\mathfrak{a}]$ is the class of \mathfrak{a} in the ideal class group $Cl(K)$, $J_K^+ \subset J_K$ is the sub-semigroup of integral ideals, and $\zeta(\cdot, [\mathfrak{a}])$ is the partial ζ -function defined by $\zeta(s, [\mathfrak{a}]) = \sum_{\mathfrak{b} \in J_K^+ \cap [\mathfrak{a}]} N_K(\mathfrak{b})^{-s}$ for $s > 1$. Since any measure satisfying (3.1) is concentrated on $J_K^+ = (\mathbb{A}_{K,f}^* \cap \widehat{\mathcal{O}}) / \widehat{\mathcal{O}}^*$, we conclude that if $\mathfrak{a}_1, \dots, \mathfrak{a}_{h_K}$ are representatives of different ideal classes, then any probability measure satisfying (3.1) is a unique convex combination of the measures $\nu_{\mathfrak{a}_n, \beta}$, $1 \leq n \leq h_K$.

Let $\mathfrak{a} \in J_K^+$ be a nonzero ideal in \mathcal{O} . Denote by $\mu_{\mathfrak{a},\beta}$ the image of the measure $m \times \nu_{\mathfrak{a},\beta}$ under the projection $\widehat{\mathcal{O}} \times \widehat{\mathcal{O}} / \widehat{\mathcal{O}}^* \rightarrow \Omega_{\mathcal{O}}$. By construction the measure $\mu_{\mathfrak{a},\beta}$ is concentrated on the set

$$\{(r, \mathfrak{b}) : \mathfrak{b} \in J_K^+ \cap [\mathfrak{a}], r \in \widehat{\mathcal{O}} / \mathfrak{b} \widehat{\mathcal{O}} = \mathcal{O} / \mathfrak{b}\} \subset \Omega_{\mathcal{O}}.$$

The partially defined action of $K \rtimes K^*$ on $\Omega_{\mathcal{O}}$ is transitive on this set. Therefore by Corollary 1.4, in order to extend $(\mu_{\mathfrak{a},\beta})_*$ to a KMS_{β} -state we have to choose a tracial state on the C^* -algebra of the stabilizer of one point in this set. We take $(0, \mathfrak{a})$ as such a point. Its stabilizer in $K \rtimes K^*$ is $\mathfrak{a} \rtimes \mathcal{O}^*$. For a tracial state τ on $C^*(\mathfrak{a} \rtimes \mathcal{O}^*)$ denote by $\varphi_{\mathfrak{a},\tau,\beta}$ the corresponding σ - KMS_{β} -state on $\mathfrak{T}[\mathcal{O}]$.

Now pick representatives $\mathfrak{a}_1, \dots, \mathfrak{a}_{h_K}$ of different ideal classes. Any measure μ on $\Omega_{\mathcal{O}}$ with Radon–Nikodym cocycle N_K^{β} is a convex combination of the measures $\mu_{\mathfrak{a}_n, \beta}$. It is concentrated on the set $\{(r, \mathfrak{a}) : \mathfrak{a} \in J_K^+, r \in \mathcal{O} / \mathfrak{a}\} \subset \Omega_{\mathcal{O}}$. The partially defined action of $K \rtimes K^*$ has h_K orbits on this set, and we can take the points $(0, \mathfrak{a}_n)$ as representatives of these orbits. To extend μ_* we need to choose a tracial state on the C^* -algebra of the stabilizer of every point $(0, \mathfrak{a}_n)$ that carries a positive measure. In other words, any σ - KMS_{β} -state on

$\mathfrak{T}[\mathcal{O}]$ is a convex combination $\sum_{n=1}^{h_K} \lambda_n \varphi_{\mathfrak{a}_n, \tau_n, \beta}$, where τ_n is a tracial state on the C^* -algebra $C^*(\mathfrak{a}_n \rtimes \mathcal{O}^*)$. The weights λ_n and the tracial states τ_n define a tracial state $\bigoplus_n \lambda_n \tau_n$ on $\bigoplus_n C^*(\mathfrak{a}_n \rtimes \mathcal{O}^*)$. We therefore get the following result.

