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# STRONG DUAL FACTORIZATION PROPERTY

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ABSTRACT. Let $A$ be a Banach algebra. We give a new characterization of the property $A^* = A^*A$, called the left strong dual factorization property when one assumes that $A$ has a bounded approximate identity. Without the assumption of the existence of a bounded approximate identity, we prove that this property implies the equivalence between the given norm of $A$ and the norm inherited from $RM(A)$, the right multiplier algebra of $A$. Secondly, we present a complete description of the strong topological centres of $N_\alpha(E)$ of $\alpha$-nuclear operators on a Banach space $E$. Using this description, we characterize the Banach spaces $E$ such that $N_\alpha(E)$ has the left and right strong dual factorization property.

KEYWORDS: *Approximable operator, dual factorization property, Banach algebra, nuclear operator.*

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## 1. INTRODUCTION

A.T.-M. Lau and A. Ülger in [15] were the first who studied the properties $A^* = A^*A$ or $A^* = AA^*$ for a Banach algebra $A$ with a BAI (bounded approximate identity), which we call here the left and right strong dual factorization property, respectively. These properties are particularly interesting if $A$ is weakly sequentially complete. In the case of the group algebra $L^1(G)$ and the Fourier algebra $A(G)$, respectively, where $G$ is a locally compact group, the left or right strong dual factorization property are equivalent to the discreteness and the compactness of $G$, respectively. Moreover, A.T.-M. Lau and A. Ülger in [15] proved that a weakly sequentially complete Banach algebra $A$ with a sequential bounded approximate identity which enjoys the left or right strong dual factorization property is unital. These two properties have also been used by A.T.-M. Lau and A. Ülger to study topological centres [15]. More recently, the author in [27], characterized these properties using the notion of strong topological centre. These new objects were the key to the study of Arens regularity of the Fourier algebra on weakly amenable groups in [25].

This paper is the first step of a program which studies the strong dual factorization property for general Banach algebras. It is suspected that the strong dual factorization property implies the existence of a bounded approximate identity. However, we are far from proving this. We prove that proper abstract Segal algebras cannot enjoy it. Consequently, the norm of a faithful Banach algebra $A$ such that $A^* = A^*A$ is equivalent to the norm inherited from $RM(A)$, the right multiplier algebra of $A$. Recently, the author studied the notion of strong topological centre [27] and linked it to the strong dual factorization property. It has been proved in [15], that $A(c_0)$, the algebra of approximable operators on $c_0$, has the left but not the right strong dual factorization property. This suggests that there might exist a Banach space $E$ such that $E^*$ does not have the bounded approximation property such that $A(E)$ has the left or right strong dual factorization property. Combining these two facts, we study the strong topological centres of the algebra of $\alpha$-nuclear operators as M. Daws did in [5] for the topological centres. In the case where $\alpha$ is the injective tensor norm, $N_\alpha(E)$ is $A(E)$. The special attention we put on $A(E)$ comes from the fact that all $\alpha$-nuclear operator algebras, except $A(E)$, are proper abstract Segal algebra. We characterize for which Banach space $E$ the algebra $A(E)$ has the left or right strong dual factorization property.

This paper is organized as follows. In the next section, we establish our notation and recall some preliminary definitions and results. In Section 3, we present the definitions of the three levels of dual factorization property, which generalize the definition by A.T.-M. Lau and A. Ülger. We obtain a new characterization of the left and right strong dual factorization property for a Banach algebra $A$ having a BAI. In Section 4, we study the strong dual factorization property for faithful Banach algebras without BAI and prove that no proper abstract Segal algebra enjoys it. Consequently, we obtain that the left, respectively right, strong dual factorization property of a Banach algebra $A$ implies the equivalence of the norm of $A$ with the norm of its right, respectively left, multiplier algebra. In Section 5, we present the definition of the strong topological centres and some related results. Following the path used by M. Daws in [5], we obtain preliminary results used in subsequent sections. In Section 6, we characterize the strong topological centres of $N_\alpha(E)$ where $E^*$ has the bounded approximation property. In Section 7, assuming that the integral operators and the nuclear operators on $E^*$ coincide, we characterize the strong topological centres of $N_\alpha(E)$. We also characterize the Banach spaces $E$ for which $N_\alpha(E)$ has the left or right strong dual factorization property.

## 2. PRELEMINAIRIES

Through out this paper, $A$ denotes a Banach algebra. Let $\Box$ and $\triangle$ denote the left and right Arens product, respectively. The left Arens product is defined by

$$\langle m\Box n, f\rangle = \langle m, n\Box f\rangle, \quad \langle n\Box f, a\rangle = \langle n, f\Box a\rangle, \quad \langle f\Box a, b\rangle = \langle f, ab\rangle.$$

Similarly, one can define the right Arens product and check that

$$m\Box n = w^*\text{-}\lim_{\alpha} w^*\text{-}\lim_{\beta} m_\alpha n_\beta \quad \text{and} \quad m\triangle n = w^*\text{-}\lim_{\beta} w^*\text{-}\lim_{\alpha} m_\alpha n_\beta,$$

where $(m_\alpha)$ and $(n_\beta)$ are nets in $A^{**}$ converging weak$^*$ to $m$ and $n$, respectively. The *left and right topological centres* are defined, respectively, by

$$Z_{\mathrm{l}}(A^{**}) = \{m \in A^{**} : m\Box n = m\triangle n,\ \forall n \in A^{**}\} \quad \text{and}$$
$$Z_{\mathrm{r}}(A^{**}) = \{m \in A^{**} : n\Box m = n\triangle m,\ \forall n \in A^{**}\}.$$

Using the definition of Arens products, it is easy to see that

$$Z_{\mathrm{l}}(A^{**}) = \{m \in A^{**} : \lambda_m : n \mapsto m\Box n \text{ is } w^*\text{-}w^*\text{-continuous}\} \quad \text{and}$$
$$Z_{\mathrm{r}}(A^{**}) = \{m \in A^{**} : \rho^m : n \mapsto n\triangle m \text{ is } w^*\text{-}w^*\text{-continuous}\}.$$

A mixed unit $\Gamma \in A^{**}$ is an element such that $m = m\Box\Gamma = \Gamma\triangle m$ for all $m \in A^{**}$. It is well known that a Banach algebra $A$ has a bounded approximate identity (BAI) if and only if $A^{**}$ has a mixed unit. We define the sets $A^*A$ and $AA^*$, respectively, by

$$A^*A = \{f \cdot a : f \in A^*, a \in A\} \quad \text{and} \quad AA^* = \{a \cdot f : f \in A^*, a \in A\}.$$

The linear span of $A^*A$ is denoted by $\langle A^*A\rangle$ and $\overline{\langle A^*A\rangle}$ denotes the closed linear span of $A^*A$. Similar definitions hold for $\langle AA^*\rangle$ and $\overline{\langle AA^*\rangle}$.

Let $E$ be a Banach space. The set $B(E)$ denotes the set of all bounded linear operators on $E$. We denote the canonical embedding of $E$ into its second dual $E^{**}$ by $\kappa_E$. Let $LM(A)$ and $RM(A)$ be the left and the right multiplier algebra of $A$, respectively. They are defined, respectively, by

$$LM(A) = \{T \in B(A) : T(ab) = T(a)b, \forall a, b \in A\} \quad \text{and}$$
$$RM(A) = \{T \in B(A) : T(ab) = aT(b), \forall a, b \in A\}.$$

Both left and right multipliers are closed subalgebras of $B(A)$ and $B(A)^{\mathrm{op}}$,the opposite algebra of $B(A)$ and are thus Banach algebras. For $a \in A$, the linear maps $b \mapsto ab$ and $b \mapsto ba$ will be denoted by $L_a$ and $R_a$. The left and right regular representations of $A$ in $B(A)$ given by $a \mapsto L_a$ and $a \mapsto R_a$ are injective if $A$ is left and right faithful respectively. Let $\Gamma$ be a fixed mixed unit, then $T \mapsto T^{**}(\Gamma)$ is an isomorphism, from $LM(A)$ into $A^{**}$, onto its image.

We shall consider $\mathcal{F}(E)$, the finite rank operators, and its closure $A(E)$, the approximable operators, in $B(E)$. We denote by $I(E)$ the integral operators with the integral norm $\|\cdot\|_I$, $N(E)$ the nuclear operators with the norm $\|\cdot\|_N$ and $W(E)$ the weakly compact operators. All those are Banach algebras with respect with their own norms and are linked as follow.

$$\mathcal{F}(E) \subseteq N(E) \subseteq A(E) \subseteq W(E) \quad \text{and} \quad A(E) \subseteq I(E) \subseteq W(E).$$

An operator ideal is an ideal of $B(E)$ containing $\mathcal{F}(E)$ which is a Banach algebra with respect to some norm. A closed operator ideal is an operator ideal which is closed in $B(E)$. For example, $A(E)$ and $W(E)$ are closed operator ideals but not $\mathcal{F}(E)$, $N(E)$ and $I(E)$. B. Johnson proved that the algebras of left and right multipliers, respectively, of operator ideals are isometrically isomorphic to $B(E)$ ([4], Theorem 2.5.13).

Since we are interested in some operator ideals of $B(E)$, the definitions and results concerning tensor norms are not given in their full generality. We will follow the notation of [28]. We refer the reader to [28] and [6] for more details and proofs on the topic.

