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C^*-ALGEBRAS WITH THE WEAK EXPECTATION PROPERTY AND A MULTIVARIABLE ANALOGUE OF ANDO'S THEOREM ON THE NUMERICAL RADIUS

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ABSTRACT. A classic theorem of T. Ando characterises operators that have numerical radius at most one as operators that admit a certain positive 2×2 operator matrix completion. In this paper we consider variants of Ando's theorem in which the operators (and matrix completions) are constrained to a given C^*-algebra. By considering $n \times n$ matrix completions, an extension of Ando's theorem to a multivariable setting is made. We show that the C^*-algebras in which these extended formulations of Ando's theorem hold true are precisely the C^*-algebras with the weak expectation property (WEP). We also show that a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ has WEP if and only if whenever a certain 3×3 (operator) matrix completion problem can be solved in matrices over $\mathcal{B}(\mathcal{H})$, it can also be solved in matrices over $\mathcal{A}$. This last result gives a characterisation of WEP that is spatial and yet is independent of the particular representation of the C^*-algebra. This leads to a new characterisation of injective von Neumann algebras. We also give a new equivalent formulation of the Connes embedding problem as a problem concerning 3×3 matrix completions.

KEYWORDS: *Weak expectation property, Ando's theorem, the Connes embedding problem, numerical radius, operator system quotient, operator system tensor product.*

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1. INTRODUCTION

The *numerical radius* of a bounded linear operator X acting on a Hilbert space $\mathcal{H}$ is the quantity $w(X)$ defined by

$$w(X) = \sup\{|\langle X\xi, \xi\rangle| : \xi \in \mathcal{H}, \|\xi\| = 1\}.$$

A classic theorem of Ando [2] gives a matricial positivity characterisation of the numerical radius of an operator: $w(X) \leqslant \frac{1}{2}$ if and only if there exist positive operators $A, B \in \mathcal{B}(\mathcal{H})$ such that $A + B = I \in \mathcal{B}(\mathcal{H})$ and $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$ is a positive operator on $\mathcal{H} \oplus \mathcal{H}$. With a little rescaling, one can see that Ando's theorem is equivalent to the statement that $w(X) < \frac{1}{2}$ if and only if there exist positive invertible operators $A, B \in \mathcal{B}(\mathcal{H})$ with $A + B = I$ such that the 2×2 operator matrix above is positive and invertible.

It is natural to wonder if this theorem remains true when $\mathcal{B}(\mathcal{H})$ is replaced by an arbitrary unital C^*-algebra. The answer is yes, as long as one requires that the inequality be strict. Thus, we have the following minor improvement of Ando's theorem, where, in its formulation below, $\mathcal{A}_+$ and $\mathcal{A}_+^{-1}$ denote the set of positive elements and the group of positive invertible elements of a unital C^*-algebra $\mathcal{A}$ respectively, and where the numerical radius $w(x)$ of $x \in \mathcal{A}$ is the maximum value of $|s(x)|$ as s ranges through all states of $\mathcal{A}$.

THEOREM 1.1. *Assume that $\mathcal{A}$ is any unital C^*-algebra. The following statements are equivalent for $x \in \mathcal{A}$:*

(i) $w(x) < \frac{1}{2}$;

(ii) *for every unitary $v \in \mathcal{B}$ in every unital C^*-algebra $\mathcal{B}$, the element $x \otimes v \in \mathcal{A} \otimes_{\min} \mathcal{B}$ satisfies $w(x \otimes v) < \frac{1}{2}$;*

(iii) *for every unital C^*-algebra $\mathcal{B}$,*

$$1_{\mathcal{A}} \otimes 1_{\mathcal{B}} + x \otimes v + x^* \otimes v^* \in (\mathcal{A} \otimes_{\min} \mathcal{B})_+^{-1}$$

for every unitary $v \in \mathcal{B}$;

(iv) $1_{\mathcal{A}} \otimes 1_{\mathcal{B}} + x \otimes u + x^* \otimes u^* \in (\mathcal{A} \otimes_{\min} \mathcal{B})_+^{-1}$, *where $\mathcal{B}$ is the universal C^*-algebra generated by a (universal) unitary u;*

(v) $1 + zx + \bar{z}x^* \in \mathcal{A}_+^{-1}$ *for every $z \in \mathbb{T}$;*

(vi) *there exist $a, b \in \mathcal{A}_+^{-1}$ such that, with $a + b = 1$,*

$$\begin{bmatrix} a & x \\ x^* & b \end{bmatrix} \in \mathcal{M}_2(\mathcal{A})_+.$$

Surprisingly, this slight extension of Ando's theorem is logically equivalent to the assertion that $C(\mathbb{T})$ is a nuclear C^*-algebra (Theorem 5.1(i)).

It is also natural to wonder if it is possible to formulate Ando's theorem for a greater number of variables. Since the numerical radius of an operator remains unchanged when one tensors with a unitary, we have been led to the following definition.

DEFINITION 1.2. The *free joint numerical radius* $w(X_1, \ldots, X_n)$ of n operators $X_1, \ldots, X_n \in \mathcal{B}(\mathcal{H})$ is

$$w(X_1, \ldots, X_n) = \sup\{w(X_1 \otimes U_1 + \cdots + X_n \otimes U_n)\},$$

where the supremum is taken over every Hilbert space $\mathcal{K}$, every choice of n unitaries $U_1, \ldots, U_n \in \mathcal{B}(\mathcal{K})$, and the tensor product is spatial.

We obtain a characterisation (Theorem 3.4) of n tuples of operators with $w(X_1, \ldots, X_n) < \frac{1}{2}$ in terms of matrix positivity, which is a natural extension of Ando's theorem. What is perhaps most surprising is that when one asks whether the entries of this operator matrix can be chosen from a given C^*-algebra $\mathcal{A}$ containing $X_1, \ldots, X_n$, we find that the extension of Ando's result to two or more variables holds if and only if the C^*-algebra $\mathcal{A}$ has the weak expectation property (WEP).

In this manner, we obtain a characterisation of the WEP property for a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ that is in the spirit of completion problems. That is, we prove in Theorem 6.1 that a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ has WEP if and only if whenever a certain 3×3 operator completion problem can be solved in $\mathcal{B}(\mathcal{H})$, then it can be solved with entries from the C^*-algebra.

This characterisation of WEP is independent of the particular faithful representation of the C^*-algebra on a Hilbert space. In this regard Theorem 6.1 departs from the original definition of WEP, which requires that *every* faithful representation of the given C^*-algebra admits a weak expectation into its double commutant. This requirement in the original definition of WEP is crucial, as every C^*-algebra has at least one faithful representation which has a weak expectation into its double commutant.

Since a von Neumann algebra has WEP if and only if it is injective, our results also give a characterisation of injective von Neumann algebras in terms of our 3×3 completion property (Corollary 6.2). In particular, this shows that the operators that solve our 3×3 completion problem for a given unital C^*-subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ without WEP will generally not even be found within the double commutant of $\mathcal{A}$.

Finally, by Kirchberg's results [10], we find in Theorem 7.1 that Connes' embedding problem is equivalent to whether or not $C^*(\mathbb{F}_2)$ has our 3×3 completion property. This might represent some progress on this conjecture, since, as noted already, it is sufficient to check our property for any chosen faithful representation.

2. OPERATOR SYSTEMS

If $\psi : \mathcal{R} \to \mathcal{T}$ is a linear map of operator systems, then for each $p \in \mathbb{N}$ the linear map $\psi^{(p)} : \mathcal{M}_p(\mathcal{R}) \to \mathcal{M}_p(\mathcal{T})$ is defined by $\psi^{(p)}([x_{ij}]_{i,j}) = [\psi(x_{ij})]_{ij}$. The positive matricial cones of an operator system $\mathcal{R}$ are denoted by $\mathcal{M}_p(\mathcal{R})_+$, and the order unit of $\mathcal{R}$ is denoted by $1_{\mathcal{R}}$ or, if there is little chance of ambiguity, by 1.