THEOREM 3.2 ([2], Theorem 7.3). *Choose representatives $\mathfrak{a}_1, \dots, \mathfrak{a}_{h_K} \in J_K^+$ of different ideal classes. Then, for every $\beta > 2$, there is an affine isomorphism between the simplex of σ -KMS $_{\beta}$ -states on $\mathfrak{T}[\mathcal{O}]$ and the simplex of tracial states on the C^* -algebra*

$$\bigoplus_{n=1}^{h_K} C^*(\mathfrak{a}_n \rtimes \mathcal{O}^*).$$

Note that the extremal KMS $_{\beta}$ -states are states of the form $\varphi_{\mathfrak{a}, \tau, \beta}$, where τ is an extremal tracial state on $C^*(\mathfrak{a} \rtimes \mathcal{O}^*)$. Such a state has type I $_{\infty}$ or II $_{\infty}$ depending on whether τ is of type I or II $_1$.

Let us now turn to the more complicated case $1 \leq \beta \leq 2$. Recall that the unique measure ν_1 satisfying (3.1) for $\beta = 1$ is the delta-measure at $0 \in \widehat{\mathcal{O}}/\widehat{\mathcal{O}}^*$. Assume $1 < \beta \leq 2$. Again it is easy to construct a probability measure satisfying (3.1): take $\nu_{\beta} = \prod_{v \in V_{K,f}} \nu_{\beta, v}$, where the measure $\nu_{\beta, v}$ on $\mathcal{O}_v/\mathcal{O}_v^*$ is defined by

$$\nu_{\beta, v} = (1 - N_K(\mathfrak{p}_v)^{-(\beta-1)}) \sum_{n=0}^{\infty} N_K(\mathfrak{p}_v)^{-(\beta-1)n} \delta_{\mathfrak{p}_v^n},$$

where \mathfrak{p}_v is the prime ideal in \mathcal{O} corresponding to the place v and where we identified $\mathcal{O}_v^{\times}/\mathcal{O}_v^*$ with the sequence $\{\mathfrak{p}_v^n\}_n$. (The measure ν_{β} is of course well-defined for all $\beta > 1$. For $\beta > 2$ we have $\nu_{\beta} = \sum_{n=1}^{h_K} \frac{\zeta(\beta-1, [a_n])}{\zeta_K(\beta-1)} \nu_{\mathfrak{a}_n, \beta}$, where ζ_K is the Dedekind ζ -function.) By Theorem 3.1(i) this is the unique probability measure satisfying (3.1). Note that the ν_{β} -measure of the set $(\mathbb{A}_{K,f}^* \cap \widehat{\mathcal{O}})/\widehat{\mathcal{O}}^*$ is zero, which follows from the divergence to zero of the product $\prod_v (1 - N_K(\mathfrak{p}_v)^{-(\beta-1)})$. It is also clear that the ν_{β} -measure of the set of points $a \in \widehat{\mathcal{O}}/\widehat{\mathcal{O}}^*$ with at least one zero coordinate is zero.

For every $1 \leq \beta \leq 2$, let μ_{β} be the image of the measure $m \times \nu_{\beta}$ under the map $\widehat{\mathcal{O}} \times \widehat{\mathcal{O}}/\widehat{\mathcal{O}}^* \rightarrow \Omega_{\mathcal{O}}$.

LEMMA 3.3. *For $1 \leq \beta \leq 2$ the set of points in $\Omega_{\mathcal{O}}$ with non-trivial stabilizers in $K \rtimes K^*$ has μ_{β} -measure zero.*

Proof. For $\beta = 1$ the measure space $(\Omega_{\mathcal{O}}, \nu_1)$ can be identified with $(\widehat{\mathcal{O}}, m)$, with the partially defined action of $K \rtimes K^*$ given by $(n, k)r = n + kr$. Clearly, every element $g \neq e$ in $K \rtimes K^*$ has at most one fixed point in $\widehat{\mathcal{O}}$, and the measure of every such point is zero.

Assume now that $1 < \beta \leq 2$. We have to show that for every element $g = (n, k) \in K \rtimes K^*$, $g \neq e$, the set of points $(r, a) \in \Omega_{\mathcal{O}}$ fixed by g has μ_{β} -measure zero. In other words, for ν_{β} -a.e. $a \in \widehat{\mathcal{O}}/\widehat{\mathcal{O}}^*$ the set $A_{g,a}$ of points $r \in \widehat{\mathcal{O}}/a\widehat{\mathcal{O}}$ such that $g(r, a) = (r, a)$ has m_a -measure zero, where m_a is the normalized Haar measure on $\widehat{\mathcal{O}}/a\widehat{\mathcal{O}}$. We will show that $m_a(A_{g,a}) = 0$ for every $a = (a_v)_v$ such that $a \notin \mathbb{A}_{K,f}^*/\widehat{\mathcal{O}}^*$ and $a_v \neq 0$ for all v . Since the ν_{β} -measure of the complement of the set of such points a is zero, this will prove the lemma.