Let $E$ be a Banach space. The *projective tensor norm* $\pi$ on $E^* \otimes E$ is defined by

$$\pi(\tau) = \inf\Big\{ \sum_{i=1}^{r} \|f_i\| \cdot \|x_i\| : \tau = \sum_{i=1}^{r} f_i \otimes x_i \Big\} \quad (\tau \in E^* \otimes E).$$

The completion of $(E^* \otimes E, \pi)$ is denoted by $E^* \widehat{\otimes}_\pi E$ and it is called the projective tensor product of $E^*$ and $E$. The *injective tensor norm* $\epsilon$ is defined on $E^* \otimes E$ by

$$\epsilon(\tau) = \sup\Big\{ \Big| \sum_{i=1}^{n} \langle m, x_i\rangle \langle g, y_i\rangle \Big| : m \in E^{**},\ g \in E^*,\ \|m\| = \|g\| = 1 \Big\},$$

where $\tau = \sum\limits_{i=1}^{n} x_i \otimes y_i \in E^* \otimes E$. The completion of $(E^* \otimes E, \epsilon)$ is denoted by $E^* \widehat{\otimes}_\epsilon E$. With the identification of $\mathcal{F}(E)$ with $E^* \otimes E$, it is easy to check that $A(E)$ is exactly $E^* \widehat{\otimes}_\epsilon E$.

Following the terminology of [28], a *tensor norm* $\alpha$ is a finitely generated uniform crossnorm. The completion of $E^* \otimes E$ with respect to $\alpha$ is denoted by $E^* \widehat{\otimes}_\alpha E$. We also denote by $\alpha^{\mathrm{t}}$ and $\alpha^*$ the transpose and the dual tensor norm of $\alpha$, respectively. The injective and projective tensor norm are tensor norms and symmetric, i.e. $\epsilon^{\mathrm{t}} = \epsilon$ and $\pi^{\mathrm{t}} = \pi$. Moreover , $\epsilon^* = \pi$ and $\pi^* = \epsilon$.

DEFINITION 2.1. Let $\alpha$ be a tensor norm. Since $\alpha(\tau) \leqslant \pi(\tau)$ for $\tau \in E^* \otimes E$, the formal identity map $I_\alpha : E^* \widehat{\otimes}_\pi E \to E^* \widehat{\otimes}_\alpha E$ is norm decreasing. Using $(I_\alpha)^*$, we identify $(E^* \widehat{\otimes}_\alpha E)^*$ with a subspace of $B(E^*)$, denoted $B_{\alpha^*}(E^*)$. The elements of $B_{\alpha^*}(E^*)$ are called *$\alpha^*$-integral operators* and the norm $\|\cdot\|_{\alpha^*}$ is given using the usual norm for a dual Banach space.

The set of $\epsilon$-integral operators on $E^*$ is $B(E^*)$ and the $\pi$-integral operators are the usual integral operators on $E^*$.

DEFINITION 2.2. Let $E$ be a Banach space and let $\alpha$ be a tensor norm. Define the norm-decreasing map $J_\alpha : E^* \widehat{\otimes}_\alpha E \to B(E)$ by

$$J_\alpha(f \otimes x)(y) = \langle f, y\rangle x$$

for $x, y \in E$ and $f \in E^*$. Equipped with the quotient norm, denoted $\|\cdot\|_{N_\alpha}$, the image of $J_\alpha$ is the set of *$\alpha$-nuclear operators* denoted by $N_\alpha(E)$.

The set of $\epsilon$-nuclear operators is $A(E)$ and the set of $\pi$-nuclear operator is $N(E)$.

A Banach space $E$ has the *bounded approximation property* if there exists $M > 0$ such that for every compact subset $K \subseteq E$ and every $\delta > 0$ there exists a finite rank operator $S : E \to E$ such that $\|S\| < M$ and $\|x - Sx\| \leqslant \delta$ for every $x \in K$. It is well known that $A(E)$ has a bounded approximate identity if and only if $E^*$ has the bounded approximation property.

We finish this section with two results that will be needed later. Their proofs can be found in [5].

PROPOSITION 2.3. *Let $E$ be a Banach space. Let $T \in B(E)$, and let $\alpha$ be a tensor norm. The following are equivalent:*

(i) *$T$ is an $\alpha$-integral operator,*
(ii) *$T^*$ is an $\alpha^{\mathrm{t}}$-integral operator,*
(iii) *$T^{**}$ is an $\alpha$-integral operator.*
*Furthermore, $\|T\|_\alpha = \|T^*\|_{\alpha^{\mathrm{t}}}$ holds.*

THEOREM 2.4. *Let $E, F, G$ be Banach spaces.*
(i) *If $T \in I(E,F)$ and $S \in W(F,G)$, then $ST \in N(E,G)$.*
(ii) *If $S \in W(E,F)$ and $T \in I(F,G)$ then $\kappa_G TS \in N(E, G^{**})$. Moreover, if $E^*$ has the approximation property, then $TS \in N(E,G)$.*

## 3. DEFINITION, EXAMPLES AND CHARACTERIZATION

In this section, we present the definitions of the three levels of the dual factorization property for a Banach algebra $A$ as well as examples. We then present a different approach than the one used in [15] to study the strong dual factorization property for Banach algebra having a BAI. From this approach, we obtain a new characterization of this property and extend a result of Baker–Lau–Pym concerning the quotient map between $A^{**}$ and $(A^*A)^*$ for a Banach algebra $A$ with a BAI.

DEFINITION 3.1. Let $A$ be a Banach algebra. We say that:
(i) $A$ has the *left strong dual factorization property* if $A^* = A^*A$;
(ii) $A$ has the *left dual factorization property* if $A^* = \overline{A^*A}$;
(iii) $A$ has the *left weak dual factorization property* if $A^* = \overline{\langle A^*A \rangle}$.
The right and two sided versions are defined analogously.

Of course, when $A$ has a BAI, (i), (ii) and (iii) are the same. Obviously, (i) implies (ii), which implies (iii).

Now, we present some examples to illustrate these properties .

EXAMPLE 3.2. (i) Any faithful reflexive Banach algebra $A$ has the left and right weak dual factorization property. We only prove the left version since the right one is similar. Suppose that $A^* \neq \overline{\langle A^*A \rangle}$. Then by the Hahn–Banach theorem, there exists a non-zero $m \in A^{**} = A$ such that $m|_{\overline{\langle A^*A \rangle}} = 0$. In particular,

$$0 = \langle m, f \cdot a \rangle \quad = \quad \langle a \Box m, f \rangle$$

for any $f \in A^*$ and $a \in A$ . This implies that $a \Box m = 0$ for all $a \in A$ which is a contradiction, since $A$ is faithful.

(ii) The Banach algebra $l^2(I)$, where $I$ is an index set, equipped with the pointwise multiplication has the dual factorization property. We have that $\overline{l^2(I) \cdot l^2(I)} = \overline{l^1(I)} = l^2(I)$.

(iii) The Banach algebra $A(c_0)$ has the left strong dual factorization property but not the right one ([15], Example 2.5).

(iv) Any Arens regular Banach algebra with a BAI has the strong dual factorization property ([29], Theorem 3.1).

The list of examples above does not cover all possibles cases. Here, we did not present a faithful Banach algebra without BAI with the strong dual factorization property. We will deal with this case in Section 4. Another missing example is a Banach algebra $A$ with the weak dual factorization property, but without the dual factorization property. Such an example is unknown at this moment.

We now pass to the study of the strong dual factorization property of Banach algebras having a BAI. This has been done first in [15] where there are many characterizations of it. However, all these characterizations rely on Lemma 2.1 of [15] which states that the left and right strong dual factorization property, respectively, of a Banach algebra $A$ with a BAI $(e_\alpha)$ is equivalent to the weak convergence of the nets $(f \triangle e_\alpha)$ and $(e_\alpha \Box f)$, respectively, to $f$. Our approach is inspired by Theorem 1.1 of [1] which states that $(A^*A)^*$ is isomorphic to the Banach algebra $\mathrm{Hom}_{A,r}(A^*, A^*)$ of bounded right $A$-module homomorphisms on $A^*$. The link between $\mathrm{Hom}_{A,r}(A^*, A^*)$ and $(A^*A)^*$ is also used by M. Neufang in [23] to prove that $RM(A)$ can be identified with the topological centre of $(A^*A)^*$, which can be defined analogously to the left topological centre of $A^{**}$, for Banach algebras with the $F_\kappa$ property and the Mazur property of level $\kappa$, where $\kappa > \aleph_0$ is a cardinal. Here, we present Proposition 4.1 of [23] which we will use later and also establish our notation through this section.

PROPOSITION 3.3 ([23], Proposition 4.1). *Let $A$ be a Banach algebra with a* BRAI $\{e_\alpha\}_{\alpha \in I}$. *Define $\rho : (A^*A)^* \to \mathrm{Hom}_{A,\mathrm{r}}(A^*, A^*)$ by*

$$\langle \rho(m)(f), a \rangle = \langle m, f \cdot a \rangle$$

*for $m \in (A^*A)^*$, $f \in A^*$ and $a \in A$. For a left unit $\Gamma$ in $(A^{**}, \Box)$, define $\tau_\Gamma$ :* $\mathrm{Hom}_{A,\mathrm{r}}(A^*, A^*) \to (A^*A)^*$ *by*

$$\langle \tau_\Gamma(\phi), g \rangle = \langle \Gamma, \phi(g) \rangle$$

*for $\phi \in \mathrm{Hom}_{A,\mathrm{r}}(A^*, A^*)$ and $g \in A^*A$. Then $\rho$ and $\tau_\Gamma$ are algebra homomorphisms which are inverse to each other.*

The following diagram illustrates the situation, where $\Gamma$ is a fixed left unit in $(A^{**}, \Box)$. In this diagram, $\pi$ is the restriction map. We define a homomorphism

(3.1)
$$\begin{tikzcd} (A^{**}, \Box) \arrow[r, "\pi"] & (A^*A)^* \arrow[d, shift left, "\rho"] \\ & \mathrm{Hom}_{A,\mathrm{r}}(A^*, A^*) \arrow[u, shift left, "\tau_\Gamma"] \end{tikzcd}$$

from $(A^{**}, \Box)$ to $\mathrm{Hom}_{A,\mathrm{r}}(A^*, A^*)$ that will make the diagram commute.