Two (classes of) operator systems have a prominent role in what follows. The first, $\mathcal{T}_n$, is an operator subsystem of the $n \times n$ complex matrix algebra $\mathcal{M}_n$. If $\{e_{ij}\}_{1 \leqslant i,j \leqslant n}$ denotes the set of standard matrix units of $\mathcal{M}_n$, then

$$\mathcal{T}_n = \mathrm{Span}\{e_{ij} : |i-j| \leqslant 1\}$$

is the operator system of tridiagonal matrices.

The second operator system of interest is denoted by $\mathcal{S}_n$. Assume that $\mathbb{F}_\infty$ is the free group on countably many generators $u_1, u_2, \dots$ and let $\mathbb{F}_n \subset \mathbb{F}_\infty$ be the free group on n generators $u_1, \dots, u_n$. In the group C^*-algebra $C^*(\mathbb{F}_\infty)$ these generators u_j are (universal) unitaries. For a fixed $n \in \mathbb{N}$, consider the operator subsystem $\mathcal{S}_n \subset C^*(\mathbb{F}_n)$ defined by

$$\mathcal{S}_n = \mathrm{Span}\{u_{-n}, u_{-n+1}, \dots, u_{-1}, u_0, u_1, \dots, u_{n-1}, u_n\},$$

where $u_0 = 1$ and $u_{-k} = u_k^*$ for $1 \leqslant k \leqslant n$.

The relationship between the two types of operator systems described above is as follows. Let $n \geqslant 2$ and let $\phi : \mathcal{T}_n \to \mathcal{S}_{n-1}$ be the linear map defined by

$$\phi(e_{ii}) = \frac{1}{n} u_0, \quad \phi(e_{j,j+1}) = \frac{1}{n} u_j, \quad \text{and} \quad \phi(e_{j+1,j}) = \frac{1}{n} u_{-j}$$

for $i = 1, \dots, n$ and $j = 1, \dots, n-1$. The map ϕ is ucp ([6], Theorem 4.2), but the most important fact about ϕ is that it is a *complete quotient map*, which we explain below.

Recall from [9] that if $\psi : \mathcal{R} \to \mathcal{T}$ is a surjective completely positive linear map of operator systems with kernel $\mathcal{J}$, then there is an operator system structure on the quotient vector space $\mathcal{R}/\mathcal{J}$ and there exists a completely positive linear isomorphism $\dot{\psi} : \mathcal{R}/\mathcal{J} \to \mathcal{T}$ such that $\psi = \dot{\psi} \circ q$, where $q : \mathcal{R} \to \mathcal{R}/\mathcal{J}$ is the canonical quotient map. If the completely positive linear isomorphism $\dot{\psi}$ is a complete order isomorphism (that is, if $\dot{\psi}^{-1}$ is completely positive), then ψ is said to be a complete quotient map.

Returning to the map $\phi : \mathcal{T}_n \to \mathcal{S}_{n-1}$ above, we have from Theorem 4.2 of [6] that ϕ is a complete quotient map; that is, the operator systems $\mathcal{T}_n/\mathcal{J}$, where $\mathcal{J} = \ker(\phi)$, and $\mathcal{S}_{n-1}$ are completely order isomorphic. The importance of this fact concerning ϕ is that strictly positive elements in the matrix space $\mathcal{M}_p(\mathcal{S}_{n-1})$ lift back to strictly positive elements in $\mathcal{M}_p(\mathcal{T}_n)$ for every $p \in \mathbb{N}$ (Proposition 3.2). This fact, when applied while tensoring these operator systems with a C^*-algebra, is at the heart of our matrix completion perspective.

The fundamental results for a theory of operator system tensor products are developed in [8], [9]. An operator system tensor product $\mathcal{R} \otimes_\tau \mathcal{T}$ is an operator system structure on the algebraic tensor product $\mathcal{R} \otimes \mathcal{T}$ satisfying a set of natural axioms. Given two operator system tensor product structures $\mathcal{R} \otimes_{\tau_1} \mathcal{T}$ and $\mathcal{R} \otimes_{\tau_2} \mathcal{T}$ on $\mathcal{R} \otimes \mathcal{T}$, we of course have equality of $\mathcal{R} \otimes_{\tau_1} \mathcal{T}$ and $\mathcal{R} \otimes_{\tau_2} \mathcal{T}$ as sets; however, we use the notation $\mathcal{R} \otimes_{\tau_1} \mathcal{T}_1 = \mathcal{R} \otimes_{\tau_2} \mathcal{T}_1$ to indicate that the identity map is a complete order isomorphism. Given two inclusions, $\mathcal{R}_1 \subseteq \mathcal{R}_2$ and $\mathcal{T}_1 \subseteq \mathcal{T}_2$ and operator system tensor product structures $\mathcal{R}_1 \otimes_{\tau_1} \mathcal{T}_1$ and $\mathcal{R}_2 \otimes_{\tau_2} \mathcal{T}_2$, we use the notation $\mathcal{R}_1 \otimes_{\tau_1} \mathcal{T}_1 \subset_{\mathrm{coi}} \mathcal{R}_2 \otimes_{\tau_2} \mathcal{T}_2$ to indicate that the tensor product of the two inclusion maps is a complete order isomorphism onto its range.

Of particular interest here are the tensor products $\otimes_{\min}$, $\otimes_{\mathrm{c}}$, and $\otimes_{\max}$ [8]. In this case, we have that the tensor products of the identity maps are ucp as maps from $\mathcal{R} \otimes_{\max} \mathcal{T}$ to $\mathcal{R} \otimes_{\mathrm{c}} \mathcal{T}$ and from $\mathcal{R} \otimes_{\mathrm{c}} \mathcal{T}$ to $\mathcal{R} \otimes_{\min} \mathcal{T}$ for all operator systems

$\mathcal{R}, \mathcal{T}$. Indeed, the matricial cones associated with $\mathcal{R} \otimes_{\max} \mathcal{T}$ lie within the matricial cones of any operator system structure $\mathcal{R} \otimes_{\tau} \mathcal{T}$, while the matricial cones of any $\mathcal{R} \otimes_{\tau} \mathcal{T}$ are contained in the corresponding matricial cones of $\mathcal{R} \otimes_{\min} \mathcal{T}$.

We identify $\mathcal{M}_p(\mathcal{R} \otimes_{\min} \mathcal{T})$ with $\mathcal{M}_p(\mathcal{R}) \otimes_{\min} \mathcal{T}$, for any operator systems $\mathcal{R}$ and $\mathcal{T}$.

When considering operator system tensor products of unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, the identity map of the operator system $\mathcal{A} \otimes_{\min} \mathcal{B}$ into the C^*-algebra tensor product $\mathcal{A} \otimes_{\min} \mathcal{B}$ is completely positive ([8], Corollary 4.10). The analogous statement for the $\otimes_{\max}$ tensor product also holds ([8], Theorem 5.12). Because of the norm-order duality in operator systems ([5], Section 4 and [11], Proposition 13.3), we have that $\mathcal{A} \otimes_{\min} \mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}$ as operator systems if and only if the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$ has a unique C^*-norm if and only if $\mathcal{A} \otimes_{\min} \mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}$ as C^*-algebras. For this reason, the equation $\mathcal{A} \otimes_{\min} \mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}$ can be treated unambiguously as a statement about operator systems or as a statement about C^*-algebras.

The following lifting property [6] for positive matrices over operator systems is a key feature of our approach. An operator system $\mathcal{R}$ is said to have *property* $(\mathfrak{S}_n)$ for a fixed $n \in \mathbb{N}$ if, for every $p \in \mathbb{N}$, every $\varepsilon > 0$, and every positive

$$\sum_{i=1-n}^{n-1} S_i \otimes u_i \in \mathcal{M}_p(\mathcal{R}) \otimes_{\min} C^*(\mathbb{F}_{n-1}),$$

there exist $R_{ij}^{\varepsilon} \in \mathcal{M}_p(\mathcal{R})$, for $1 \leqslant i, j \leqslant n$, such that:

(i) $R_{\varepsilon} = [R_{ij}^{\varepsilon}]_{1 \leqslant i,j \leqslant n}$ is positive in $\mathcal{M}_n(\mathcal{M}_p(\mathcal{R}))$,

(ii) $R_{ij}^{\varepsilon} = 0$ for all $|i-j| \geqslant 2$, $R_{i,i+1}^{\varepsilon} = S_i$, and $R_{i+1,i}^{\varepsilon} = S_{-i}$ for all i, and

(iii) $\sum_{i=1}^{n} R_{ii}^{\varepsilon} = S_0 + \varepsilon 1_{\mathcal{M}_p(\mathcal{R})}$.