The set $A_{g,a}$ can be nonempty only when $ka = a$, whence $k \in \mathcal{O}^*$ as $a_v \neq 0$ for all v by assumption. Then g acts on $\mathbb{A}_{K,f}/a\widehat{\mathcal{O}}$, and the set $A_{g,a}$ consists of points $r \in \widehat{\mathcal{O}}/a\widehat{\mathcal{O}}$ such that $(k-1)r = -n$. If $k = 1$, this means that $A_{g,a}$ is nonempty only when $n \in a\widehat{\mathcal{O}}$. But then $g = (0,1)$, since $K \cap a\widehat{\mathcal{O}} = \{0\}$ by our assumption that $a \notin \mathbb{A}_{K,f}^*$. Since we assumed that $g \neq e$, we conclude that $A_{g,a}$ can be nonempty only when $k \neq 1$. In this case $k-1$ is invertible in \mathcal{O}_v for v outside a finite set $F \subset V_{K,f}$. Hence, if $r = (r_v)_v \in A_{g,a}$, then $r_v \in \mathcal{O}_v/a_v\mathcal{O}_v$ is uniquely determined for $v \notin F$. It follows that

$$m_a(A_{g,a}) \leq \prod_{v \in V_{K,f} \setminus F} |\mathcal{O}_v/a_v\mathcal{O}_v|^{-1}.$$

The latter product diverges to zero, since by assumption there are infinitely many places v such that $a_v \neq \mathcal{O}_v^*$. ■

By Theorem 1.3 and Corollary 1.2 the only way to extend the state μ_{β^*} on $C(\Omega_{\mathcal{O}})$ to a KMS_{β} -state on $\mathfrak{T}[\mathcal{O}]$ is by composing it with the canonical conditional expectation $E: \mathfrak{T}[\mathcal{O}] \rightarrow C(\Omega_{\mathcal{O}})$. Thus we have proved the following result.

THEOREM 3.4 ([2], Theorem 6.7). *For every $1 \leq \beta \leq 2$ there exists a unique σ - KMS_{β} -state on $\mathfrak{T}[\mathcal{O}]$.*

Note that using Corollary 3.2 in [10] and the same arguments as in the proof of Theorem 3.2 in [6], it is easy to show that these KMS -states have type III₁.

We would like to finish by making a few remarks about the uniqueness of measures satisfying (3.1) for $1 < \beta \leq 2$, which is the most non-trivial part in the above analysis. When the field K has class number one, this uniqueness is quite simple and a proof can be obtained using the same arguments as for $K = \mathbb{Q}$. This goes back to [1] and is equivalent to the uniqueness of $\text{KMS}_{\beta-1}$ -states on the symmetric part of the Bost–Connes system for \mathbb{Q} . The situation changes drastically when $h_K > 1$. In this case, as we saw, the uniqueness can be deduced from Theorem 3.1(i). Although an equivalent form of this theorem is explicitly stated in [7], it is based on a result on the Bost–Connes systems established in [4]. Let us briefly describe the arguments in [7] applied to the classification problem of measures satisfying (3.1).

The action of K^* on $\mathbb{A}_{K,f}/\widehat{\mathcal{O}}^*$ defines an action of the group of principal ideals. Induce this action to an action of the whole group J_K of fractional ideals. By a general result on Morita equivalent systems, see Theorem 3.2 in [5], the set of KMS -weights remains the same under induction. In the present case, when everything is formulated in terms of measures, this is quite obvious: a Radon measure with a given Radon–Nikodym cocycle with respect to a group action is completely determined by its restriction to any compact open subset intersecting every orbit. From this it can be deduced that classification of measures satisfying (3.1) is equivalent to classifying probability measures ν on $\text{Cl}(K) \times \widehat{\mathcal{O}}/\widehat{\mathcal{O}}^*$

such that

$$(3.3) \quad \nu(\mathfrak{a}Y) = N_K(\mathfrak{a})^{-(\beta-1)}\nu(Y) \quad \text{for Borel } Y \subset \text{Cl}(K) \times \widehat{\mathcal{O}}/\widehat{\mathcal{O}}^* \text{ and } \mathfrak{a} \in J_K.$$

Here the action of $J_K = \mathbb{A}_{K,f}^*/\widehat{\mathcal{O}}^*$ on $\text{Cl}(K) \times \mathbb{A}_{K,f}/\widehat{\mathcal{O}}^*$ is diagonal, so for every v the element $\mathfrak{p}_v \in J_K$ changes only two coordinates, one corresponds to $\text{Cl}(K)$, the other to the place v .