Define $\Phi : (A^{**}, \Box) \to \mathrm{Hom}_{A,\mathrm{r}}(A^*, A^*)$ by $\Phi(m) = {}_mT$ where ${}_mT(f) = m\Box f$ for $m \in A^{**}$ and $f \in A^*$. We now add the map $\Phi$ to the previous diagram and obtain the following.

(3.2)
$$\begin{tikzcd} A^{**} \arrow[r, "\pi"] \arrow[dr, "\Phi"] & (A^*A)^* \arrow[d, shift left, "\rho"] \\ & \mathrm{Hom}_{A,\mathrm{r}}(A^*, A^*) \arrow[u, shift left, "\tau_\Gamma"] \end{tikzcd}$$

Next, we prove that this diagram commutes.

LEMMA 3.4. *Let $A$ be a Banach algebra with a* BRAI. *Then $\Phi = \rho \circ \pi$.*

*Proof.* Let $m \in A^{**}$ and ${}_mT : A^* \to A^*$ be defined by ${}_mT(f) = m\Box f$. Let $\rho$ and $\tau_\Gamma$ as in Proposition 3.3. We have

$$\begin{aligned} \langle \tau_\Gamma(\Phi(m)), f \cdot a \rangle &= \langle \Gamma, (\Phi(m))(f \cdot a) \rangle = \langle \Gamma, {}_mT(f \cdot a) \rangle = \langle \Gamma, m\Box(f \cdot a) \rangle \\ &= \langle \Gamma, (m\Box f)\triangle a \rangle = \langle a\Box\Gamma, m\Box f \rangle = \langle a\triangle m, f \rangle \\ &= \langle m, f \cdot a \rangle = \langle \pi(m), f \cdot a \rangle \end{aligned}$$

for $f \in A^*$ and $a \in A$. ∎

The definition of $\Phi$ holds for general Banach algebras, since we only used the module action of $A^{**}$ on $A^*$. In this generality, the following proposition addresses the injectivity and surjectivity of $\Phi$. These two properties of $\Phi$ characterize natural properties of the Banach algebra $A$.

PROPOSITION 3.5. *Let $A$ be a Banach algebra. Then*

(i) $\ker(\Phi) = \overline{\langle A^*A \rangle}^{\perp}$*; in particular, $\Phi$ is injective if and only if $A^* = \overline{\langle A^*A \rangle}$.*

(ii) *$\Phi$ is surjective if and only if $A$ has a* BRAI.

*Proof.* The first assertion is obvious from the definition of $\Phi$. Now, let us prove the second assertion. Suppose that $\Phi$ is surjective. In particular, there is

$\Gamma \in A^{**}$ such that $\Phi(\Gamma) = \mathrm{id}$. Thus, $f =_\Gamma T(f) = \Gamma \Box f$ for any $f \in A^*$. For all $n \in A^{**}$,

$$\langle n, f\rangle = \langle n, \Gamma \Box f\rangle = \langle n \Box \Gamma, f\rangle.$$

So, $\Gamma$ is a left unit for $(A^{**}, \Box)$; therefore $A$ has a bounded right approximate identity.

If $A$ has a BRAI, then Lemma 3.4 and Theorem 1.1 of [1] give the conclusion. ∎

Proposition 3.5(ii) gives us a lot of information about $\mathrm{Hom}_{A,\mathrm{r}}(A^*, A^*)$ when $A$ has a BRAI. If we are interested in particular properties of $(A^*A)^*$ or in the projection map $\pi$, it is useful to look at $\mathrm{Hom}_{A,\mathrm{r}}(A^*, A^*)$. We have that $RM(A)$ is embedded into different algebras, for example we always have a canonical injective anti-homomorphism between $RM(A)$ and $\mathrm{Hom}_{A,\mathrm{r}}(A^*, A^*)$. Furthermore, the injective anti-homomorphism between $RM(A)$ and $(A^{**}, \Box)$ is independent of the choice of a left unit $\Gamma$ in $A^{**}$. The following diagram, which is easily seen to commute, shows the complete picture of the situation when the existence of a BRAI is assumed.

$$\begin{tikzcd}
(A^{**},\Box) \arrow[rr, "\pi"] \arrow[drr, "\Phi"] & & (A^*A)^* \arrow[d, shift left, "\rho"] \\
& & \mathrm{Hom}_{A,\mathrm{r}}(A^*,A^*) \arrow[u, shift left, "\tau_\Gamma"] \\
& RM(A) \arrow[uul, hook] \arrow[ur, hook] & \\
A \arrow[uuu, hook, "\kappa_A"] \arrow[ur, hook] & &
\end{tikzcd}$$

Next, we generalize Theorem 1.3(i) of [1], which affirms that, if $A$ has a BAI, $\pi$ is injective on $A$ regarded as a subalgebra of $A^{**}$.

PROPOSITION 3.6. *Let $A$ be a Banach algebra with a* BRAI. *Then the map $\pi : A^{**} \to (A^*A)^*$ is injective on $Z_\mathrm{r}(A^{**})$ and on $RM(A)$, regarded as a subalgebras of $(A^{**}, \Box)$.*

*Proof.* By Lemma 3.4, we only need to show that $\Phi$ is injective on $Z_\mathrm{r}(A^{**})$ and $RM(A)$. First, consider the case of $Z_\mathrm{r}(A^{**})$. Let $m_1, m_2 \in Z_\mathrm{r}(A^{**})$ be such that $\Phi(m_1) = \Phi(m_2)$. Let $n \in A^{**}$. We have that ${}_{m_1}T(f) = {}_{m_2}T(f)$ for all $f \in A^*$. Thus,

$$\langle n\triangle m_1, f\rangle = \langle n\Box m_1, f\rangle = \langle n, {}_{m_1}T(f)\rangle = \langle n, {}_{m_2}T(f)\rangle = \langle n\Box m_2, f\rangle = \langle n\triangle m_2, f\rangle.$$

Since this is true for an arbitrary $f \in A^*$, we conclude that $n\triangle m_1 = n\triangle m_2$. In particular for $n = \Gamma$, where $\Gamma$ is a left unit, and so we have that $m_1 = m_2$. The second assertion is clear from the definition of $\Phi$ and the discussion before the previous diagram. ∎

We are grateful to the referee who indicated us that the first conclusion of Proposition 3.6 can also be deduced from Proposition 3.15 of [14].

Lau–Ülger mention, in [15], that the characterization of Proposition 2.11 in [15] may be not easy to use, since it is hard to determine the set $M_1 := \{m \in A^{**} : A \cdot m \subseteq A\}$. One checks directly that $RM(A) \subseteq M_1$. Therefore, a natural question is: When does $M_1$ coincide with $RM(A)$? In fact, the answer is a new characterization of the left strong dual factorization property.

THEOREM 3.7. *Let $A$ be a Banach algebra with a* BRAI. *Then $A^*A = A^*$ if and only if $M_1 = RM(A)$ in $(A^{**}, \Box)$.*

*Proof.* We only need to show that $M_1 \subseteq RM(A)$. Since $\Phi$ is injective by Proposition 3.5 and since the diagram commutes, it is enough to verify that $\Phi(M_1) \subseteq RM(A)$ in $\mathrm{Hom}_{A,\mathrm{r}}(A^*, A^*)$. But this holds by Lemma 3.2.11 of [27].

Now, suppose that $M_1 = RM(A)$. Note that

$$(A^*A)^\perp \subseteq M_1 = RM(A).$$

Hence, the conclusion is obtained easily by Proposition 3.6. ∎

The approach used in this section to study the strong dual factorization property can also be used to study $(A^*A)^*$ (see Chapter 3 of [27]).

## 4. STRONG DUAL FACTORIZATION PROPERTY WITHOUT BAI

In this section, we first recall the definition of an abstract Segal algebra and present some preliminary results. After that, we show that there is no proper right abstract Segal algebra with the left strong dual factorization property. As a consequence of this theorem, we obtain that the left or right, respectively, strong dual factorization property of a Branch algebra $A$ implies the equivalence of the norm of $A$ with its right and left multiplier algebra norm, respectively.

DEFINITION 4.1. Let $(A, \|\cdot\|_A)$ be a Banach algebra. A Banach algebra $(B, \|\cdot\|_B)$ is a *left or right abstract Segal algebra*, respectively, in $A$ if the following conditions are satisfied:

(i) The algebra $B$ is a dense left or right ideal of $A$, respectively;
(ii) There is a constant $C > 0$ such that $\|b\|_A \leqslant C\|b\|_B$ for each $b \in B$;

(iii) There is a constant $M > 0$ such that, respectively,

$$\|ab\|_B \leqslant M\|a\|_A\|b\|_B, \tag{4.1}$$

$$\|ba\|_B \leqslant M\|a\|_A\|b\|_B, \tag{4.2}$$

for all $a \in A$ and for all $b \in B$.