To conclude this section, we connect the notions described above using the operator systems $\mathcal{T}_n$ and $\mathcal{S}_{n-1}$, the linear map ϕ that links them, and property $(\mathfrak{S}_n)$.

THEOREM 2.1 ([6]). *Assume that $n \geqslant 2$ and $\mathcal{R}$ is an arbitrary operator system.*

(i) *The map* $\mathrm{id}_{\mathcal{R}} \otimes \phi : \mathcal{R} \otimes_{\max} \mathcal{T}_n \to \mathcal{R} \otimes_{\max} \mathcal{S}_{n-1}$ *is a complete quotient map.*

(ii) *The map* $\mathrm{id}_{\mathcal{R}} \otimes \phi : \mathcal{R} \otimes_{\min} \mathcal{T}_n \to \mathcal{R} \otimes_{\min} \mathcal{S}_{n-1}$ *is a complete quotient map if and only if $\mathcal{R}$ has property $(\mathfrak{S}_n)$.*

3. PROPERTY $(\mathfrak{S}_n)$ AND A MULTIVARIABLE ANALOGUE OF ANDO'S THEOREM

In this section we use the fact that $\mathcal{B}(\mathcal{H})$ has property $(\mathfrak{S}_n)$ ([6], Theorem 6.2) to derive a multivariable analogue of Ando's theorem.

DEFINITION 3.1. If $\mathcal{R}$ is an operator system with order unit $1_{\mathcal{R}}$, then $s \in \mathcal{R}$ is *strictly positive* if there is a real number $\delta > 0$ such that $s \geqslant \delta 1_{\mathcal{R}}$.

This terminology is not entirely standard. It is not hard to see that an element $s \in \mathcal{R}$ is strictly positive if and only if for every unital C^*-algebra $\mathcal{A}$ and every ucp map $\psi : \mathcal{R} \to \mathcal{A}$ we have that $\psi(s)$ is positive and invertible. Thus, in our terminology $P \in \mathcal{B}(\mathcal{H})$ is strictly positive if and only if P is positive and invertible.

PROPOSITION 3.2. *The following statements are equivalent for a* ucp *surjection* $\psi : \mathcal{R} \to \mathcal{T}$ *of operator systems:*

(i) *for every* $p \in \mathbb{N}$ *and every strictly positive* $y \in \mathcal{M}_p(\mathcal{T})$ *there is a strictly positive* $x \in \mathcal{M}_p(\mathcal{R})$ *such that* $\psi^{(p)}(x) = y$*;*

(ii) ψ *is a complete quotient map.*

Proof. The proof in the case of general $p \in \mathbb{N}$ is no different than the proof in the case $p = 1$, and so we settle on this case for simplicity of notation.

(i) $\Rightarrow$ (ii) Assume that $y \in \mathcal{T}$ is strictly positive. Let $\dot{x} \in (\mathcal{R}/\ker\psi)$ be the unique preimage of y under $\dot{\psi}$; note that $\psi(x) = y$.

We aim to show that $\dot{x}$ is positive. By definition of positivity in the quotient ([9], Section 3), we are to show that for every $\varepsilon > 0$ there is a $k_\varepsilon \in \ker\psi$ such that $\varepsilon 1_{\mathcal{R}} + x + k_\varepsilon \in \mathcal{R}_+$.

Fix $\varepsilon > 0$. Because $y + \varepsilon 1_{\mathcal{T}}$ is strictly positive, there is a strictly positive $x_\varepsilon \in \mathcal{R}$ such that $\psi(x_\varepsilon) = y + \varepsilon 1_{\mathcal{T}}$. Thus, if $k_\varepsilon = x_\varepsilon - (x + \varepsilon 1_{\mathcal{R}})$, then $k_\varepsilon \in \ker\psi$ and $\varepsilon 1_{\mathcal{R}} + x + k_\varepsilon = x_\varepsilon$ is stricly positive in $\mathcal{R}$. Hence, $\dot{x}$ is positive.

(ii) $\Rightarrow$ (i) Assume that $a \in \mathcal{T}$ is strictly positive: $a \geqslant \delta 1_{\mathcal{T}}$ for some $\delta > 0$. Thus, $y = a - \delta 1_{\mathcal{T}} \in \mathcal{T}_+$ and so $z = y + \frac{\delta}{2} 1_{\mathcal{T}} \in \mathcal{T}_+$.

By hypothesis, $\dot{\psi}$ is a complete order isomorphism and so $z = \dot{\psi}(\dot{h})$ for some positive $\dot{h} \in (\mathcal{R}/\ker\psi)$. By definition of positivity in the quotient, there is a $k \in \ker\psi$ such that $\frac{\delta}{4} + h + k \in \mathcal{R}_+$. Let $q = \frac{\delta}{2} + h + k$; thus, $q \geqslant \frac{\delta}{4} 1_{\mathcal{R}}$ and $\psi(q) = \dot{\psi}(\dot{q}) = \frac{\delta}{2} 1_{\mathcal{T}} + z = a$. ∎

Note that for the implication (i) $\Rightarrow$ (ii) in Proposition 3.2 above, it is enough to check that strictly positive elements have positive lifts. Also, by adding and subtracting fractions of δ, a positive lift of a strictly positive element can be replaced by a strictly positive lift.

We now apply Proposition 3.2 to obtain a characterisation of property $(\mathfrak{S}_n)$ in terms of a strictly positive matrix completion condition: namely, in the formulation below, for every $p \in \mathbb{N}$, variables $x_1, \ldots, x_{n-1} \in \mathcal{M}_p(\mathcal{A})$ determine a strictly positive element (3.1) of $\mathcal{M}_p(\mathcal{A}) \otimes_{\min} C^*(\mathbb{F}_{n-1})$ if and only if the partially specified hermitian tridiagonal matrix $\widetilde{T}$ with superdiagonal given by $x_1, \ldots, x_{n-1}$ and diagonal entries unspecified can be completed to a strictly positive matrix T in $\mathcal{M}_{pn}(\mathcal{A})$ with diagonal entries that sum to the identity of $\mathcal{M}_p(\mathcal{A})$.

THEOREM 3.3. *For a fixed* $n \geqslant 2$*, the following statements are equivalent for a unital* C^**-algebra* $\mathcal{A}$*:*

(i) $\mathcal{A}$ *has property* $(\mathfrak{S}_n)$*;*

(ii) *for every* $p \in \mathbb{N}$ *and every* $x_1, \ldots, x_{n-1} \in \mathcal{M}_p(\mathcal{A})$ *for which*

$$1 \otimes 1 + \sum_{j=1}^{n-1} (x_j \otimes u_j) + \sum_{j=1}^{n-1} (x_j^* \otimes u_j^*) \tag{3.1}$$

is strictly positive in $\mathcal{M}_p(\mathcal{A}) \otimes_{\min} C^*(\mathbb{F}_{n-1})$, *the matrix*

$$\begin{bmatrix} a_1 & x_1 & 0 & \cdots & 0 \\ x_1^* & a_2 & x_2 & & \vdots \\ 0 & x_2^* & \ddots & \ddots & 0 \\ \vdots & & \ddots & a_{n-1} & x_{n-1} \\ 0 & \cdots & 0 & x_{n-1}^* & a_n \end{bmatrix} \tag{3.2}$$

is strictly positive in $\mathcal{M}_{pn}(\mathcal{A})$ *for some* $a_1, \ldots, a_{n-1} \in \mathcal{M}_p(\mathcal{A})$ *such that* $a_1 + a_2 + \cdots + a_n = 1 \in \mathcal{M}_p(\mathcal{A})$.