The Bost–Connes system for K , in turn, is defined using the partially defined diagonal action of J_K on the space

$$Y_K = \text{Gal}(K^{ab}/K) \times_{\widehat{\mathcal{O}}^*} \widehat{\mathcal{O}}.$$

Here $\widehat{\mathcal{O}}^*$ acts on $\text{Gal}(K^{ab}/K)$ via the Artin map $r_K: \mathbb{A}_K^* \rightarrow \text{Gal}(K^{ab}/K)$. We have

$$\text{Gal}(K^{ab}/K)/r_K(K_\infty^*\widehat{\mathcal{O}}^*) = \text{Gal}(H(K)/K) \cong \text{Cl}(K),$$

where $H(K)$ is the Hilbert class field of K . Thus

$$\text{Cl}(K) \times \widehat{\mathcal{O}}/\widehat{\mathcal{O}}^* = Y_K/r_K(K_\infty^*\widehat{\mathcal{O}}^*).$$

Therefore the uniqueness of measures satisfying (3.3), or (3.1), is equivalent to the uniqueness of $r_K(K_\infty^*\widehat{\mathcal{O}}^*)$ -invariant measures on Y_K satisfying the same scaling property. In other words, Theorem 3.4 is essentially equivalent to the uniqueness of $r_K(K_\infty^*\widehat{\mathcal{O}}^*)$ -invariant $\text{KMS}_{\beta-1}$ -states on the Bost–Connes system for K for $1 < \beta \leq 2$. That a $\text{KMS}_{\beta-1}$ -state for the Bost–Connes system is unique for every inverse temperature in this region is proved in Theorem 2.1 in [4].

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REFERENCES

- [1] J.-B. BOST, A. CONNES, Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory, *Selecta Math. (N.S.)* **1**(1995), 411–457.
- [2] J. CUNTZ, C. DENINGER, M. LACA, C^* -algebras of Toeplitz type associated with algebraic number fields, preprint arXiv:1105.5352v1 [math.OA].
- [3] A. KUMJIAN, J. RENAULT, KMS states on C^* -algebras associated to expansive maps, *Proc. Amer. Math. Soc.* **134**(2006), 2067–2078.
- [4] M. LACA, N.S. LARSEN, S. NESHVEYEV, On Bost–Connes type systems for number fields, *J. Number Theory* **129**(2009), 325–338.
- [5] M. LACA, S. NESHVEYEV, KMS states of quasi-free dynamics on Pimsner algebras, *J. Funct. Anal.* **211**(2004), 457–482.
- [6] M. LACA, S. NESHVEYEV, Type III₁ equilibrium states of the Toeplitz algebra of the affine semigroup over the natural numbers, *J. Funct. Anal.* **261**(2011), 169–187.

- [7] M. LACA, S. NESHVEYEV, M. TRIFKOVIĆ, Bost–Connes systems, Hecke algebras, and induction, *J. Noncommut. Geom.* **7**(2013), 525–546.
- [8] M. LACA, I. RAEBURN, Phase transition on the Toeplitz algebra of the affine semi-group over the natural numbers, *Adv. Math.* **225**(2010), 643–688.
- [9] F.J. MURRAY, J. VON NEUMANN, On rings of operators. II, *Trans. Amer. Math. Soc.* **41**(1937), 208–248.
- [10] S. NESHVEYEV, Von Neumann algebras arising from Bost–Connes type systems, *Int. Math. Res. Not.* **1**(2011), 217–236.
- [11] S. NESHVEYEV, Traces on crossed products, preprint arXiv:1010.0600v1 [math.OA].
- [12] J. RENAULT, *A Groupoid Approach to C^* -Algebras*, Lecture Notes in Math., vol. 793, Springer, Berlin 1980.
- [13] J. RENAULT, Représentation des produits croisés d’algèbres de groupoïdes, *J. Operator Theory* **18**(1987), 67–97.
- [14] M. TAKESAKI, *Theory of Operator Algebras*. I, Encyclopaedia Math. Sci., vol. 124, Operator Algebras and Non-commutative Geometry, vol. 5, Springer-Verlag, Berlin 2002.

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