A Banach algebra $(B, \|\cdot\|_B)$ is a *symmetric abstract Segal algebra* in $A$, if it is a left and right abstract Segal algebra in $A$.

EXAMPLE 4.2. The following are abstract Segal algebras.

(i) Any proper Segal algebra of $L^1(G)$ where $G$ is an infinite non-discrete locally compact group.

(ii) Any $p^{\text{th}}$-Schatten class, in particular $N(H)$, the algebra of nuclear operators and the algebra of Hilbert–Schmidt operators, $HS(H)$, over an infinite dimensional Hilbert space $H$, are abstract Segal algebras in their closures in $B(H)$.

(iii) The Figa-Talamanca–Herz Lebesgue algebra $A_p^{\mathrm{r}}(G)$, where $G$ is an infinite locally compact group, is an abstract Segal algebra in the Figa-Talamanca–Herz algebra $A_p(G)$.

The following is due to S. McKilligan, however, her argument can be easily adapted to the cases of left and right abstract Segal algebras.

LEMMA 4.3 ([21], Theorem 2.2). *Let $(A, \|\cdot\|_A)$ be a Banach algebra and $(B, \|\cdot\|_B)$ a right abstract Segal algebra in $A$. Then each element of $B^*B$ extends uniquely to an element of $A^*$. Thus, $B^*B$ can be identified with a subset of $A^*$.*

*Proof.* Let $\phi \in B^*$ and $b, c \in B$. Observe that

$$|\langle \phi \cdot b, c\rangle| = |\langle \phi, bc\rangle| \leqslant \|\phi\|_{B^*}\|bc\|_B \leqslant M\|\phi\|_{B^*}\|b\|_B\|c\|_A,$$

where $M$ is a constant. This inequality shows that $\phi \cdot b$ is bounded on $B$ with respect to the norm of $A$. Since $B$ is dense in $A$, $\phi \cdot b$ can be extended in a unique way to $A^*$. ∎

A natural question is: When can $B^*B$ be identified with a subset of $A^*A$? The proof of Lemma 1 in [22] gives a partial answer, and the argument still holds for abstract symmetric Segal algebras.

COROLLARY 4.4. *Let $(A, \|\cdot\|_A)$ be a Banach algebra with a* BRAI. *Let $(B, \|\cdot\|_B)$ be an abstract Segal algebra in $A$. Then $B^*B$ can be identified with a subset of $A^*A$.*

*Proof.* By Cohen's factorization theorem, $B = BA$. So, $B^*B = B^*(BA) = (B^*B)A$. Let $h \in B^*B$. Then $h = f \cdot a$, for some $f \in B^*B$ and $a \in A$. By Lemma 4.3, there is an extension $\widetilde{f} \in A^*$ such that $\widetilde{f}|_B = f$. One checks that $\widetilde{f} \cdot a \in A^*A$ extends $h$ and the extension is unique by Lemma 4.3. ∎

With Lemma 4.3, we have all we need to prove the main theorem of this section. This result is a new fact on the strong dual factorization property.

THEOREM 4.5. *Let $(A, \|\cdot\|_A)$ be a Banach algebra. Then there is no proper right abstract Segal algebra $(B, \|\cdot\|_B)$ in $A$ such that $B^* = B^*B$.*

*Proof.* Suppose there is a proper right abstract Segal algebra $(B, \|\cdot\|_B)$ in $A$ such that $B^* = B^*B$. Let $i : (B, \|\cdot\|_B) \to (A, \|\cdot\|_A)$ be the natural inclusion map which is continuous by the definition of a right abstract Segal algebra. Let $P : A^* \to B^*$ be the adjoint of $i$. So, $P(f) = f|_B$ for $f \in A^*$. By the density of $B$, $P$ is injective. In view of Lemma 4.3 and the hypothesis that $B^* = B^*B$, $P$ is surjective and thus, a homeomorphism. The operator $P^* : B^{**} \to A^{**}$ is also a homeomorphism. Hence, there are constants $D_1$ and $D_2$ such that

$$D_1 \|m\|_{B^{**}} \leqslant \|P^*(m)\|_{A^{**}} \leqslant D_2 \|m\|_{B^{**}}$$

for all $m \in B^{**}$. But, $P^*(b) = i^{**}(b) = i(b)$ for all $b \in B$. ∎

All the algebras listed in Example 4.2 do not have the left or the right strong dual factorization property. We would like to mention, however, that the $p^{\text{th}}$-Schatten classes being reflexive do have the weak dual factorization property.

The strong dual factorization property is, indeed, very a strong property. For example, for the Fourier algebra $A(G)$, the strong dual factorization property implies that $A(G)$ is unital. However, in general, we can only show the following.

THEOREM 4.6. *Let $A$ be a faithful Banach algebra. If $A$ has the left, respectively right, strong dual factorization property, then the norm of $A$ is equivalent to the norm of $RM(A)$, respectively $LM(A)$.*

*Proof.* Suppose that the norm of $A$ is not equivalent to the norm of its right multiplier algebra. Therefore, $A$ is a proper right abstract Segal algebra in $\overline{R}_A$, the closure of its right regular representation, and so we obtain a contradiction due to Theorem 4.5. ∎

Note that for a discrete group $G$, $L^1(G)$ is unital, and so has the strong dual factorization property. In this case, $L^1(G)$ does not have proper Segal algebras. It is tempting to suspect that a Banach algebra $A$ enjoying the strong dual factorization property does not have proper abstract Segal algebras. However, this is false. If $A = K(H)$, then as a $C^*$-algebra, it is Arens regular and has a BAI; it thus has the strong dual factorization property ([29], Theorem 3.1). But the algebra of Hilbert–Schmidt operators is a proper abstract Segal algebra in $K(H)$ when $H$ is an infinite dimensional Hilbert space.

COROLLARY 4.7. *Let $E$ be a Banach space. Let $\alpha$ be a tensor norm. If $N_\alpha(E)$ has the left or right strong dual factorization property, then $\alpha = \varepsilon$, i,e, $N_\alpha(E) = A(E)$.*

*Proof.* We only prove the right version, the left one is identical. By Theorem 4.6, the $\alpha$-norm has to be equivalent to the norm of $LM(N_\alpha(E)) = B(E)$ by Theorem 2.5.13 of [4]. Thus, $N_\alpha(E)$ is norm closed in $B(E)$ which is the case only for $A(E) = N_\varepsilon(E)$. ∎

REMARK 4.8. It is clear that the set $B^*B$ can be replaced by $\langle B^*B\rangle$ in Lemma 4.3, Corollary 4.4, Theorem 4.5 and Theorem 4.6.

## 5. STRONG TOPOLOGICAL CENTRES OF $\alpha$-NUCLEAR OPERATOR ALGEBRA

In this section, we present briefly the notion of the left and right strong topological centre and some related results. We then restrict our study of these two objects to the class of $\alpha$-nuclear operator algebras. By representing $N_\alpha(E)^{**}$ in $B(E^{**})$, we obtain inclusions which are the start of our approach in the two next sections.

The strong topological centres were first introduced in [27] to compare two class of Banach algebras in terms of Arens regularity. As we will see in Theorem 5.4, they can be used to characterize the left and right strong dual factorization properties. These objects have been used recently in [25] to prove that the Fourier algebra over any infinite locally compact weakly amenable group cannot be Arens regular. For information on strong topological centres, we refer the reader to [24], [25] and [27].

DEFINITION 5.1. Let $A$ be a Banach algebra. The *left strong topological centre* and the *right strong topological centre* of $A$ are defined, respectively, by

$$SZ_{\mathrm{l}}(A^{**}) = \{m \in A^{**} : \lambda_m = T^{**} \text{ for some } T \in B(A)\} \quad \text{and}$$
$$SZ_{\mathrm{r}}(A^{**}) = \{m \in A^{**} : \rho^m = T^{**} \text{ for some } T \in B(A)\}.$$

REMARK 5.2. We would like to mention that another pair of strong topological centres, denoted by $SZ_t(A^{**},\Box)$ and $SZ_t(A^{**},\triangle)$, were introduced and studied in [14].

From the definition of the topological centres, it is clear that $SZ_{\mathrm{l}}(A^{**})$ and $SZ_{\mathrm{r}}(A^{**})$ are, generally proper, closed subalgebras of $Z_{\mathrm{l}}(A^{**})$ and $Z_{\mathrm{r}}(A^{**})$, respectively, since the condition $\lambda_m = T^{**}$, for $m \in Z_{\mathrm{l}}(A^{**})$ is more restrictive. Moreover, this condition hides an algebraic one as the following lemma shows.

LEMMA 5.3 ([13], Theorem 17). *Let $A$ be a Banach algebra. Then*

$$SZ_{\mathrm{l}}(A^{**}) = Z_{\mathrm{l}}(A^{**}) \cap \{m \in A^{**} : m\Box A \subseteq A\}, \quad \textit{and}$$
$$SZ_{\mathrm{r}}(A^{**}) = Z_{\mathrm{r}}(A^{**}) \cap \{m \in A^{**} : A\triangle m \subseteq A\}.$$

Note that in the case where $A$ has a BAI, we have that $SZ_{\mathrm{l}}(A^{**}) = Z_{\mathrm{l}}(A^{**}) \cap LM(A)$ and $SZ_{\mathrm{r}}(A^{**}) = Z_{\mathrm{r}}(A^{**}) \cap RM(A)$ ([27], Theorem 4.2.3). We now present the result that motivates the use of strong topological centres in the study of the left and right strong dual factorization properties.