Proof. (i) $\Rightarrow$ (ii) Because $\mathcal{A}$ has property $(\mathfrak{S}_n)$, Theorem 2.1 implies that the linear map $\mathrm{id}_{\mathcal{M}_p(\mathcal{A})} \otimes \phi : \mathcal{M}_p(\mathcal{A}) \otimes_{\min} \mathcal{T}_n \to \mathcal{M}_p(\mathcal{A}) \otimes_{\min} \mathcal{S}_{n-1}$ is a complete quotient map. Hence, if $z = 1 \otimes 1 + \sum_{j=1}^{n-1} (x_j \otimes u_j) + \sum_{j=1}^{n-1} (x_j^* \otimes u_{-j})$ is strictly positive in $\mathcal{A} \otimes_{\min} C^*(\mathbb{F}_{n-1})$, then there is a strictly positive lift $y \in \mathcal{M}_p(\mathcal{A}) \otimes_{\min} \mathcal{T}_n$ with $[\mathrm{id}_{\mathcal{M}_p(\mathcal{A})} \otimes \phi](y) = z$, by Proposition 3.2. The element y necessarily has the form

$$y = \sum_{j=1} h_j \otimes e_{jj} + \sum_{i=1}^{n-1} k_i \otimes e_{i,i+1} + \sum_{i=1}^{n-1} k_i^* \otimes e_{i+1,i},$$

for some $k_1, \ldots, k_{n-1} \in \mathcal{M}_p(\mathcal{A})$ and (strictly) positive $h_1, \ldots, h_n \in \mathcal{M}_p(\mathcal{A})$. Let $\delta > 0$ be such that $\delta 1 \leqslant y$. Thus,

$$\begin{aligned} \delta(1 \otimes 1) \leqslant z &= \sum_{j=1} h_j \otimes \Big(\frac{1}{n} 1\Big) + \sum_{i=1}^{n-1} k_i \otimes \Big(\frac{1}{n} u_i\Big) + \sum_{i=1}^{n-1} k_i^* \otimes \Big(\frac{1}{n} u_i\Big) \\ &= \Big(\sum_{j=1} \frac{1}{n} h_j\Big) \otimes 1 + \sum_{i=1}^{n-1} \Big(\frac{1}{n} k_i\Big) \otimes u_i + \sum_{i=1}^{n-1} \Big(\frac{1}{n} k_i^*\Big) \otimes u_i. \end{aligned}$$

The linear independence of $u_{1-n}, \ldots, u_0, \ldots, u_{n-1}$ implies that z is the image of y if and only if $k_i = n x_i$ for $1 \leqslant i \leqslant n$ and $\sum_{j=1}^{n} \frac{1}{n} h_j = 1$. Now since y is strictly positive, so is $\frac{1}{n} y$. Therefore, if $a_j = \frac{1}{n} h_j$ for each j, then $a_1 + \cdots + a_n = 1$ and the strictly positive element $\frac{1}{n} y$ is given by the matrix (3.2).

(ii) $\Rightarrow$ (i) By Theorem 2.1 and Proposition 3.2, to show that $\mathcal{A}$ has property $(\mathfrak{S}_n)$ it is enough to show that for every strictly positive $g \in \mathcal{M}_p(\mathcal{A}) \otimes_{\min} \mathcal{S}_{n-1}$ there exists a strictly positive $h \in \mathcal{M}_p(\mathcal{A}) \otimes_{\min} \mathcal{T}_n$ such that $[\mathrm{id}_{\mathcal{M}_p(\mathcal{A})} \otimes \phi](h) = g$.

Assuming that $g \in \mathcal{M}_p(\mathcal{A}) \otimes_{\min} \mathcal{S}_{n-1}$ is strictly positive, there exist $\delta > 0$ and $x_k \in \mathcal{M}_p(\mathcal{A})$ such that

$$\delta(1 \otimes 1) \leqslant g = x_0 \otimes 1 + \sum_{j=1}^{n-1} (x_j \otimes u_j) + \sum_{j=1}^{n-1} (x_j^* \otimes u_{-j}).$$

If α is an arbitrary state on $\mathcal{M}_p(\mathcal{A})$ and β is a state on $C^*(\mathbb{F}_{n-1})$ such that $\beta(u_k) = 0$ for all $k \neq 0$, then $\delta \leqslant \alpha \otimes \beta(g) = \alpha(x_0)$. Thus, x_0 is strictly positive. Thus, $z = (x_0^{-1/2} \otimes 1) g (x_0^{-1/2} \otimes 1)$ is strictly positive and has the form (3.1); therefore, by hypothesis, z has a strictly positive lift some strictly positive $y \in \mathcal{M}_p(\mathcal{A}) \otimes \mathcal{T}_n$. The proof of the implication (i) $\Rightarrow$ (ii) shows how the entries of the matrix y are determined by those of the matrix z. But since $x_0^{-1/2} x_i x_0^{-1/2}$ are the tensor factors of u_i in the expansion of z as a sum of elementary tensors, one sees immediately (using the computations in (i) $\Rightarrow$ (ii) that $[\mathrm{id}_{\mathcal{M}_p(\mathcal{A})} \otimes \phi](h) = g$ for some strictly positive $h \in \mathcal{M}_p(\mathcal{A}) \otimes_{\min} \mathcal{T}_n$. ∎

We can now state our multivariable analogue of Ando's theorem.

THEOREM 3.4. *Let $X_1, \ldots, X_{n-1} \in \mathcal{B}(\mathcal{H})$. Then $w(X_1, \ldots, X_{n-1}) < \frac{1}{2}$ if and only if there exist $A_1, \ldots, A_n \in \mathcal{B}(\mathcal{H})_+^{-1}$ with $A_1 + \cdots + A_n = I$ such that the following is positive and invertible in $M_n(\mathcal{B}(\mathcal{H})) = \mathcal{B}(\mathcal{H}^{(n)})$:*

$$\begin{bmatrix} A_1 & X_1 & 0 & \cdots & 0 \\ X_1^* & A_2 & X_2 & & \vdots \\ 0 & X_2^* & \ddots & \ddots & 0 \\ \vdots & & \ddots & A_{n-1} & X_{n-1} \\ 0 & \cdots & 0 & X_{n-1}^* & A_n \end{bmatrix}. \tag{3.3}$$

Proof. Since $\mathcal{B}(\mathcal{H})$ is injective it has WEP and thus, by Theorem 6.2 of [6], $\mathcal{B}(\mathcal{H})$ has property $(\mathfrak{S}_n)$ for all n. (Alternatively, Proposition 3.5 of [6] gives a direct proof that $\mathcal{B}(\mathcal{H})$ has the lifting property $(\mathfrak{W})$, which is easily seen to imply the lifting property $(\mathfrak{S}_n)$.)

We have that $w(X_1 \otimes U_1 + \cdots + X_{n-1} \otimes U_{n-1}) < \frac{1}{2}$ for all unitaries if and only if $1 \otimes 1 + \sum_{j=1}^{n-1} (X_j \otimes u_j) + \sum_{j=1}^{n-1} (X_j \otimes u_j)^*$ is strictly positive in $\mathcal{B}(\mathcal{H}) \otimes_{\min} C^*(\mathbb{F}_{n-1})$. Thus, by applying Theorem 3.3 we have the desired lifting.

Conversely, assume that, for a given sequence $X_1, \ldots, X_{n-1} \in \mathcal{B}(\mathcal{H})$, there exist operators $A_1, \ldots, A_n$ satisfying the conditions above. Tensoring the complete quotient map $\phi : \mathcal{T}_n \to \mathcal{S}_{n-1}$ with the identity map on $\mathcal{B}(\mathcal{H})$ yields a ucp map $\mathrm{id}_{\mathcal{B}(\mathcal{H})} \otimes \phi : \mathcal{B}(\mathcal{H}) \otimes_{\min} \mathcal{T}_n \to \mathcal{B}(\mathcal{H}) \otimes_{\min} \mathcal{S}_{n-1}$. The image of the operator matrix (3.3) under this map is

$$\Big(\sum_{j=1}^{n} A_j \Big) \otimes 1 + \sum_{j=1}^{n-1} (X_j \otimes u_j + X_j^* \otimes u_j^*) = I \otimes 1 + \sum_{j=1}^{n-1} (X_j \otimes u_j + X_j^* \otimes u_j^*),$$

which will be strictly positive. Hence, $w(X_1 \otimes U_1 + \cdots + X_{n-1} \otimes U_{n-i}) < \frac{1}{2}$, for any set of unitaries, $U_1, \ldots, U_{n-1}$. ∎

The definition of the free joint numerical radius involves a supremum, but Theorem 3.4 allows us to also characterise it as an infimum.

COROLLARY 3.5. *If $X_1, \ldots, X_{n-1} \in \mathcal{B}(\mathcal{H})$, then*

$$w(X_1, \ldots, X_{n-1}) = \inf\{\|A_1 + \cdots + A_n\|\},$$

where the infimum is taken over all sets of operators $A_1, \ldots, A_n \in \mathcal{B}(\mathcal{H})$ for which the operator matrix (3.3) *is positive in $\mathcal{B}(\mathcal{H}^{(n)})$.*

When $n = 2$ we recover the original version of Ando's theorem. Thus, we see that Ando's theorem can be derived as a consequence of the fact that $\mathcal{T}_n \to \mathcal{S}_{n-1}$ is a complete quotient map and that $\mathcal{B}(\mathcal{H})$ has the lifting property $(\mathfrak{S}_n)$.

4. PROPERTY $(\mathfrak{S}_n)$ FOR C^*-ALGEBRAS

If, in the universal representation $\mathcal{A} \subset \mathcal{B}(\mathcal{H}_u)$ of a unital C^*-algebra $\mathcal{A}$, there exists a ucp map $\Phi : \mathcal{B}(\mathcal{H}_u) \to \mathcal{A}^{**}$ such that $\Phi(a) = a$ for all $a \in A \subset A^{**} \subset \mathcal{B}(\mathcal{H}_u)$, then $\mathcal{A}$ is said to have the *weak expectation property* (WEP). Equivalently, $\mathcal{A}$ has WEP if for every faithful representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\pi)$ there is a ucp $\Phi_\pi : \mathcal{B}(\mathcal{H}_\pi) \to \pi(\mathcal{A})''$ such that $\Phi(\pi(a)) = \pi(a)$ for all $a \in \pi(A) \subset \pi(A)'' \subset \mathcal{B}(\mathcal{H}_\pi)$.

An important feature of the C^*-algebras of free groups is that they may be used to detect C^*-algebras with WEP. This is achieved through an important theorem of Kirchberg ([10], Proposition 1.1(iii)): a C^*-algebra has WEP if and only if there is a unique C^*-norm on the algebraic tensor product $\mathcal{A} \otimes C^*(\mathbb{F}_\infty)$. We prove below (in Theorem 4.3) that individual operator systems $\mathcal{S}_n$, assuming n is at least 3, also detect C^*-algebras with WEP.

The main results of this section can be derived as consequences of results that appear in the thesis of the second author. See especially Section 5 of [7]. We give an independent derivation of these facts below.

We begin with two preliminary lemmas.

LEMMA 4.1. *$\mathcal{R} \otimes_{\mathrm{c}} \mathcal{S}_n \subset_{\mathrm{coi}} \mathcal{R} \otimes_{\mathrm{c}} C^*(\mathbb{F}_n)$ for every $n \in \mathbb{N}$ and every operator system $\mathcal{R}$.*

Proof. By definition of $\otimes_{\mathrm{c}}$ ([8], Section 6), it suffices to show that, for every pair of ucp maps $\phi : \mathcal{R} \to \mathcal{B}(\mathcal{H})$ and $\psi : \mathcal{S}_n \to \mathcal{B}(\mathcal{H})$ with communing ranges, there is a ucp extension $\widetilde{\psi} : C^*(\mathbb{F}_n) \to \mathcal{B}(\mathcal{H})$ of ψ such that $\widetilde{\psi}$ and ϕ have commuting ranges.

To this end, let ϕ and ψ be such a pair. Dilate each contraction $\psi(u_i)$ to a unitary $w_i \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$:

$$w_i = \begin{bmatrix} \psi(u_i) & (1-\psi(u_i)\psi(u_{-i}))^{1/2} \\ (1-\psi(u_{-i})\psi(u_i))^{1/2} & -\psi(u_{-i}) \end{bmatrix}.$$

Let $\widetilde{\phi} : \mathcal{R} \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ be given by $\widetilde{\phi}(r) = \phi(r) \oplus \phi(r)$. Because $\phi(r)\psi(u_i) = \psi(u_i)\phi(r)$ for all $-n \leqslant i \leqslant n$, functional calculus yields $\widetilde{\phi}(r)w_i = w_i\widetilde{\phi}(r)$ for all $r \in \mathcal{R}$ and $-n \leqslant i \leqslant n$. Since $C^*(\mathbb{F}_n)$ is universal, there is a unital homomorphism $\pi : C^*(\mathbb{F}_n) \to \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ such that $\pi(u_i) = w_i$ for each $1 \leqslant i \leqslant n$, whence $\pi(u_i) = w_i$ for all $-n \leqslant i \leqslant n$. Let $p = \begin{bmatrix} 1_{\mathcal{H}} & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ and define $\widetilde{\psi} : C^*(\mathbb{F}_n) \to \mathcal{B}(\mathcal{H})$ by $\widetilde{\phi}(x) = p\pi(x)p_{|\mathcal{H}}$. Thus, $\widetilde{\psi}$ is a ucp extension of ψ. Moreover, as the range of $\widetilde{\phi}$ commutes with p and with the range of π, $\widetilde{\psi}$ and ϕ have commuting ranges. ∎

LEMMA 4.2. *The following statements are equivalent for a unital C^*-algebra $\mathcal{A}$:*
(i) $\mathcal{A} \otimes_{\min} C^*(\mathbb{F}_n) = \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_n)$ *for some* $n \geqslant 2$*;*
(ii) $\mathcal{A} \otimes_{\min} C^*(\mathbb{F}_n) = \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_n)$ *for every* $n \geqslant 2$*;*
(iii) $\mathcal{A} \otimes_{\min} C^*(\mathbb{F}_\infty) = \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty)$*.*

Proof. The free group $\mathbb{F}_\infty$ is a subgroup of $\mathbb{F}_2$ and, hence, of $\mathbb{F}_n$, for any fixed $n \geqslant 2$. Therefore, $\mathcal{A} \otimes_{\min} C^*(\mathbb{F}_\infty) \subset \mathcal{A} \otimes_{\min} C^*(\mathbb{F}_n)$ is an inclusion of C^*-algebras. By Proposition 8.8 of [12] there is a ucp projection $\psi : C^*(\mathbb{F}_n) \to C^*(\mathbb{F}_n)$ with range $C^*(\mathbb{F}_\infty)$ (considered as a unital C^*-subalgebra of $C^*(\mathbb{F}_n)$). Thus,

$$\mathcal{A} \otimes_{\min} C^*(\mathbb{F}_\infty) \subset \mathcal{A} \otimes_{\min} C^*(\mathbb{F}_n) = \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_n) \to \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty)$$

yields a ucp map $\mathcal{A} \otimes_{\min} C^*(\mathbb{F}_\infty) \to \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty)$, which implies that $\mathcal{A} \otimes_{\min} C^*(\mathbb{F}_\infty) = \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty)$. This proves the implication (i) $\Rightarrow$ (iii). But using the fact that any two countable free groups are subgroups of each other, the same arguments apply to obtain the equivalence of statements (i), (ii), and (iii). ∎