THEOREM 5.4 ([27], Theorem 4.2.12). *Let $A$ be a Banach algebra with a* BAI. *Then $A$ has the left, respectively right, strong dual factorization property if and only if $SZ_{\mathrm{l}}(A^{**}) = LM(A)$, respectively, $SZ_{\mathrm{r}}(A^{**}) = RM(A)$.*

By Corollary 4.7, we can restrict our study to the injective tensor norm. However, we prefer to present a general approach to studying the strong topological centres of the algebras $N_\alpha(E)$ as this requires almost no extra work.

Our approach to determine the strong topological centres of $N_\alpha$ is heavily inspired by [5]. Let $E$ be a Banach space and let $A = N_\alpha(E)$. Then $E$ is a natural left $A$-module, and $E^*$ and $E^{**}$ become right and left $A$-module, respectively. Therefore, we define a module action of $A$ on $E^* \otimes E$ and on $E^{**} \otimes E^*$ by

$$(f \otimes x) \cdot a = f \cdot a \otimes x, \quad a \cdot (f \otimes x) = f \otimes a \cdot x,$$
$$(m \otimes f) \cdot a = m \otimes f \cdot a, \quad a \cdot (m \otimes f) = a \cdot m \otimes f,$$

for $a \in A$, $x \in E$, $f \in E^*$ and $m \in E^{**}$.

We define the Arens representation as detailed in Section 1.4 of [26] for the general case and in Section 2.1 of [5] for the specific case of the Banach algebras $N_\alpha(E)$. Let $\phi_1 : E^{**}\widehat{\otimes}_\pi E^* \to A^*$ be defined by $\langle \phi_1(m \otimes f), a\rangle = \langle a \cdot m, f\rangle$ for $a \in A$, $m \in E^{**}$ and $f \in E^*$ and extend it by continuity and linearity. Let $\theta_1 = \phi_1^*$. By Theorem 2.3 of [5], $\theta_1$ is a norm-decreasing homomorphism between $(A^{**}, \Box)$ and $(B(E^{**}), \circ)$.

We expose the situation in the following diagram. Let $\iota$ be the inclusion map and $\nu(T) = T^{**}$ for $T \in B(E)$.

$$\begin{tikzcd}
(N_\alpha(E))^{**} \arrow[r, "\theta_1"] & B(E^{**}) \\
N_\alpha(E) \arrow[u, hook, "\kappa_{N_\alpha}"] \arrow[r, hook, "\iota"] & B(E) \arrow[u, hook, "\nu"']
\end{tikzcd}$$

By the density of $\mathcal{F}(E)$ in $N_\alpha(E)$ and the continuity of each map in the previous diagram, one can show easily that this diagram commutes. We state this observation in the following lemma for easier reference late on.

LEMMA 5.5. *Let $E$ be a Banach space. Let $\alpha$ be a tensor norm and $T \in N_\alpha(E)$. Then the previous diagram commutes, i.e. $\theta_1(\kappa_{N_\alpha}(T)) = \nu(T)$.*

To simplify the notation, if $S$ is a subset of $B(E)$, we define $S^a := \{T^* \in B(E^*) : T \in S\}$. With this notation, $\nu(S) = S^{aa}$.

LEMMA 5.6. *Let $E$ be a Banach space. Let $T \in B(E^*)$. If $U^* \circ T$ is weak*-weak*-continuous for every $U \in \mathcal{F}(E)$, then $T \in B(E)^a$.*

*Proof.* Let $\{f_\beta\}_{\beta\in I} \in E^*$ converge to $f$ in the weak* topology. Let $x, x_0 \in E \setminus \{0\}$ such that $x \neq x_0$. By the Hahn–Banach theorem there exists $g \in E^*$ such that $\langle g, x_0\rangle = 1$. Then

$$\begin{aligned}\lim_\beta \langle T(f_\beta), x\rangle &= \lim_\beta \langle T(f_\beta), (g \otimes x)x_0\rangle = \lim_\beta \langle ((g \otimes x)^* \circ T)(f_\beta), x_0\rangle \\ &= \langle ((g \otimes x)^* \circ T)(f), x_0\rangle = \langle T(f), x\rangle.\end{aligned}$$

Thus, $T$ is weak*-weak*-continuous. ∎

We start by studying the left strong topological centre. Observe that we can represent $(A^{**}, \triangle)$ in the Banach algebra of bounded left $A$-module homomorphisms on $A^*$ denoted by $\mathrm{Hom}_{A,1}(A^*, |A^*)$ via the anti-homomorphism $\Psi$, and we have

$$\Psi(SZ_1(A^{**})) \subseteq LM(A) \quad \text{in } \mathrm{Hom}_{A,1}(A^*, A^*),$$

where $\Psi : A^{**} \to \mathrm{Hom}_{A,1}(A^*, A^*)$ is given by $\Psi(m)(f) = f \triangle m$ for $m \in A^{**}$ and $f \in A^*$. For $N_\alpha(E)$, we represent its second dual in $B(E^{**})$ by $\theta_1$ , which also contains $LM(N_\alpha(E)) = B(E)$ ([4], Theorem 2.5.13), via $\nu$. We should expect that $\theta_1(SZ_1(N^\alpha(E)^{**})) \subseteq \nu(B(E))$. By Lemma 5.3 and 5.5, our goal is to identify the elements in $B(E^{**})$ which consider $N_\alpha(E)^{aa}$ as a left or right ideal. An answer to this question is unknown to the author in the case of a non-reflexive Banach space $E$, not even for $B(E)$. However, we present next a necessary condition, which seems to be the first result about this problem.

LEMMA 5.7. *Let $E$ be a non reflexive Banach space, and let $\alpha$ be a tensor norm. Then*

$$\{T \in B(E^{**}) : T \circ N_\alpha(E)^{aa} \subseteq N_\alpha(E)^{aa}\} \subseteq \{T \in B(E^{**}) : T(\kappa_E(E)) \subseteq \kappa_E(E)\} \quad \textit{and}$$
$$\{T \in B(E^{**}) : N_\alpha(E)^{aa} \circ T \subseteq N_\alpha(E)^{aa}\} \subseteq B(E^*)^a.$$

*Moreover, the inclusions are equalities if $\alpha = \varepsilon$, i.e. $N_\alpha(E) = A(E)$.*

*Proof.* Let $T \in B(E^{**})$, $f \in E^*$ and $x \in E$. It is easy to obtain that

$$T \circ (f \otimes x)^{**} = \kappa_{E^*}(f) \otimes T(\kappa_E(x)) \quad \text{and} \quad (f \otimes x)^{**} \circ T = T^*(\kappa_{E^*}(f)) \otimes \kappa_E(x).$$

If $T$ is such that $T \circ N_\alpha(E)^{aa} \subseteq N_\alpha(E)^{aa}$, then $T \circ (f \otimes x)^{**} \in \mathcal{F}(E)^{aa}$. Thus, from the previous equalities, $T(E) \subseteq E$, which proves the first assertion. The second assertion is obtained similarly. Finally suppose that $N_\alpha(E) = A(E)$. We only need to prove the reverse inclusion. First, let $T \in B(E^{**})$ be such that $T(\kappa_E(E)) \subseteq \kappa_E(E)$. Let $U_\beta \in \mathcal{F}(E)$ converging in norm to $U \in A(E)$. Then $T \circ U_\beta$ converges to $T \circ U$, hence $T \circ U \in A(E)^{aa}$, as the norm of $B(E)$ is the same as the norm of $B(E^{**})$. We obtain the second equality with a similar argument. ∎

In the proof of Proposition 5.8 of [5], it is proved that

$$\theta_1(Z_1((N_\alpha(E))^{**})) \subseteq B_{\alpha^t}(E^*)^a \subseteq B(E^*)^a.$$

A combination of this fact with the Lemma 5.3 and Lemma 5.7 gives that

$$\theta_1(SZ_1((N_\alpha(E))^{**})) \subseteq B(E^*)^a \cap \{T \in B(E^{**}) : T(\kappa_E(E)) \subseteq \kappa_E(E)\} = B(E)^{aa}.$$

A priori, it seems that we lost information, but this is not the case as the next proposition shows. Before, let us define the following sets. Let $E$ be a Banach space and $\alpha$ be a tensor norm. Then

$$Z_1^0(E,\alpha) = \{T^* \in B_{\alpha^t}(E^*)^a : T \circ \kappa_E^* \circ S^{**} = \kappa_E^* \circ T^{**} \circ S^{**} \ (S \in N_\alpha(E)^*)\} \quad \text{and}$$
$$Z_2^0(E,\alpha) = \{T \in B_\alpha(E^{**}) : T^{**}(\kappa_E(E)) \subseteq \kappa_E(E),\ T \circ S^* \in W(E)^{aa} \ (S \in N_\alpha(E)^*)\}.$$

PROPOSITION 5.8. *Let $E$ be a Banach space, and let $\alpha$ be a tensor norm. Then*

$$\theta_1(SZ_{\mathrm{l}}((N_\alpha(E))^{**})) \subseteq B_\alpha(E)^{aa};$$
$$\theta_1(SZ_{\mathrm{r}}(N_\alpha(E)^{**})) \subseteq Z_2^0(E,\alpha) \cap W(E)^{aa}.$$