THEOREM 4.3. *For a fixed $n \geqslant 2$, the following statements are equivalent for a unital C^*-algebra $\mathcal{A}$.*
(i) $\mathcal{A}$ *has* $(\mathfrak{S}_n)$*.*
(ii) $\mathcal{A} \otimes_{\min} \mathcal{S}_{n-1} = \mathcal{A} \otimes_{\max} \mathcal{S}_{n-1}$*.*
(iii) $\mathcal{A} \otimes_{\min} C^*(\mathbb{F}_{n-1}) = \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_{n-1})$*.*
If $n = 2$, then the equivalent statements (i), (ii) *and* (iii) *hold for every $\mathcal{A}$. If $n \geqslant 3$, then the equivalent statements* (i), (ii) *and* (iii) *hold if and only if $\mathcal{A}$ has* WEP*.*

Proof. (i) $\Leftrightarrow$ (ii) The complete quotient map $\phi : \mathcal{T}_n \to \mathcal{S}_{n-1}$ induces a complete quotient map $\mathrm{id}_{\mathcal{R}} \otimes \phi : \mathcal{R} \otimes_{\max} \mathcal{T}_n \to \mathcal{R} \otimes_{\max} \mathcal{S}_{n-1}$ for every operator system $\mathcal{R}$ ([6], Proposition 1.6). Such is the case in particular for $\mathcal{R} = \mathcal{A}$. Because $\otimes_{\min} = \otimes_{\mathrm{c}}$ if one of the tensor factors is $\mathcal{T}_n$ ([6], Proposition 4.1) and because $\otimes_{\mathrm{c}} = \otimes_{\max}$ if one of the tensor factors is a unital C^*-algebra ([8], Theorem 6.6), we deduce that $\mathcal{A} \otimes_{\min} \mathcal{T}_n = \mathcal{A} \otimes_{\max} \mathcal{T}_n$. Now consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A} \otimes_{\min} \mathcal{T}_n & \xrightarrow{\cong} & \mathcal{A} \otimes_{\max} \mathcal{T}_n \\ {\scriptstyle \mathrm{id}_{\mathcal{A}} \otimes \phi} \big\downarrow & & \big\downarrow {\scriptstyle \mathrm{id}_{\mathcal{A}} \otimes \phi} \\ \mathcal{A} \otimes_{\min} \mathcal{S}_{n-1} & \xrightarrow[\theta = \mathrm{id}]{} & \mathcal{A} \otimes_{\max} \mathcal{S}_{n-1} \end{array} .$$

The top arrow is a complete order isomorphism, the rightmost arrow is a complete quotient map, while the bottom arrow is a linear isomorphism θ in which θ^{-1} is completely positive. By Lemma 5.1 of [6], $\mathrm{id}_{\mathcal{A}} \otimes \phi : \mathcal{A} \otimes_{\min} \mathcal{T}_n \to \mathcal{A} \otimes_{\min} \mathcal{S}_{n-1}$ is a complete quotient map if and only if θ is a complete order isomorphism. But since $\mathrm{id}_{\mathcal{A}} \otimes \phi : \mathcal{A} \otimes_{\min} \mathcal{T}_n \to \mathcal{A} \otimes_{\min} \mathcal{S}_{n-1}$ is a complete quotient map if and only if $\mathcal{A}$ has $(\mathfrak{S}_n)$ (by Theorem 2.1(ii)), we deduce the equivalence (i) $\Leftrightarrow$ (ii).

(ii) $\Rightarrow$ (iii) By Lemma 4.1, $\mathcal{A} \otimes_{\mathrm{c}} \mathcal{S}_{n-1} \subset_{\mathrm{coi}} \mathcal{A} \otimes_{\mathrm{c}} C^*(\mathbb{F}_{n-1})$. Thus, $\mathcal{A} \otimes_{\max} \mathcal{S}_{n-1} \subset \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_{n-1})$ since $\otimes_{\mathrm{c}} = \otimes_{\max}$ if one of the tensor factors is a C^*-algebra. By the hypothesis that $\mathcal{A} \otimes_{\min} \mathcal{S}_{n-1} = \mathcal{A} \otimes_{\max} \mathcal{S}_{n-1}$, the inclusion map $\iota : \mathcal{A} \otimes_{\min} \mathcal{S}_{n-1} \to \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_{n-1})$ is ucp. Assuming that $\mathcal{A} \otimes_{\max} C^*(\mathbb{F}_{n-1})$ is faithfully represented as a unital C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, there is a ucp extension $\widetilde{\iota} : \mathcal{A} \otimes_{\min} C^*(\mathbb{F}_{n-1}) \to \mathcal{B}(\mathcal{H})$ of ι. For each $-(n-1) \leqslant k \leqslant (n-1)$ and every $a \in \mathcal{A}$,

$$\begin{aligned} \widetilde{\iota}((a \otimes u_k)^*(a \otimes u_k)) &= \widetilde{\iota}(a^* a \otimes u_k^* u_k) = \widetilde{\iota}(a^* a \otimes 1) = \iota(a^* a \otimes 1) \\ &= \iota(a \otimes u_k)^* \iota(a \otimes u_k) = \widetilde{\iota}(a \otimes u_k)^* \widetilde{\iota}(a \otimes u_k). \end{aligned}$$

Likewise, $\widetilde{\iota}((a \otimes u_k)(a \otimes u_k)^*) = \widetilde{\iota}(a \otimes u_k)\widetilde{\iota}(a \otimes u_k)^*$. Thus, the multiplicative domain of $\widetilde{\iota}$ contains $\{\sum_{k=-(n-1)}^{n-1} a_k \otimes u_k : a_k \in \mathcal{A}\} = \mathcal{A} \otimes \mathcal{S}_{n-1}$, and therefore it also contains the C^*-algebra $\mathcal{A} \otimes \mathcal{S}_{n-1}$ generates, namely $\mathcal{A} \otimes_{\min} C^*(\mathbb{F}_{n-1})$. Thus, $\widetilde{\iota}$ is a unital homomorphism and, hence, $\mathcal{A} \otimes_{\min} C^*(\mathbb{F}_{n-1}) = \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_{n-1})$.

(iii) $\Rightarrow$ (ii) As before, Lemma 4.1 yields $\mathcal{A} \otimes_{\max} \mathcal{S}_{n-1} \subset_{\mathrm{coi}} \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_{n-1})$. By hypothesis, the inclusion map $\iota : \mathcal{A} \otimes_{\min} \mathcal{S}_{n-1} \to \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_{n-1})$ is ucp with range $\mathcal{A} \otimes_{\max} \mathcal{S}_{n-1}$. Thus, $\mathcal{A} \otimes_{\min} \mathcal{S}_{n-1} = \mathcal{A} \otimes_{\max} \mathcal{S}_{n-1}$. This completes the proof of the equivalence of statements (i)–(iii).

To prove the additional assertions, first let $n = 2$. In this case $C^*(\mathbb{F}_{n-1}) = C(\mathbb{T})$. Because $\mathcal{A} \otimes_{\min} C(\mathbb{T}) = \mathcal{A} \otimes_{\max} C(\mathbb{T})$ for every C^*-algebra $\mathcal{A}$, condition (iii) holds for every $\mathcal{A}$.

Next, assume $n \geqslant 3$. If $\mathcal{A}$ has WEP, then $\mathcal{A}$ has property $(\mathfrak{S}_n)$ for every $n \geqslant 2$ ([6], Theorem 6.2), and in particular for the fixed n is the statement of the theorem. Conversely, if $\mathcal{A}$ has property $(\mathfrak{S}_n)$, then using the equivalence of (i) and (ii) we have $\mathcal{A} \otimes_{\min} C^*(\mathbb{F}_{n-1}) = \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_{n-1})$. Lemma 4.2 yields $\mathcal{A} \otimes_{\min} C^*(\mathbb{F}_\infty) = \mathcal{A} \otimes_{\max} C^*(\mathbb{F}_\infty)$. Therefore, by Kirchberg's theorem ([10], Proposition 1.1(iii)), $\mathcal{A}$ has WEP. ∎

The following fact was noted (for a fixed $n \geqslant 3$ rather than for $n = 3$ as below) in the proof of Theorem 4.3, but is important enough to isolate and state separately.

COROLLARY 4.4. *The following statements are equivalent for a unital C^*-algebra $\mathcal{A}$:*

(i) *$\mathcal{A}$ has property $(\mathfrak{S}_3)$;*
(ii) *$\mathcal{A}$ has* WEP*;*
(iii) *$\mathcal{A}$ has property $(\mathfrak{S}_n)$ for every $n \geqslant 2$.*

Proof. The equivalence of (ii) and (iii) is proven in Theorem 6.2 of [6]. Clearly (iii) implies (i). Theorem 4.3 shows that (i) implies (ii). ∎

5. ANDO'S THEOREM FOR C^*-ALGEBRAS

Consider the following two pairs of logical assertions concerning $x_1, \dots, x_n$ in a unital C^*-algebra $\mathcal{A}$.

STRICT FORM OF ANDO'S THEOREM. (Case $n = 1$).

(A1) for every unitary $v \in \mathcal{B}$ in every unital C^*-algebra $\mathcal{B}$, the element $x \otimes v \in \mathcal{A} \otimes_{\min} \mathcal{B}$ satisfies $w(x \otimes v) < \frac{1}{2}$;

(A2) there exist strictly positive $a, b \in \mathcal{A}$ such that

$$\begin{bmatrix} a & x \\ x^* & b \end{bmatrix}$$

is strictly positive in $\mathcal{M}_2(\mathcal{A})$ and $a + b = 1$.

The strict form of Ando's theorem is the assertion that (A1) and (A2) are logically equivalent.

MULTIVARIABLE VERSION OF THE STRICT FORM OF ANDO'S THEOREM. (Case $n \geqslant 2$).

(A1′) for every unital C^*-algebra $\mathcal{B}$ and unitaries $v_1, \dots, v_n \in \mathcal{B}$, the numerical radius of $\sum\limits_{j=1}^{n} x_j \otimes v_j$ in $\mathcal{A} \otimes_{\min} \mathcal{B}$ satisfies

$$w(x_1 \otimes v_1 + \cdots + x_n \otimes v_n) < \frac{1}{2};$$

(A2′) there exist strictly positive $a_1, \ldots, a_{n+1} \in \mathcal{A}$ such that

$$\begin{bmatrix} a_1 & x_1 & 0 & \cdots & 0 \\ x_1^* & a_2 & x_2 & & \vdots \\ 0 & x_2^* & \ddots & \ddots & 0 \\ \vdots & & \ddots & a_n & x_n \\ 0 & \cdots & 0 & x_n^* & a_{n+1} \end{bmatrix}$$

is a strictly positive element of $\mathcal{M}_n(\mathcal{A})$ and $a_1 + \cdots + a_{n+1} = 1$.

The multivariable version of the strict form of Ando's theorem is the assertion that (A1′) and (A2′) are logically equivalent.

THEOREM 5.1 (Ando's theorem, nuclearity, and WEP). (i) *The strict form of Ando's theorem holds for every C^*-algebra $\mathcal{A}$ if and only if $C(\mathbb{T})$ is a nuclear C^*-algebra.*

(ii) *The multivariable version of the strict form of Ando's theorem holds for a C^*-algebra $\mathcal{A}$ if and only if $\mathcal{A}$ has* WEP.

Proof. An element z of a unital C^*-algebra B satisfies $w(z) < \frac{1}{2}$ if and only if $1 + 2\Re(z)$ is strictly positive. Because $C(\mathbb{T})$ is the universal C^*-algebra generated by a unitary, Theorems 4.3 and 3.3 complete the proof. ∎

Because $C(\mathbb{T})$ is (by a fairly simple argument) known to be nuclear, Theorem 5.1 explains why the strict form of Ando's theorem holds for every unital C^*-algebra $\mathcal{A}$.

Once one knows that the strict form of Ando's theorem holds, the remaining equivalences of Theorem 1.1 are easily verified.

REMARK 5.2. It is also possible to use the proof of Ando's theorem given by Bunce [4] to prove the strict form of Ando's theorem. Bunce's proof has the added benefit of giving us a concrete formula for the entries a and b satisfying (A2). Bunce's proof shows that if we let $Q = (q_{i,j})_{i,j\in\mathbb{N}}$ be the infinite matrix with entries from $\mathcal{A}$ given by $q_{i,i} = 1, q_{i,i+1} = x, q_{i+1,i} = x^*$ and $q_{i,j} = 0$ for all other pairs, then one may take $a = S_0(Q)$ where $S_0(Q)$ is the "short", in the language of Anderson and Trapp [1], of the operator Q to the first entry. It is not difficult to compute that in this case

$$a = S_0(Q) = 1 - x^*(Q^{-1})_{1,1}x,$$

where $(Q^{-1})_{1,1}$ denotes the $(1,1)$-entry of the matrix defining the inverse of Q. Finally, since we are assuming that $w(x) < \frac{1}{2}$ the matrix Q is strictly positive and tri-diagonal so that the entries of Q^{-1} can be seen to be in the unital C^*-algebra generated by x and x^*.

Combining Remark 5.2 with Theorem 5.1 leads to a rather perverse proof that $C(\mathbb{T})$ is a nuclear C^*-algebra.

6. C^*-SUBALGEBRAS OF $\mathcal{B}(\mathcal{H})$ WITH WEP

Corollary 4.4 shows that a unital C^*-algebra $\mathcal{A}$ has WEP if and only if $\mathcal{A}$ has property $(\mathfrak{S}_3)$, while Theorem 3.3 shows that $\mathcal{A}$ has $(\mathfrak{S}_3)$ if and only if strictly positive elements of $\mathcal{A} \otimes_{\min} \mathcal{S}_2$ lift to strictly positive elements of $\mathcal{A} \otimes_{\min} \mathcal{T}_3$. Therefore, these results yield some new characterisations of WEP, which we explore in this section.

THEOREM 6.1. *The following statements are equivalent for a unital C^*-subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$:*

(i) *$\mathcal{A}$ has* WEP;

(ii) *whenever, for arbitrary $p \in \mathbb{N}$, there exist $X_1, X_2 \in \mathcal{M}_p(\mathcal{A})$ and $A, B, C \in \mathcal{M}_p(\mathcal{B}(\mathcal{H}))$ such that $A + B + C = I$ and*

$$\begin{bmatrix} A & X_1 & 0 \\ X_1^* & B & X_2 \\ 0 & X_2^* & C \end{bmatrix} \tag{6.1}$$

is strictly positive in $\mathcal{M}_{3p}(\mathcal{B}(\mathcal{H}))$, there also exist $\widetilde{A}, \widetilde{B}, \widetilde{C} \in \mathcal{M}_p(\mathcal{A})$ with the same property.

As noted in the Introduction, one important aspect of Theorem 6.1 is that this characterisation of WEP requires a specific property of just one of the many faithful representations that an abstract C^*-algebra $\mathcal{A}$ can have. Consequently, if one faithful representation has this property, then all do. In contrast, Lance's definition of WEP requires that every faithful representation of $\mathcal{A}$ satisfy a certain property (namely, that there exist a weak expectation into the double commutant) or equivalently, that a special representation, namely the universal representation have this property. In further contrast, the reduced atomic representation of a C^*-algebra always possesses a weak expectation into the double commutant. So one faithful representation possessing a weak expectation is not enough to characterise WEP.

Theorem 6.1 is also a new characterisation of injectivity for von Neumann algebras.

COROLLARY 6.2. *If $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra, then $\mathcal{A}$ is injective if and only if statement* (ii) *of Theorem* 6.1 *holds for $\mathcal{A}$.*

Proof. A von Neumann algebra has WEP if and only if it is injective. ∎

We do not know if in the results above it is sufficient to consider only the case $p = 1$.

To conclude this section, we present a second characterisation of WEP in terms of strict row contractions.

LEMMA 6.3. *For any unital C^*-algebra $\mathcal{A}$, the matrix*

$$\begin{bmatrix} 1 & x_1 & 0 \\ x_1^* & 1 & x_2 \\ 0 & x_2^* & 1 \end{bmatrix}$$

is strictly positive if and only if $1 - x_1^* x_1 - x_2 x_2^*$ is strictly positive.