*Proof.* From the discussion before this proposition, we have that

$$\theta_1(Z_{\mathrm{l}}((N_\alpha(E))^{**})) \subseteq B_{\alpha^{\mathrm{t}}}(E^*)^a \cap B(E)^{aa}.$$

It follows immediately, from Proposition 2.3, that

$$\theta_1(Z_{\mathrm{l}}((N_\alpha(E))^{**})) \subseteq B_\alpha(E)^{aa}.$$

Let us prove the inclusion involving the right strong topological centre. By Proposition 5.8 of [5],

$$\theta_1(Z_{\mathrm{r}}(N_\alpha(E)^{**})) \subseteq Z_2^0(E,\alpha).$$

Therefore, by Lemma 5.3 and Lemma 5.7, we have that

$$\theta_1(SZ_{\mathrm{r}}(N_\alpha(E)^{**})) \subseteq Z_2^0(E,\alpha) \cap B(E^*)^a.$$

Hence, $T \in W(E)^{aa}$ by Lemma 5.10 of [5]. ∎

Note that, if we add the hypothesis that $N_\alpha(E)^* \subseteq W(E^*)$, then by Theorem 5.11 of [5]

$$Z_1^0(E,\alpha) \cap B(E)^{aa} = B_{\alpha^{\mathrm{t}}}(E^*)^a \cap B(E)^{aa} = B_{\alpha^{\mathrm{t}}}(E).$$

In the particular case of $\alpha = \varepsilon$, we have trivially that $A(E)^* = I(E^*) \subseteq W(E^*)$, but $B_{\alpha^{\mathrm{t}}}(E) = B(E)$, which shows that we cannot be more specific than Proposition 5.8.

## 6. WHEN THE DUAL HAS THE BOUNDED APPROXIMATION PROPERTY

In this section, our aim is to present a complete description of the left and right strong topological centre of $N_\alpha(E)^{**}$ with the extra conditions that $E^*$ has the bounded approximation property and $\alpha$ is an accessible tensor norm. Moreover, we characterize the Banach spaces $E$ such that $A(E)$ has the strong right dual factorization with the same assumption on $E^*$ and $\alpha$.

Let $E$ be a Banach space. For $T \in B(E^{**})$, define $\nu(T) \in B(E^*)$ and $\mathcal{Q}(T) \in B(E^{**})$, respectively, by

$$\nu(T) = \kappa_E^* \circ T^* \circ \kappa_{E^*}, \quad \mathcal{Q}(T) = \nu(T)^*.$$

We define a bilinear operator $\star$ on $B(E^{**})$ by $T \star S = \mathcal{Q}(T) \circ S$ for $T, S \in B(E^{**})$. Note that $B(E^{**})$ equipped with $\star$ is a Banach algebra by Proposition 2.5 of [5]. The assumption of $\alpha$ to be accessible mentioned above is due to the following theorem.

THEOREM 6.1 ([5], Theorem 5.17). *Let $E$ be a Banach space such that $E^*$ has the bounded approximation property. Let $\alpha$ be an accessible tensor norm, and let $A = N_\alpha(E)$. There exists a homomorphism, $\psi_1 : (B_\alpha(E^{**}), \circ) \to (A^{**}, \Box)$, which is also an isomorphism onto its range, such that $\theta_1 \circ \psi_1 = \mathrm{Id}_{B_\alpha(E^{**})}$. There also exists a bounded homomorphism $\psi_2 : (B_\alpha(E^{**}), \star) \to (A^{**}, \triangle)$ such that $\theta_1 \circ \psi_2 = \mathcal{Q}$. For $i = 1, 2$, $\psi_i(T^{**}) = \kappa_A(T)$ for $T \in A = N_\alpha(E)$.*

The maps $\psi_i$, for $i = 1, 2$, are defined as follow. Let $\Gamma \in A(E)^{**} = I(E^*)^*$ with $\theta_1(\Gamma) = \mathrm{Id}_{E^{**}}$. For $T \in B_\alpha(E^{**})$ and $S \in B_{\alpha^*}(E^*) = A^*$,

$$\langle \psi_1(T), S\rangle = \langle \Gamma, \eta(T \circ S^*)\rangle, \quad \langle \psi_2(T), S\rangle = \langle \Gamma, \eta(T) \circ S\rangle,$$

where $\eta(T) = (\kappa_E)^* \circ T^* \circ \kappa_{E^*}$ for $T \in B(E^{**})$. Define:

$$\begin{aligned} Z_1(E,\alpha) = \{&T^* \in B_{\alpha^t}(E^*)^a : T \circ S \in N_{\alpha^*}(E^*),\\ &\kappa_{E^*} \circ T \circ (\kappa_E)^* \circ S^{**} = T^{**} \circ S^{**}, S \in B_{\alpha^*}(E^*)\};\\ Z_2(E,\alpha) = \{&T \in B_\alpha(E^{**}) : T(E^{**}) \subseteq \kappa_E(E), T \circ S^* \in N_{\alpha^*}(E^*)^a, S \in B_{\alpha^*}(E^*)\};\\ X_1(E,\alpha) = \overline{\mathrm{lin}}&\{\eta(T \circ S^*), S \in B_{\alpha^*}(E^*), T \in B_\alpha(E^{**})\};\\ X_2(E,\alpha) = \overline{\mathrm{lin}}&\{T \circ S : S \in B_{\alpha^*}(E^*), T \in B_{\alpha^t}(E^*)\}. \end{aligned}$$

In Theorem 5.18 and Theorem 5.19 of [5], M. Daws uses these sets to describe the topological centres as follow:

$$\begin{aligned} Z_l(N_\alpha(E)^{**}) &= \{\psi_2(T) + \phi : T \in Z_1(E,\alpha), \phi \in X_1(E,\alpha)^\perp\},\\ Z_r(N_\alpha(E)^{**}) &= \{\psi_1(T) + \phi : T \in Z_2(E,\alpha), \phi \in X_2(E,\alpha)^\perp\}. \end{aligned}$$

LEMMA 6.2. *Let $E$ be a Banach space such that $E^*$ has the bounded approximation property, and let $\alpha$ be an accessible tensor norm. Then for $A = N_\alpha(E)$, $(X_1(E,\alpha)) = \overline{\langle A^{**} \Box A^*\rangle}$ and $X_2(E,\alpha) = \overline{\langle A^* \triangle A^{**}\rangle}$.*

*Proof.* Since $E^*$ has the bounded approximation property,

$$N_\alpha(E)^{**} = \psi_1(B_\alpha(E^{**})) \oplus \ker(\theta_1).$$

Recall that $N_\alpha(E)^* = B_{\alpha^*}(E^*)$, again as $E^*$ has the bounded approximation property. By Proposition 5.12 of [5], $N_\alpha(E)^{**} \Box \ker(\theta_1) = 0$ which implies that

$$\ker(\theta_1) \Box B_{\alpha^*}(E^*) = 0.$$

Thus, $N_\alpha(E)^{**} \Box B_{\alpha^*}(E^*) = \psi_1(B_\alpha(E^{**})) \Box B_{\alpha^*}(E^*)$. Let $T \in B_\alpha(E^{**})$ and $S \in B_{\alpha^*}(E^*)$. Then

$$\psi_1(T) \Box S = \eta(\theta_1(\psi_1(T)) \circ S^*) = \eta(T \circ S^*).$$

Let us prove the other statement. For $\Phi \in N_\alpha(E)^{**}$, we have $\theta_1(\Phi)^* \in B_{\alpha^t}(E^{***})$. Thus, $\nu(\theta_1(\Phi)) \in B_{\alpha^t}(E^*)$. In particular for $S \in B_{\alpha^*}(E^*)$, $S \triangle \Phi = \nu(\theta_1(\Phi)) \circ S$. Hence, we get that $\overline{\langle A^* \triangle A^{**}\rangle} \subseteq X_2(E,\alpha)$. Moreover, for any $T \in B_{\alpha^t}(E^*)$, there exists a $\Phi \in \mathrm{Im}(\theta_1)$ such that $T^* = \theta_1(\Phi)$, as $B_{\alpha^t}(E^*)^a \subseteq B_\alpha(E^{**}) = \mathrm{Im}(\theta_1)$. Then $\nu(\theta_1(\Phi)) = \nu(T^*) = T$ which gives us the missing inclusion. ∎

It is an easy consequence of the definition of the Arens products that, for a Banach algebra $A$, $(A^{**}\Box A^{*})^{\perp}$ and $(A^{*}\triangle A^{**})^{\perp}$ are, respectively, in $Z_{\mathrm{l}}(A^{**})$ and $Z_{\mathrm{r}}(A^{**})$, which explains, by Lemma 6.2, why $X_1^{\perp}$ and $X_2^{\perp}$ appear in the description of topological centres given by M. Daws.