Proof. Let y denote the 3×3 matrix in question and factor y as

$$y = \begin{bmatrix} 1 & 0 & 0 \\ x_1^* & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 - x_1^* x_1 & x_2 \\ 0 & x_2^* & 1 \end{bmatrix} \begin{bmatrix} 1 & x_1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, y is strictly positive if and only if the middle factor is, which in turn is strictly positive if and only if $\begin{bmatrix} 1 - x_1^* x_1 & x_2 \\ x_2^* & 1 \end{bmatrix}$ is. But in $\mathcal{M}_2(\mathcal{A})$ this matrix is unitarily equivalent to

$$\begin{bmatrix} 1 & 0 \\ x_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 - x_1^* x_1 - x_2 x_2^* \end{bmatrix} \begin{bmatrix} 1 & x_2^* \\ 0 & 1 \end{bmatrix},$$

which is strictly positive if and only if $1 - x_1^* x_1 - x_2 x_2^*$ is strictly positive. ∎

For $X_1, X_2 \in \mathcal{B}(\mathcal{H})$, the condition that $I - X_1^* X_1 - X_2 X_2^*$ be strictly positive is equivalent to the condition that $(X_1^*, X_2) : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H}$, where $(X_1^*, X_2)(\xi \oplus \eta) = X_1^* \xi + X_2 \eta$, be a strict (row) contraction, namely

$$\|(X_1^*, X_2)\| < 1.$$

THEOREM 6.4. *The following statements are equivalent for a unital C^*-subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$:*

(i) *$\mathcal{A}$ has* WEP*;*

(ii) *whenever, for arbitrary $p \in \mathbb{N}$, $X_1, X_2 \in \mathcal{M}_p(\mathcal{A})$ are operators for which there exist strictly positive $A, B, C \in \mathcal{M}_p(\mathcal{B}(\mathcal{H}))$ such that $A + B + C = I$ and*

$$\|(B^{-1/2} X_1 A^{-1/2}, B^{-1/2} X_2 C^{-1/2})\| < 1,$$

there also exist strictly positive $\widetilde{A}, \widetilde{B}, \widetilde{C} \in \mathcal{M}_p(\mathcal{A})$ with the same property.

Proof. Suppose that $X_1, X_2 \in \mathcal{M}_p(\mathcal{A})$ and that $A, B, C \in \mathcal{M}_p(\mathcal{B}(\mathcal{H}))$ are strictly positive operators such that $A + B + C = I$.

Let Y denote the 3×3 matrix (6.1) and let $D = A \oplus B \oplus C$, which is strictly positive. Observe that Y is strictly positive if and only if $D^{-1/2} Y D^{-1/2}$ is strictly positive, which by Lemma 6.3 occurs precisely when

$$(B^{-1/2} X_1^* A^{-1/2}, B^{-1/2} X_2 C^{-1/2})$$

is a strict row contraction. Relabeling X_1 by X_1^* yields the result. ∎

COROLLARY 6.5. *If $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra, then $\mathcal{A}$ is injective if and only if statement* (ii) *of Theorem* 6.4 *holds for $\mathcal{A}$.*

Corollary 6.5 shows that, in Theorem 6.4, if a unital C^*-subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ has WEP, then one should not expect to replace the original $A, B, C \in \mathcal{B}(\mathcal{H})$ with operators $A, B, C \in \mathcal{A}''$ if the von Neumann algebra $\mathcal{A}''$ is non-injective.

7. THE CONNES EMBEDDING PROBLEM

Perhaps the most outstanding open problem in operator algebra theory at present is Connes' embedding problem: is every II_1-factor with separable predual a subfactor of the ultrapower R^ω of the hyperfinite II_1-factor R? Kirchberg's equivalent formulation of Connes' embedding problem is the problem of whether $C^*(\mathbb{F}_\infty)$ has WEP ([3], Theorem 13.3.1 and [10], Proposition 8.1). Below we state a new equivalent form of the problem.

THEOREM 7.1. *The following statements are equivalent:*
(i) *the Connes embedding problem has an affirmative solution;*
(ii) *for every $p \in \mathbb{N}$ and every $x_1, x_2 \in \mathcal{M}_p(C^*(\mathbb{F}_2))$ for which*

$$1 \otimes 1 + (x_1 \otimes u_1) + (x_2 \otimes u_2) + (x_1^* \otimes u_1^*) + (x_2^* \otimes u_2^*)$$

is strictly positive in $\mathcal{M}_p(C^*(\mathbb{F}_2)) \otimes_{\min} C^*(\mathbb{F}_2)$, there are strictly positive $a, b, c \in \mathcal{M}_p(C^*(\mathbb{F}_2))$ such that $a + b + c = 1 \in \mathcal{M}_p(C^*(\mathbb{F}_2))$ and

$$\begin{bmatrix} a & x_1 & 0 \\ x_1^* & b & x_2 \\ 0 & x_2^* & c \end{bmatrix}$$

is strictly positive in $\mathcal{M}_{3p}(C^*(\mathbb{F}_2))$.

Proof. Statement (ii) is equivalent to the assertion that $C^*(\mathbb{F}_2)$ has property $(\mathfrak{S}_3)$, which in turn is equivalent to the assertion that $C^*(\mathbb{F}_2)$ has WEP. But $C^*(\mathbb{F}_2)$ has WEP if and only if $C^*(\mathbb{F}_2) \otimes_{\min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_2) \otimes_{\max} C^*(\mathbb{F}_\infty)$ [10].

By Lemma 4.2, $C^*(\mathbb{F}_2) \otimes_{\min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_2) \otimes_{\max} C^*(\mathbb{F}_\infty)$ if and only if $C^*(\mathbb{F}_\infty) \otimes_{\min} C^*(\mathbb{F}_\infty) = C^*(\mathbb{F}_\infty) \otimes_{\max} C^*(\mathbb{F}_\infty)$, which is equivalent to the assertion that the Connes embedding problem has an affirmative solution [10]. ∎

Alternatively, if one fixes a faithful, unital representation of $C^*(\mathbb{F}_2)$ on some Hilbert space, then one may also use Theorem 6.1 or Theorem 6.4 to give other equivalences of Connes' embedding problem. We state these below.

THEOREM 7.2. *The following statements are equivalent for a fixed faithful unital representation of $C^*(\mathbb{F}_2)$ on some Hilbert space $\mathcal{H}$:*
(i) *the Connes embedding problem has an affirmative solution;*

(ii) *for arbitrary* $p \in \mathbb{N}$, *whenever* $X_1, X_2 \in \mathcal{M}_p(C^*(\mathbb{F}_2))$ *are operators for which there exist strictly positive* $A, B, C \in \mathcal{M}_p(\mathcal{B}(\mathcal{H}))$ *such that* $A + B + C = I$ *and*

$$\begin{bmatrix} A & X_1 & 0 \\ X_1^* & B & X_2 \\ 0 & X_2^* & C \end{bmatrix}$$

is strictly positive in $\mathcal{M}_{3p}(\mathcal{B}(\mathcal{H}))$, *there also exist* $\widetilde{A}, \widetilde{B}, \widetilde{C} \in \mathcal{M}_p(C^*(\mathbb{F}_2))$ *with the same property;*

(iii) *for arbitrary* $p \in \mathbb{N}$, *whenever* $X_1, X_2 \in \mathcal{M}_p(C^*(\mathbb{F}_2))$ *are operators for which there exist strictly positive* $A, B, C \in \mathcal{M}_p(\mathcal{B}(\mathcal{H}))$ *such that* $A + B + C = I$ *and*

$$\|(B^{-1/2}X_1A^{-1/2}, B^{-1/2}X_2C^{-1/2})\| < 1,$$

then there also exist strictly positive $\widetilde{A}, \widetilde{B}, \widetilde{C} \in \mathcal{M}_p(C^*(\mathbb{F}_2))$ *with the same property.*

In the special case of $C^*(\mathbb{F}_2)$, we also do not know if it is sufficient to check the properties above in the case $p = 1$.

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