THEOREM 6.3. *Let $E$ be a Banach space such that $E^*$ has the bounded approximation property. Let $\alpha$ be an accessible tensor norm. Then*

$$SZ_{\mathrm{l}}(N_\alpha(E)^{**}) = \{\psi_2(T)+\Phi : \Phi \in (X_1(E,\alpha))^{\perp}, T \in Z_1(E,\alpha)\cap B_\alpha(E)^{aa}\} \quad \textit{and}$$
$$SZ_{\mathrm{r}}(N_\alpha(E)^{**}) = \{\psi_1(T)+\phi : T \in Z_2(E,\alpha)\cap W(E)^{aa},\ \phi \in X_2(E,\alpha)^{\perp}\}.$$

*Proof.* To simplify the notation, let $A = N_\alpha(E)$. By Theorem 5.18 of [5], we have that

$$Z_{\mathrm{l}}(A^{**}) = \{\psi_2(T)+\Phi, \Phi \in (X_1(E,\alpha))^{\perp}, T \in Z_1(E,\alpha)\}.$$

By Lemma 6.2, $(X_1(E,\alpha))^{\perp} = (A^{**}\Box A^{*})^{\perp}$ which is in $SZ_{\mathrm{l}}(A^{**})$. By Lemma 5.3, it is enough to determine what is

$$\psi_2(Z_1(E,\alpha)) \cap \{m \in A^{**} : m\cdot A \subseteq A\}.$$

Let $m$ be an element of this set. Then there exists $T \in B_{\alpha^{\mathrm{t}}}(E^*)$ such that $m = \psi_2(T^*)$. Moreover, for each $U \in N_\alpha(E)$, there is $S \in N_\alpha(E)$ such that $m\Box\kappa_A(U) = \kappa_A(S)$. Hence,

$$S^{**} = \theta_1(\kappa_A(S)) = \theta_1(m\Box\kappa_A(U)) = \theta_1(\psi_2(T^*))\circ U^{**} = (Q)(T^*)\circ U^{**} = T^*\circ U^{**}.$$

The last equality is obtained by Lemma 5.7 of [5]. In particular, $U^*\circ T$ is $w^*$-continuous for every $U \in \mathcal{F}(E)$, and so $T \in B_\alpha(E)^a$ by Lemma 5.6 and Lemma 2.3. We deduce that

$$\begin{aligned} SZ_{\mathrm{l}}(A^{**}) &= Z_{\mathrm{l}}(A^{**}) \cap \{m \in A^{**} : m\cdot A \subseteq A\} \\ &\subseteq \{\psi_2(T)+\Phi, \Phi \in (X_1(E,\alpha))^{\perp}, T \in Z_1(E,\alpha)\cap B_\alpha(E)^{aa}\}. \end{aligned}$$

We can reverse this argument since $N_\alpha(E)$ is an operator ideal.

Now, we prove the equality concerning $SZ_{\mathrm{r}}(A^{**})$. Since,

$$Z_{\mathrm{r}}(A^{**}) = \{\psi_1(T)+\phi : T \in Z_2(E,\alpha),\ \phi \in X_2^{\perp}\}$$

by Theorem 5.19 of [5] and $X_2(E,\alpha) = \overline{\langle A^*\triangle A^{**}\rangle}$ by Lemma 6.2, we need to determine for which $T \in Z_2(E,\alpha)$, the inclusion $N_\alpha(E)\Box\psi_1(T) \subseteq N_\alpha(E)$ holds. By Lemma 5.7, we have that a such $T$ is in $B(E^*)^a \cap Z_2(E,\alpha)$. We deduce that $T \in W(E)^{aa}$ by Lemma 5.10 of [5]. Therefore,

$$SZ_{\mathrm{r}}(N_\alpha(E)^{**}) \subseteq \{\psi_1(T)+\phi : T \in Z_2(E,\alpha)\cap W(E)^{aa},\ \phi \in X_2(E,\alpha)^{\perp}\}.$$

One can easily show the reverse inclusion using the fact that $N_\alpha(E)$ is an operator ideal in $B(E)$. ∎

COROLLARY 6.4. *Let $E$ be a Banach space such that $E^*$ has the bounded approximation property. Then*

$$SZ_{\mathrm{l}}(A(E)^{**}) = \{\psi_2(T^*) : T \in B(E)^a \text{ and } T \circ S \in N(E^*), \forall S \in I(E^*)\} \quad \text{and}$$
$$SZ_{\mathrm{r}}(A(E)^{**}) = \{\psi_1(T) : T \in W(E)^{aa}\}.$$

*Proof.* Using Theorem 6.3, we get that

$$SZ_{\mathrm{r}}(A(E)^{**}) = \{\psi_1(T) + \phi : T \in B(E^{**}) \cap W(E)^{aa}, T \circ S^* \in N(E^*)^a \text{ for } S \in I(E^*),\ \phi \in X_2(E,\alpha)^{\perp}\}.$$

Since $E^*$ has the bounded approximation property, $A(E)$ has a BAI and so $X_2(E,\alpha)^{\perp} = 0$ by Lemma 6.2. Moreover, for $S \in I(E^*)$ and $T \in W(E)^{aa}$, we automatically have that $T \circ S^* \in N(E^*)^a$ by Theorem 2.4. The proof for the left strong topological centre is similar. ∎

Corollary 6.4 is a generalization of Theorem 4.1.8 of [27] which proves under the extra assumption $I(E^*) = N(E^*)$ that $SZ_{\mathrm{r}}(A(E)^{**}) = W(E)^{aa}$. In view of Corollary 4.7, the right strong dual factorization property of $N_\alpha(E)$ is only possible when $\alpha = \varepsilon$, i.e. for $A(E)$.

COROLLARY 6.5. *Let $E$ be a Banach space such that $E^*$ has the bounded approximation property. Then $A(E)$ has the right strong dual factorization property if and only if $E$ is reflexive.*

*Proof.* Fix a mixed unit $\Gamma$ in $A(E)^{**}$. Then $\theta_1(\Gamma) = \mathrm{Id}_{E^{**}}$. Suppose that $A(E)$ has the right strong dual factorization property. Then $\Gamma \in SZ_{\mathrm{r}}(A(E)^{**})$ by Theorem 5.4. Hence by Corollary 6.4, there is a $T \in W(E)^{aa}$ such that $\psi_1(T) = \Gamma$. However,

$$(\mathrm{Id}_E)^{**} = \mathrm{Id}_{E^{**}} = \theta_1(\Gamma) = \theta_1(\psi_1(T)) = T.$$

Thus, $\mathrm{Id}_E \in W(E)$ and so $E$ is reflexive. The converse is trivial. ∎

## 7. WHEN EVERY INTEGRAL OPERATORS ARE NUCLEAR

In this section, we slightly modify our assumption by assuming that $\alpha$ is totally accessible and $E^*$ is such that $I(E^*) = N(E^*)$. With these assumptions, we completely describe $SZ_{\mathrm{l}}(N_\alpha(E)^{**})$ and $SZ_{\mathrm{r}}(N_\alpha(E)^{**})$. We also characterize for which Banach space $E$, $A(E)$ has the left strong and left weak dual factorization property, respectively. The condition $I(E^*) = N(E^*)$ is satisfied by many Banach spaces, in particular if $E^*$ has the Radon–Nikodym property. However, the converse is false as it is shown in Example 5.32 of [5] with the dual of the James tree space.

THEOREM 7.1. *Let $E$ be a Banach space such that $I(E^*) = N(E^*)$. Let $\alpha^*$ be a totally accessible tensor norm. Then*

$$SZ_{\mathrm{l}}(N_\alpha(E)^{**}) = \theta_1^{-1}(Z_1^0(E,\alpha) \cap B_\alpha(E)^{aa}) \quad \textit{and}$$
$$SZ_{\mathrm{r}}(N_\alpha(E)^{**}) = \theta_1^{-1}(Z_2^0(E,\alpha) \cap W(E)^{aa}).$$

*Proof.* By Theorem 5.34 of [5], we have

$$Z_{\mathrm{l}}(N_\alpha(E)^{**}) = \theta_1^{-1}(Z_1^0(E,\alpha)) \quad \text{and} \quad Z_{\mathrm{r}}(N_\alpha(E)^{**}) = \theta_1^{-1}(Z_2^0(E,\alpha)).$$

Let $m \in N_\alpha(E)^{**}$ such that $\theta_1(m) \in Z_1^0(E,\alpha) \cap B_\alpha(E)^{aa}$. Then there is $T \in B_\alpha(E)$ such that $\theta_1(m) = T^{**}$. Let $U \in N_\alpha(E)$. Since $N_\alpha(E)$ is an ideal in $B(E)$, $T \circ U = S \in N_\alpha(E)$ and so

$$\theta_1(m \Box \kappa_{N_\alpha}(U)) = T^{**} \circ U^{**} = (T \circ U)^{**}.$$

Hence, $m \Box U \in N_\alpha(E)$, because $\theta_1$ is an isometry by the assumption that $\alpha$ is totally accessible. We obtain the conclusion of the first assertion by Proposition 5.8.

Let us prove the equality involving the strong right topological centre. It is enough, by Proposition 5.8, to prove that

$$\theta_1^{-1}(Z_2^0(E,\alpha) \cap W(E)^{aa}) \subseteq SZ_{\mathrm{r}}(N_\alpha(E)^{**}).$$

Let $m$ be an element of the set on the left side in the previous inclusion. Then $m \in Z_{\mathrm{r}}(N_\alpha(E)^{**})$ as mentioned at the beginning of the proof. Let $T \in W(E)$ such that $T^{**} = \theta_1(m)$ and $U \in N_\alpha(E)$. We have

$$\theta_1(\kappa_{N_\alpha}(U) \triangle m) = \theta_1(\kappa_{N_\alpha}(U) \Box m) = U^{**} \circ T^{**} = (U \circ T)^{**}.$$

Since $N_\alpha(E)$ is an operator ideal and $\theta_1$ is an isometry, we have that $\kappa_{N_\alpha}(U) \triangle m$ is an element of $N_\alpha(E)$ and obtain the conclusion by Lemma 5.3. ∎

THEOREM 7.2. *Let $E$ be a Banach space such that $I(E^*) = N(E^*)$. Then*

$$SZ_{\mathrm{l}}((A(E))^{**}) = X \cap B(E)^{aa}$$

*where* $X = \{T \in B(E^{**}) : \langle T, u \rangle = 0,\ u \in E^{**} \widehat{\otimes}_\pi E^*, J_\pi(u) = 0\}$.

*Proof.* By Corollary 5.36 of [5], we have that

$$Z_{\mathrm{l}}((A(E))^{**}) = X \cap B(E^*)^a.$$

Note that we identify isometrically $A(E)^{**}$ with $X$ via $\theta_1$. By Proposition 5.8, we obtain that $SZ_{\mathrm{l}}(A(E)^{**}) \subseteq X \cap B(E^*)^a \cap B(E)^{aa} = X \cap B(E)^{aa}$. To prove the reverse inclusion, remark that $LM(A(E)) = B(E)$, and so $X \cap B(E)^{aa} \subseteq SZ_{\mathrm{l}}(A(E)^{**})$. ∎

Before we characterize the left strong dual factorization property of $A(E)$, we need the following lemma.

LEMMA 7.3. *Let $E$ be a Banach space such that $E^*$ has the bounded approximation property. Then the following diagram commutes:*

$$\begin{tikzcd} (A(E))^{**} \arrow[r, "\theta_1"] & B(E^{**}) \\ LM(A(E)) \arrow[u, hook, "\Omega"] \arrow[r, leftrightarrow] & B(E) \arrow[u, hook, "\nu"'] \end{tikzcd}$$

*Proof.* Since $E^*$ has the bounded approximation property, $A(E)$ has a BAI. Let $\Gamma$ be a fixed mixed unit such that $\theta_1(\Gamma) = \mathrm{Id}_{E^{**}}$ and let $T \in B(E)$. Define $L_T \in LM(A(E))$ by letting $L_T(S) = T \circ S$ for $S \in A(E)$. Then $\Omega(T) = (L_T)^{**}(\Gamma)$. In particular, $\Omega(\mathrm{Id}) = L_{\mathrm{Id}}^{**}(\Gamma) = \Gamma$. To prove that this diagram commutes, we need to prove that $\theta_1((L_T)^{**}(\Gamma)) = T^{**}$. This will be done in 4 steps. Let $m \in E^{**}$, $f, g \in E^*$ and $x, e \in E$.

*Step* 1.

$$\begin{aligned} \langle f \cdot (g \otimes T(x)), e\rangle &= \langle f, (g \otimes T(x)) \cdot e\rangle = \langle f, \langle g, e\rangle T(x)\rangle = \langle f, T(x)\rangle\langle g, e\rangle \\ &= \langle T^*(f), x\rangle\langle g, e\rangle = \langle T^*(f), (g \otimes x) \cdot e\rangle = \langle T^*(f) \cdot (g \otimes x), e\rangle. \end{aligned}$$

*Step* 2.

$$\langle T^*(f) \cdot (g \otimes x), m\rangle = \langle (g \otimes x) \cdot m, T^*(f)\rangle = \langle \phi_1(m \otimes T^*(f)), g \otimes x\rangle.$$

*Step* 3.

$$\begin{aligned} \langle (L_T)^*(\phi_1(m \otimes f)), g \otimes x\rangle &= \langle \phi_1(m \otimes f), L_T(g \otimes x)\rangle = \langle \phi_1(m \otimes f), g \otimes T(x)\rangle \\ &= \langle (g \otimes T(x)) \cdot m, f\rangle = \langle m, f \cdot (g \otimes T(x))\rangle \\ &= \langle \phi_1(m \otimes T^*(f)), g \otimes x\rangle \quad \text{by Steps 1 and 2.} \end{aligned}$$

*Step* 4.

$$\begin{aligned} \langle \theta_1((L_T)^{**}(\Gamma))(m), f\rangle &= \langle (L_T)^{**}(\Gamma), \phi_1(m \otimes f)\rangle = \langle \Gamma, (L_T)^*(\phi_1(m \otimes f))\rangle \\ &= \langle \Gamma, \phi_1(m \otimes T^*(f))\rangle \quad \text{by Step 3} \\ &= \langle \theta_1(\Gamma)(m), T^*(f)\rangle = \langle \mathrm{Id}_{E^{**}}(m), T^*(f)\rangle \\ &= \langle m, T^*(f)\rangle = \langle T^{**}(m), f\rangle. \quad \blacksquare \end{aligned}$$

COROLLARY 7.4. *Let $E$ be a Banach space such that $E^*$ has the bounded approximation property. Then $A(E)$ has the left strong dual factorization property if and only if $I(E^*) = N(E^*)$.*

*Proof.* Since $E^*$ has the bounded approximation property, $A(E)$ has a BAI. If $A(E)$ has the left strong dual factorization property, then by Theorem 5.4 and Lemma 7.3, we have

$$SZ_1(A(E)^{**}) = LM(A(E)) = B(E)^{aa}.$$

In particular, $\mathrm{Id}^* \circ S \in N(E^*)$ for all $S \in I(E^*)$ by Corollary 6.4, which gives $I(E^*) = N(E^*)$. The converse is obtained by combining Theorem 7.2, Lemma 7.3 and Theorem 5.4. ∎

The previous result is a generalization of Example 2.5 in [15], where A.T.-M. Lau and A. Ülger proved that $A(c_0)$ has the left strong dual factorization property. Their proof uses the fact that $l_1$ has the Radon–Nikodym property to conclude that $I(l_1) = N(l_1)$, which is a sufficient and necessary condition by Corollary 7.4.

THEOREM 7.5. *Let $E$ be a Banach space. Then $A(E)$ has the left weak dual factorization property if and only if $I(E^*) = \overline{N(E^*)}^{\|\cdot\|_I}$.*

*Proof.* Let $U \in A(E)$ and $T \in I(E^*)$. Then $T \square U = U^* \circ T \in N(E^*)$ by Theorem 2.4.

Suppose that $A^* = \overline{\langle A^* A \rangle}$. Then

$$I(E^*) = \overline{\langle I(E^*) \square A(E) \rangle}^{\|\cdot\|_I} \subseteq \overline{N(E^*)}^{\|\cdot\|_I} \subseteq I(E^*).$$

To prove the converse, suppose that $I(E^*) = \overline{N(E^*)}^{\|\cdot\|_I}$. By the density of $\mathcal{F}(E^*)$ in $N(E^*)$ and $\|\cdot\|_I \leqslant \|\cdot\|_N$, we have that $I(E^*) = \overline{\mathcal{F}(E^*)}^{\|\cdot\|_I}$. The equality

$$F(E^*) = F(E^*) \square F(E)$$

holds since $(n \otimes f) \square (g \otimes x) = f(x)(n \otimes g)$ for $n \in E^{**}$, $f, g \in E^*$ and $x \in E$. Hence,

$$\begin{aligned} I(E^*) = \overline{\mathcal{F}(E^*)} = \overline{\mathcal{F}(E^*) \square \mathcal{F}(E)}^{\|\cdot\|_I} &\subseteq \overline{I(E^*) \square A(E)}^{\|\cdot\|_I} \\ &\subseteq \overline{\langle I(E^*) \square A(E) \rangle}^{\|\cdot\|_I} \subseteq \overline{N(E^*)}^{\|\cdot\|_I} = I(E^*). \end{aligned}$$ ∎

Under the assumption that $E^*$ has the bounded approximation property, Theorem 7.5 gives us directly that $A(E)$ has the left strong dual factorization property if and only if $I(E^*) = \overline{N(E^*)}^{\|\cdot\|_I}$. This is less than Corollary 7.4, where we obtained that every integral operator is nuclear.

Theorem 7.5 allows us to give the first non-reflexive Banach algebra with the weak dual factorization property.

EXAMPLE 7.6. In [8], the authors construct a Banach space $Z$ such that $Z$ has the approximation property but not the bounded approximation property, thus $Z^*$ also does not have the bounded approximation property. Consequently, $A(Z)$ does not have bounded approximation identities. Moreover, $Z$ can be such that $Z^*$ is separable. In particular, $Z^*$ has the Radon–Nikodym property. Thus, $I(Z^*) = N(Z^*)$. Theorem 7.5 gives us directly that $A(Z)$ has the weak dual factorization property.

## 8. OPEN PROBLEMS

Let $A$ be a Banach algebra. Concerning the problems tackled in this paper a certain number of questions remain unsolved. Here we have collected some of them in the form of remarks and questions.

(1) The only concrete examples of Banach algebras with the dual factorization property are reflexive. Of course, Theorem 7.5 gives a condition on a Banach space $E$ such that $A(E)$ has the weak dual factorization property. However, we do not know any Banach space $E$ such that $N(E^*)$ is a proper subspace of $I(E^*)$ and $I(E^*) = \overline{N(E^*)}^{\|\cdot\|_I}$.

(2) Is there a Banach algebra with the weak dual factorization property, but without the dual factorization property? An affirmative answer to (2) would yield an affirmative answer to (1) as well.

(3) Does the strong dual factorization property of a Banach algebra $A$ imply the existence of a BAI ?

We would like to mention here that, so far, we only have that the norm of $A$ must be equivalent to $RM(A)$-norm. However, this is not enough to ensure the existence of a BAI, as shown by Willis in [30]. One can easily prove that the strong dual factorization property of $A(E)$ implies that $I(E^*) = N(E^*)$ using a similar argument as in proof of Theorem 7.5. However, we do not know if this is sufficient.

(4) If the answer to (3) is "no", what are the conditions on $A$ to get a positive answer?

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